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GLOBAL STRUCTURE OF WEBS IN CODIMENSION ONE

(état du 17/03/2008)

Vincent Cavalier and Daniel Lehmann

Abstract

In this paper, we describe the global structure of webs in codimension one, and study in particular their singularities (the *caustic*). We define and study different concepts which have no interest locally near regular points, such as the *type*, the *reducibility*, the *quasi-smoothness*, the *CI-property* (complete intersection), the *dicriticality*.... As an example of application, we explain how the algebraicity of a global holomorphic web in codimension one on the complex n -dimensional projective space \mathbb{P}_n (algebraicity that we prove by the way to be equivalent to the linearity) depends only on the behaviour of this web near its caustic, at least for quasi-smooth webs with CI irreducible components.

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Key words: Global webs, caustics, quasi-smoothness, dicriticality, linearizability, algebraicity.

Global structure of webs in codimension one

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1 Introduction

In this paper, we want for instance to explain how the algebraicity of a global holomorphic web in codimension one on the complex n -dimensional projective space \mathbb{P}_n reflects on the behaviour of this web near its singularities, and -in some cases- can be readen on this behaviour, generalizing a theorem already proved in [CL1] for $n = 2$. By the way, we shall need to describe the global structure of webs in codimension one, which may be interesting in itself. We shall define in particular the *quasi-smoothness* and the *dicriticality*, both concepts which can be readen on the singular part of the web (its *caustic*). In fact, we shall see easily that any linear web on \mathbb{P}_n is dicritical (in fact, any globally defined linear web is algebraic). Conversely, at least for webs whose any irreducible component is a *complete intersection* (CI), we shall prove that the quasi-smoothness (a generically satisfied condition) and the dicriticality (generically not satisfied) imply the algebraicity. The interest of such a result is to provide a situation where properties of the web on the caustic imply properties everywhere else.

The general method will be to define a web of codimension one on a n -dimensional manifold M by the data of a n -dimensional subvariety W of the manifold \tilde{M} of the contact elements of M , the tautological contact form on \tilde{M} being integrable on the regular part W' of W , and inducing consequently a foliation $\tilde{\mathcal{F}}$ of codimension one. This foliation is nothing else but a kind of desingularization (“decrossing”) of the web: above the regular part M_0 of the web (the points of M where the leaves of the local foliations defined by the web are not tangent), the leaves of $\tilde{\mathcal{F}}$ project locally onto the leaves of the local foliations in M , the part W_0 of W above M_0 being simply a covering space of M_0 . But we shall be also interested by looking what happens above the part Γ_W of W which projects onto the caustic $M \setminus M_0$. The quasi-smoothness means that any irreducible component C of W is smooth ($C' = C$), so that we may use on C the general tools for foliations on smooth manifold, and in particular the vanishing theorem of Bott : dicriticality means in fact that the foliation $\tilde{\mathcal{F}}|_C$ induced by $\tilde{\mathcal{F}}$ on C has no singularity ; in this case, the square $(c_1)^2(N(\tilde{\mathcal{F}}|_C))$ of the Chern class of the bundle $N(\tilde{\mathcal{F}}|_C)$ normal to the foliation $\tilde{\mathcal{F}}|_C$, must vanish. In particular, when $M = \mathbb{P}_n$, this will imply that all partial degrees δ_α of the CI-web are necessarily zero : this characterizes algebraic CI-webs.

Let us be now more precise. Let d be an integer ≥ 1 . Locally, on an open set U of \mathbb{C}^n with coordinates $x = (x_\lambda)_\lambda$, a codimensional one holomorphic d -web on U is a family of d one codimensional holomorphic foliations \mathcal{F}_i , ($1 \leq i \leq d$), on U . The regular part of the web is the open subset U_0 of U where these foliations are all non-singular and where their leaves are not tangent. We shall not require in this definition of U_0 that any k of these d foliations are in general position for $k \leq n$. Each foliation \mathcal{F}_i may be given by a holomorphic 1-form $\omega_i = \sum_{\lambda=1}^n a_i^\lambda(x) dx_\lambda$ which is integrable ($\omega_i \wedge d\omega_i \equiv 0$), defined up to multiplication by a unit (holomorphic nowhere-vanishing function u), and whose kernel is the Lie algebra of vector fields tangent to \mathcal{F}_i . If we assume $a_i^n(x) \neq 0$, the leaves of \mathcal{F}_i will be the hypersurfaces which are the graphs $x_n = \varphi(x_1, \dots, x_{n-1})$ of the common solutions φ to the $n - 1$ partial differential equations $a_i^\alpha(x) + a_i^n(x) \frac{\partial \varphi}{\partial x_\alpha} = 0$, ($1 \leq \alpha \leq n - 1$).

We can still say that a local d -web is defined by the union $W = \bigcup_{i=1}^d U_i$ of the n -dimensional U_i 's, each U_i being defined by the $n - 1$ equations $a_i^\alpha(x) + a_i^n(x) p_\alpha = 0$ ($1 \leq \alpha \leq n - 1$) in $U \times \mathbb{C}^{n-1}$ with coordinates $(x = (x_\lambda), p = (p_\alpha))$, ($1 \leq \lambda \leq n$, $1 \leq \alpha \leq n - 1$). Observe that the local contact form $dx_n - \sum_\alpha p_\alpha dx_\alpha$ has a restriction to W which is integrable, and that the foliation $\tilde{\mathcal{F}}$ defined

by it on W has a restriction to U_i which projects onto \mathcal{F}_i by the map $(x, p) \mapsto x$. Of course, this transcription in $U \times \mathbb{C}^{n-1}$ may seem pedantic and needless while we remain locally on U . But when we wish to define a d -web globally on a holomorphic n -dimensional manifold M which is no more necessarily equal to an open set U of \mathbb{C}^n , it may happen, as already observed in the case $n = 2$ (see [CL1]), that the local foliations \mathcal{F}_i are not globally distinguishable. Thus, it will be useful to handle them all together, (x, p) becoming local coordinates on the manifold \widetilde{M} of contact elements of M , and W a n -dimensional subvariety of \widetilde{M} , on the smooth part W' of which the tautological contact form of \widetilde{M} becomes integrable, $W_0 \rightarrow M_0$ becoming a d -fold covering space which is no more necessarily trivial.

In the next section 2, we recall some basic facts, more or less known, on the $2n - 1$ -dimensional manifold \widetilde{M} of contact elements in M .

In section 3, we give the general definition of a global d -web on M as a n dimensional subvariety W of \widetilde{M} , globalizing the previous considerations, and precisizing the *critical set* Γ_W and its projection onto M which is the caustic. Various concepts such as the *type* of a d -web, the reducibility (decomposition of the web as the juxtaposition of a finite number of global “irreducible” webs), the “smoothness” and the “quasi-smoothness”, the “dicriticality”, etc... are defined.

After defining a global PDE in section 4 as a particular hypersurface S in \widetilde{M} , we study in section 5 the particular case of the so-called *CI-webs* (when W is the complete intersection $W = \cap_{\alpha=1}^{n-1} S_\alpha$ of $n - 1$ distinct global PDE's S_α), and *LCI-webs* (locally complete intersections). Many calculations are much easier in these cases. In particular, after defining the *critical scheme*, whose underlying analytical set is Γ_W , we give a practical criterium in this case for a web to be dicritical.

In section 6, we study some properties specific to the case $M = \mathbb{P}_n$. Let \mathbb{P}'_n be the dual projective space of hyperplanes in \mathbb{P}_n . Both $\widetilde{\mathbb{P}}_n$ and $\widetilde{\mathbb{P}}'_n$ are equal to the sub-manifold of $\mathbb{P}_n \times \mathbb{P}'_n$ whose points are the pairs (m, H) of a point $m \in \mathbb{P}_n$ and a hyperplane H such that $m \in H$. An interesting duality exists between webs on \mathbb{P}_n and on \mathbb{P}'_n . In particular, some sub-variety W 's of $\widetilde{\mathbb{P}}_n$ may define simultaneously a non-dicritical web on \mathbb{P}_n and a non-dicritical web on \mathbb{P}'_n (the so called *bi-webs*). We shall study also the degree of a web and the multi-degree of a CI-web: in particular, the CI-webs which are algebraic are those of muti-degree 0. Their leaves are also the hyperplanes tangent to an algebraic developable hypersurface (hypersurface equal to the union of a one-parameter family of $(n - 2)$ -dimensional projective subspaces of \mathbb{P}_n , along which the tangent hyperplane remains the same). The *linear* webs are those whose all leaves are hyperplanes of \mathbb{P}_n : they all are dicritical. The cohomology of $\widetilde{\mathbb{P}}_n$ is then computed.

We have now all the needed tools for giving in section 7 the properties of the dicritical webs on \mathbb{P}_n .

In a second part of our study which will be published somewhere else (see preprint [CL2]), we focus on abelian relations. We study there the webs satisfying to a condition which is generically satisfied, the so called *regularity*:

- the rank of such a regular web at a non-singular point (i.e. the dimension of the vector space of germs of abelian relations at that point) is upper-bounded by some number $\pi'(n, d)$ which, for $n \geq 3$, is strictly smaller than the number $\pi(n, d)$ of Castelnuovo (the maximal geometrical genus of an algebraic irreducible and non-degenerate curve of degree d in \mathbb{P}_n , which Chern proved to be the upper-bound of the rank in the general case ([C]));

- we define on M_0 , when d is equal for some r to the the dimension $c(n, r)$ of the vector space of all homogeneous polynomials of degree r in n variables with scalar coefficients, a generalization of the Blaschke curvature; this is the curvature of some connection. The non-vanishing of this curvature is an obstruction for the rank of the web to reach the maximal value $\pi'(n, d)$.

[In the case $n = 2$, any web is regular, any d may be written $c(2, d - 1)$, the numbers $\pi(2, d)$ and $\pi'(2, d)$ are equal; we then recover, globalizing them on M_0 , the Blaschke curvature defined locally in [H] for $n = 2$, $d \geq 3$, generalizing the case $n = 2, d = 3$ of Blaschke ([B]).]

2 Background on the manifold of contact elements

Let M be a holomorphic (non-singular) manifold, not necessarily compact. Denote by n its complex dimension, TM its complex tangent bundle, and \widetilde{M} the $2n-1$ -dimensional manifold equal to the total space of the grassmanian bundle $\mathbb{G}_{n-1}M \xrightarrow{\pi} M$ (a point of \widetilde{M} over a point $m \in M$ is a $n-1$ -dimensional sub-vector space of T_mM , and is called a *contact element* of M at $m = \pi(\widetilde{m})$).

In the sequel, indices such as λ, μ, \dots will denote integers runing from 1 to n , while α, β, \dots will denote integers runing from 1 to $n-1$.

For all family $v = (v_\alpha)$ of $n-1$ vectors linearly independant in T_mM , $[v]$ or $[v_1, \dots, v_{n-1}]$ or $[(v_\alpha)_\alpha]$ will denote the contact element generated by v in \widetilde{M}_m .

Since any \widetilde{m} over m is the kernel of a non-vanishing 1-form on T_mM well defined up to multiplication by a scalar unit, we can also identify \widetilde{M} with the total space of $\mathbb{P}T^*M \rightarrow M$ (the projectivisation of the complex cotangent bundle T^*M).

2.1 Canonical bundles and tautological contact form on \widetilde{M} :

Let $\mathcal{T} \subset \pi^{-1}(TM)$ be the tautological vector-bundle over \widetilde{M} (whose fiber over $[v_1, \dots, v_{n-1}]$ is the sub-vector space of T_mM generated by (v_1, \dots, v_{n-1}) , identified with the line in T_m^*M of all 1-forms vanishing on the v_α 's ($1 \leq \alpha \leq n-1$)).

Lemma 2.1 *The dual \mathcal{L}^* of the quotient bundle $\mathcal{L} = \pi^{-1}(TM)/\mathcal{T}$ is the tautological complex line-bundle of $\mathbb{P}(T^*M)$.*

The proof is obvious.

Let (*) and (**) the two obvious exact sequences of vector bundles

$$(*) \quad 0 \rightarrow \mathcal{T} \rightarrow \pi^{-1}(TM) \rightarrow \mathcal{L} \rightarrow 0.$$

and

$$(**) \quad 0 \rightarrow \mathcal{V} \rightarrow T\widetilde{M} \rightarrow \pi^{-1}(TM) \rightarrow 0,$$

where \mathcal{V} denotes the sub-bundle of tangent vectors to \widetilde{M} which are "vertical" (i.e. tangent to the fiber of π).

By composition of the projection $T\widetilde{M} \rightarrow \pi^{-1}(TM)$ of (**) with the projection $\pi^{-1}(TM) \rightarrow \mathcal{L}$ of (*), we get a canonical holomorphic 1-form on \widetilde{M} , with coefficients in \mathcal{L}

$$\omega : T\widetilde{M} \rightarrow \mathcal{L},$$

which is called *the tautological contact form*. This terminology will be justified below by lemma 2-3.

Remark 2.2 *We imbed \mathcal{L}^* into $\pi^{-1}(T^*M)$ by duality of (*), and $\pi^{-1}(T^*M)$ into $T^*\widetilde{M}$ by duality of (**). We may also imbed \mathcal{T} into $T\widetilde{M}$, when we identify it to the kernel of ω .*

2.2 Local coordinates on \widetilde{M}

Let $x = (x_1, x_2, \dots, x_n)$ be local holomorphic coordinates on an open set U of M , and m a point of U . Let $u = (u_1, \dots, u_n)$ denote the coordinates in T_m^*M with respect to the basis $(dx_1)_m, \dots, (dx_n)_m$; let $[u] = [u_1, \dots, u_n]$ denote the point in $\mathbb{P}T_m^*M$ of homogeneous coordinates u with respect to the same basis in T_m^*M , and let $p = (p_1, p_2, \dots, p_{n-1})$ be the system of affine coordinates on the affine subspace $u_n \neq 0$ of $\mathbb{P}T_m^*M$ defined by $p_\alpha = -\frac{u_\alpha}{u_n}$ ($1 \leq \alpha \leq n-1$).

Therefore, $(x, p) = (x_1, x_2, \dots, x_n, p_1, p_2, \dots, p_{n-1})$ is a system of local holomorphic coordinates on the open set \tilde{U} of contact elements above U which are not parallel to $(\frac{\partial}{\partial x_n})$: the point of coordinates (x, p) is the $(n-1)$ -dimensional subspace of T_n^*M which is the kernel of the 1-form $\eta = (dx_n)_m - \sum_{\alpha=1}^{n-1} p_\alpha (dx_\alpha)_m$; it is generated by the the vectors $(X_\alpha)_m = \left(\frac{\partial}{\partial x_\alpha}\right)_m + p_\alpha \left(\frac{\partial}{\partial x_n}\right)_m$, $(1 \leq \alpha \leq n-1)$.

Obviously, the vector fields $X_\alpha = \left(\frac{\partial}{\partial x_\alpha}\right) + p_\alpha \left(\frac{\partial}{\partial x_n}\right)$, $(1 \leq \alpha \leq n-1)$, define a holomorphic local trivialization $\sigma_{\mathcal{T}}$ of \mathcal{T} above \tilde{U} , while the image $\sigma_{\mathcal{L}} = \left[\frac{\partial}{\partial x_n}\right]$ of $\frac{\partial}{\partial x_n}$ by the projection $\pi^{-1}(TM) \rightarrow \mathcal{L}$ of $(*)$ defines a holomorphic local trivialization of \mathcal{L} .

Remark 2.3 *By permutation of the order of the coordinates x_i , such that the new last one be no more the former last one (writing for example $x'_n = x_1$, $x'_1 = x_n$ and $x'_\lambda = x_\lambda$ for $\lambda \neq 1, n$), we get a new system of local coordinates on some open set \tilde{U}' containing contact elements parallel to $(\frac{\partial}{\partial x_n})_m$.*

Lemma 2.4

(i) *The local form $\eta = dx_n - \sum_{\alpha=1}^{n-1} p_\alpha dx_\alpha$ is a contact form on \tilde{U} , and defines a local holomorphic trivialization of \mathcal{L}^* . Moreover, the trivializations η and $\sigma_{\mathcal{L}} = \left[\frac{\partial}{\partial x_n}\right]$ are dual to each other.*

(ii) *The 1-form $\eta \otimes \sigma_{\mathcal{L}}$ is equal to the restriction of ω to \tilde{U} .*

Proof : Both forms $\eta \otimes \sigma_{\mathcal{L}}$ and ω vanish when applied to the vertical vectors and to the vectors $\frac{\partial}{\partial x_\alpha} + p_\alpha \frac{\partial}{\partial x_n}$ (with $\alpha \leq n-1$), and take the value $\sigma_{\mathcal{L}}$ when applied to $\frac{\partial}{\partial x_n}$, hence are equal. Moreover η belongs to the dual \mathcal{L}^* (identified to the set of 1-forms on \tilde{M} vanishing on \mathcal{T}). Since $\eta(\frac{\partial}{\partial x_n}) \equiv 1$, $\sigma_{\mathcal{L}}$ and η are dual to each other. We check easily that $\eta \wedge (d\eta)^{n-1}$ is a volume form.

QED

2.3 Change of local coordinates and transition functions

Let (x', p') (or (x', u')) be the local coordinates associated to $x' = (x'_1, \dots, x'_n)$ above some open set U' of M such that $U \cap U' \neq \emptyset$. Let J (or $J(x, x')$ in case of ambiguity) be the jacobian matrix $J = \frac{D(x'_1, \dots, x'_n)}{D(x_1, \dots, x_n)}$ with coefficients $J_\mu^\lambda = \frac{\partial x'_\lambda}{\partial x_\mu}$. Denote by Δ_n^M (or $\Delta_n^M(x, x')$) its determinant, and let K (or $K(x, x')$) be the matrix $K = \Delta_n^M \cdot J^{-1}$, with coefficients $K_\mu^\lambda = \Delta_n^M \cdot \frac{\partial x_\lambda}{\partial x'_\mu}$. In the sequel $u = (u_1, u_2, \dots, u_n)$ will also be understood as a line-matrix $(u_1 \ u_2 \ \dots \ u_n)$.

Proposition 2.5

(i) *The coordinates u and u' are related by the formulae $u' = (1/\Delta_n^M) u \circ K$ and $u = u' \circ J$, while for homogeneous coordinates $[u']$ (resp. $[u]$), defined up to multiplication by a non-zero scalar, we may write $[u'] = [u \circ K]$ (resp. $[u] = [u' \circ J]$). The following formulae hold :*

$$p'_\alpha = - \frac{\sum_\beta K_\beta^\alpha p_\beta - K_n^\alpha}{\sum_\beta K_\beta^n p_\beta - K_n^n} \text{ and } p_\alpha = - \frac{\sum_\beta J_\beta^\alpha p'_\beta - J_n^\alpha}{\sum_\beta J_\beta^n p'_\beta - J_n^n}.$$

(ii) *The expressions $\Delta_{n-1}^T(x, x') = -(\sum_\beta K_\beta^n p_\beta - K_n^n)$ constitute a system of transition functions for the bundle $\bigwedge^{n-1} \mathcal{T}$:*

$$X_1 \wedge \dots \wedge X_{n-1} = \Delta_{n-1}^T(x, x') (X'_1 \wedge \dots \wedge X'_{n-1}).$$

(iii) *Let $\eta' = dx'_n - \sum_\alpha p'_\alpha dx'_\alpha$. The following formula holds :*

$$\eta' = (\Delta_n^M(x, x') / \Delta_{n-1}^T(x, x')) \eta,$$

and the functions $\Delta_n^M / \Delta_{n-1}^T$ constitute a system of transition functions for \mathcal{L} :

Proof :

In fact the formula $u = u' \circ J$ is equivalent to the definition of the jacobian matrix. Interchanging x and x' , we get $u' = (u \circ J^{-1})$; but, at the level of the projective space J^{-1} and K induce the same automorphism, and $[u']$ is also equal to $[u \circ K]$.

We then deduce the following formula, by writing $u_n = u'_n = -1$, $u_\alpha = p_\alpha$ and $u'_\alpha = p'_\alpha$. Interchanging J and J^{-1} , we get also the p_α 's in function of the p'_α 's.

Let Δ_{n-1}^T be a transition function for the bundle $\bigwedge^{n-1} \mathcal{T}$. From the exact sequence (*), we deduce the isomorphism $\mathcal{L} \cong \bigwedge^n \pi^{-1}(TM) \otimes \bigwedge^{n-1} \mathcal{T}^*$. Therefore, $\Delta_n^M / \Delta_{n-1}^T$ is a system of transition functions for \mathcal{L} .

Moreover, since both 1-forms η and η' are colinear, we deduce the equality

$$\eta = \frac{-1}{\Delta_n^M} \left(\sum_{\beta} K_{\beta}^n p_{\beta} - K_n^n \right) \eta'$$

from the equality $\Delta_n^M . dx_{\lambda} = \sum_{\mu} K_{\mu}^{\lambda} dx'_{\mu}$. This proves that $\Delta_{n-1}^T = -(\sum_{\beta} K_{\beta}^n p_{\beta} - K_n^n)$ and achieves the proof of the proposition.

3 Webs of codimension one on a holomorphic manifold M

Let W be a sub-variety of \widetilde{M} having pure dimension n . Let $\pi_W : W \rightarrow M$ be the restriction to W of the projection $\pi : \widetilde{M} \rightarrow M$. Denote by

W' the regular part of W ,

Γ_W the set of points $\tilde{m} \in W$ where

- either W is singular,
- or the differential $d\pi_W : T_{\tilde{m}}W' \rightarrow T_{\pi(\tilde{m})}M$ is not an isomorphism.

W_0 the complementary part (included into W') of Γ_W in W ,

$M_0 = \pi(W_0)$.

Let d be an integer ≥ 1 .

Definition 3.1 *We shall say that W is a d -web if*

(o) *The map $\pi_W : W \rightarrow M$ is surjective*¹.

(i) *The restriction ω_W to W' of the tautological contact form $\omega : T\widetilde{M} \rightarrow \mathcal{L}$ satisfies to the integrability condition: this means that, given a local trivialization $\sigma_{\mathcal{L}}$ of \mathcal{L} , the restriction η_W to $W' \cap \widetilde{U}$ of the local contact form η such that $\omega = \eta \otimes \sigma_{\mathcal{L}}$ satisfies to the condition $\eta_W \wedge d\eta_W \equiv 0$ (this condition not depending on the local trivialization $\sigma_{\mathcal{L}}$).*

(ii) *The restriction $\pi_W : W_0 \rightarrow M_0$ of π_W to W_0 is a d -fold covering [The integer d will be called the **weight** of the web].*

(iii) *The analytical set Γ_W has complex dimension at most $n - 1$, or is empty.*

(iv) *For any $m \in M$, the set $(\pi_W)^{-1}(m) = W \cap (\pi)^{-1}(m)$ is an algebraic subset of degree d and dimension 0 in $\mathbb{P}T_m^*(M)$.*

The analytical set Γ_W is called the critical set of the web, its projection $\pi(\Gamma_W)$ the caustic or the singular part, and its complementary part M_0 the regular part of the web.

¹This condition is automatic when M is compact, because of (ii) and (iii).

Remark 3.2

- (i) Any d -web W on M induces a d -web $W|_U = W \cap \pi^{-1}(U)$ on any open set U of M .
- (ii) In this definition, we do not require for $d \geq n$ that any n -uple of points in a same fiber of $W_0 \rightarrow M_0$ be in general position : it may exist some point $m \in M_0$ and some n -uple $(\tilde{m}_1, \dots, \tilde{m}_n)$ of points in $(\pi_W)^{-1}(m)$ which are in a same $(n-2)$ -dimensional sub-vector space of the $(n-1)$ -dimensional vector space $(\mathbb{G}_{n-1}M)_m$.

Definition 3.3 The irreducible components C of the variety W in \tilde{M} are called the components of the web. They all are webs. The web is said to be irreducible if it has only one component. It is said to be smooth if W is smooth ($W' = W$, which implies that it is irreducible), and quasi-smooth if any irreducible component is smooth.

3.1 The foliation $\tilde{\mathcal{F}}$

The integrability condition (i) of the definition of a web implies that the distribution of vector fields on W' belonging to the kernel of $\omega_W : TW' \rightarrow \mathcal{L}$ is involutive: the integral submanifolds of this distribution defines therefore a foliation $\tilde{\mathcal{F}}$ on W' . This foliation may have singularities, but not on W_0 : in fact, the restrictions \tilde{x}_i of x_i to W_0 ($1 \leq i \leq n$) define local coordinates on every sheet of the covering $W_0 \rightarrow M_0$. With respect to these coordinates, η_{W_0} is written $d\tilde{x}_n - \sum_{\alpha=1}^{n-1} p_\alpha(\tilde{x}) d\tilde{x}_\alpha$ and does not vanish.

By projection π_W of the restriction of this foliation to W_0 , we get locally, near any point $m \in M_0$, d distinct one-codimensional regular foliations \mathcal{F}_i , and $\tilde{\mathcal{F}}$ may be understood as a “decrossing” of these d foliations.

Definition 3.4 We call leaf of the web any hypersurface in M whose intersection with M_0 is locally a leaf of one of the local foliations \mathcal{F}_i .

Conversely, given d distinct regular foliations \mathcal{F}_i ($1 \leq i \leq d$) of codimension one on some open set U of M , assume moreover that these foliations are mutually transverse to each other at any point of U , and that there exists holomorphic coordinates (x_1, \dots, x_n) on U such that $\frac{\partial}{\partial x_n}$ be tangent to none of the foliations $\tilde{\mathcal{F}}_i$: we may define \mathcal{F}_i by an integrable non-vanishing 1-form $\eta_i = dx_n - \sum_{\alpha=1}^{n-1} p_\alpha^i(x) dx_\alpha$. The sub-manifold U_i of \tilde{M} defined by the $n-1$ equations $p_\alpha = p_\alpha^i(x)$ ($1 \leq \alpha \leq n-1$) does not depend on the choice of the local coordinates. The union $W_U = \coprod_{i=1}^d U_i$ is then a d -fold trivial covering space of U , and defines a d -web (everywhere regular, with no critical set). Denoting by $\pi_i : U_i \rightarrow U$ the restriction of π to the sheet U_i , the form $(\pi_i)^*(\eta_i)$ is equal to the restriction of $d\tilde{x}_n - \sum_{\alpha=1}^{n-1} p_\alpha(\tilde{x}) d\tilde{x}_\alpha$ to U_i .

3.2 Dicriticality :

Since TW_0 is naturally isomorphic to $(\pi_W)^{-1}(TM_0)$, the natural injection $\mathcal{T} \rightarrow (\pi)^{-1}(TM)$ defines an obvious injective map $\mathcal{T}|_{W_0} \rightarrow TW_0$, and the image of this map is the tangent bundle to the foliation $\tilde{\mathcal{F}}|_{W_0}$ since it is annihilated by ω_W : the foliation $\tilde{\mathcal{F}}$ is non-singular on W_0 .

Let now C be a irreducible component of W and C' its regular part. In general, $\tilde{\mathcal{F}}$ will have singularities on C' .

Definition 3.5 We shall say that the web is dicritical on C if the restriction of the foliation $\tilde{\mathcal{F}}$ to C' is non-singular. We shall say that the web is dicritical if it is dicritical on every irreducible component.

The dicriticality on C is equivalent to the possibility of extending $\mathcal{T}|_{W_0} \rightarrow TW_0$ as an injective morphism $\mathcal{T}|_{C'} \rightarrow TC'$. We shall give below a practical criterium for that, in the case of LCI-webs.

We shall see also that any totally linearizable web on a manifold M (for example any linear web on \mathbb{P}_n) is dicritical.

3.3 Irreducibility of webs and indistinguishability of the foliations :

The results of this subsection will not be used below.

Let's consider

- the number $N_1 (\geq 1)$ of irreducible components of W ,
- the number $N_2 (\geq 1)$ of connected components of W_0 ,
- the number $N_3 (\geq 0)$ of global distinct foliations on M_0 whose leaves coincide locally near a point with a leaf of one of the d local foliations \mathcal{F}_i near this point.

It is clear that $N_3 \leq N_2$. [We may have $N_3 < N_2$; consider for instance, the 2-web on $M = \mathbb{P}_2$ whose leaves are the tangent lines to a given proper conic X . Then, the caustic is X , and the 2-fold covering $W_0 \rightarrow M_0$ is not trivial ; however, $N_3 = 0$, while $N_1 = N_2 = 1$].

In particular, when M_0 is connected, the d -foliations \mathcal{F}_i are globally distinguishable (i.e. $N_3 = d$) iff the d -fold covering $W_0 \rightarrow M_0$ is trivial (i.e. $N_2 = d$).

On the other hand, $N_2 \geq N_1$, and we can prove in fact that $N_2 = N_1$ if W is compact and quasi-smooth. (see [CL1] in the case $n = 2$).

4 Global first order homogeneous polynomial partial differential equations

In view of the the next section where we shall study the CI-webs whose solutions satisfy to $n-1$ distinct global first order homogeneous polynomial partial differential equations, we want first to precise in this section what is a global first order homogeneous polynomial PDE (sometimes, we shall say in short PDE in the sequel of this paper).

Above some open set U of \mathbb{C}^n , a local first order polynomial PDE homogeneous of degree d is a differential operator D which is a holomorphic section of $S^d(TU)$ (the d^{th} -symetric power of the tangent space TU), which can always be written

$$D = \sum_{|I|=d} A_I(x) \partial_I,$$

where $I = (i_1, \dots, i_n)$ denotes a multi-index of non-negative integers, $|I| = i_1 + \dots + i_n$, the coefficients A_I are holomorphic functions of x , and ∂_I denotes the symmetric product $\prod_{\lambda} \left(\frac{\partial}{\partial x_{\lambda}} \right)^{i_{\lambda}}$.

Solutions of this PDE are hypersurfaces Σ of equation $f(x_1, \dots, x_n) = 0$ in U , such that $Df = 0$, i.e.:

$$\sum_{|I|=d} A_I(x) \cdot \partial_I f = 0.$$

For avoiding extra-solutions defined by the vanishing of a common factor to all of the coefficients A_I , we shall also assume that, at any point of U , the germs of the A_I 's are relatively prime.

In particular, if Σ is a graph-hypersurface of equation $\varphi(x_1, \dots, x_{n-1}) - x_n = 0$, φ is solution of the equation

$$F\left(x_1, \dots, x_{n-1}, \varphi(x_1, \dots, x_{n-1}), \frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_{n-1}}\right) = 0,$$

where $F : \tilde{U} \rightarrow \mathbb{C}$ is the holomorphic function of $2n - 1$ variables $(x_1, \dots, x_{n-1}, x_n, p_1, \dots, p_{n-1})$ defined as

$$F(x, p) = \sum_{|I|=d} (-i)^{i_n} A_I(x) p^{\hat{I}},$$

where $\hat{I} = (i_1, \dots, i_{n-1})$ will be written shortly $\hat{I} = I \setminus \{i_n\}$, and $p^{\hat{I}} = (p_1)^{i_1} \dots (p_{n-1})^{i_{n-1}}$.

Setting

$$\begin{aligned} \tilde{F}(x, u_1, \dots, u_{n-1}, u_n) &= (-u_n)^d F(x, -\frac{u_1}{u_n}, \dots, -\frac{u_{n-1}}{u_n}), \\ &= \sum_{|I|=d} A_I(x) \cdot u^I, \text{ where } u^I = \prod_{\lambda} (u_{\lambda})^{i_{\lambda}}, \end{aligned}$$

we still can write

$$F(x, p) = \tilde{F}(x, p_1, \dots, p_{n-1}, -1).$$

Let S be the hypersurface of equation $F(x, p) = 0$ in \tilde{U} . We shall assume that this equation is reduced. Hence, if $G(x, p) = 0$ is another equation of S , obtained from another differential operator D_1 , there is a unique unit $\rho : U \rightarrow \mathbb{C}^*$, such that $D_1 = \rho D$. The two equations $Df = 0$ and $D_1f = 0$ having the same solutions, we are mainly interested by the surface S .

If we get G and \tilde{G} from G associated to $D_1 = \rho D$ when using other local coordinates (x') on another open set U' , we must have $\tilde{G}(x', u') = \rho(x) \tilde{F}(x, u)$ above $U \cap U'$. Since $\frac{u'_n}{u_n} = \frac{\Delta_{n-1}^T(x, x')}{\Delta_n^M(x, x')}$ after proposition 2-5, we get:

Lemma 4.1 *There exists some unit $\rho : U \cap U' \rightarrow \mathbb{C}^*$, uniquely defined, such that*

$$G(x', p') = \left(\frac{\Delta_{n-1}^T(x, x')}{\Delta_n^M(x, x')} \right)^{-d} \cdot \rho(x) \cdot F(x, p).$$

We shall therefore define a global first order homogeneous polynomial PDE of degree d on a holomorphic n -dimensional manifold M by the data of a family (U_a, D_a) such that:

- the U_a 's are open sets in M making a covering of M ,
- for any a , D_a is a local homogeneous polynomial PDE's of degree d over U_a ,
- for any pair (a, b) such that $U_a \cap U_b$ be not empty, there exists a (necessarily unique) unit $\rho_{ab} : U_a \cap U_b \rightarrow \mathbb{C}^*$ such that $D_b = \rho_{ab} D_a$.

Because of their uniqueness, the functions ρ_{ab} satisfy to the cocycle condition, and are therefore a system of transition functions for a holomorphic line-bundle E over M . Hence, the sections D_a glue together to define a holomorphic section D of the vector bundle $E \otimes S^d(TM)$ over M .

Let $(^a x$ be a system of holomorphic local coordinates on U_a . From the previous lemma, we deduce that the corresponding functions F_a and F_b are related on $\tilde{U}_a \cap \tilde{U}_b$ by the formula

$$F_b = \left(\frac{\Delta_{n-1}^T(^a x, ^b x)}{\Delta_n^M(^a x, ^b x)} \right)^{-d} \cdot \rho_{ab} \cdot F_a.$$

Hence, the functions F_a glue together, defining a section s of the bundle $\pi^{-1}(E) \otimes \mathcal{L}^d$ over \tilde{M} . This justifies the following

Definition 4.2 *A holomorphic line-bundle E on M and an integer $d \geq 1$ being given, a global first order polynomial PDE homogeneous of degree d and type E on the holomorphic n -dimensional manifold M is a holomorphic section D of $E \otimes S^d(TM)$, the corresponding local equations $F = 0$ being reduced, and having coefficients $A_I(x)$ not depending² on the coordinates p , and having germs at any point relatively prime (these conditions not depending on the local holomorphic coordinates on M , and on a local holomorphic trivialization σ_E of E). The integer d is also called the weight of the PDE.*

Remark 4.3 *It is equivalent to give S , or (up to multiplication by a global unit on M , i.e. a constant if M is compact) the section s , or the differential operator D with coefficients in E .*

²Notice that any section of $\pi^{-1}(E) \otimes \mathcal{L}^d$ on \tilde{U} may be written locally $\sum_{|I|=d} A_I(x, p) p^I (\sigma_{\mathcal{L}})^d \otimes \pi^{-1}(\sigma_E)$, once given a local trivialization σ_E of E and local coordinates, with the convention $p_n = -1$. Their coefficients A_I depend in general on p , but here they don't.

4.1 Linearizability and co-critical set of a polynomial PDE

Let (D, s, S) be a global polynomial PDE on M as above.

Definition 4.4 *This PDE is said to be linearizable if it satisfies to the following assumption :*

For any point $\tilde{m}_0 \in S$ above a point $m_0 \in M$, there exists local coordinates $x = (x_1, \dots, x_n)$ near m_0 , such that if \tilde{m}_0 has coordinates $(x_0, p_0) = (x_1^0, \dots, x_n^0, p_1^0, \dots, p_{n-1}^0)$, then the hypersurface of equation

$$x_n - x_n^0 = \sum_{\alpha} p_{\alpha}^0 (x_{\alpha} - x_{\alpha}^0)$$

is a solution of the PDE.

Definition 4.5 *We call co-critical set of the previous PDE the analytical set Δ_S of points $\tilde{m} \in S$ where either S is singular, or $\mathcal{T}_{\tilde{m}}$ (seen as a subspace of $T_{\tilde{m}}\tilde{M}$) is included into $T_{\tilde{m}}S$. Locally, Δ_S is defined by the equations $F = 0$ and $\frac{\partial F}{\partial x_{\alpha}} + p_{\alpha} \frac{\partial F}{\partial x_n} = 0$ for any α .*

Proposition 4.6 *Assume a global polynomial PDE to be linearizable. Then its co-critical set Δ_S is all of S . This condition may be written locally, relatively to any system of local coordinates, in the following way:*

For any α , there exists a (local) holomorphic function C_{α} such that

$$\frac{\partial F}{\partial x_{\alpha}} + p_{\alpha} \frac{\partial F}{\partial x_n} = C_{\alpha} F.$$

Proof: Assume the web on M to be locally defined by the equation $F(x, p) = 0$ and let $(x_0, p_0) = (x_1^0, \dots, x_n^0, p_1^0, \dots, p_{n-1}^0)$ be such that $F(x_0, p_0) = 0$. If the hypersurface of equation

$$x_n - x_n^0 = \sum_{\alpha} p_{\alpha}^0 (x_{\alpha} - x_{\alpha}^0)$$

is a solution of the PDE, then :

$$F\left(x_1, \dots, x_{n-1}, x_n^0 + \sum_{\alpha} p_{\alpha}^0 (x_{\alpha} - x_{\alpha}^0), p_1^0, \dots, p_{n-1}^0\right) \equiv 0.$$

By derivation of this identity, with respect to each x_{α} , we get : $\frac{\partial F}{\partial x_{\alpha}} + p_{\alpha} \frac{\partial F}{\partial x_n} = 0$ at the point (x_0, p_0) . Hence

$$\frac{\partial F}{\partial x_{\alpha}} + p_{\alpha} \frac{\partial F}{\partial x_n} \equiv 0 \text{ on } S \text{ for any } \alpha.$$

But this means precisely that the vector fields $X_{\alpha} = \frac{\partial}{\partial x_{\alpha}} + p_{\alpha} \frac{\partial}{\partial x_n}$, which generate \mathcal{T} locally are tangent to S at any point of S . Since this property has an intrinsic meaning, not depending on the local coordinates, the same equations will be satisfied when written relatively to any other system of local coordinates.

QED

Proposition 4.7 *Let (D, s, S) be a global polynomial PDE on M as above. Let ∇ be any connection on $\pi^{-1}(E) \otimes \mathcal{L}^d$, and $Ds : \mathcal{T} \rightarrow \pi^{-1}(E) \otimes \mathcal{L}^d$ the restriction of the covariant derivative $\nabla s : T\tilde{M} \rightarrow \pi^{-1}(E) \otimes \mathcal{L}^d$ to the sub-bundle \mathcal{T} of $T\tilde{M}$. Then, the morphism $Ds|_S : \mathcal{T}|_S \rightarrow \pi^{-1}(E) \otimes \mathcal{L}^d|_S$ does not depend on the connection ∇ , and the cocritical set of the PDE is exactly the set of points $\tilde{m} \in S$ where $Ds|_S$ vanishes.*

Proof : The section s may be written locally $F \cdot \sigma$ for a convenient trivialization σ of the line-bundle $\pi^{-1}(E) \otimes \mathcal{L}^d$, hence $\nabla s = dF \cdot \sigma + F \cdot \nabla \sigma$. This formula proves already that the restriction $\nabla s|_S$ of ∇s to S (where F vanishes) does not depend on ∇ . Moreover, since \mathcal{T} is generated by the vector fields $X_{\alpha} = \frac{\partial}{\partial x_{\alpha}} + p_{\alpha} \frac{\partial}{\partial x_n}$, the condition $\frac{\partial F}{\partial x_{\alpha}} + p_{\alpha} \frac{\partial F}{\partial x_n} = 0$ for any α or $Ds|_S = 0$ are equivalent.

QED

5 Complete intersection webs

These webs are those whose leaves are the solutions of a system of $n - 1$ global polynomial partial differential equations: $W = \bigcap_{\alpha=1}^{n-1} S_\alpha$.

Definition 5.1 A d -web W on M is said to be a complete intersection web (CI-web) if there exists a family $(S_\alpha)_\alpha$ of $n - 1$ global polynomial PDE's, of respective type E_α and weight d_α , such that $d = \prod_\alpha d_\alpha$, $W = \bigcap_{\alpha=1}^{n-1} S_\alpha$, the sheaf of ideals of germs of holomorphic functions vanishing on W being generated by the germs of functions vanishing on one of the S_α 's.

If s_α is a holomorphic section of $\pi^{-1}(E_\alpha) \otimes \mathcal{L}^{d_\alpha}$, such that $S_\alpha = (s_\alpha)^{-1}(0)$, s_α may be locally written $s_\alpha = F_\alpha \cdot \sigma_\alpha$, where σ_α denotes a local trivialization of E_α (holomorphic section nowhere zero), F_α denoting a holomorphic function of the shape $F_\alpha(x, p) = \sum_{|I|=d_\alpha} A_I^\alpha(x) p^I$ (with the convention $p_n = -1$), the germs of the coefficients $A_I^\alpha(x)$ being relatively primes, and the equation $F_\alpha = 0$ being reduced. Thus the ideal of germs of functions vanishing on W is generated by the F_α 's.

Definition 5.2 A d -web W on M is said to be a locally complete intersection web (LCI-web) if there exists a covering of M by open sets U_λ such that every induced web $W|_{U_\lambda}$ is a CI-web.

Remark 5.3 The property for a web to be CI (resp. LCI) is stronger than the property for the underlying analytical space W to be CI (resp. LCI) in \widetilde{M} . For instance, given d distinct points (p_i, q_i) on a proper conic in the (p, q) -plane, let \mathcal{F}_i be the regular foliation on \mathbb{C}^3 (with coordinates (x, y, z)) defined by the 1-form $dz - p_i dx - q_i dy$. If d is odd, the regular d -web \mathbb{C}^3 defined by these d foliations cannot be a CI-web, not even a LCI-web, while the corresponding manifold W is LCI since it has no singularity.

5.1 Critical scheme of a CI web :

In view of the study below of the dicriticality, we shall be interested in defining Γ_W not only as an analytical set, but as a scheme. This will be done now in the case of a CI web.

If a LCI-web W is locally defined by the $n - 1$ equations $F_\alpha(x, p) = 0$, ($1 \leq \alpha \leq n - 1$), the points of W' are those at which the jacobian matrix $\frac{D(F_1, \dots, F_{n-1})}{D(x_1, \dots, x_n, p_1, \dots, p_{n-1})}$ has rank $n - 1$, and W_0 is the subset of W' where the determinant Δ_p of $\frac{D(F_1, \dots, F_{n-1})}{D(p_1, \dots, p_{n-1})}$ does not vanish.

If we change local coordinates and trivializations of the E_α 's, we get for any α , after lemma 4-1 :

$$G_\alpha \left(x', -\frac{\sum_\beta K_\beta^1 p_\beta - K_n^1}{\sum_\beta K_\beta^n p_\beta - K_n^n}, \dots, -\frac{\sum_\beta K_\beta^{n-1} p_\beta - K_n^{n-1}}{\sum_\beta K_\beta^n p_\beta - K_n^n} \right) = \left(\frac{\Delta_{n-1}^T(x, x')}{\Delta_n^M(x, x')} \right)^{-d_\alpha} \cdot \rho_\alpha(x) \cdot F_\alpha(x, p).$$

Denoting by $\Delta'_{p'}$ the determinant of $\frac{D(G_1, \dots, G_{n-1})}{D(p'_1, \dots, p'_{n-1})}$, and by v the determinant of $\frac{D(p'_1, \dots, p'_{n-1})}{D(p_1, \dots, p_{n-1})}$, we deduce the equality

$$\Delta_p = \left(\prod_\alpha \rho_\alpha \right)^{-1} \cdot \left(\frac{\Delta_{n-1}^T(x, x')}{\Delta_n^M(x, x')} \right)^{-\sum_\alpha d_\alpha} \cdot v \cdot \Delta'_{p'} + \text{terms vanishing on } W.$$

Hence, for a CI web, the Δ_p 's glue together above W , defining a section s_Γ over W of the bundle

$$\pi^{-1}(\otimes_\alpha E_\alpha) \otimes \mathcal{L}^{\sum_\alpha d_\alpha} \otimes \bigwedge^{n-1} \mathcal{V}^*,$$

the zero set of which is Γ_W .

While the system $(F_\alpha = 0)$ of local equations for W is reduced, we cannot at all assert that the system $(F_\alpha = 0, \Delta_p = 0)$ of local equations for Γ_W has the same property.

Definition 5.4 We call critical scheme the scheme defined by the vanishing of the above section s_Γ .

5.2 Dicriticality of a CI web :

Still assuming the CI-web W locally defined by the $n - 1$ equations $F_\alpha(x, p) = 0$, we first want to precise the natural morphism $\mathcal{T}|_{W_0} \rightarrow TW_0$ defining $\tilde{\mathcal{F}}$ on W_0 : hence, we are looking for the matrix $((R_\alpha^\beta))$ such that, for any α , the vector field $\frac{\partial}{\partial x_\alpha} + p_\alpha \frac{\partial}{\partial x_n} + \sum_\beta R_\alpha^\beta \frac{\partial}{\partial p_\beta}$ be tangent to W_0 . The R_α^β 's are then given by the solution of the linear system

$$\sum_\beta R_\alpha^\beta \frac{\partial F_\gamma}{\partial p_\beta} = - \left(\frac{\partial F_\gamma}{\partial x_\alpha} + p_\alpha \frac{\partial F_\gamma}{\partial x_n} \right), \quad \text{for any } \alpha, \gamma.$$

This can still be written :

$$\frac{D(F_1, \dots, F_{n-1})}{D(p_1, \dots, p_{n-1})} \circ {}^t R = -\Theta,$$

where Θ denotes the matrix with coefficients $\Theta_\alpha^\gamma = \left(\frac{\partial F_\gamma}{\partial x_\alpha} + p_\alpha \frac{\partial F_\gamma}{\partial x_n} \right)$, hence the solution

$${}^t R = -L \circ \Theta, \quad \text{where } L = \Delta_p \cdot \left(\frac{D(F_1, \dots, F_{n-1})}{D(p_1, \dots, p_{n-1})} \right)^{-1},$$

the matrix $\frac{D(F_1, \dots, F_{n-1})}{D(p_1, \dots, p_{n-1})}$ being invertible on W_0 .

Assuming for the moment W to be irreducible, the only case where it is possible to extend $\mathcal{T}|_{W_0} \rightarrow TW_0$ as an injective morphism $\mathcal{T}|_{W'} \rightarrow TW'$ is the case where all coefficients of the matrix $L \circ \Theta$ contain Δ_p as a common factor, i.e. vanish on the critical scheme. In the general case, the same can be done for any irreducible component. We thus get the

Proposition 5.5 *For a LCI-web W to be dicritical (resp. dicritical on an irreducible component C of W), it is necessary and sufficient that the matrix $L \circ \Theta$ vanishes on the critical scheme (resp. on the critical scheme of C).*

Remark 5.6 *We shall see below that, in case of a linear web on the complex projective space \mathbb{P}_n , the matrix Θ itself vanishes on the critical scheme : such a property will be called hyper-dicriticality. In fact, in this case, s_Γ can be defined on all of $\tilde{\mathbb{P}}_n$, Θ vanishes on all of W , on the condition to use only affine coordinates as local coordinates.*

Definition 5.7 *A linearizable web on M is a web locally isomorphic to a linear web : this means that there exists local holomorphic coordinates near any point of M_0 , with respect to which the leaves of the web have an affine equation. If there exists local holomorphic coordinates near any point of M (not only of M_0), with respect to which the leaves of the web have an affine equation, we shall say that the web is totally linearizable.*

Proposition 5.8

Any totally linearizable web on a manifold M is dicritical.

Proof: Assume a web on M to be locally defined by the equations $F_\alpha(x, p) = 0$, ($1 \leq \alpha \leq n - 1$), and let $(x_0, p_0) = (x_1^0, \dots, x_n^0, p_1^0, \dots, p_{n-1}^0)$ be such that, for any α , $F_\alpha(x_0, p_0) = 0$. If the leaves of the web have an affine equation with respect to the local coordinates x , the hypersurface of equation $x_n - x_n^0 = \sum_\alpha p_\alpha^0 (x_\alpha - x_\alpha^0)$ is a leaf, so that, for any α ,

$$F_\alpha(x_1, \dots, x_{n-1}, x_n^0 + \sum_\alpha p_\alpha^0 (x_\alpha - x_\alpha^0), p_1^0, \dots, p_{n-1}^0) \equiv 0.$$

By derivation of this identity with respect to each x_β , we get $\Theta_\alpha^\beta = 0$ at the point (x_0, p_0) , hence $\Theta \equiv 0$; a fortiori $L \circ \Theta \equiv 0$. Since, on the critical scheme, this equation does not depend on the local coordinates, the web is dicritical as far as the local coordinates are available on Γ_W , i.e. if the web is hyperlinearizable.

QED

6 Webs on the complex projective space \mathbb{P}_n .

6.1 The manifold $\widetilde{\mathbb{P}}_n$ of contact elements :

Denote by (X_0, \dots, X_n) the homogeneous coordinates on \mathbb{P}_n , and (u_0, \dots, u_n) the homogeneous coordinates on the dual projective space \mathbb{P}'_n of the projective hyperplanes in \mathbb{P}_n : the hyperplane of coordinates (u_0, \dots, u_n) is the hyperplane of equation $u_0X_0 + u_1X_1 + \dots + u_nX_n = 0$ in \mathbb{P}_n .

Lemma 6.1 *The manifold $\widetilde{\mathbb{P}}_n$ is naturally bi-holomorphic to the submanifold of points $([X_0, \dots, X_n], [u_0, \dots, u_n])$ in $\mathbb{P}_n \times \mathbb{P}'_n$ such that $u_0X_0 + u_1X_1 + \dots + u_nX_n = 0$: a contact element is a pair (m, h) made of a point $m \in \mathbb{P}_n$ and of a hyperplane $h \in \mathbb{P}'_n$ going through this point ($m \in h$). According to this identification, the projection π becomes the restriction to $\widetilde{\mathbb{P}}_n$ of the first projection of $\mathbb{P}_n \times \mathbb{P}'_n$.*

Proof : We identify obviously the pair (m, h) ($m \in h$) to the sub vector-space $T_m h$ of $T_m \mathbb{P}_n$ (which is a point in $\mathbb{G}_{n-1} \mathbb{P}_n$).

Remark 6.2

(i) *If m has affine coordinates $(x_\lambda = \frac{X_\lambda}{X_0})$ on the open set $X_0 \neq 0$, and h has for equation $x_n - \sum_{\alpha=1}^{n-1} p_\alpha x_\alpha = 0$ with respect to these coordinates, we identify $((x_\lambda)_\lambda, (p_\alpha)_\alpha)$ to the point $([X_0, \dots, X_n], [u_0, \dots, u_n])$ with $X_0 = 1$, $X_\lambda = x_\lambda$, $u_0 = x_n - \sum_{\alpha=1}^{n-1} p_\alpha x_\alpha$, $u_\alpha = p_\alpha$ and $u_n = -1$.*

(ii) *Conversely, we identify the point $([X_0, \dots, X_n], [u_0, \dots, u_n])$ in the open set $X_0 \cdot u_n \neq 0$ of $\mathbb{P}_n \times \mathbb{P}'_n$, such that $u_0X_0 + u_1X_1 + \dots + u_nX_n = 0$, with the point of coordinates $(x_\lambda = \frac{X_\lambda}{X_0}, p_\alpha = -\frac{u_\alpha}{u_n})$.*

The spaces \mathbb{P}_n et \mathbb{P}'_n play the same role, so that $\widetilde{\mathbb{P}}_n = \widetilde{\mathbb{P}}'_n$: the second projection $\pi' : \widetilde{\mathbb{P}}_n \rightarrow \mathbb{P}'_n$ is also a fiber space with fiber \mathbb{P}_{n-1} , with which we can do the same constructions as with π . Let \mathcal{V}' , \mathcal{T}' and \mathcal{L}' be the bundles built from π' in the same way as \mathcal{V} , \mathcal{T} and \mathcal{L} are built from π . Writting $p_n = x_n - \sum_{\alpha} p_\alpha x_\alpha$, notice that, the 1-form $dx_n - \sum_{\alpha} p_\alpha dx_\alpha$ may be written $dp_n - \sum_{\alpha} x_\alpha dp_\alpha$ on the open set $X_0 \cdot u_n \neq 0$ of $\widetilde{\mathbb{P}}_n$, with respect to the coordinates $(-x_\alpha, p_n, p_\alpha)$. The tautological contact form ω is then the same when $\widetilde{\mathbb{P}}_n$ is seen as the manifold of contact elements of $\widetilde{\mathbb{P}}_n$ or of $\widetilde{\mathbb{P}}'_n$. Therefore, when a n -dimensional subvariety W of $\widetilde{\mathbb{P}}_n$ is simultaneously a web on \mathbb{P}_n and on \mathbb{P}'_n ("bi-web"), the foliation $\widetilde{\mathcal{F}}$ on the regular part of W is the same for both webs.

Denote respectively by $\mathcal{O}(-1)$ and $\mathcal{O}'(-1)$ the tautological line bundles of \mathbb{P}_n and \mathbb{P}'_n , $\mathcal{O}(1)$ and $\mathcal{O}'(1)$ their dual bundle. Set $\ell = \pi^{-1}(\mathcal{O}(-1))$ and $\ell' = \pi'^{-1}(\mathcal{O}'(-1))$. More generally, ℓ^k (resp. ℓ^{-k} or $\check{\ell}^k$) will denote, for any integer $k \geq 0$, the k^{th} tensor power of ℓ (resp. of its dual bundle $\check{\ell}$), and the same goes for $(\ell')^k$ and $(\ell')^{-k} = (\check{\ell}')^k$.

Lemma 6.3

(i) *The line bundle \mathcal{L} is isomorphic to $\check{\ell} \otimes \ell'$, which is also the normal bundle $N_{\widetilde{\mathbb{P}}_n}$ of $\widetilde{\mathbb{P}}_n$ in $\mathbb{P}_n \times \mathbb{P}'_n$.*

(ii) *The line bundle $\pi^{-1}(\bigwedge^n T\mathbb{P}_n)$ is isomorphic to $(n+1)\check{\ell}$.*

(iii) *The sub-bundles \mathcal{V} and \mathcal{T}' of $T\widetilde{\mathbb{P}}_n$ the same goes, as well as \mathcal{V}' and \mathcal{T} .*

Proof :

On the open set $(X_0 \neq 0) \cap (X_n \neq 0)$ of \mathbb{P}_n , we get $p'_\alpha = \frac{-p_\alpha}{x_n - \sum_{\alpha} p_\alpha x_\alpha}$ by the change of affine coordinates $(x'_\alpha = \frac{x_\alpha}{x_n}, x'_n = \frac{1}{x_n})$, hence the formula

$$dx_n - \sum_{\alpha} p_\alpha x_\alpha = -x_n(x_n - \sum_{\alpha} p_\alpha x_\alpha) \left(dx'_n - \sum_{\alpha} p'_\alpha x'_\alpha \right).$$

Since $-x_n(x_n - \sum_{\alpha} p_{\alpha} x_{\alpha}) = \frac{-X_n}{X_0} \left(\frac{\sum_{\lambda} u_{\lambda} X_{\lambda}}{u_n X_0} \right)$, and since $\sum_{\lambda} u_{\lambda} X_{\lambda} = -u_0 X_0$ on $\widetilde{\mathbb{P}}_n$, we get finally $-x_n(x_n - \sum_{\alpha} p_{\alpha} x_{\alpha}) = \frac{X_n u_0}{X_0 u_n}$. Thus, $\frac{X_n u_0}{X_0 u_n}$ is the transition function for \mathcal{L}^* on the open set $(X_0.u_n \neq 0) \cap (X_n.u_0 \neq 0)$; but this is also the transition function for the bundle $\ell \otimes \ell'$, hence part (i) the lemma.

We know in general that the bundles TM and $\bigwedge^n TM$ have the same Chern class c_1 . Here the formula $T\mathbb{P}_n \oplus 1 = (n+1)\mathcal{O}(1)$ implies that $c_1(T\mathbb{P}_n)$ is equal to $c_1((n+1)\mathcal{O}(1))$. But this Chern class is still equal to that of $\mathcal{O}(n+1)$. Hence, the bundles $\bigwedge^n T\mathbb{P}_n$ and $\mathcal{O}(n+1)$ are isomorphic, since the isomorphism class of a complex line-bundle is characterized by its Chern class. This proves part (ii) of the lemma.

Using the coordinates $(x_{\lambda} = \frac{X_{\lambda}}{X_0}, p_{\alpha} = -\frac{u_{\alpha}}{u_n})$ on the open subset $X_0.u_n \neq 0$ of $\widetilde{\mathbb{P}}_n$, we make the change of coordinates :

$$x'_{\alpha} = p_{\alpha}, \quad x'_n = x_n - \sum_{\alpha} p_{\alpha} x_{\alpha}, \quad p'_{\alpha} = x_{\alpha}.$$

Notice that x'_n is also equal to $-\frac{u_0}{u_n}$. We get the formulae:

$$\frac{\partial}{\partial p'_{\alpha}} = \frac{\partial}{\partial x_{\alpha}} + p_{\alpha} \frac{\partial}{\partial x_n} \quad \text{and} \quad \frac{\partial}{\partial p_{\alpha}} = \frac{\partial}{\partial x'_{\alpha}} + p'_{\alpha} \frac{\partial}{\partial x'_n}.$$

Then, any vector of \mathcal{V}' is a vector of \mathcal{T} and conversely, when $X_0.u_n \neq 0$. Any point of $\widetilde{\mathbb{P}}_n$ belonging to some open subset $X_i.u_j \neq 0$ (with $i \neq j, 0 \leq i, j \leq n$, the previous identification may be done above all of $\widetilde{\mathbb{P}}_n$. We have of course the similar identification between \mathcal{V} and \mathcal{T}' , hence the part (iii) of the lemma.

6.2 Degree of a web on \mathbb{P}_n , and multi-degree of a CI-web :

6.2.1 Bi-degree of a global PDE on \mathbb{P}_n :

Let $H(X_0, \dots, X_n; u_0, \dots, u_n)$ be a polynomial with respect to the variables $(X_0, \dots, X_n; u_0, \dots, u_n)$ homogeneous of degree δ with respect to the variables X_0, \dots, X_n , and homogeneous of degree d with respect to the variables u_0, \dots, u_n . We shall call (δ, d) the *bi-degree* of H . Let S be the hyper-surface of equations $(H = 0, \sum_{\rho=0}^n u_{\rho} X_{\rho} = 0)$ in $\widetilde{\mathbb{P}}_n$. Any other polynomial \overline{H} defining the same hyper-surface S must have the same bi-degree.

Such a hypersurface S is the zero set of a holomorphic section of $\check{\ell}^{\delta} \otimes \check{(\ell')^d}$, or equivalently (after lemma 6-3-(i)) of $\pi^{-1}(E) \otimes \mathcal{L}^d$, with $E = \mathcal{O}(\delta - d)$. It defines a global first order PDE on \mathbb{P}_n , homogeneous of degree d .

The integer δ (resp. d) is in fact the degree of the algebraic $(n-2)$ -dimensional subvariety of the points $m \in \mathbb{P}_n$ (resp. of the points $h \in \mathbb{P}'_n$ such that (m, h) is a formal solution of the given PDE at order 1, h denoting a given generic hyperplane of \mathbb{P}_n (resp. m denoting a given generic point in \mathbb{P}_n).

By restriction to the set $X_0.u_n \neq 0$, and relatively to the corresponding affine coordinates $x_{\lambda} = X_{\lambda}/X_0, p_{\alpha} = -u_{\alpha}/u_n$, S has equation

$$H(1, x_1, \dots, x_n; x_n - \sum_{\alpha} p_{\alpha} x_{\alpha}, p_1, \dots, p_{n-1}, -1) = 0,$$

the first member of which is a polynomial of degree d in p , with polynomial coefficients $a_i(x)$ of degree $\leq \delta + d$ in x .

Conversely, any global PDE on \mathbb{P}_n may be defined by this procedure from a polynomial H . Identifying \mathbb{C}^n to the affine open set $X_0 \neq 0$ in \mathbb{P}_n , with affine coordinates $x_{\lambda} = \frac{X_{\lambda}}{X_0}$.

Lemma 6.4

(i) A global PDE on \mathbb{P}_n is completely defined by its restriction to \mathbb{C}^n .

(ii) For a global (first order, homogeneous polynomial) PDE on \mathbb{C}^n to be the restriction of a PDE on \mathbb{P}_n , it is necessary and sufficient that the coefficients a_i be all polynomial in (x_1, \dots, x_n) .

The proof is completely similar to that given in [CL1] for $n = 2$ (notice that, when $n = 2$, any homogeneous first order PDE is automatically a web).

6.2.2 Algebraic PDE's on \mathbb{P}_n :

Definition 6.5 A global PDE (S, D, s) , homogeneous of degree d on \mathbb{P}_n , is said to be algebraic, if there exists an algebraic hypersurface \overline{S} of degree d in \mathbb{P}'_n (necessarily unique), such that S be the set of all points (m, h) with $m \in h$ and $h \in \overline{S}$.

Lemma 6.6 The algebraic polynomial PDE S 's, homogeneous of degree d , are the PDE's of bi-degree $(0, d)$.

Proof : Let $\Phi(u_0, u - 1, \dots, u_n) = 0$ be the equation of an algebraic hypersurface of degree d in \mathbb{P}'_n (well defined up to multiplication by a non-zero scalar). Then, the PDE S defined by the polynomial $H(X_0, \dots, X_n; u_0, \dots, u_n) = \Phi(u_0, u - 1, \dots, u_n)$ is algebraic, and the map so defined is an obvious bijection onto the set of global PDE's of bi-degree $(0, d)$. [Observe that the polynomial H of such a PDE of bi-degree $(0, d)$ is well defined, since $H(X_0, \dots, X_n; u_0, \dots, u_n) \cdot \sum_{\rho} u_{\rho} X_{\rho}$ cannot have bi-degree $(0, d)$ if $K \neq 0$.

QED

6.2.3 Multi-degree of a CI-web on \mathbb{P}_n :

A CI-web W is the complete intersection of $n - 1$ global PDE's S_{α} ($1 \leq \alpha \leq n - 1$). Let $(\delta_{\alpha}, n_{\alpha})$ be the bi-degree of S_{α} for such a CI-web.

Definition 6.7 The family $(\delta_{\alpha})_{\alpha}$ is called the multi-degree of the CI-web, and the number $\delta = \prod_{\alpha} \delta_{\alpha}$ its degree.

6.3 Linear and algebraic webs:

Definition 6.8

(i) A linear web on an open set of \mathbb{P}_n is a web all leaves of which are pieces of hyperplanes.

(ii) A algebraic d -web on \mathbb{P}_n is a web whose leaves are the hyperplanes belonging to some algebraic curve of degree d in \mathbb{P}'_n .

Theorem 6.9 The algebraic webs on \mathbb{P}_n are the linear webs globally defined on all of \mathbb{P}_n

Proof : If a web W is linear, and if a point (m_0, h_0) belongs to W , then all points (m, h_0) such that m belongs to h_0 are still in W . Therefore, the web is completely defined by the projection $\overline{W} = \pi'(W)$. It is then sufficient to prove that \overline{W} is an algebraic set, because it will be then automatically one-dimensional since W has dimension n .

It is then possible to generate the ideal of W by functions on $\widetilde{\mathbb{P}}_n$ going to the quotient modulo π' ; we get therefore analytical functions on \mathbb{P}'_n defining \overline{W} . Hence, \overline{W} is analytic, and consequently algebraic³.

QED

³We have already proved this theorem when $n = 2$ by another method (see [CL1]). The principle of the method used here has been suggested to us by L. Pirio.

Remark 6.10 *Because of the duality between the curves in \mathbb{P}'_n and the developable hypersurfaces in \mathbb{P}_n , the hyperplanes of an algebraic web are also the hyperplanes tangent to some algebraic developable hypersurface \mathcal{C} in \mathbb{P}_n . This means that there exists an algebraic curve γ (i.e. analytical set of pure dimension 1), a holomorphic fiber-bundle $\widehat{\mathcal{C}} \rightarrow \gamma$ with base the analytical set γ and fiber \mathbb{P}_{n-2} , and an immersion $\Phi : \widehat{\mathcal{C}} \rightarrow \mathbb{P}_n$ of the total space $\widehat{\mathcal{C}}$ of this bundle into \mathbb{P}_n ,*

- whose image $\Phi(\widehat{\mathcal{C}})$ is \mathcal{C} ,
- whose restriction to any fiber of $\widehat{\mathcal{C}}$ is a bi-holomorphism onto some $(n-2)$ - dimensional sub projective space of \mathbb{P}_n ,
- and such that the tangent hyperplane to \mathcal{C} at some regular point $\Phi(a)$ of \mathcal{C} depends only on the fiber of $\widehat{\mathcal{C}}$ to which a belongs.

Proposition 6.11 *The algebraic CI-webs on \mathbb{P}_n are the webs $W = \cap_{\alpha} S_{\alpha}$ of multi-degree $(\delta_{\alpha})_{\alpha}$ with all δ_{α} equal to 0.*

Proof: It is clear that, if $W = \cap_{\alpha} S_{\alpha}$ such that all δ_{α} 's are zero, then W is algebraic (it corresponds to the curve in \mathbb{P}'_n defined by the intersections of the \overline{S}_{α} 's. Conversely, assume that $W = \cap_{\alpha} S_{\alpha}$ is algebraic. Assume that there exists at least one of the S_{α} 's (let us say S_{α_0} which is not algebraic, and let (m_0, h_0) be a point of W . Since $\delta_{\alpha_0} \geq 1$, there exists $m \in \mathbb{P}_n$, such that (m, h_0) does not belong to S_{α_0} , and a fortiori does not belong to W . But there is a contradiction with the algebraicity of W (since (m_0, h_0) belongs to W , all points (m, h_0) should belong to W).

6.4 Cohomology of $\widetilde{\mathbb{P}}_n$:

Let us denote respectively by $\xi = c_1(\check{\ell})$ and $\xi' = c_1(\check{\ell}')$ the Chern classes in $H^2(\widetilde{\mathbb{P}}_n, \mathbb{Z})$ of the bundles $\pi^{-1}(\mathcal{O}(1))$ and $(\pi')^{-1}(\mathcal{O}'(1))$.

Theorem 6.12 *The cohomology algebra of $\widetilde{\mathbb{P}}_n$ is the quotient of the free algebra $\mathbb{Z}[\xi, \xi']$ by the 3 relations $\xi^{n+1} = 0$, $(\xi')^{n+1} = 0$ and $\sum_{j=0}^n (-1)^j \xi^j (\xi')^{n-j} = 0$:*

$$H^*(\widetilde{\mathbb{P}}_n, \mathbb{Z}) = \mathbb{Z}[\xi, \xi'] / (\xi^{n+1}, (\xi')^{n+1}, \sum_{i=0}^n (-1)^i \xi^i (\xi')^{n-i}).$$

Proof : Since the base \mathbb{P}_n and the fiber \mathbb{P}_{n-1} of the fibration π have only even cohomology, the spectral sequence of this fibration collapses, so that $H^*(\widetilde{\mathbb{P}}_n, \mathbb{Z})$ is isomorphic -as a graded vector space- to the E_2 - term $\left(\mathbb{Z}[\xi] / \xi^{n+1} \right) \otimes \left(\mathbb{Z}[\eta] / \eta^n \right)$, where η denotes the generator in $H^2(\mathbb{P}_{n-1}, \mathbb{Z})$ of the cohomology algebra of the fiber.

Since \mathcal{L}^* is the tautological line bundle of $\widetilde{\mathbb{P}}_n = \mathbb{P}(T^*M)$, the Chern class $\eta = c_1(\check{\mathcal{L}}^*)$ induces on every fiber of π the generator of the cohomology of this fiber. Therefore, $H^*(\widetilde{\mathbb{P}}_n, \mathbb{Z})$ is generated by ξ and η as an algebra, or also by ξ and ξ' since we can deduce the equality $\eta = -(\xi + \xi')$ from the isomorphism $\mathcal{L}^* \cong \ell \otimes \ell'$: hence $H^*(\widetilde{\mathbb{P}}_n, \mathbb{Z})$ is a quotient of the free algebra $\mathbb{Z}[\xi, \xi']$.

The relations $\xi^{n+1} = 0$ and $(\xi')^{n+1} = 0$ are obvious for dimensional reasons.

Lemma 6.13 *The Chern class $c_j(\mathcal{T})$ is given by the formula :*

$$c_j(\mathcal{T}) = \sum_{i=0}^j (-1)^i \binom{n+1}{j-i} \xi^{j-i} (\xi + \xi')^i.$$

Proof : From the identity on the total Chern classes $c(\mathcal{T})(1 + \xi + \xi') = (1 + \xi)^{n+1}$ induced by the exact sequence (*) of vector bundles, we can compute $c_j(\mathcal{T})$ by induction on j with the formula $c_j(\mathcal{T}) = \binom{n+1}{j} \xi^j - (\xi + \xi')c_{j-1}(\mathcal{T})$.

In particular, since $c_n(\mathcal{T}) = 0$, we get the relation $\sum_{i=0}^n (-1)^i \binom{n+1}{n-i} \xi^{n-i} (\xi + \xi')^i = 0$, which can still be formally written $\left(((\xi + \xi') - \xi)^{n+1} + \xi^{n+1} \right) / (\xi + \xi') = 0$, i.e.

$$\sum_{i=0}^n (-1)^i \xi^i (\xi')^{n-i} = 0.$$

Since $\mathbb{Z}[\xi, \xi'] / (\xi^{n+1}, (\xi')^{n+1}, \sum_{i=0}^n (-1)^i \xi^i (\xi')^{n-i})$ and E_2 are isomorphic as graded vector spaces, we get the formula of the theorem.

QED

Theorem 6.14 *Any CI-web W of codimension one on \mathbb{P}_n has a non-empty critical set Γ_W . Equivalently, such a web has a non-empty caustic.*

Proof :

If Γ_W was empty, W would be in particular non-singular, the foliation $\tilde{\mathcal{F}}$ defined by the morphism $\omega_W : TW \rightarrow \mathcal{L}$ would have no singularity, so that $(c_1)^2(\mathcal{L}) \frown [W]$ would be zero after the Bott vanishing theorem.

Since the CI web W is the zero set of a section of the bundle $\tilde{N}_W = \bigoplus_{\alpha} (\check{\ell}^{\delta_{\alpha}} \otimes (\check{\ell}')^{d_{\alpha}})$ above $\tilde{\mathbb{P}}_n$, the fundamental class $[W]$ is the Poincaré dual of the Chern class $c_{n-1}(\tilde{N}_W) = \prod_{\alpha} (\delta_{\alpha} \xi + d_{\alpha} \xi')|_W$. Hence,

$$(c_1)^2(\mathcal{L}) \frown [W] = \left((c_1)^2(\mathcal{L}) \smile c_{n-1}(\tilde{N}_W) \right) \frown [\tilde{\mathbb{P}}_n] \text{ in } H_{2(n-2)}(\tilde{\mathbb{P}}_n).$$

Since $\tilde{\mathbb{P}}_n$ is the zero set of a section of \mathcal{L} on $\mathbb{P}_n \times \mathbb{P}'_n$, we still have

$$(c_1)^2(\mathcal{L}) \frown [W] = \left((c_1)^3(\mathcal{L}) \smile c_{n-1}(\tilde{N}_W) \right) \frown [\mathbb{P}_n \times \mathbb{P}'_n] \text{ in } H_{2(n-2)}(\mathbb{P}_n \times \mathbb{P}'_n).$$

Denote $\bar{\xi}$ and $\bar{\xi}'$ the Chern classes of $\mathcal{O}(1)$ and $\mathcal{O}'(1)$ in $H^2(\mathbb{P}_n)$ and $H^2(\mathbb{P}'_n)$ respectively, or in $H^*(\mathbb{P}_n \times \mathbb{P}'_n) = \left(\mathbb{Z}[\bar{\xi}] / \bar{\xi}^{n+1} \right) \otimes \left(\mathbb{Z}[\bar{\xi}'] / \bar{\xi}'^{n+1} \right)$.

But $(c_1)^3(\mathcal{L}) \smile c_{n-1}(\tilde{N}_W) = (\bar{\xi} + \bar{\xi}')^3 \prod_{\alpha=1}^{n-1} (\delta_{\alpha} \bar{\xi} + d_{\alpha} \bar{\xi}')$ may be written $\sum_{i=2}^n a_i \bar{\xi}^i (\bar{\xi}')^{n+2-i}$ with coefficients a_i all strictly positive. Hence $(c_1)^2(\mathcal{L}) \frown [W]$ may not vanish.

QED

7 Dicriticality and algebraicity of global CI-webs

While all results of this section could be given for webs which are not necessarily CI or LCI, we shall restrict ourselves to CI-webs for simplicity.

Proposition 7.1 *Any linear web on an open set of \mathbb{P}_n (in particular any algebraic web on \mathbb{P}_n) is totally linearizable, hence dicritical.*

Proof: If a web on \mathbb{P}_n is linear, the equation of the leaves are affine with respect to any system of affine coordinates. Since there are such affine coordinates near any point of \mathbb{P}_n , and in particular near any point of the critical set, this proves that a linear web is hyperlinearizable, hence part (ii).
QED

For webs which are globally defined on \mathbb{P}_n , we have therefore the implications

$$\text{algebraic} \iff \text{linear} \implies \text{dicritical}.$$

We shall prove now conversely, at least in the case of webs whose each irreducible component is CI, that in fact

$$(\text{dicritical} + \text{quasi-smooth}) \implies \text{algebraic}.$$

Theorem 7.2 *For $d \geq 3$, any d -web on \mathbb{P}_n which is quasi-smooth and dicritical, and whose each irreducible component is CI, is algebraic.*

Proof : Since quasi-smoothness and dicriticality are defined on each irreducible component, we may assume W irreducible and smooth. Dicriticality means then that the foliation $\tilde{\mathcal{F}}$ has no singularity. After Bott $(c_1)^2(N(\tilde{\mathcal{F}}))$ must vanish.

The total Chern class $c(N(\tilde{\mathcal{F}}))$ is equal to $c(W).c(\mathcal{T}|_W)^{-1}$.

After the exact sequence (*) of section 2, $c(\mathcal{T}) = (\pi^*c(\mathbb{P}_n)).c(\mathcal{L})^{-1}$.

Assume the CI-web to be defined by the $n - 1$ global PDE's S_α of bi-degree $(\delta_\alpha \geq 0, d_\alpha \geq 1)$, with $d = \prod_\alpha d_\alpha$ and $\delta = \prod_\alpha \delta_\alpha$. Then

$$c(W) = c(\tilde{\mathbb{P}}_n|_W) \cdot \left(\prod_\alpha (1 + \delta_\alpha \xi + d_\alpha \xi') \Big|_W \right)^{-1}.$$

Since $\tilde{\mathbb{P}}_n$ has \mathcal{L} for normal bundle in $\mathbb{P}_n \times \mathbb{P}'_n$, and since $(\pi')^{-1}(T\mathbb{P}'_n) \oplus 1 = (n + 1)\check{\ell}$, we get finally :

$$c(N(\tilde{\mathcal{F}})) = c(\pi'^{-1}T\mathbb{P}'_n|_W) \smile \left(\prod_\alpha (1 + \delta_\alpha \xi + d_\alpha \xi') \Big|_W \right)^{-1},$$

and in particular

$$c_1(N(\tilde{\mathcal{F}})) = \left((n + 1)\xi' - \sum_\alpha (\delta_\alpha \xi + d_\alpha \xi') \right) \Big|_W.$$

Let us write $\bar{\delta} = \sum_\alpha \delta_\alpha$ and $\bar{d} = \sum_\alpha d_\alpha$. We thus get

$$(c_1)^2(N(\tilde{\mathcal{F}})) = \left[(\bar{\delta})^2 \xi^2 + (n + 1 - \bar{d})^2 (\xi')^2 - 2\bar{\delta}(n + 1 - \bar{d}) \xi \xi' \right] \Big|_W.$$

Since the CI web W is the zero set of a section of the bundle $\tilde{N}_W = \bigoplus_\alpha (\check{\ell}^{\delta_\alpha} \otimes (\check{\ell}')^{d_\alpha})$, the fundamental class $[W]$ is the Poincaré dual of the Chern class $c_{n-1}(\tilde{N}_W) = \prod_\alpha (\delta_\alpha \xi + d_\alpha \xi') \Big|_W$, so that

$$(c_1)^2(N(\tilde{\mathcal{F}})) \smile [W] = \left(\left[(\bar{\delta})^2 \xi^2 + (n + 1 - \bar{d})^2 (\xi')^2 - 2\bar{\delta}(n + 1 - \bar{d}) \xi \xi' \right] \smile \prod_\alpha (\delta_\alpha \xi + d_\alpha \xi') \right) \smile [\tilde{\mathbb{P}}_n]$$

in $H^{2n+2}(\tilde{\mathbb{P}}_n)$. If $(c_1)^2(N(\tilde{\mathcal{F}})) = 0$, then

$$\left[(\bar{\delta})^2 \xi^2 + (n + 1 - \bar{d})^2 (\xi')^2 - 2\bar{\delta}(n + 1 - \bar{d}) \xi \xi' \right] \smile \left(\prod_\alpha (\delta_\alpha \xi + d_\alpha \xi') \right) \smile \xi^{n-2}$$

must vanish in $H^{4n-2}(\tilde{\mathbb{P}}_n)$. Using the relations $\xi^{n+1} = 0$, $(\xi')^{n+1} = 0$, and $\xi^n (\xi')^{n-1} = \xi^{n-1} (\xi')^n$, we deduce that the number

$$\mathcal{N} = \bar{\delta}^2 - 2\bar{\delta}(n + 1 - \bar{d})(1 + \sigma_1) + (n + 1 - \bar{d})^2(\sigma_1 + \sigma_2)$$

must vanish, where σ_1 (resp. σ_2) denotes the first (resp the second) elementary symmetric function $\sigma_1 = \sum_{\alpha} \delta_{\alpha}/d_{\alpha}$ (resp. $\sigma_2 = \sum_{\alpha\beta} \delta_{\alpha}\delta_{\beta}/d_{\alpha}d_{\beta}$) of the numbers $\delta_{\alpha}/d_{\alpha}$.

Since every d_{α} is at least equal to 1, and since $\prod_{\alpha} d_{\alpha}$ ($= d$) is at least 3, $n + 1 - \bar{d}$ is always non-positive. Thus \mathcal{N} is non-negative, and may be zero only if all δ_{α} 's vanish. This is what we want.

QED

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