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Semi-classical analysis and passive imaging

Yves Colin de Verdière *

March 17, 2008

Abstract

Passive imaging is a new technique which has been proved to be very efficient, for example in seismology: the correlation of the noisy fields, computed from the fields recorded at different points, is strongly related to the Green function of the wave propagation. The aim of this paper is to provide a mathematical context for this approach and to show, in particular, how the methods of semi-classical analysis can be used in order to find the asymptotic behaviour of the correlations.

Introduction

Passive imaging is a way to solve inverse problems: it has been successful in seismology and acoustics [5, 6, 17, 22, 23, 28, 29, 31, 4]. The method is as follows: let us assume that we have a medium X (a smooth manifold) and a smooth, deterministic (no randomness in it), linear wave equation in X . We hope to recover (part of) the geometry of X from the wave propagation. We assume that there is somewhere in X a source of noise $\mathbf{f}(x, t)$ which is a stationary random field. This source generates, by the (linear) wave propagation, a field $\mathbf{u}(x, t) = (u^\alpha(x, t))_{\alpha=1, \dots, N}$. This field \mathbf{u} is recorded at different points A, B, \dots on long time intervals. We want to get some information on the propagation of waves from B to A in X from the correlation matrix¹

$$C_{A,B}(\tau) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbf{u}(A, t) \otimes \mathbf{u}(B, t - \tau)^* dt$$

(equivalently

$$C_{A,B}^{\alpha\beta}(\tau) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T u^\alpha(A, t) \overline{u^\beta(B, t - \tau)} dt)$$

*Institut Fourier, Unité mixte de recherche CNRS-UJF 5582, BP 74, 38402-Saint Martin d'Hères Cedex (France); <http://www-fourier.ujf-grenoble.fr/~ycolver/>

¹For every matrix (a_{ij}) , we write $(a_{ij})^* := (\overline{a_{ji}})$.

which can be computed numerically from the fields recorded at A and B . It turns out that $C_{A,B}(\tau)$ is closely related to the deterministic *Green's function* of the wave equation in X . It means that one can hope to recover, using Fourier analysis, the propagation speeds of waves between A and B as a function of the frequency, or, more precisely, the so-called *dispersion relation*.

If the wave dynamics is time reversal symmetric, the correlation admits also a symmetry by change of τ into $-\tau$; this observation has been used for clock synchronisation, see [20].

The goal of this paper is to give precise formulae for $C_{A,B}(\tau)$ in the high frequency limit assuming a rapid decay of the correlations of the source \mathbf{f} outside the diagonal. More precisely, we have two small parameters, one of them entering into the decorrelation distance of the source noise, the other one in the high frequency propagation (the *ray* method). The fact that both are of the same order of magnitude is crucial for the method.

Let us also mention on the technical side that, rather than using mode decompositions, we prefer to work directly with the dynamics; in other words, we need really a *time dependent* rather than a *stationary approach*. Mode decompositions are often useful, but they are of no much help for general operators with no particular symmetry.

For clarity, we will first discuss the non-physical case of a first order wave equation like the Schrödinger equation, then the case of a more usual wave equations (acoustics, elasticity).

The main result says that, for $\tau > 0$, $C_{A,B}(\tau)$ is close to the Schwartz kernel of $\Omega(\tau) \circ \Pi$ where Π is a suitable *pseudo-differential operator* (a Ψ DO), whose principal symbol can be explicitly computed, and $\Omega(\tau)$ is the (semi-)group of the (damped) wave propagation. This closeness is in general only in the L^2 sense, but can be pointwise if the attenuation time is small enough. It implies that we can recover the dispersion relation, i.e. the classical dynamics, from the knowledge of all two-points correlations.

In order to make the paper readable by a large set of people, we have tried to make it self-contained by including sections on pseudo-differential operators and on random fields.

- In Section 1, we start with a quite general setting and discuss a basic formula for the correlation (Equation (5)).
- Section 2 is devoted to exact formulae in the case of an homogeneous white noise.
- In Section 3, we discuss the important property of time reversal symmetry which plays a prominent part in the applications and is also useful as a numerical test.
- In Section 4, we introduce a large family of anisotropic random fields and show the relation between their power spectra and the Wigner measures.

- Section 5 contains the main result expressing the correlation in the case of a Schrödinger wave equation in the semi-classical limit. We show moreover that pointwise estimates hold for the correlation if the attenuation time is small enough.
- Section 6 does the same in case of wave equations.
- In Section 7, we discuss shortly the case where the source noise is located on a submanifold.
- In Section 8, we focus on the case of seismology and discuss the remarkable fact that the correlation of the noise created by surface waves is enough to image the inner crust of the earth.
- Section 9 is about a quite independent issue relative to correlations of scattered waves.
- Finally, there are four appendices:
 - Appendix A on pseudo-differential operators,
 - Appendix B on classical dynamics, dispersion relation and generating functions,
 - Appendix C on Egorov Theorem for long times,
 - Appendix D on time averages versus ensemble averages, i.e. ergodicity of the source field.

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1 A general formula for the correlation

1.1 The model

We will first consider the following damped wave equation:

$$\frac{d\mathbf{u}}{dt} + \hat{H}\mathbf{u} = \mathbf{f} \quad (1)$$

- X is a *smooth manifold* of dimension d with a smooth measure $|dx|$
- $\mathbf{u}(x, t)$, $x \in X$, $t \in \mathbb{R}$ is *the field* (scalar or vector valued) with values in \mathbb{C}^N (or \mathbb{R}^N).

- The linear operator \hat{H} is the *Hamiltonian*, acting on $\mathcal{H} = L^2(X, \mathbb{C}^N)$. It satisfies some attenuation property: if we define the semi-group $\Omega(t) = \exp(-t\hat{H}), t \geq 0$, there exists $T_{\text{att}} > 0$, called the *attenuation time*, so that we have an operator norm estimate $\|\Omega(t)\| = O(e^{-t/T_{\text{att}}})$.
- The *source* \mathbf{f} is a *stationary random field* on $X \times \mathbb{R}$ with values in \mathbb{C}^N (or \mathbb{R}^N) whose matrix valued correlation kernel is given by²

$$\langle\langle \mathbf{f}(x, s) \otimes \mathbf{f}^*(y, s') \rangle\rangle = K(x, y, s - s') . \quad (2)$$

We will usually assume that $K(x, y, t)$ vanishes for large t , say $|t| \geq t_0 > 0$.

We will assume that \mathbf{f} is *Gaussian* only in order to check ergodicity: ensemble averages can then be replaced by time averages (see Appendix D).

1.2 Examples

1.2.1 Schrödinger equation

Let X be a smooth Riemannian manifold with Laplace-Beltrami operator Δ . Let us give $a : X \rightarrow \mathbb{R}$ a smooth non negative function, V a smooth real valued function on X , \hbar a non negative constant, and $\hat{K} = -\hbar^2\Delta + V(x)$, and take:

$$\frac{\hbar}{i}(u_t + a(x)u) + \hat{K}u = g .$$

It is a particular case of Equation (1) where $\hat{H} = \frac{i}{\hbar}\hat{K} + a(x)$ and $f = \frac{i}{\hbar}g$. Let us note for future use that, if $\hbar \rightarrow 0$, the principal symbol of our equation is $\omega + \|\xi\|^2 + V(x)$ and $a(x)$ is only a sub-principal term entering into the transport equation, but not in the classical dynamics. The attenuation time can be taken as $T_{\text{att}} = 1/\inf_{x \in X} a(x)$.

1.2.2 Wave equations

Let us start with

$$u_{tt} + 2au_t - \Delta u = f, \quad (3)$$

(a is a positive smooth function and Δ is the Laplace-Beltrami operator of a Riemannian metric on X) which corresponds to Equation (1) with

$$\mathbf{u} = \begin{pmatrix} u \\ u_t \end{pmatrix}, \quad \mathbf{f} := \begin{pmatrix} 0 \\ f \end{pmatrix} .$$

and

$$\hat{H} = \begin{pmatrix} 0 & -\text{Id} \\ -\Delta & 2a \end{pmatrix} .$$

²The expectation value (ensemble average) of a random variable f will be denoted $\langle\langle f \rangle\rangle$

1.2.3 Pseudo-differential equations

We can assume that the dynamics is generated by a (matrix of) pseudo-differential operator(s) (Ψ DO's, see Appendix A). Our equation looks then like:

$$\frac{\varepsilon}{i} \mathbf{u}_t + \hat{H}_\varepsilon \mathbf{u} = \mathbf{f}$$

with

$$\hat{H}_\varepsilon = \text{Op}_\varepsilon(H_0 + \varepsilon H_1) .$$

This allows to include

- An effective surface Hamiltonian associated to stratified media (included in the H_0 term) [8]. They are usually Ψ DO's with a non trivial dispersion relation, namely the (group) speed $|\partial H_0 / \partial \xi|$ is not constant.
- Frequency dependent damping included in H_1 : this is usually the case for seismic waves.

1.3 The correlation

Definition 1 *Let us define, for $t \geq 0$, $\Omega(t) := \exp(-t\hat{H})$ and the propagator P by the formula:*

$$(\Omega(t)\mathbf{v})(x) = \int_X P(t, x, y) \mathbf{v}(y) |dy| .$$

The propagator P satisfies

$$\int_X P(t, x, y) P(s, y, z) |dy| = P(t + s, x, z)$$

which comes from: $\Omega(t + s) = \Omega(t) \circ \Omega(s)$. The causal solution of Equation (1) is then given by

$$\mathbf{u}(x, t) = \int_0^\infty ds \int_X P(s, x, y) \mathbf{f}(t - s, y) |dy| .$$

Physicists would define the correlation as the following limit:

$$C_{A,B}(\tau) := \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbf{u}(A, t) \otimes \mathbf{u}(B, t - \tau)^* dt .$$

We will see in Appendix D, that under some mild assumptions (\mathbf{f} Gaussian...), this limit exists almost surely and is equal to the expectation value

$$C_{A,B}(\tau) = \langle\langle \mathbf{u}(A, 0) \otimes \mathbf{u}(B, -\tau)^* \rangle\rangle . \quad (4)$$

We will take Formula (4) as a definition.

We get by a simple calculation:

Theorem 1 *If P is defined as in Definition 1 and K by Equation (2), we have, for $\tau > 0$:*

$$C_{A,B}(\tau) = \int_0^\infty ds \int_{-\infty}^{s+\tau} d\sigma \int_{X \times X} |dx||dy| P(s + \tau - \sigma, A, x) K(x, y, \sigma) P^*(s, B, y) \quad (5)$$

and $C_{A,B}(-\tau) = C_{B,A}(\tau)^*$.

Equation (5) can be written for $\tau > t_0$ as follows³:

$$C_{A,B}(\tau) = [\Omega(\tau)\Pi](A, B) \quad (6)$$

with

$$\Pi = \int_0^\infty \Omega(s)\mathcal{L}\Omega^*(s)ds \quad (7)$$

and

$$\mathcal{L} = \int_{|t| \leq t_0} \Omega(-t)\hat{K}(t)dt .$$

If we assume that $K(x, y, \sigma) = L(x, y)\delta(\sigma)$, we get the simpler formula

$$\mathcal{L} = \hat{L} .$$

These equalities hold between operators on $L^2(X, \mathbb{C}^N)$, hence as distributions on $X \times X$. If $\hat{K}(t)$ is Hilbert-Schmidt, the equality holds in $L^2(X \times X)$ and hence almost everywhere.

2 Exact formulae for white noises

2.1 Vector valued white noise

Let us assume that $L(x, y) = \delta(x - y)\text{Id}$ meaning that \mathbf{f} is a vector valued *white noise* on $X \times \mathbb{R}$. We get, for $\tau > 0$,

$$\Pi = \int_0^\infty \Omega(s)\Omega^*(s)ds$$

and, assuming $\hat{H} = i\hat{A} + 1/T_{\text{att}}$, with \hat{A} self-adjoint, the following simple formula:

$$C_{A,B}(\tau) = \frac{T_{\text{att}}}{2} P(\tau, A, B) ,$$

which is an exact relation between the correlation and the propagator P of the wave equation without attenuation.

³If R is an operator we will denote by $[R](x, y)$ its Schwartz kernel; \hat{L} is the integral operator whose kernel is $L(x, y)$

2.2 Twisted white noise

This section was motivated by a question of Philippe Roux.

Definition 2 *A twisted white noise is a random field given by $\mathbf{f} = L_0 \mathbf{w}$ with \mathbf{w} a white noise as defined in Section 2.1 and $L_0 \in \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N)$. Its correlation is $\delta(x - y)\delta(s - s')K_0$ with $K_0 = L_0 L_0^*$.*

We have then

$$C_{A,B}(\tau) = [\Omega(\tau)\Pi](A, B)$$

with

$$\Pi = \int_0^\infty \Omega(s)K_0\Omega^*(s)ds .$$

In the particular case of the scalar wave equation with a constant damping a and the dynamics described in Section 2.3 by Equation (8), we get using a gauge transform where

$$\Omega(t) = e^{-at} \begin{pmatrix} e^{itQ} & 0 \\ 0 & e^{-itQ} \end{pmatrix}$$

$$\Pi = \frac{1}{2a}K_{0,\text{diag}} + R$$

where $K_{0,\text{diag}}$ is the diagonal part of K_0 and R is going to vanish in the high frequency limit⁴.

2.3 Wave equations

Let us take the case of a scalar wave equation with constant damping; the closest results were derived in [30, 19].

We will consider the wave equation (3) with

- $a > 0$ is a constant damping coefficient
- Δ a Riemannian laplacian in some Riemannian manifold X , possibly with boundary:

$$\Delta = g^{ij}(x)\partial_{ij} + b_i(x)\partial_i$$

which is self-adjoint with respect to $|dx|$ and appropriate boundary conditions; in fact we could replace the Laplacian by any self-adjoint operator on X !

- $f = f(x, t)$ the source of the noise which will be assumed to be a scalar white noise (homogeneous diffuse field):

$$\langle\langle f(x, s)f(y, s') \rangle\rangle = \delta(s - s')\delta(x - y)$$

⁴ R is a pseudo-differential operator of degree -1 : $R = \text{Op}_{\varepsilon=1}(r)$ with $r \in \Sigma_{-1}$

Let us compute the “causal solution”, i.e. the solution given by $u = \mathbf{G}f$ with \mathbf{G} linear and satisfying $u(., t) = 0$ if $f(., s)$ vanishes for $s \leq t$.

We introduce the vector

$$\mathbf{u} = \begin{pmatrix} u \\ \partial_t u + au \end{pmatrix}$$

which satisfies:

$$\partial_t \mathbf{u} + a\mathbf{u} + \hat{H}\mathbf{u} = \begin{pmatrix} 0 \\ f \end{pmatrix} \quad (8)$$

with

$$\hat{H} = - \begin{pmatrix} 0 & \text{Id} \\ \Delta + a^2 & 0 \end{pmatrix}$$

In order to get a readable expression, it is convenient to introduce $Q = \sqrt{-(\Delta + a^2)}$. We get then easily:

$$u(x, t) = \int_0^\infty ds \int_X e^{-as} \left[\frac{\sin sQ}{Q} \right] (x, y) f(y, t - s) |dy|$$

where $\sin sQ$ is defined from the spectral decomposition of Q (any choice of the square root gives the same result for $\frac{\sin sQ}{Q}$). The meaning of the brackets is “Schwartz kernel of”. We define

$$G_a(t, x, y) = Y(t) \left[e^{-at} \frac{\sin tQ}{Q} \right] (x, y)$$

with Y the Heaviside function. We will call G_a the (*causal*) *Green function*. We can rewrite:

$$u(x, t) = \int_{\mathbb{R}} ds \int_X G_a(t - s, x, y) f(y, s) |dy| .$$

Let us assume now that f is an homogeneous white noise and compute the correlation $C_{A,B}(\tau)$.

We get quite easily, using

$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta)) :$$

$$C_{A,B}(\tau) = \frac{e^{-a|\tau|}}{4a} \left[(Q^2 + a^2)^{-1} \left(\cos \tau Q + \frac{a \sin |\tau| Q}{Q} \right) \right] (A, B) . \quad (9)$$

Taking the τ derivative, we get the simpler formula:

$$\frac{d}{d\tau} C_{A,B}(\tau) = \begin{cases} \frac{-1}{4a} G_a(\tau, A, B) & \text{for } \tau > 0 \\ \frac{1}{4a} G_a(-\tau, A, B) & \text{for } \tau < 0 \end{cases} \quad (10)$$

The Fourier transform of $C_{A,B}(\tau)$ is:

$$\int_{\mathbb{R}} e^{-i\tau\omega} C_{A,B}(\tau) d\tau = \frac{-1}{2a\omega} [\Im ((\omega^2 + 2ia\omega + \Delta)^{-1})] (A, B) .$$

Low frequency filtering: from Equation (9), we see for each eigenmode $\Delta u_j = \omega_j^2 u_j$ a prefactor, which in the limit of large τ is $\approx e^{|\tau|(r_j-a)}$ for $\omega_j^2 = a^2 - r_j^2 < a^2$. This acts as low frequency filter as observed in [19].

3 Time reversal symmetry

Definition 3 1. A dispersion relation $\{\mathbb{D}(x, \xi, \omega) = 0\} \subset T^*(X \times \mathbb{R})$ is said to be time reversal symmetric (TRS) if it is invariant by $\alpha : (x, \xi) \rightarrow (x, -\xi)$.

2. A linear wave equation (with no attenuation!) is said to be time reversal symmetric (TRS) if for any solution $\mathbf{u}(x, t)$ the field $\bar{\mathbf{u}}(x, -t)$ is also a solution. It implies that the eigenmodes of the Hamiltonian can be chosen to be real-valued.

Remark 1 If the wave equation $Q\mathbf{u} = 0$ satisfies 2., the associated dispersion relation (the determinant of the principal symbol) satisfies 1.

Example 3.1 • Schrödinger equations without magnetic fields

• Acoustic and elastic wave equations.

Lemma 1 If \mathbb{D} is TRS and $\gamma(t) = (x(t), \xi(t), \omega_0)$ is a solution of Hamilton's equations, $\alpha(\gamma(-t)) = (x(-t), -\xi(-t), \omega_0)$ too.

We have the following:

Proposition 1 The correlation satisfies the general identity:

$$C_{A,B}(\tau) = C_{B,A}^*(-\tau)$$

and, in case of a white noise and a time reversible wave dynamics modified by a constant attenuation (as in Section 2.1):

$$C_{A,B}(-\tau) = C_{A,B}(\tau) . \tag{11}$$

Approximations of Equation (11) turn out to be important in applications to clocks synchronisation as in [19].

4 Random fields and pseudo-differential operators

4.1 Goal

Our aim in this section is to build quite general random fields with correlation distances given by a small parameter ε . It seems to be natural for that purpose to use ε -pseudo-differential operators. We will see how to compute the generalised power spectrum using Wigner measures.

4.2 White noises

Let $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ be an Hilbert space. There exists a canonical *Gaussian random field* on it, called the *white noise* and denoted by $\mathbf{w}_{\mathcal{H}}$ (or simply \mathbf{w} if there is no possible confusion). This random field is defined by the properties that:

- For all $\vec{e} \in \mathcal{H}$,

$$\langle\langle \mathbf{w} | \vec{e} \rangle\rangle = 0$$

- For all $\vec{e}, \vec{f} \in \mathcal{H}$,

$$\langle\langle \mathbf{w} | \vec{e} \rangle \overline{\langle \mathbf{w} | \vec{f} \rangle} \rangle\rangle = \langle \vec{e} | \vec{f} \rangle$$

Unfortunately, \mathbf{w} is not a random vector in \mathcal{H} unless $\dim \mathcal{H} < \infty^5$, but only a random *Schwartz distribution*.

We have nevertheless the following useful:

Proposition 2 *If A is an Hilbert-Schmidt operator on \mathcal{H} , the random field $A\mathbf{w}$ is almost surely in \mathcal{H} .*

Proof.–

$\langle\langle A\mathbf{w} | A\mathbf{w} \rangle\rangle = \langle\langle A^*A\mathbf{w} | \mathbf{w} \rangle\rangle = \text{Trace}(A^*A)$ which is finite, by definition, exactly for Hilbert-Schmidt operators.

□

⁵If \mathbf{w} were a vector in \mathcal{H} , we would have $\mathbf{w} = \sum \langle \mathbf{w} | \vec{e}_j \rangle \vec{e}_j$ for any orthonormal basis (\vec{e}_j) and we see that

$$\langle\langle \|\mathbf{w}\|^2 \rangle\rangle = \sum \langle\langle \mathbf{w} | \vec{e}_j \rangle^2 \rangle\rangle = \dim \mathcal{H} .$$

4.3 Examples

Example 4.1 Stationary noise on the real line: *let us take a random field on the real line which is given by the convolution product of the scalar white noise w with a fixed smooth compactly supported function F : $f = F \star w$. Then f is stationary: it means that the correlation kernel $K(t, t') = \langle\langle f(t)\overline{f(t')} \rangle\rangle$ is a function of $t - t'$. On the level of Fourier transforms $\hat{f} = \hat{F}\hat{w}$, $\langle\langle \hat{f}(\omega)\overline{\hat{f}(\omega')} \rangle\rangle = |\hat{F}|^2(\omega)\delta(\omega - \omega')$ and the positive function $|\hat{F}|^2(\omega)$ is usually called the power spectrum of the stationary noise.*

Example 4.2 *If X is a d -dimensional bounded domain. Let us denote the Sobolev spaces on X by $H^s(X)$. If $Q : L^2(X) \rightarrow H^s(X)$ with $s > d/2$, Qw is in $L^2(X)$.*

Example 4.3 Brownian motions: *if $X = \mathbb{R}$, w is the derivative of the Brownian motion: if $b(t) = \int_0^t w(s)ds$, $b : [0, +\infty[\rightarrow \mathbb{R}$ is the Brownian motion which is in $L^2([0, T])$ for all finite T .*

Example 4.4 *If X is a smooth compact manifold or domain and Q is smoothing, meaning that Q is given by an integral smooth kernel*

$$Qf(x) = \int_X [Q](x, y)f(y)|dy| ,$$

$F = Qw$ is a random smooth function. Its correlation kernel

$$C(x, y) := \langle\langle F(x)\overline{F(y)} \rangle\rangle$$

is given by:

$$[QQ^*](x, y) = \int_X [Q](x, z)\overline{[Q](y, z)}|dz| .$$

Example 4.5 Random vector fields: *let us consider $\mathcal{H} = L^2(X, \mathbb{R}^N)$. For example, in the case of elasticity, X is a 3D domain and $N = 3$. The fields here are just fields of infinitesimal deformations (vector fields).*

4.4 Modelling the noise using pseudo-differential operators

The main goal of the present section is to build natural random fields which are non homogeneous with small distances of correlation of the order of $\varepsilon \rightarrow 0$. The noise is non homogeneous in X , but could also be non isotropic w.r. to directions.

4.4.1 Noises from pseudo-differential operators

It is therefore natural to take for noise on a manifold Z the image of an homogeneous white noise by a pseudo-differential operator N of smooth compactly supported symbol $n(z, \zeta)$. The correlation $C(z, z')$ will then be given as the Schwartz kernel of NN^* which is a ΨDO of principal symbol $|n|^2$. We have

$$C(z, z') \sim \varepsilon^{-d} k \left(z, \frac{z' - z}{\varepsilon} \right) ,$$

where $k(z, \cdot)$ is the Fourier transform with respect to ζ of $|n|^2(z, \zeta)$, the “power spectrum” of the noise at the point z .

This construction gives smooth random fields which can be localised in some very small domains of the manifold Z , which are non isotropic and which have small distance of correlations. Moreover it will allow to use techniques of microlocal analysis with the small parameter given by ε .

4.4.2 Power spectrum and Wigner measures

Definition 4 *If $f = (f_\varepsilon)$ is a suitable family of functions on Z , the Wigner measures W_f^ε of f are the signed measures on the phase space T^*Z defined by*

$$\int adW_f^\varepsilon := \langle \text{Op}_\varepsilon(a) f_\varepsilon | f_\varepsilon \rangle .$$

The measures dW_f^ε are the phase space densities of energy of the functions f_ε .

We now define:

Definition 5 *The power spectrum of the random field $f = (f_\varepsilon)$ is the phase space density P_f^ε defined by:*

$$P_f^\varepsilon = \langle\langle W_f^\varepsilon \rangle\rangle :$$

the power spectrum of a random field is the average of its Wigner measure.

Proposition 3 *The power spectrum P of $f_\varepsilon = \text{Op}(n)w$, satisfies:*

$$P_f^\varepsilon \sim (2\pi\varepsilon)^{-d} |n|^2(x, \xi) |dx d\xi| .$$

Proof. –

Let us put $N = \text{Op}(n)$, we have

$$\langle \text{Op}(a) N w | N w \rangle = \langle N^* \text{Op}(a) N w | w \rangle$$

and $\langle\langle A w | w \rangle\rangle = \text{trace}(A)$. We get

$$\langle\langle \int adW_f^\varepsilon \rangle\rangle = \text{trace}(N^* \text{Op}(a) N)$$

which can be evaluated using the Ψ DO calculus as

$$\langle\langle \int adW_f^\varepsilon \rangle\rangle \sim (2\pi\varepsilon)^{-d} \int a|n|^2 dx d\xi .$$

□

4.4.3 Space-time noises

If $Z = X \times \mathbb{R}$ is the space-time, we will take our noise as before $f = Lw$; we will assume the noise *homogeneous in time*, the symbol l of L is assumed to be given by $l(x, \xi, \omega)$.

In this case, the correlation is given by:

$$K(x, y; t) = [LL^*](x, y; 0, t) \tag{12}$$

which is the Schwartz kernel of a Ψ DO of principal symbol $ll^*(x, \xi; \omega)$.

5 High frequency limit of the correlation: ”Schrödinger equations”

The main result is easier to derive in the case of a scalar field governed by a wave equation which gives the first order time derivative of the field: it is a generalisation of the Schrödinger equation.

5.1 Assumptions

Let us start with the semi-classical Schrödinger like equation

$$\frac{\varepsilon}{i}u_t + \hat{H}u = \frac{\varepsilon}{i}f$$

where

- \hat{H} is **admissible** (Definition 11 in Appendix C): \hat{H} is an ε -pseudo-differential operator:

$$\hat{H} := \text{Op}(H_0 + \varepsilon H_1)$$

with

- The **principal symbol** $H_0(x, \xi) : T^*X \rightarrow \mathbb{R}$, which gives the classical (“rays”) dynamics, is elliptic of degree m .
- The **sub-principal symbol** $H_1(x, \xi)$ admits some positivity property which controls the attenuation: there exists $k > 0$, such that

$$h_1 := \Im H_1 \leq -k .$$

- The **random field** f is given by $f = \text{Op}(l(x, \xi, \omega))w$ with w the white noise on $X \times \mathbb{R}$ and with l smooth, compactly supported w.r. to (x, ξ) and whose Fourier transform w.r. to ω is compactly supported. The power spectrum of f is $(2\pi\varepsilon)^{-(d+1)}|l|^2(x, \xi, \omega)$.

The previous assumptions will be used everywhere inside Section 5.

5.2 Subprincipal symbols and attenuation

Lemma 2 *Under the assumptions of Section 5.1, we have, for all $t \geq 0$, the estimate*

$$\|\Omega(t)\| = O(e^{-t/T_{\text{att}}})$$

with $\|\cdot\|$ the operator norm in $L^2(X)$, any $T_{\text{att}} > 1/k$ and ε small enough.

Proof.–

$$\frac{d}{dt}\langle v(t)|v(t)\rangle = 2\Re\langle v(t)| - i\hat{H}_1(t)v(t)\rangle$$

and we use Gårding inequality (see [10]): if $a \geq 0$, $\text{Op}_\varepsilon(a) \geq -C$ for any $C > 0$ and ε small enough.

□

5.3 Main result

We get the main result:

Theorem 2 *With the assumptions of Section 5.1, the correlation is given, for $\tau > 0$, by*

$$C_{A,B}(\tau) = [\Omega(\tau) \circ \Pi](A, B)$$

where $\Pi = \text{Op}(\pi) + R$, $\pi \equiv \sum_{j=0}^{\infty} \varepsilon^j \pi_j$, with⁶:

$$\pi_0(x, \xi) = \int_0^{T_{\text{Ehrenfest}}} \exp\left(2 \int_{-t}^0 h_1(\Phi_s(x, \xi)) ds\right) |l|^2(\Phi_{-t}(x, \xi), -H_0(x, \xi)) dt .$$

and R the remainder term is small. More precisely:

1. Let us consider $C_{A,B}(\tau)$ as the Schwartz kernel of an operator $\hat{C}(\tau)$. This operator is Hilbert-Schmidt⁷ with an Hilbert-Schmidt norm of the order of ε^{-d} . We have

$$\|\hat{R}\|_{\text{H-S}} = O(\varepsilon^{\alpha-d}) ,$$

with some $\alpha > 0$.

⁶The Ehrenfest time $T_{\text{Ehrenfest}}$ is given by Definition 10.7 in Appendix C

⁷An Hilbert-Schmidt operator A is an operator whose Schwartz kernel $[A](x, y)$ is in $L^2(X \times X)$ and the Hilbert-Schmidt norm $\|A\|_{\text{H-S}}$ of A is the L^2 norm of $[A]$.

2. For any N, k , there exists $C_{N,k}$ so that, if $T_{\text{att}} \leq C_{N,k} T_{\text{Ehrenfest}} / |\log \varepsilon|$, we have:

$$\|[\Omega(\tau)R](A, B)\|_{C^k(X \times X)} = O(\varepsilon^N) .$$

The proof of Theorem 2 will be given in Section 5.4.

5.4 Proof of Theorem 2

We split the integral $\Pi = \Pi_1 + \Pi_2$ with $\Pi_1 := \int_0^{T_\gamma} \Omega(t) A \Omega^*(t) dt$ and $\Pi_2 = \int_{T_\gamma}^\infty \Omega(t) A \Omega^*(t) dt$. The operator Π_1 is estimated using Theorem 5. The Hilbert-Schmidt norm of Π_2 is bounded by

$$\int_{T_\gamma}^\infty e^{-2t/T_{\text{att}}} \|A\|_{\text{H-S}} dt .$$

The Schwartz kernel of Π_1 is the kernel of a Ψ DO while the kernel of Π_2 is estimated as follows: it is enough to estimate, for any $l \in \mathbb{N}$, the Sobolev $H^{-ml} \rightarrow H^{ml}$ norm of $\Omega^*(t) A \Omega(t)$. This is done using the same trick than in Section 10.9.

5.5 Applications

From Appendix B, we know that, under some genericity assumption, the Green's function admits a WKB representation. The correlation admits also a WKB expansion which is obtained by changing the amplitude, but not the phase.

Assuming still $\tau > 0$, we see that the correlation $C_{A,B}(\tau)$ is close to the kernel of a Fourier integral operator associated to the canonical transformation Φ_τ . It is given as a sum over all classical trajectories γ from B to A in time τ of Cauchy data (B, ξ_B) with $H_0(B, \xi_B) = -\omega$ for which the backward trajectories crosses the support of the power spectrum $ll^*(\cdot, \cdot, \omega)$. If γ is such a trajectory and B and A are non conjugated along it, this contribution is given by the well known Van Vleck formula⁸ multiplied by $\pi(B, \xi_B)$.

Corollary 1 *Let K be the support of $l(x, \xi, -H_0(x, \xi))$ and K_∞ the smallest closed set of T^*X invariant by the Hamiltonian flow of H_0 and containing K . The Hamiltonian H_0 restricted to K_∞ can be recovered from the knowledge of $\hat{C}(\tau)$ for $0 < |\tau| \leq \tau_0$.*

In particular, if there exists (x, ξ) with $H_0(x, \xi) = E$ and $l(x, \xi, -E) \neq 0$ and if Φ_t is ergodic on $H_0^{-1}(E)$, then we can recover the flow Φ_t on $H_0^{-1}(E)$.

⁸The Van Vleck formula expresses the propagator $P(\tau, A, B)$ as a sum of $p_\gamma = (2\pi i \varepsilon)^{-d/2} a_\gamma(\varepsilon) \exp(iS(\gamma)/\varepsilon)$ with $a_\gamma(\varepsilon)$ a formal power series in ε with a first term explicitly computable

6 High frequency limit of the correlation: wave equations

6.1 General wave equations

We want to derive results similar to those of Theorem 2 in the case of wave equations. For that, we introduce a matrix version of what is done in Section 5:

$$\frac{\varepsilon}{i}\mathbf{u}_t + \hat{H}\mathbf{u} = \frac{\varepsilon}{i}\mathbf{f} \quad (13)$$

where

- \hat{H} is a matrix of ε -pseudo-differential operators:

$$\hat{H} := \text{Op}(H_0 + \varepsilon H_1)$$

with

- The **principal symbol** $H_0(x, \xi) : T^*X \rightarrow \text{Herm}(\mathbb{C}^N)$, elliptic of degree 1. The eigenvalues of $H_0(x, \xi)$ are $\lambda_1(x, \xi) \leq \lambda_j(x, \xi) \leq \lambda_N(x, \xi)$; the corresponding eigenspaces $E_j(x, \xi) \subset \mathbb{C}^N$ are called the *polarisations*.
- The **sub-principal symbol** $H_1(x, \xi)$ admits some positivity property which controls the attenuation: there exists $k > 0$, such that the quadratic forms

$$h_{1;x,\xi}(v) := \Im \langle H_1(x, \xi)v | v \rangle$$

restricted to the polarisation bundles are $\leq -k\|v\|^2$ with $k > 0$.

- The **random field** \mathbf{f} is given by $\mathbf{f} = \text{Op}(l(x, \xi, \omega))\mathbf{w}$:
 - $\mathbf{w} = (w_1, \dots, w_N)$ are independent white noises on $X \times \mathbb{R}$
 - the matrix l is smooth, compactly supported w.r. to (x, ξ) and its Fourier transform w.r. to ω is compactly supported. The power spectrum of f is $(2\pi\varepsilon)^{-(d+1)}l^*(x, \xi, \omega)$.

Example 6.1

$$u_{tt} + 2au_t - \Delta u = f$$

where $a > 0$ and $-\Delta$ is the Laplace-Beltrami operator on a smooth complete Riemannian manifold X . We take

$$\mathbf{u} = \begin{pmatrix} \varepsilon\sqrt{-\Delta}u \\ -i\varepsilon u_t \end{pmatrix}$$

and

$$\hat{H} = \begin{pmatrix} 0 & -\varepsilon\sqrt{-\Delta} \\ \varepsilon\sqrt{-\Delta} & 2i\varepsilon a(x) \end{pmatrix}$$

and

$$H_0 = \begin{pmatrix} 0 & -\|\xi\| \\ \|\xi\| & 0 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & 0 \\ 0 & -2ia \end{pmatrix}.$$

6.2 Normal forms

Let us choose $z_0 = (x_0, \xi_0) \in T^*X$ be so that $\lambda_1(z_0) < \lambda_2(z_0) < \dots < \lambda_N(z_0)$. Then using a Ψ DO gauge transform, we can reduce \hat{H} to

$$\hat{H}_{\text{normal}} \equiv \text{Diag}(\text{Op}(\lambda_j + \varepsilon\lambda_j^1 + \dots)) \quad (14)$$

near z_0 , with

$$\lambda_j^1(x, \xi) = \pi_j H_1 \pi_j + iK_j,$$

where K is anti-Hermitian. This is done for example in the paper [12], see also [25].

6.3 The main result

We will use two Assumptions:

1. **No mode conversions in the frequency window:** We will assume that we work in a frequency window $[a, b] \subset \mathbb{R}$ so that for any (x, ξ) and for any $1 \leq j \leq N$ so that $\lambda_j(x, \xi) \in I$, we have, for $k \neq j$, $\lambda_k(x, \xi) \neq \lambda_j(x, \xi)$.
2. **Support of l :** we will assume that $l(x, \xi, \omega)$ vanishes if $\omega \notin I$ and also if $\omega \in I$ and (x, ξ) is close to B .

Theorem 3 *With the two previous Assumptions, the correlations $C_{A,B}^{j,k}(\tau)$ are given*

- If $j = k$, as in Theorem 2 with $H_0 = \lambda_j$
- If $j \neq k$, $C_{A,B}^{j,k}(\tau) = O(\varepsilon^\infty)$

In particular, for the elastic wave equation, the correlation between S-waves and P-waves vanishes.

7 Source of the noise on a submanifold

We want to discuss the case where the source noise is located on a submanifold Y of the manifold X . We will assume that our dynamics is in a normal form as in Section 6.2, Equation (14). In this case, the non-diagonal part of the correlation may not vanish. If we consider $C_{A,B}^{j,k}(\tau)$ with $\tau > 0$, the following pairs of rays (γ_j, γ_k) contribute in a non trivial way: there exists $t_Y < 0$ with $y_0 = \gamma_j(t_Y) = \gamma_k(t_Y) \in Y$, $\gamma_j(0) = B$, $\gamma_k(\tau) = A$ and the restrictions of the momenta of both trajectories to the tangent space $T_{y_0}Y$ coincide.

This result could be useful as a way to get informations on the source noise: for example, in the case of seismology, is the source of oceanic noise located on the coast or in the middle of the ocean.

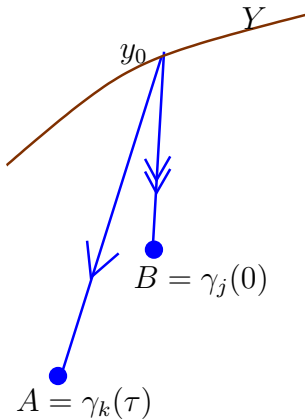


Figure 1: Rays involved in the correlation $C_{A,B}^{j,k}(\tau)$

8 Using surface waves in order to image the inner crust

8.1 Correlation at the boundary

This is specific to the case of seismology. The part of the Green's function which enters into the correlations is dominated by the *surface waves* contributions. The earth crust acts as a *wave guide* and the surface waves are driven by *effective Hamiltonians* given by a vertical Sturm-Liouville equation. Solving an *inverse spectral problem* for these Sturm-Liouville equation allows imaging of the crust. Details are given in [8]. The inverse spectral problem is solved in [9].

8.2 The model

We work locally in $X = \{(\mathbf{x}, z) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid z \leq 0\}$. We will consider the very simple case of an acoustic wave equation near the origin of X :

$$\begin{cases} u_{tt} - \operatorname{div}(n \operatorname{gradu}) = 0 \\ u(\mathbf{x}, 0) = 0 \end{cases} \quad (15)$$

with

$$n(\mathbf{x}, z) = N(\mathbf{x}, z, \frac{z}{\varepsilon})$$

and $N(\mathbf{x}, z, Z) : \mathbb{R}^{d-1} \times \mathbb{R}_- \times \mathbb{R}_- \rightarrow \mathbb{R}_+$ a non negative function which is independent of Z for $Z \leq Z_0 < 0$. We will assume that N is smooth.

We plan to see that Equation (15) admits, as $\varepsilon \rightarrow 0$, asymptotic solutions of frequency of order ε^{-1} located near the boundary. Moreover these solutions are determined by solving an effective pseudo-differential equation on the boundary $\partial X = \mathbb{R}^{d-1} \times \{0\}$. We will assume that

$$0 < \inf_{Z \leq 0} N(\mathbf{x}, 0, Z) < N(\mathbf{x}, 0, -\infty) .$$

Physically, it means that the propagation speed at some points very close to the boundary is smaller than inside the medium. This kind of assumption is usually satisfied in seismology where the speed of elastic waves into surface sediments layers is smaller than the speed inside the rocks below the sediments. As a consequence, the crust will act as a *wave guide*.

8.3 A Sturm-Liouville operator:

let us consider, for each $(\mathbf{x}, \xi) \in T^*\partial X$, the self-adjoint differential operator $L_{\mathbf{x}, \xi}$ on the half line $Z \leq 0$, with Dirichlet boundary condition at $Z = 0$, defined by:

$$L_{\mathbf{x}, \xi} v := -\frac{d}{dZ}(N(\mathbf{x}, 0, Z) \frac{dv}{dZ}) + N(\mathbf{x}, 0, Z) |\xi|^2 v . \quad (16)$$

The spectrum of $L_{\mathbf{x}, \xi}$ consists of a finite discrete spectrum and a continuous spectrum $[N(\mathbf{x}, 0, Z_0) |\xi|^2, +\infty[$. $L_{\mathbf{x}, \xi}$ admits, for ξ large enough, a non empty discrete spectrum of simple eigenvalues

$$\inf_Z N(\mathbf{x}, 0, Z) |\xi|^2 < \lambda_1(\mathbf{x}, \xi) < \dots < \lambda_j(\mathbf{x}, \xi) < \dots < \lambda_k(\mathbf{x}, \xi) < N(\mathbf{x}, 0, Z_0) |\xi|^2 ,$$

which depend smoothly of (\mathbf{x}, ξ) . In order to see that, we can interpret $|\xi|^{-2} L_{\mathbf{x}, \xi}$ as a semi-classical Schrödinger type operator with an effective Planck constant $|\xi|^{-1}$ and a principal symbol

$$p_{\mathbf{x}}(Z, \zeta) = N(\mathbf{x}, 0, Z) (\zeta^2 + 1)$$

which admits a well near $(Z, \zeta) = (0, 0)$. We should however take care of the fact that the number k depends on (\mathbf{x}, ξ) and goes to ∞ as ξ does.

8.4 WKB solutions of the stationary wave equation:

we want to build the effective surface Hamiltonians which describe the surface waves. Let us start with the:

Lemma 3 *Let us consider the operator \hat{H} defined by:*

$$\hat{H}u := -\varepsilon^2 \operatorname{div}(n \operatorname{grad} u) \quad (17)$$

acting on functions on X vanishing at $z = 0$ (Dirichlet boundary conditions).

Let us choose $\lambda(\mathbf{x}, \xi)$ an eigenvalue of $L_{\mathbf{x}, \xi}$ depending smoothly of $(\mathbf{x}, \xi) \in U$, where U is a bounded open set of $T^\partial X$, and $\varphi(\mathbf{x}, \xi, \cdot)$ a normalised associated eigenfunction. There exists:*

- *An asymptotic expansion*

$$\Phi_\varepsilon = \sum_{m=0}^{\infty} \varphi_m(\mathbf{x}, \xi, \frac{z}{\varepsilon}) \varepsilon^m$$

with $\varphi_0 = \varphi$ and the $\varphi_m(\mathbf{x}, \xi, \cdot)$'s are smoothly dependent of (\mathbf{x}, ξ) with values in the domain of $L_{\mathbf{x}, \xi}$. The φ_m 's ($m \geq 1$) are unique if they are assumed to be orthogonal to φ .

- *A symbol*

$$a_\varepsilon(\mathbf{x}, \xi) = \sum_{m=0}^{\infty} a_m(\mathbf{x}, \xi) \varepsilon^m$$

with $a_0 = \lambda$

such that we have the following identity of formal power series in ε :

$$\hat{H} \left(\Phi_\varepsilon(\mathbf{x}, \xi, \frac{z}{\varepsilon}) e^{i\langle \mathbf{x} | \xi \rangle / \varepsilon} \right) = a_\varepsilon(\mathbf{x}, \xi) \Phi_\varepsilon(\mathbf{x}, \xi, \frac{z}{\varepsilon}) e^{i\langle \mathbf{x} | \xi \rangle / \varepsilon} . \quad (18)$$

8.5 Interpretation: effective dynamics

Lemma 3 can be reformulated as saying that the effective surface wave equations are, for each eigenvalue $\lambda_j(\mathbf{x}, \xi)$ of $L_{\mathbf{x}, \xi}$, of the form

$$\varepsilon^2 u_{tt} + \operatorname{Op}(a_\varepsilon(\mathbf{x}, \xi))u = 0 .$$

9 Random scattered waves

In this last independent section, we will revisit what was may be the starting point of this story by Keiiti Aki in the fifties: he wanted to measure the speed of propagation of seismic plane waves by averaging over the incidence directions. It turns out that we get nice formulae even for non homogeneous media.

9.1 Introduction

Let us consider the propagation of waves outside a compact domain D in the Euclidian space \mathbb{R}^d . Let us put $\Omega = \mathbb{R}^d \setminus D$. We can assume for example Neumann boundary conditions. We will denote by Δ_Ω the previous self-adjoint operator. So our stationary wave equation is the Helmholtz equation $\Delta_\Omega f + k^2 f = 0$ with the boundary conditions. We consider a bounded interval $I = [E_-, E_+] \subset]0, +\infty[$ and the Hilbert subspace \mathcal{H}_I of $L^2(\Omega)$ which is the image of the spectral projector P_I of our Laplace operator Δ_Ω .

Let us compute the integral kernel $\Pi_I(x, y)$ of P_I defined by:

$$P_I f(x) = \int_\Omega \Pi_I(x, y) f(y) |dy|$$

into 2 different ways:

1. From general spectral theory
2. From scattering theory.

Taking the derivatives of $\Pi_I(x, y)$ w.r. to E_+ , we get a simple general and exact relation between the correlation of scattered waves and the Green's function confirming the calculations from [21] in the case where D is a disk.

9.2 $\Pi_I(x, y)$ from spectral theory

Using the resolvent kernel (Green's function) $G(k, x, y) = [(k^2 + \Delta_\Omega)^{-1}](x, y)$ for $\Im k > 0$ and the Stone formula, we have:

$$\Pi_I(x, y) = -\frac{2}{\pi} \Im \left(\int_{k_-}^{k_+} G(k + i0, x, y) k dk \right)$$

Taking the derivative w.r. to k_+ of $\Pi_{[E_-, k^2]}(x, y)$, we get

$$\frac{d}{dk} \Pi_{[E_-, k^2]}(x, y) = \frac{-2k}{\pi} \Im(G(k + i0, x, y)) . \quad (19)$$

9.3 Short review of scattering theory

They are many references for scattering theory: for example [18].

Let us define the plane waves

$$e_0(x, \mathbf{k}) = e^{i\langle \mathbf{k} | x \rangle} .$$

We are looking for solutions

$$e(x, \mathbf{k}) = e_0(x, \mathbf{k}) + e^s(x, \mathbf{k})$$

of the Helmholtz equation in Ω where e^s , the scattered wave satisfies the so-called Sommerfeld radiation condition:

$$e^s(x, \mathbf{k}) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}} \left(e^\infty\left(\frac{x}{|x|}, \mathbf{k}\right) + O\left(\frac{1}{|x|}\right) \right), \quad x \rightarrow \infty.$$

The complex function $e^\infty(\hat{x}, \mathbf{k})$ is usually called the *scattering amplitude*.

It is known that the previous problem admits a unique solution. In more physical terms, $e(x, \mathbf{k})$ is the wave generated by the full scattering process from the plane wave $e_0(x, \mathbf{k})$. Moreover we have a generalised Fourier transform:

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(\mathbf{k}) e(x, \mathbf{k}) |d\mathbf{k}|$$

with

$$\hat{f}(\mathbf{k}) = \int_{\mathbb{R}^d} \overline{e(y, \mathbf{k})} f(y) |dy|.$$

From the previous generalised Fourier transform, we can get the kernel of any function $\Phi(-\Delta_\Omega)$ as follows:

$$[\Phi(-\Delta_\Omega)](x, y) = (2\pi)^{-d} \int_{\mathbb{R}^d} \Phi(k^2) e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\mathbf{k}|. \quad (20)$$

9.4 $\Pi_I(x, y)$ from scattering theory

Using Equation (20) with $\Phi = 1_I$ the characteristic functions of some bounded interval I , we get:

$$\Pi_I(x, y) = (2\pi)^{-d} \int_{E_- \leq k^2 \leq E_+} e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\mathbf{k}|.$$

Using polar coordinates and defining $|d\sigma|$ as the usual measure on the unit $(d-1)$ -dimensional sphere, we get:

$$\Pi_I(x, y) = (2\pi)^{-d} \int_{E_- \leq k^2 \leq E_+} \int_{\mathbf{k}^2=E} e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} k^{d-1} dk |d\sigma|.$$

We will denote by σ_{d-1} the total volume of the unit sphere in \mathbb{R}^d : $\sigma_0 = 2$, $\sigma_1 = 2\pi$, $\sigma_2 = 4\pi, \dots$.

Taking the same derivative as before, we get:

$$\frac{d}{dk} \Pi_{[E_-, k^2]}(x, y) = \frac{k^{d-1}}{(2\pi)^d} \int_{\mathbf{k}^2=E} e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\sigma|.$$

This integral can be interpreted, using the correlation $C_E^{\text{scatt}}(x, y)$ of random scattered waves of energy E defined by

$$C_E^{\text{scatt}}(x, y) = \frac{1}{\sigma_{d-1}} \int_{\mathbf{k}^2=E} e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\sigma|,$$

as

$$\frac{d}{dk} \Pi_{[E_-, k^2]}(x, y) = \frac{k^{d-1} \sigma_{d-1}}{(2\pi)^d} C_E^{\text{scatt}}(x, y). \quad (21)$$

9.5 Correlation of scattered plane waves and Green's function: the scalar case

From Equations (19) and (21), we get:

$$\frac{k^{d-1} \sigma_{d-1}}{(2\pi)^d} C_E^{\text{scatt}}(x, y) = -\frac{2k}{\pi} \Im(G(k + i0, x, y)).$$

Hence

$$C_E^{\text{scatt}}(x, y) = \frac{-2^{d+1} \pi^{d-1}}{k^{d-2} \sigma_{d-1}} \Im(G(k + i0, x, y)).$$

For later use, we put

$$\gamma_d(k) = \frac{2^{d+1} \pi^{d-1}}{k^{d-2} \sigma_{d-1}}. \quad (22)$$

9.6 The case of elastic waves

We will consider the vectorial stationary elastic wave equation in the domain Ω :

$$\hat{H}\mathbf{u} - \omega^2 \mathbf{u} = 0,$$

with symmetric boundary conditions, where

$$\hat{H}\mathbf{u} = -a \Delta \mathbf{u} - b \text{grad div } \mathbf{u}.$$

where a and b are constant:

$$a = \frac{\mu}{\rho}, \quad b = \frac{\lambda + \mu}{\rho}$$

with λ , μ the Lamé's coefficients and ρ the density of the medium.

- *The case $\Omega = \mathbb{R}^d$*

We want to derive the spectral decomposition of \hat{H} from the Fourier inversion formula. Let us choose, for $\mathbf{k} \neq 0$, by $\hat{\mathbf{k}}, \hat{\mathbf{k}}_1, \dots, \hat{\mathbf{k}}_{d-1}$ an orthonormal basis of \mathbb{R}^d with $\hat{\mathbf{k}} = \frac{\mathbf{k}}{k}$ such that these vectors depends in a measurable way of \mathbf{k} . Let us introduce $P_P^{\mathbf{k}} = \hat{\mathbf{k}}\hat{\mathbf{k}}^*$ the orthogonal projector onto $\hat{\mathbf{k}}$ and $P_S^{\mathbf{k}} = \sum_{j=1}^{d-1} \hat{\mathbf{k}}_j \hat{\mathbf{k}}_j^*$ so that $P_P + P_S = \text{Id}$. Those projectors correspond respectively to the polarisations of P - and S -waves.

We have

$$\Pi_I(x, y) = (2\pi)^{-d} \int_{\omega^2 \in I} \omega^{d-1} d\omega \left((a+b)^{-d/2} \int_{k^2 = \omega^2 / (a+b)^2} e^{i\mathbf{k}(x-y)} P_P^{\mathbf{k}} d\sigma + a^{-d/2} \int_{k^2 = \omega^2 / a^2} e^{i\mathbf{k}(x-y)} P_S^{\mathbf{k}} d\sigma \right) .$$

using the plane waves

$$e_P^O(x, \mathbf{k}) = e^{i\mathbf{k}x} \hat{\mathbf{k}}$$

and

$$e_{S,j}^O(x, \mathbf{k}) = e^{i\mathbf{k}x} \hat{\mathbf{k}}_j$$

we get the formula:

$$\Pi_I(x, y) = (2\pi)^{-d} \int_{\omega^2 \in I} \omega^{d-1} d\omega \left((a+b)^{-d/2} \int_{k^2 = \omega^2 / (a+b)^2} e_P^O(x, \mathbf{k}) (e_P^O(y, \mathbf{k}))^* d\sigma + a^{-d/2} \sum_{j=1}^{d-1} \int_{k^2 = \omega^2 / a^2} e_{S,j}^O(x, \mathbf{k}) (e_{S,j}^O(y, \mathbf{k}))^* d\sigma \right) .$$

- *Scattered plane waves*

There exists scattered plane waves

$$e_P(x, \mathbf{k}) = e_P^O(x, \mathbf{k}) + e_P^s(x, \mathbf{k})$$

$$e_{S,j}(x, \mathbf{k}) = e_{S,j}^O(x, \mathbf{k}) + e_{S,j}^s(x, \mathbf{k})$$

satisfying the Sommerfeld condition and from which we can deduce the spectral decomposition of \hat{H} .

- *Correlations of scattered plane waves and Green's function*

Following the same path as for scalar waves, we get an identity which holds now for the full Green's tensor $\mathfrak{S}\mathbf{G}(\omega + i0, x, y)$:

$$\mathfrak{S}\mathbf{G}(\omega + i0, x, y) = -\gamma_d(\omega) \left((a+b)^{-d/2} \int_{k^2 = \omega^2 / (a+b)^2} e_P(x, \mathbf{k}) (e_P(y, \mathbf{k}))^* d\sigma + a^{-d/2} \sum_{j=1}^{d-1} \int_{k^2 = \omega^2 / a^2} e_{S,j}(x, \mathbf{k}) (e_{S,j}(y, \mathbf{k}))^* d\sigma \right) ,$$

with $\gamma_d(\omega)$ defined by Equation (22).

This formula expresses the fact that the correlation of scattered plane waves, randomised with the appropriate weights, is proportional to the Green's tensor.

10 Appendix A: a review about pseudo-differential operators

10.1 Basic calculus

We will define the pseudo-differential operators (Ψ DO's) on \mathbb{R}^d . Ψ DO's on manifolds are defined locally by the same formulæ. More details can be found in [7, 10, 11, 27].

Definition 6 (Classical symbols) • The space Σ_k of symbols of degree k is the space of smooth functions $p : T^*\mathbb{R}^d \rightarrow \mathbb{C}$ which satisfy

$$\forall \alpha, \beta, |D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{k - |\beta|} .$$

- A classical symbol of degree m and order l is a family of functions

$$p_\varepsilon : T^*\mathbb{R}^d \rightarrow \mathbb{C}$$

which admits an asymptotic expansion

$$p_\varepsilon \sim \sum_{j=0}^{\infty} \varepsilon^{j+l} p_j(x, \xi)$$

with $p_j \in \Sigma_{m-j}$. We will denote this space by $S_{\text{class}}^{m,l}$.

Definition 7 (Pseudo-differential operators) An ε -pseudo-differential operator P (a Ψ DO) of degree m and order l on \mathbb{R}^d is given locally by the kernel

$$[P](z, z') = (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} e^{i(z-z'|\zeta)/\varepsilon} p_\varepsilon \left(\frac{z+z'}{2}, \zeta \right) |d\zeta|$$

where $p_\varepsilon(z, \zeta)$, the so-called (total) symbol of P , is in $S_{\text{class}}^{m,l}$.

We will denote $P = \text{Op}(p_\varepsilon)$.

The kernel of P is then given by:

$$[P](z, z') = \varepsilon^{-d} \tilde{p} \left(\frac{z+z'}{2}, \frac{z'-z}{\varepsilon} \right) \quad (23)$$

with \tilde{p} the partial Fourier transform of $p_\varepsilon(z, \zeta)$ w.r. to ζ . Very often, one is only able to compute the symbol p_0 which is called the *principal symbol* of P .

The most basic fact about Ψ DO's is the fact they can be composed: if $P = \text{Op}(p)$ and $Q = \text{Op}(q)$, we have $PQ = \text{Op}(pq + O(\varepsilon))$. The composition formula for the total symbol is given by the *Moyal \star -product* $\text{Op}(a) \circ \text{Op}(b) = \text{Op}(a \star b)$ with

$$a \star b \equiv ab + \frac{\varepsilon}{2i} \{a, b\} + \sum_{j=2}^{\infty} \varepsilon^j P_j(a, b)$$

where

- $\{a, b\}$ is the Poisson bracket
- $P_j(a, b)$ is a bi-linear bi-differential operator, homogeneous of degree j with respect to a and b .

The *Moyal bracket* of a and b

$$\{\{a, b\}\} \cong -i \sum_{j=1}^{\infty} \varepsilon^j \{a, b\}_j$$

gives the symbol of the commutator of 2 pseudo-differential operators in terms of their symbols. $\{., .\}_1$ is the Poisson bracket.

10.2 The symbols $S_\delta^{m,l}$

In order to formulate the Egorov Theorem for long times, we will need more sophisticated classes of symbols.

Definition 8 ($S_\delta^{m,l}$) For any real numbers m, l and any δ with $0 \leq \delta < \frac{1}{2}$, a symbol $a \in S_\delta^{m,l}$ is a smooth function on T^*X depending on ε which satisfies

$$\forall \alpha \in \mathbb{N}^{2d}, \exists C_\alpha > 0, \forall (x, \xi) \in T^*X, |D^\alpha a(x, \xi)| \leq C_\alpha \varepsilon^{l-\delta|\alpha|} \langle \xi \rangle^m,$$

where $\langle \xi \rangle := 1 + \|\xi\|$.

We have $S_{\text{class}}^{m,l} \subset S_0^{m,l}$.

We can associate to $a \in S_\delta^{m,l}$ a pseudo-differential operator $\text{Op}(a)$ using formula (23). Such operators obey a nice pseudo-differential calculus, see [10, 13]: if, for $j = 1, 2$, $a_j \in S_{\delta_j}^{m_j, l_j}$, $A_j := \text{Op}(a_j)$ and $\delta := \max(\delta_1, \delta_2)$, we have $A_1 \circ A_2 = \text{Op}(a_1 \star a_2)$ with $a_1 \star a_2 \in S_\delta^{m_1+m_2, l_1+l_2}$ and admits the asymptotic expansion given by the Moyal \star -product. We have $P_j(a_1, a_2) \in S_\delta^{m_1+m_2, l_1+l_2-j(\delta_1+\delta_2)}$. In particular $[A_1, A_2] = \frac{\varepsilon}{i} \text{Op}(\{a_1, a_2\}) + \text{Op}(b)$ with $b \in S_\delta^{m_1+m_2, l_1+l_2+2(1-\delta_1-\delta_2)}$.

10.3 Asymptotic expansions

The basic tool in order to find a ΨDO is an inductive construction based on the realisation of any asymptotic expansion: given a sequence $a_j \in S_\delta^{m, l_j}$ with $l_1 < l_2 < \dots < l_j < \dots$ and $l_j \rightarrow +\infty$ as $j \rightarrow \infty$, there exists $a \in S_\delta^{m, l_1}$ such that

$$a \cong \sum_{j=1}^{\infty} a_j$$

meaning that

$$\forall l, a - \sum_{j=1}^{l-1} a_j \in S_\delta^{m, l_l}.$$

Such an a is uniquely defined modulo a remainder term r which lies in $S_\delta^{m,\infty}$. The corresponding ΨDO $\text{Op}(a)$ is well defined modulo $O(\varepsilon^\infty)$. This implies, if $m \leq 0$ and a compactly supported in x , that the $L^2 \rightarrow L^2$ norm of $\text{Op}(r)$ is $O(\varepsilon^\infty)$.

10.4 Some Lemmas

Lemma 4 *Let us consider the operator \mathcal{L} given by*

$$\mathcal{L} := \frac{1}{2\pi\varepsilon} \int \int \Omega(-t) e^{it\omega/\varepsilon} \text{Op}_x(L(x, \xi, \omega)) d\omega dt$$

with $L \in \Sigma_{-\infty}(X \times \mathbb{R})$ with a with an ω Fourier transform compactly supported. Then \mathcal{L} is a ΨDO of principal symbol $l_0(x, \xi) = L(x, \xi, -H_0(x, \xi))$.

he proof is a simple corollary of

Lemma 5 *If $A = \text{Op}(a)$ with a compactly supported, the operator $B = \exp(it\hat{H})A$ is a ΨDO of principal symbol $b = \exp(itH_0)a$.*

which itself is a consequence of the functional calculus for ΨDO 's.

Appendix B: asymptotics of the Green function and classical dynamics

10.5 The scalar case

To the Hamiltonian $H_0 : T^*X \rightarrow \mathbb{R}$, we associate the (ray) dynamics defined by

$$\frac{dx_j}{dt} = \frac{\partial H_0}{\partial \xi_j}, \quad \frac{d\xi_j}{dt} = -\frac{\partial H_0}{\partial x_j}. \quad (24)$$

Let us denote by ϕ_t the flow of the Hamiltonian vector field X_{H_0} defined in Equation (24). If $x_0, y_0 \in X$ and $\phi_t(y_0, \eta_0) = (x_0, \xi_0)$, we say that x_0 and y_0 are *not conjugate* along the ray $\gamma(s) = \phi_s(y_0, \eta_0)$, $0 \leq s \leq t$, if $\delta\eta \rightarrow d\phi_t(0, \delta\eta)$ is invertible at the point (y_0, η_0) . In this case the action $S(t, x_0, y_0)$ is defined by $S(t, x_0, y_0) = \int_\gamma \xi dx - H_0(x, \xi) dt$. $S(t, \cdot, \cdot)$ is a generating function of ϕ_t meaning that $\phi_t(y, -\partial S/\partial y) = (x, \partial S/\partial x)$ for (y, η) close to (y_0, η_0) .

The action S is useful in order to describe a WKB expansion of the propagator $P(t, x, y)$ for (x, y) close to (x_0, y_0) : $P(t, x, y)$ is a sum of contributions $P_\gamma(t, x, y)$ of rays going from y to x in the time t . Assuming that x and y are not conjugate along γ , we have

$$P_\gamma(t, x, y) \equiv (2\pi i\varepsilon)^{-d/2} \left(\sum_{j=0}^{\infty} \varepsilon^j a_{j,\gamma}(t, x, y) \right) e^{iS(t,x,y)/\varepsilon}.$$

10.6 The matrix case

If we start with a dispersion relation

$$\det(\omega \text{Id} - \mathbf{H}_0(x, \xi)) = \mathbb{D}(x, \xi, \omega) = 0$$

of degree N , we can (at least outside a *mode conversion set*, rewrite

$$\mathbb{D}(x, \xi, \omega) = \prod_{j=0}^M (\omega - \lambda_j(x, \xi))^{d_j}$$

which gives M ray dynamics λ_j associated to different polarisations

$$P_j = \ker (\mathbf{H}_0(x, \xi) - \lambda_j(x, \xi) \text{Id}) .$$

Appendix C: long time Egorov Theorem

The main reference for this part is [3]. They get only an operator norm result, while, under some ellipticity assumption, we get a pointwise result.

10.7 Ehrenfest time

Definition 9 (Ehrenfest time) *Let X be a smooth manifold of dimension d . Let us consider a smooth proper Hamiltonian H_0 on T^*X . We fix also some compact interval I .*

- *The Liapounov exponent $\Lambda_I = \Lambda \in [0, +\infty[$ is the infimum of the real numbers λ for which the differential of the Hamiltonian flow Φ_t of H_0 satisfies the following uniform estimate:*

$$\exists C > 0, \forall z \in T^*X \text{ with } H_0(z) \in I, \forall t \geq 0, \|d\Phi_t(z)\| \leq Ce^{\lambda t} .$$

- *Let us denote by $\varepsilon > 0$ the semi-classical parameter. The Ehrenfest time, $T_{\text{Ehrenfest}} \in]0, +\infty]$ is defined as*

$$T_{\text{Ehrenfest}} := |\log \varepsilon| / \Lambda .$$

We will assume in what follows that $T_{\text{Ehrenfest}}$ is finite. For any time smaller than $T_{\text{Ehrenfest}}$, any cell of diameter ε is not expanded to the whole phase space. We will need a smaller time:

Definition 10 *Let us give $0 < \gamma << \frac{1}{2}$. The time T_γ is defined as*

$$T_\gamma := \left(\frac{1}{2} - \gamma \right) T_{\text{Ehrenfest}} .$$

The previous definition will be used via the

Lemma 6 *Assume that we have a flow Φ_t on a compact manifold X with a Liapounov exponent Λ , then we have the following uniform estimates*

$$\forall \alpha, |D^\alpha \Phi_t(z)| = O^+(e^{\Lambda|t|}) .$$

10.8 Egorov theorem

Definition 11 An admissible Hamiltonian of degree m is $\hat{H} := \text{Op}(H_0 + \varepsilon H_1)$ with

- $H_0 \in \Sigma_m$ a real valued function which is elliptic:

$$\exists C > 0, \forall (x, \xi) \in T^*X, H_0(x, \xi) \geq C \langle \xi \rangle^m .$$

- $H_1 \in \Sigma_{m-1}$ and $h_1 = \Im H_1 \leq -k$ with $k \geq 0$.

Example 10.1 A typical admissible Hamiltonian is the Schrödinger operator $\hat{H} = -\varepsilon^2 \Delta + V(x)$ with $V(x) \geq 0$.

We define the dynamics, for $t \geq 0$, by:

$$\Omega(t) = \exp(-it\hat{H}/\varepsilon) .$$

Theorem 4 (Egorov Theorem) Let us give \hat{H} an admissible Hamiltonian and $A(0) = \text{Op}(a)$ with $a \in C_o^\infty(H_0^{-1}(I))$. Then the operator $A(t) = \Omega(t)A(0)\Omega^*(t)$ is, for t fixed, a pseudo-differential operator whose symbol $a(t) \equiv \sum_{j=0}^\infty \varepsilon^j a_j(t)$ belongs to $S_0^{-\infty, 0}$ and satisfies

$$a_0(t)(z) = \exp\left(2 \int_{-t}^0 h_1(\Phi_s(z)) ds\right) a(\Phi_{-t}(z)) . \quad (25)$$

Let us remark that the previous result is a priori *uniform* only in a fixed interval $t \in [0, T_0]$ independent of ε .

Let us describe the inductive construction of the whole symbol $a(t)$ in terms of the Moyal brackets. We compute

$$\frac{d}{ds} \Omega(t-s)B(s)\Omega^*(t-s) = \Omega(t-s) \left(B'(s) + \frac{i}{\varepsilon} (\hat{H}B(s) - B(s)\hat{H}^*) \right) \Omega^*(t-s) .$$

If $\frac{d}{ds} \Omega(t-s)B(s)\Omega^*(t-s) \equiv 0$ with $B(0) = A(0)$ as formal powers series in ε , we have $B(t) = \Omega(t)A(0)\Omega^*(t)$ modulo a smoothing operator, hence Egorov Theorem.

We get the following equations by looking at the ε^j term:

$$\frac{d}{dt} b_j + \{H_0, b_j\}_1 - 2h_1 b_j = c_j ,$$

with $c_0 = 0$ and, for $j \geq 1$, $c_j = \sum_{k=0}^{j-1} P_{k,j}(b_k)$ where the $P_{k,j}$'s are linear differential operators constructed by using the Moyal products with H_0 and H_1 .

10.9 Long time Egorov Theorem

We want to extend Egorov Theorem to times going to infinity with ε . From Lemma 6, we get

Lemma 7 *Uniformly, for $t \in [0, T_\gamma(\varepsilon)]$, $a_0(t)$ belongs to $S_\delta^{-\infty, 0}$ with $\delta = \frac{1}{2} - \gamma < \frac{1}{2}$ and the constants in the estimates of the derivatives of the symbol are uniform in t .*

The main result is:

Theorem 5 (Long time Egorov Theorem) *Let us give \hat{H} admissible (Section 10.8) and $a \in C_o^\infty(T^*X)$, for any t with $t \in [0, T_\gamma]$ and with uniform symbol estimates w.r. to t , the operator $A(t) := \Omega(t)A\Omega^*(t)$ is a pseudo-differential operator whose total symbol is given as in the classical Egorov Theorem 4 by*

$$A(t) \equiv \text{Op} \left(\sum_{j=0}^{\infty} \varepsilon^j a_j(t) \right)$$

where $\varepsilon^j a_j(t) \in S_\delta^{-\infty, j(1-2\delta)}$.

More precisely, for any M and k , there exists N so that if $R_N(t) = A(t) - \text{Op} \left(\sum_{j=0}^N \varepsilon^j a_j(t) \right)$, the C^k norm of the kernel of $R_N(t)$ satisfies, uniformly for $t \in [0, T_\gamma]$, the bound

$$\| [R_N(t)] \|_{C^k(X \times X)} = O(\varepsilon^M) .$$

Remark 2 *The long time Egorov Theorem was first proved by Bambusi, Graffi and Paul [1] and improved by Bouzouina and Robert [3].*

Our estimate of the remainder term is better than the one given in [3] which is only an L^2 operator norm. We need the ellipticity of \widehat{H}_0 .

Proof.–

In [3], it is proved that, for any M' , there exists N so that:

$$\frac{d}{ds} \Omega(t-s) \text{Op} \left(\sum_{j=0}^N \varepsilon^j a_j(s) \right) \Omega^*(t-s) = \Omega(t-s) \text{Op}(r_N(s)) \Omega^*(t-s)$$

with $r_N(s) \in S_\delta^{-\infty, M'}$ uniformly in $[0, T_\gamma]$. It is then enough to use ellipticity in the following way: we introduce $P := \widehat{H} + C$ with C large enough. We get

$$P^l \Omega(t-s) \text{Op} \left(\sum_{j=0}^N \varepsilon^j a_j(s) \right) \Omega^*(t-s) (P^*)^l = \Omega(t-s) P^l \text{Op} \left(\sum_{j=0}^N \varepsilon^j a_j(s) \right) (P^*)^l \Omega^*(t-s)$$

and the derivative

$$\frac{d}{ds} \left(P^l \Omega(t-s) \text{Op} \left(\sum_{j=0}^N \varepsilon^j a_j(s) \right) \Omega^*(t-s) (P^*)^l \right) = \Omega(t-s) P^l \text{Op}(r_N(s)) (P^*)^l \Omega^*(t-s) .$$

Defining q_N by $P^l \text{Op}(r_N(s)) (P^*)^l = \text{Op}(q_N)$, the product rules of ΨDO 's imply that q_N belongs to $S_\delta^{-\infty, M'}$. Ellipticity of \hat{H} implies that the Sobolev norms $\|\cdot\|_{lm}$ satisfy:

$$\|f\|_{lm} \asymp \varepsilon^{-lm} \|P^l f\|_{L^2} .$$

So we get the estimate of the $H^{-ml} \rightarrow H^{ml}$ norm:

$$\left\| \frac{d}{ds} \Omega(t-s) \text{Op} \left(\sum_{j=0}^N \varepsilon^j a_j(s) \right) \Omega^*(t-s) \right\|_{-lm, lm} = O(\varepsilon^{M'-2lm}) .$$

□

10.10 Integrals with two dynamics

Lemma 8 *Let us assume that we have two Hamiltonians $\hat{H}_\pm = \text{Op}(H_{0,\pm} + \varepsilon H_{1,\pm})$ satisfying the Assumptions of Section 5.1. Let us define the operator $J := \int_0^\infty \Omega_+(t+\tau) \text{Op}(a) \Omega_-^*(t) dt$, with $a \in C_o^\infty(T^*X)$. If we assume that, for $z \in \text{Support}(a)$, $H_{0,+}(z) \neq H_{0,-}(z)$ (no mode conversions in the support of a), then*

$$J = \varepsilon \Omega_+(\tau) K ,$$

with K a ΨDO of principal symbol $k_0 = -ia/(H_{0,+} - H_{0,-})$.

In particular, if a vanishes in some neighbourhood of B , $[J](A, B) = 0(\varepsilon^\infty)$.

Proof.–

There exists some ΨDO K so that $A = -i(\hat{H}_+ K - K \hat{H}_-)$. J can be rewritten as

$$J = \varepsilon \int_0^\infty \frac{d}{dt} (\Omega_+(t+\tau) K \Omega_-^*(t)) dt .$$

Integration by part gives the result.

□

11 Appendix D: time average versus ensemble average

This section is inspired from [2]. Let us define

$$C_T(\tau, A, B) := \frac{1}{T} \int_0^T u(A, t) \otimes \overline{u(B, t - \tau)} dt ,$$

we have

Theorem 6 *Let us assume that $f(\cdot, t) = Kw(\cdot, t)$ where $w(x, t)$ is a (Gaussian) white noise on $X \times \mathbb{R}$ and $K : \mathcal{H} \rightarrow \mathcal{H}$. Then*

1. *If K is Hilbert-Schmidt, $C_T(\tau, A, B) \rightarrow \langle\langle C(\tau, A, B) \rangle\rangle$, as $T \rightarrow \infty$, almost surely for almost every pair (A, B) .*
2. *If K is smoothing and \hat{H} elliptic, $C_T(\tau, A, B) \rightarrow \langle\langle C(\tau, A, B) \rangle\rangle$, as $T \rightarrow \infty$, almost surely in the $C^\infty(X \times X)$ topology.*

Proof.–

1. We will assume w.l.o.g. that u is scalar valued and real and use the following classical identity: if $f = (f_1, f_2, f_3, f_4) : \Omega \rightarrow \mathbb{R}^4$ is a Gaussian random variable,

$$\langle\langle f_1 f_2 f_3 f_4 \rangle\rangle = \langle\langle f_1 f_2 \rangle\rangle \langle\langle f_3 f_4 \rangle\rangle + \langle\langle f_1 f_3 \rangle\rangle \langle\langle f_2 f_4 \rangle\rangle + \langle\langle f_1 f_4 \rangle\rangle \langle\langle f_2 f_3 \rangle\rangle . \quad (26)$$

We have

$$\langle\langle \|C_T(\tau)\|_{\mathbb{H-S}}^2 \rangle\rangle = \frac{1}{T^2} \int_0^T dt \int_0^T dt' \int_{X \times X} dAdB \langle\langle u(A, t) u(B, t - \tau) u(A, t') u(B, t' - \tau) \rangle\rangle .$$

We apply the identity (26) and get:

$$\langle\langle \|C_T(\tau)\|_{\mathbb{H-S}}^2 \rangle\rangle = \langle\langle C_T(\tau) \rangle\rangle_{\mathbb{H-S}}^2 + \frac{1}{T^2} \int_0^T dt \int_0^T dt' \int_{X \times X} dAdB (II + III) ,$$

with

$$II = \langle\langle u(A, t) u(A, t') \rangle\rangle \langle\langle u(B, t - \tau) u(B, t' - \tau) \rangle\rangle$$

and

$$III = \langle\langle u(A, t) u(B, t' - \tau) \rangle\rangle \langle\langle u(A, t - \tau) u(B, t') \rangle\rangle .$$

For example, we have

$$\langle\langle u(A, t) u(B, t' - \tau) \rangle\rangle = \left[\int_0^\infty ds \int_0^\infty ds' \Omega(s) K K^* \Omega^*(s') \delta(t - s = t' - s' - \tau) \right] (A, B)$$

whose L^2 norm is estimated as $O(\exp(-|t - t'|))$. Using Cauchy-Schwarz inequality and the fact that a product of two Hilbert-Schmidt operators is trace class, we get

$$\left| \frac{1}{T^2} \int_0^T dt \int_0^T dt' \int_{X \times X} dAdB(II + III) \right| = O(1/T) .$$

Standards argument involving Tchebichev inequality allow to conclude.

2. The second assertion is proved in a similar way using the trick of Section 5.4.

□

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