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Jean-Luc Fouquet, Jean-Marie Vanherpe. On $(P_5, \overline{P_5})$ -sparse graphs and other families. 2008. hal-00264104

HAL Id: hal-00264104

<https://hal.science/hal-00264104>

Preprint submitted on 14 Mar 2008

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On $(P_5, \overline{P_5})$ -sparse graphs and other families

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We extend the notion of P_4 -sparse graphs previously introduced by HOÀNG in [11] by considering \mathcal{F} -sparse graphs where \mathcal{F} denotes a finite set of graphs on p vertices. Thus we obtain some results on $(P_5, \overline{P_5})$ -sparse graphs already known on $(P_5, \overline{P_5})$ -free graphs. Finally we completely describe the structure of $(P_5, \overline{P_5}, \text{bull})$ -sparse graphs, it follows that those graphs have bounded clique-width.

1. Introduction

P_4 -free graphs, also called *Cographs*, were designed to be completely decomposable by complementation and motivated researchers for studying graph classes characterized with forbidden configurations. In addition, a number of optimization problems on a graph can be reduced to their weighted version on the set of prime graphs also called the set of representative graphs (recall that the representative graph of graph G is obtained from G by contracting every maximal proper module of G into a single vertex)(see [13]). Thus sub-classes of P_5 - free graphs were intensively studied (see e.g. [2, 3, 4]), in particular Fouquet in [7] consider $(P_5, \overline{P_5})$ -free graphs and the subclass of $(P_5, \overline{P_5}, \text{Bull})$ -free graphs (see Figure 1). Later Giakoumakis and Rusu [10] provide efficient solutions for some optimization problems on $(P_5, \overline{P_5})$ -free graphs.

Hoàng introduced in [11] the P_4 -sparse graphs (every induced subgraph on 5 vertices contains at most one P_4) and several extensions of this notion have arisen in the literature (see for examples [1, 8, 9, 14]). We are concerned here with $(P_5, \overline{P_5})$ -graphs and $(P_5, \overline{P_5}, \text{Bull})$ -graphs where these classes of graphs are defined in the same way (every subgraph on 6 vertices contains at most one

subgraph in the family).

In this paper, we extend the notion of P_4 -sparse in the following way : A graph G is said to be \mathcal{F} -sparse, where \mathcal{F} denotes a set of graphs of order p , whenever any induced subgraph of G on $p + 1$ vertices contains at most one graph of \mathcal{F} as induced subgraph. We study \mathcal{F} -sparse when $\mathcal{F} = \{P_5, \overline{P_5}\}$ and when $\mathcal{F} = \{P_5, \overline{P_5}, \text{Bull}\}$. Those graphs classes are defined with configurations which are prime with respect to modular decomposition (see Figure 1) and which properly intersect graphs classes such that PL -graphs or some (q, t) -graphs classes.

We obtain some results on $(P_5, \overline{P_5})$ -sparse graphs already known on $(P_5, \overline{P_5})$ -free graphs and we completely describe the structure of $(P_5, \overline{P_5}, \text{bull})$ -sparse graphs. This shows that those graphs have bounded clique-width.

Basics

Let $G = (V, E)$ be a graph, the complementary graph of G is denoted \overline{G} . If x and y are two adjacent vertices of G , x is said adjacent to y and y is a *neighbor* of x . A graph on $2n$ vertices such that all of them have exactly one neighbor is a nK_2 .

Let X be a set of vertices and x be a vertex such that $x \notin X$, the set of neighbors of x that belong to X is said the *neighborhood* of

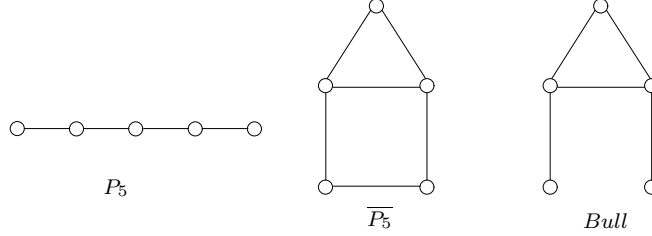


Figure 1. The forbidden configurations in a $(P_5, \overline{P_5}, Bull)$ -free graph

x in X and is denoted $N_X(x)$, if $N_X(x) = \emptyset$ x is said *independent* of X and *total* for X when $N_X(x) = X$, if x is not independent of X nor total for X , x is said *partial* for X . If x is independent of X (resp. total for X), x is said isolated in $X \cup \{x\}$ (universal for $X \cup \{x\}$).

Let X and Y be two disjoint sets of vertices, the set $\bigcup_{y \in Y} N_X(y)$ is denoted $N_X(Y)$ and called the *neighborhood of Y in X* . If there is no edge connecting a vertex of X to a vertex of Y , the sets X and Y are independent while X is total for Y when there is all possible edges connecting vertices of X to vertices of Y .

2. On $(P_5, \overline{P_5})$ -sparse graphs.

In this section we consider \mathcal{F} -sparse graphs when $\mathcal{F} = \{P_5, \overline{P_5}\}$ and we call those graphs $(P_5, \overline{P_5})$ -sparse. Recall that in a such graph every induced subgraph on 6 vertices contains at most one P_5 or $\overline{P_5}$.

Theorem 2.1 *A prime $(P_5, \overline{P_5})$ -sparse graph is either C_5 -free or isomorphic to a C_5 .*

Proof Let G be a prime \mathcal{F} -sparse graph having at least 6 vertices.

Observe first that a vertex, say x , which is partial to a C_5 of G is either adjacent to exactly two non-adjacent vertices of the C_5 or to three consecutive vertices of the C_5 . In all other cases of adjacencies the subgraph induced by the vertices of the C_5 together with x contains two P_5 or $\overline{P_5}$, a contradiction.

Let $abcde$ be a C_5 of G , since G is prime there must exist in G a vertex, say x which is partial to $abcde$. Without loss of generality we can assume that x is adjacent to a and c and independent of d and e . Let A be the set of vertices of G which are adjacent to a and c and independent of d and e . Since A contains at least two vertices ($\{b, x\} \subseteq A$) and G is prime there must be a vertex, say y , outside of A which distinguishes two vertices of A say b_1 and b_2 . But now, the vertex y cannot be outside of A and satisfy the above observation with both C_5 ab_1cde and ab_2cde , a contradiction. \square

Welsh-Powell perfect graphs are perfectly orderable and are characterized with 17 forbidden configurations (see [5]). It is a straightforward exercise to see that $(P_5, \overline{P_5})$ -sparse graphs which are also C_5 -free are Welsh-Powell perfect. In [12], Hoàng, gives algorithms to solve the *Maximum Weighted Clique* problem as well as the *Minimum Weighted Coloring* problem on perfectly or-

derable graphs within $O(nm)$ time complexity.

Thus, as well as for $(P_5, \overline{P_5})$ -free graphs (see [10]), there exists algorithms running in $O(nm)$ time, for computing a *Maximum Weighted Clique* and a *Minimum Weighted Coloring* in a weighted $(P_5, \overline{P_5})$ -sparse graph. Since the class of $(P_5, \overline{P_5})$ -sparse graphs is auto-complementary the parameters *Maximum Weighted Stable Set* and *Minimum Weighted Clique Cover* can be computed within the same time complexity.

3. $(P_5, \overline{P_5}, \text{Bull})$ -sparse graphs.

In this section we will study \mathcal{F} -sparse graphs where $\mathcal{F} = \{P_5, \overline{P_5}, \text{Bull}\}$, namely the $(P_5, \overline{P_5}, \text{Bull})$ -sparse graphs. We will characterize the prime graphs of this family and give some consequences.

Let's first recall a main result on $(P_5, \overline{P_5}, \text{Bull})$ -free graphs.

Theorem 3.1 ([7]) *A prime graph G is $(P_5, \overline{P_5}, \text{bull})$ -free if and only if one of the following holds :*

1. G is isomorphic to a C_5
2. G or its complement is a bipartite P_5 -free graph.

Since Theorem 2.1 also holds for $(P_5, \overline{P_5}, \text{Bull})$ -sparse graphs we consider henceforth only C_5 -free graphs.

Theorem 3.2 *Let G be a prime C_5 -free which contains an induced P_5 (resp. $\overline{P_5}$). G is $(P_5, \overline{P_5}, \text{Bull})$ -sparse if and only if G (resp. \overline{G}) is isomorphic to one of the graphs depicted in Figure 2.*

Proof of Theorem 3.2. It is easy to see that the graphs depicted in Figure 2

are prime $(P_5, \overline{P_5}, \text{Bull})$ -sparse graphs, consequently in the following we consider the only if part of the theorem.

Assume without loss of generality that G contains a P_5 , namely $abcde$. Observe first that a vertex partial to this P_5 can only be adjacent to c , all other adjacency cases lead to a contradiction. Let's denote C the set of vertices in G whose neighborhood in $\{a, b, c, d, e\}$ is $\{c\}$, in addition we denote I the set of vertices of G which have no neighbor in $\{a, b, c, d, e\}$ while T denotes the set of vertices of G which are total for $\{a, b, c, d, e\}$, note that $V(G) = \{a, b, c, d, e\} \cup C \cup T \cup I$. Moreover we suppose henceforth that C is not empty, otherwise by the primality assumption, G would be the $P_5 abcde$ itself (one of the graphs depicted in Figure 2).

Claim 1 *If I has a neighbor in C then $N_C(I) \cup N_I(C)$ is a nK_2 , the vertices of $N_I(C)$ are isolated in I and the vertices of $N_C(I)$ are isolated in C . Moreover T is total for C and $N_I(C)$.*

Proof Let's assume that $x \in C$ has a neighbor $i \in I$, so $\{a, b, c, x, i\}$ is a P_5 . Then x (resp. i) has no other neighbor in I (resp. C). Moreover $N_I(C)$ is isolated in I because if i has a neighbor i' in I , then $\{a, b, c, x, i, i'\}$ is a P_6 , a contradiction; and $N_C(I)$ is isolated in C because if x has a neighbor x' in C then $\{a, b, c, x, x', i\}$ induces a P_5 and a *bull*, a contradiction. Let $t \in T$, assume that t isn't a neighbor of x or i . Let first t isn't a neighbor of i then $\{i, x, c, t, a, e\}$ induces 2 P_5 or 2 *bull*. Otherwise if t isn't a neighbor of x then $\{i, x, c, t, b, d\}$ induces 2 $\overline{P_5}$, a contradiction. Let $x' \in C - N_C(I)$, recall that x isn't adjacent to x' ; if x' is not adjacent to t , the graph $G[\{a, t, c, x', x, d\}]$ contains two induced *bulls*, a contradiction, then the vertices of T are all adjacent to the vertices of

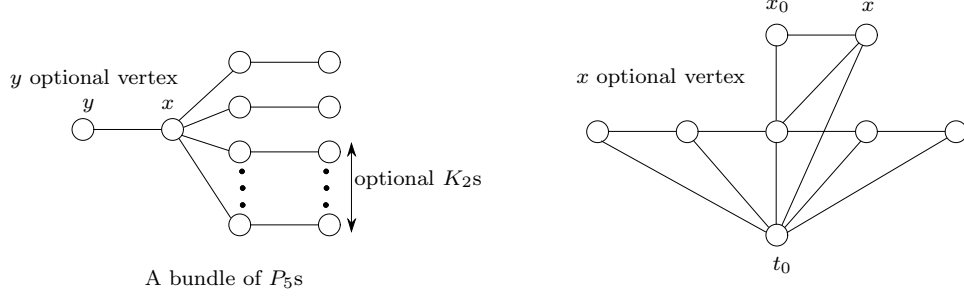


Figure 2. The 2 types of prime $(P_5, \overline{P_5}, \text{Bull})$ -sparse graphs which are C_5 -free and contain a P_5 .

$C \setminus N_C(I)$. ■

Since G is a prime graph, when I has a neighbor in C it follows that the sets T and $I \setminus N_I(C)$ are empty or $\{a, b, c, d, e\} \cup C \cup N_I(C)$ would be a non trivial module of G . Similarly $C \setminus N_C(I)$ contains at most one vertex and thus G is a *bundle of P_5 's*, one of the graphs depicted in Figure 2.

From now on, we assume that I has no neighbor in C , moreover we may assume that a vertex of T has a non-neighbor in C otherwise the set T would be empty (G is a prime graph) and once again, G would be a bundle of P_5 's.

Claim 2 *There is a unique non-edge c_0t_0 such that $c_0 \in C$ and $t_0 \in T$, c_0 is adjacent to all other vertices of C , t_0 is adjacent to all other vertices of T and has no neighbor in I .*

Proof Observe first that a vertex of T cannot have two non-neighbors in T , otherwise a such vertex say t together with two non-neighbors in C , say c_1 and c_2 and the vertices a, c and d would induce two bulls, a contradiction. Similarly, a vertex of C , say x cannot have two non-neighbors t_1 and t_2 in T or two bulls would be induced with the

vertices x, c, d, a, t_1 and t_2 , a contradiction. If there is two non-edges c_1t_1 and c_2t_2 such that $c_1, c_2 \in C$ and $t_1, t_2 \in T$, those vertices together with a and e would induce two P_5 's or two bulls or two $\overline{P_5}$'s or two C_5 's according to the connections between c_1 and c_2 and between t_1 and t_2 , a contradiction. If t_0 would have a non-neighbor in T , say t , the vertices c_0, c, t_0, t, a and e would induce two $\overline{P_5}$, a contradiction. A neighbor i of t_0 in I together with c_0 and the vertices b, c and d would induce two bulls in G , a contradiction. If c_0 is independent of another member of C say x , the graph induced by the vertices x_0, x, t_0, c and a induces a bull, as well as $G[\{x_0, x, t_0, c, e\}]$, a contradiction. ■

No vertex of $I \cup T \setminus \{t_0\}$ can distinguish the vertices of $\{a, b, c, d, e\} \cup C \cup \{t_0\}$, consequently $I \cup T \setminus \{t_0\} = \emptyset$. Moreover, $C \setminus \{c_0\}$ contains at most one vertex, it follows that G has either 7 or 8 vertices according to the fact that $C \setminus \{c_0\}$ is empty or not and is isomorphic to a graph depicted in Figure 2. □

Theorem 3.3 *Let G be a prime*

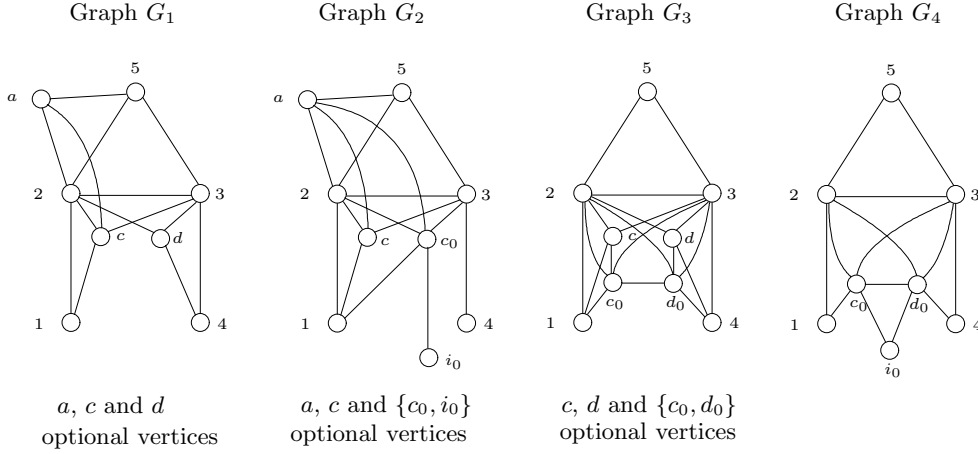


Figure 3. The 4 types of prime $(P_5, \overline{P_5}, Bull)$ -sparse graphs which are $(P_5, \overline{P_5}, C_5)$ -free and contain a *Bull*.

$(P_5, \overline{P_5}, C_5)$ -free graph which contains an induced bull.

G is $(P_5, \overline{P_5}, Bull)$ -sparse if and only if G or \overline{G} is isomorphic to one of the graphs depicted in Figure 3.

Proof It is easy to check that all graphs in Figure 3 are $(P_5, \overline{P_5}, Bull)$ -sparse.

Let's consider an induced bull in G whose vertices are numbered 1, 2, 3, 4, 5 in such a way that $\{1, 2, 3, 4\}$ induces a P_4 whose endpoints are 1 and 4 and 5 is precisely adjacent to 2 and 3 and not to 1 nor 4.

We consider the 6 following subsets of $V \setminus \{1, 2, 3, 4, 5\}$.

Let T be the set of vertices which are adjacent to all the members of $\{1, 2, 3, 4, 5\}$ and I be the set of vertices having no neighbor among $\{1, 2, 3, 4, 5\}$. Let A be the set of vertices being adjacent to 2 and 5 and independent of 1, 3 and 4, while B denotes the set of vertices which are adjacent to 3 and to 5 and independent of 1, 2, and 4. Let C be set of

vertices which are adjacent to 1, 2 and 3 and independent of 4 and 5. D denotes the set of vertices being adjacent to 2, 3 and 4 and independent of 1 and 5.

It is easy to see that a vertex x which is partial with respect to $\{1, 2, 3, 4, 5\}$ must belong to $A \cup B \cup C \cup D$, in other cases of adjacency the subgraph induced by $\{1, 2, 3, 4, 5, x\}$ would not be $(P_5, \overline{P_5}, Bull)$ -sparse. Consequently $V(G) = \{1, 2, 3, 4, 5\} \cup T \cup I \cup A \cup B \cup C \cup D$.

Claim 1 C is total for A and T , C is independent of B .

Proof Let $c \in C$. When c has a non-neighbor in A , say a , the set $\{a, 5, 3, c, 1\}$ induces a P_5 , a contradiction since G is assumed to be P_5 -free. The vertex c cannot have a neighbor in B , or this neighbor together with c , 1, 2, and 5 would induce a $\overline{P_5}$, a contradiction. When c has a non-neighbor t in T , $43ct1$ is a $\overline{P_5}$, a contradiction. ■

Let f be an edge preserving mapping such

that $f(1) = 4$, $f(4) = 1$, $f(2) = 3$, $f(3) = 2$ and $f(5) = 5$, we have $f(A) = B$, $f(B) = A$, $f(C) = D$, $f(D) = C$ while $f(T) = T$ and $f(I) = I$. It follows that we can derive from Claims 1 to Claim 5 below many analogous results by considering the mapping f and/or the complementary graph of G . For example the assertion *C is total for A* becomes *D is total for B* when considering the mapping f , while *C is total for T* becomes *B is independent of I* when applied in \overline{G} and *A is independent of I* when considering the mapping f in \overline{G} .

Let's now examine the connections between vertices of C and D and between vertices of C and I .

Claim 2 *If $A \neq \emptyset$ then there is no edge connecting a vertex of C to a vertex of D , nor a vertex of D to a vertex of I .*

Proof Let a be a vertex of A .

Assume that cd is an edge ($c \in C$ and $d \in D$), the vertices $c, d, 3, 5, a$ induce a $\overline{P_5}$, a contradiction.

Suppose that $d \in D$ has a neighbor i in I , then $id35a$ is a P_5 , a contradiction ■

Claim 3 *If a vertex of C has a neighbor in I then the vertices of $N_I(C)$ are isolated in I , the vertices of $N_C(I)$ are isolated in C , T is total for $A \cup N_I(C)$, in addition there is a unique edge c_0i_0 connecting a vertex of C to a vertex of I and i_0 is isolated in I .*

Proof Let $c \in C$ and $i \in I$ be adjacent vertices.

If i has a neighbor in I , say i' , $i'ic34$ is a P_5 when i' is independent of c while $\{i, i', c, 2, 3, 4\}$ induces 2 bulls when i is adjacent to c , a contradiction.

If c has a neighbor in C , say c' , the vertices

$i, c, c', 2, 3, 4$ induces two bulls, a contradiction.

Let at ($a \in A, t \in T$) be a non edge of G , then $ict5a$ is a P_5 of G , a contradiction.

Moreover, observe that a vertex of C cannot have two neighbors i and i' in I , otherwise the vertices $c, i, i', 2, 3, 4$ would induce two bulls, a contradiction. On the same manner, a vertex in I cannot have two neighbors in C , say c and c' or once again two bulls are induced in $G[\{i, c, c', 2, 3, 4\}]$, contradiction. Consequently, according to Claim 3, two edges connecting vertices of C to vertices of I would induce a $2K_2$ and thus this $2K_2$ together with the vertex 2 would induce a P_5 in G , a contradiction.

Let c_0i_0 be the unique edge connecting a vertex of C to a vertex of I , if i_0 has a neighbor, say i' in I , $i'i_0c_034$ would be P_5 of G , a contradiction. ■

Claim 4 *If C has a neighbor in D then $N_D(C)$ is universal in D , there is a unique edge connecting a vertex of C to a vertex of D , the vertices of I do not distinguish c_0 from d_0 and $D \setminus \{d_0\}$ is independent of I .*

Proof Assume that a vertex $c \in C$ has two neighbors in D , namely d and d' . In this case the graph induced by the vertices $c, d, d', 1, 3, 5$ contains two bulls, a contradiction. Symmetrically, a member of D cannot have two neighbors in C .

Moreover, if d is independent of some other vertex of D , namely d' , the set $\{c, 3, d, 1, 5, d'\}$ induces two bulls, a contradiction, thus $N_D(C)$ is universal in D , similarly $N_C(D)$ is universal in C .

If cd and $c'd'$ ($c, c' \in C, d, d' \in D$) are two distinct edges, $G[\{c, c', d, d', 4\}]$ is a $\overline{P_5}$, a contradiction which proves the uniqueness of an edge connecting C to D .

Assume without loss of generality that $i \in I$ is adjacent to c_0 and not to d_0 . the subgraph induced by $1, i, c_0, d_0, 3$ and 5 contains two bulls, a contradiction.

Finally, suppose that $d \in D$, distinct from d_0 is adjacent to $i \in I$, then G contains a P_5 (idd_035 if i is adjacent to d_0 and dic_025 if i is not adjacent to d_0), a contradiction. ■

Claim 5 *At least one of the sets A, B, C, D is empty.*

Proof Let $a \in A, b \in B, c \in C, d \in D$. We know by Claim 1 that a is connected to c and not to d and that b is connected to d and not to c , Claim 2 asserts that c is not adjacent to d while a and b are connected. Consequently $1cabd$ is a P_5 , a contradiction. ■

According to Claim 5 we will now discuss on the number of empty sets among A, B, C and D and prove that G or \overline{G} is isomorphic to one of the graphs depicted in Figure 3.

Case 1 : The sets A, B, C and D are all empty.

Recall that G is prime, thus the sets T and I are also empty, for otherwise $\{1, 2, 3, 4, 5\}$ would be a non-trivial module. Consequently G is a bull, a graph isomorphic to G_1 in Figure 3 when a, c and d are missing.

Case 2 : Three of the sets A, B, C and D are empty.

Assume without loss of generality that $C \neq \emptyset$. We know by Claim 1 that C is total for T

If C has no neighbor in I , no vertex of $T \cup I$ can distinguish the members of $\{1, 2, 3, 4, 5\} \cup C$ and by the primality of G the sets T and I are empty while C is reduced to a single vertex. In this case G is isomorphic to G_1 where a and d are missing.

If C has a neighbor in I we know by Claim 3 that there is a unique edge, namely c_0i_0 connecting C to I . We consider the following decomposition of C and I : $C = \{c_0\} \cup (C \setminus \{c_0\})$, $I = \{i_0\} \cup (I \setminus \{i_0\})$.

By construction $C \setminus \{c_0\}$ is independent of I and $I \setminus \{i_0\}$ is independent of C while i_0 has no neighbor in $I \setminus \{i_0\}$ and c_0 has no neighbor in $C \setminus \{c_0\}$ (Claim 3). Moreover i_0 is completely adjacent to T (Claim 3).

Consequently $T \cup (I \setminus \{i_0\}) = \emptyset$ or the set $\{1, 2, 3, 4, 5, c_0, i_0\} \cup (C \setminus \{c_0\})$ would be a non trivial module of G , a contradiction. In addition $C \setminus \{c_0\}$ is either a singleton, say $\{c\}$ or empty and G is isomorphic to G_2 without the vertex a and where c is possibly missing if $C \setminus \{c_0\} = \emptyset$ (see Figure 3).

Case 3 : Among A, B, C and D exactly two sets are empty.

Due to symmetries we only consider three different situations.

Let first suppose that $B = C = \emptyset$.

We know (Claim 1) that A and D are independent, D is total for T and A is independent of I . Moreover D is independent of I (Claim 2) and thus A is total for T . Because of the primality of G the set $T \cup I$ is empty and A as well as D is a singleton. Consequently G is isomorphic to G_2 without the vertex c (Figure 3).

Assume in a second stage that $B = D = \emptyset$. We know by Claim 1 that C is total for T and A is independent of I .

If there is an edge between C and I , it is unique (Claim 3), let's denote this edge c_0i_0 . In this case A is totally adjacent to T (Claim 3), the set $C \setminus \{c_0\}$ is independent of I and by construction c_0 is independent of $I \setminus \{i_0\}$, i_0 is independent of $I \setminus \{i_0\}$ and c_0 has no neighbor in $C \setminus \{c_0\}$ (Claim 3 again). It follows that the prime graph G is isomorphic to G_2 (Figure 3)

where c can miss if $C \setminus \{c_0\}$ is empty.

If there is no connection between C and I , some vertex of A can have a non neighbor in T , we are then in a similar situation than above in the complementary graph of G .

When C is independent of I and A is total for T the graph is isomorphic to graph G_1 in Figure 3.

Finally let's study the case $A = B = \emptyset$. We know that C and D are totally adjacent to T (Claim 1).

If C and D are not connected it is easy to see that C and D are not adjacent to I . As a matter of fact, suppose on the contrary that c_0i_0 is an edge ($c_0 \in C$ and $i_0 \in I$) and that d is some vertex in D . If d and i_0 are not connected, i_0c_02d4 is a P_5 and i_0c_03d4 is a \overline{P}_5 if d and i_0 are adjacent, a contradiction in both cases. Consequently, G being prime is isomorphic to the graph G_1 in Figure 3 where d misses.

When C has a neighbor in D , we consider the unique edge connecting C to D , namely c_0d_0 ($c_0 \in C, d_0 \in D$). By Claim 4, c_0 is universal in C and d_0 is universal in D . We know (Claim 4) that only c_0 and d_0 can have a neighbor in I .

If it is not the case G is isomorphic to G_3 in Figure 3 without c or d if $C \setminus \{c_0\}$ or $D \setminus \{d_0\}$ is empty. If, on the contrary, c_0 and d_0 have a neighbor, say i_0 in I , $C = \{c_0\}$ (or $\{1, c, c_0, i_0, d_0, 4\}$ where c is a vertex of C distinct from c_0 induces two bulls, a contradiction) and similarly $D = \{d_0\}$. Consequently G is isomorphic to the graph G_4 of Figure 3.

Case 4 : Among A, B, C and D exactly one set is empty.

For convenience we will suppose that $B = \emptyset$.

By Claim 1, A is completely adjacent to C and independent of D . There is no edge

connecting a vertex of C to a vertex of D (Claim 2).

Moreover C and D are completely adjacent to T and A is independent of I (Claim 1). In addition, there is no connection between D and I (Claim 2) and similarly A is total for T .

If there is no edge between C and I , the sets T and I must be empty (or $\{1, 2, 3, 4, 5\} \cup A \cup C \cup D$ would be a non trivial module of G) and A, C, D are singletons. In this case G is isomorphic to G_2 in Figure 3.

When there is a unique edge c_0i_0 between C and I ($c_0 \in C, i_0 \in I$), once again $I \setminus \{i_0\}$ is completely independent of $C \cup \{i_0\}$ while $\{c_0, i_0\}$ has no connections with $C \setminus \{c_0\}$ (Claim 3). Consequently, G is isomorphic to G_2 where c misses if $C = \{c_0\}$. \square

It follows from Theorem 3.1, 2.1, 3.1 and 3.3 that a prime $(P_5, \overline{P}_5, Bull)$ -sparse graph or its complement is either a C_5 or a P_5 -free bipartite graph or a bundle of P_5 's (see Figure 2) or is a graph on less than 10 vertices. This leads to a linear time recognition algorithm for $(P_5, \overline{P}_5, Bull)$ -sparse graphs, moreover those graphs have bounded clique-width (see [6]).

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