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Constants of concentration for a simple recurrent random walk on random environment

Pierre Andreatti *

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Abstract: We precise the asymptotic of the limsup of the size of the neighborhood of concentration of Sinai's walk. Also we get the almost sure limits of the number of points visited more than a fixed proportion of a given amount of time.

1 Introduction, and Results

The one-dimensional recurrent random walk on random environment (thereafter abbreviated RWRE) we treat here, also called Sinai's walk, has the property to be localized at an instant n in a neighborhood of a point of the lattice, this point can be described from the random environment and n (see [Sin82], [Gol86], [Zei01], [And05]), its limit distribution is also known (see [Gol84], [Kes86]).

In addition to this aspect of localization, this random walk has the property to spend a large amount of time in a single point of the lattice, this property was first en-lighted by [R89]. To be more precise, let us define the local time (\mathcal{L}) of the random walk ($X_n, n \in \mathbb{N}$). Let $k \in \mathbb{Z}$ and $n \in \mathbb{N}^*$

$$\begin{aligned}\mathcal{L}(k, n) &= \sum_{i=1}^n \mathbb{I}_{X_i=k}, \\ \mathcal{L}(A, n) &= \sum_{k \in A} \mathcal{L}(k, n), \quad A \subset \mathbb{Z}, \\ \mathcal{L}^*(n) &= \sup_{k \in \mathbb{Z}} \mathcal{L}(k, n).\end{aligned}$$

[R89] shows that $\limsup_n \frac{\mathcal{L}^*(n)}{n} \geq \text{const} > 0, \mathbb{P} \text{ a.s.}$ and more recently [GPS] get the more precise result $\limsup_n \frac{\mathcal{L}^*(n)}{n} = c_1, \mathbb{P} \text{ a.s.}$ giving an explicit formula for c_1 as a function of the support of the distribution of the random environment.

In this paper we mainly answer one of the question asked in [And07], more precisely we are interested in

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random variables related to the local time: let $x \in \mathbb{Z}$, $0 \leq \beta \leq 1$, $r \in \mathbb{N}$, $I_r(x) = \{x - r, \dots, x, \dots, x + r\}$, we define

$$Y_n(\beta) = \inf_{x \in \mathbb{Z}} \inf(r > 0, \mathcal{L}(I_r, n) \geq \beta n).$$

$Y_{n,\beta}$ measure the size of a neighborhood where the walk spend more than a proportion β of the total amount of time n . The first asymptotics for $Y_{n,\beta}$ can be found in [And06], more recently [And07] get the following

$$\liminf_n Y_{n,\beta} \leq \text{const}(1 - \beta)^{-2}, \mathbb{P} \text{ a.s.} \quad (1.1)$$

We call $Y_{n,\beta}$ the concentration variable of $(X_n, n \in \mathbb{N})$, the concentration is the property of the random walk to spend a large amount of time in a negligible interval comparing to a typical fluctuation of the walk. Some more work was needed to improve the inequality 1.1 into an equality, we get it here by applying the method of [GPS].

In this paper we are also interested in a second variable that we define below, let $\delta > 0$:

$$Z_{n,\delta} = \sum_{x \in \mathbb{Z}} \mathbb{I}_{\{\mathcal{L}(k,n) \geq \delta n\}}.$$

$Z_{n,\delta}$ counts the number of points of the lattice visited more than δn times. Our second result give the function m that satisfies for δ small enough $\limsup_n Z_{n,\delta} = m(\delta), \mathbb{P} \text{ a.s.}$

It is time now to define the model and present the results. The simple random walk on random environment we are dealing with is composed of two processes, the first one, called random environment, is a sequence of i.i.d. random variables $\alpha \equiv (\alpha_i, i \in \mathbb{Z})$ with distribution P , each of the α_i belonging to the interval $(0, 1)$, and the second one $(X_n, n \in \mathbb{N})$ is a birth and death process for all fixed α , with transition of probabilities given by $\mathbb{P}^\alpha[X_{n+1} = i + 1 | X_n = i] = 1 - \mathbb{P}^\alpha[X_{n+1} = i - 1 | X_n = i] = \alpha_i$. The whole process is defined on the probability space $(\Omega \times \mathbb{Z}^{\mathbb{N}}, \mathcal{F} \times \mathcal{G}, \mathbb{P})$ where $\Omega := (0, 1)^{\mathbb{Z}}$ is the state space of the random environment equipped with its Borel σ -field \mathcal{F} , \mathcal{G} is the Borel σ -field associated to $\mathbb{Z}^{\mathbb{N}}$ and for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, $\mathbb{P}[F \times G] = P \otimes \mathbb{P}^\alpha[F \times G] = \int_F P(dw_1) \int_G \mathbb{P}^{\alpha(w_1)}(dw_2)$. The one-dimensional RWRE we are interested in is the almost surely recurrent for almost all environment, [Sol75] proved that the necessary and sufficient hypothesis to get such a process is

$$E[\epsilon_0] = 0, \quad (1.2)$$

where $(\epsilon_i \equiv \log \frac{1-\alpha_i}{\alpha_i}, i \in \mathbb{Z})$. We also add the hypothesis

$$\text{Var}[\epsilon_0] > 0 \quad (1.3)$$

to get a non-trivial RWRE, and

$$P(|\epsilon_0| \leq \eta) = 1, \eta > 0 \quad (1.4)$$

for simplicity. We can now give the results which depend on the following constants,

$$\begin{aligned} \bar{A} &:= \sup\{x : x \in \text{supp}(\alpha_0)\} \in (1/2, 1], \text{ and } \bar{\alpha} := \inf\{x : x \in \text{supp}(\alpha_0)\} \in [0, 1/2), \\ \tilde{\alpha} &= \bar{\alpha}/(1 - \bar{\alpha}) \text{ and } \tilde{A} = (1 - \bar{A})/\bar{A}. \end{aligned}$$

First let us recall one of the results of [GPS], and give the explicit formula for c_1 that will be useful in the sequel:

Theorem 1.1. ([GPS]) Assume that 1.2, 1.3 and 1.4 are verified, then

$$\mathbb{P} \text{ a.s. } \liminf_n \frac{\mathcal{L}^*(n)}{n} = c_1,$$

where $c_1 = (2\bar{A} - 1)(1 - 2\bar{\alpha}) / (2(\bar{A} - \bar{\alpha}) \min(\bar{A}, 1 - \bar{\alpha}))$.

Our first result is the following,

Theorem 1.2. Assume that 1.2, 1.3 and 1.4 are verified, for all $0 \leq \beta \leq 1$,

$$\mathbb{P} \text{ a.s. } \liminf_n Y_n(\beta) = \begin{cases} 0 & \text{if } 0 \leq \beta \leq c_1, \\ f(\beta) & \text{if } c_1 < \beta < 1, \\ +\infty & \text{if } \beta = 1, \end{cases} \quad (1.5)$$

where $f(\beta)$ is the smallest integer that satisfies

$$(2x - 1) \left(\frac{y}{1 - y} \right)^{f(\beta)} + (1 - 2y) \left(\frac{1 - x}{x} \right)^{f(\beta)+1} \leq 2(\bar{A} - \bar{\alpha})(1 - \beta), \quad (1.6)$$

with $x = \bar{A} \wedge (1 - \bar{\alpha})$ and $y = (1 - \bar{A}) \wedge \bar{\alpha}$.

Remark 1.3. Notice that when β gets close to 1

$$f(\beta) = \log(1 - \beta) \left(\frac{1}{\log \bar{A}} \vee \frac{1}{\log \bar{\alpha}} + o(1) \right), \quad (1.7)$$

where $o(1)$ is such that $\lim_{\beta \rightarrow 1} o(1) = 0$, so the upper bound obtained in [And07] was pretty far from reality. Now if $\bar{\alpha} = 1 - \bar{A}$, $f(\beta)$ satisfies:

$$f(\beta) \geq \frac{\log(1 - \beta)}{\log \bar{A}} + \frac{\log(2\bar{A} - 1)}{\log(\bar{A})}.$$

Finally notice that the case $0 \leq \beta \leq c_1$ is a direct consequence of Theorem 1.1 the case $\beta = 1$ can be deduce from the 1.7.

The second theorem deals with $Z_{n,\delta}$

Theorem 1.4. Assume that 1.2, 1.3 and 1.4 are verified, for all $\delta > 0$ small enough,

$$\mathbb{P} \text{ a.s. } \limsup_n Z_{n,\delta} = \frac{1}{\delta} - \frac{1}{1 - \bar{\alpha}} \wedge \frac{1}{\bar{A}} + O(\delta) \quad (1.8)$$

Remark 1.5. Notice that $\limsup_n Z_{n,0} = +\infty$, $\mathbb{P}.a.s.$, thanks to Theorem 1.1 for all $\delta > c_1$ $\limsup_n Z_{n,\delta} = 0$, $\mathbb{P}.a.s.$, also regarding to (1.12) in [GPS], $\liminf_n Z_{n,\delta} = 0$, $\mathbb{P}.a.s.$

2 Proof of the results

A fundamental notion for the study of Sinai's random walk is the random potential associated with the random environment:

$$S_k := \begin{cases} \sum_{1 \leq i \leq k} \epsilon_i, & \text{if } k = 1, 2, \dots \\ \sum_{k+1 \leq i \leq 0} \epsilon_i, & \text{if } k = -1, -2, \dots \end{cases}$$

$$S_0 := 0,$$

recall that $\epsilon_i = \frac{1-\alpha_i}{\alpha_i}$ for all $i \in \mathbb{Z}$. Let $(W^+(x), x \in \mathbb{Z})$ be a collection of random variables distributed as S conditioned to stay non-negative for all $x > 0$ and strictly positive for $x < 0$, and in the same way $(W^-(x), x \in \mathbb{Z})$ is a collection of random variables distributed as S but conditioned to stay strictly positive for all $x > 0$ and non-negative for $x < 0$. Notice that $\sum_{x \in \mathbb{Z}} \exp(-W^\pm(x)) < \infty$, *a.s.* (see [Ber93] or [Gol84]). From W^+ and W^- we define two random probability measures, μ^+ and μ^- as follows, let $x \in \mathbb{Z}$,

$$\mu^\pm(x) = \mu^\pm(x, W^\pm) := \frac{\exp(-W^\pm(x-1)) + \exp(-W^\pm(x))}{2 \sum_{x \in \mathbb{Z}} \exp(-W^\pm(x))}. \quad (2.1)$$

Let Q^+ (resp. Q^-) the distributions of μ^+ (resp. μ^-), define $Q = 1/2(Q^+ + Q^-)$ the distribution of a random measure called μ . Let $l^1 = \{l : \mathbb{Z} \rightarrow \mathbb{R}, \|l\| := \sum_{x \in \mathbb{Z}} |l(x)| < +\infty\}$, the main ingredient in the proof of the Theorems is the following result ([GPS] Corollary 1.1), for all functions, $f : l^1 \rightarrow \mathbb{R}$ which is shift-invariant,

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[f \left(\frac{\mathcal{L}(x, n)}{n}, x \in \mathbb{Z} \right) \right] = \mathbb{E}[f(\{\mu(x), x \in \mathbb{Z}\})]. \quad (2.2)$$

2.1 Proof of Theorem 1.2

Define

$$R_n(r) = \sup_{x \in \mathbb{Z}} \sum_{k \in I_r(x)} \frac{\mathcal{L}(k, n)}{n},$$

from 2.2 we get

$$\sup_{x \in \mathbb{Z}} R_n(r) \rightarrow \sup_{x \in \mathbb{Z}} \sum_{k \in I_r(x)} \mu(x), \text{ in law,} \quad (2.3)$$

where $I_r(x)$ is defined just above 1.1. Moreover it was proven in [And06] (see also [GS02]) that

$$\limsup_n R_n(r) = \text{const} \in [0, +\infty), \mathbb{P}.a.s. \quad (2.4)$$

then 2.3 and 2.4 yields the, easy to get, fact

$$\limsup_n R_n(r) \geq g(r) := \sup \left\{ z, z \in \text{supp} \left(\sup_{x \in \mathbb{Z}} \sum_{k \in I_r(x)} \mu(k) \right) \right\}, \mathbb{P}.a.s. \quad (2.5)$$

The next step is to determine which environment maximize $g(r)$, it is not surprising that this environment is the same that maximises $\sup \{z, z \in \text{supp}(\sup_{x \in \mathbb{Z}} \mu(x))\}$ used in [GPS]. We will recall this

special environment, and restrict our analysis to μ^+ (the restriction of μ to Q^+), and shortly discuss what happens under Q^- at the end of the proof.

First, for all $x \in \mathbb{Z}_+^*$ define $\bar{\alpha}_x^+ := \bar{\alpha}$, for all $x \in \mathbb{Z}_-^*$, $\bar{\alpha}_x^+ := \bar{A}$ and $\bar{\alpha}_0^+ = \bar{\alpha} \mathbb{I}_{\{\bar{A} \geq 1 - \bar{\alpha}\}} + \bar{A} \mathbb{I}_{\{\bar{A} < 1 - \bar{\alpha}\}}$, with this choice we have the following expression for the exponential of the potential W^+ called \bar{W}^+ :

$$\exp(-\bar{W}^+(x)) = \begin{cases} \bar{\alpha}^x & \text{if } x > 0, \\ 1 & \text{if } x = 0, \\ \bar{A}^{-x-1} \left(\bar{A} \mathbb{I}_{\{\bar{A} > \bar{\alpha}\}} + \frac{1}{\bar{\alpha}} \mathbb{I}_{\{\bar{A} \leq \bar{\alpha}\}} \right) & \text{if } x < 0. \end{cases} \quad (2.6)$$

From 2.6 we easily get the following expression for $g(r)$:

$$g(r) = 1 - \frac{(2\bar{A} - 1)}{2(\bar{A} - \bar{\alpha})} \bar{\alpha}^r - \frac{(1 - 2\bar{\alpha})}{2(\bar{A} - \bar{\alpha})} \left(\bar{A} \mathbb{I}_{\{\bar{A} > \bar{\alpha}\}} + \frac{1}{\bar{\alpha}} \mathbb{I}_{\{\bar{A} \leq \bar{\alpha}\}} \right) \bar{A}^r. \quad (2.7)$$

To turn the inequality 2.5 into an equality we have to show that

$$\limsup_n \sup_{x \in \mathbb{Z}} \sum_{k \in I_\beta(x)} \frac{\mathcal{L}(k, n)}{n} \leq g(r) \quad Q^+ \otimes \mathbb{P}^\alpha .a.s.$$

To get this last inequality we use the same method as [GPS]. We just recall that the proof is based on two facts, the first one says that the distribution of $(\mathcal{L}(x, n), n \in \mathbb{N})$ under \mathbb{P}^α is stochastically dominated by the distribution of $(\mathcal{L}(0, n), n \in \mathbb{N})$ under $\mathbb{P}^{\bar{W}^+}$ (Lemma 3.2 in [GPS]). The second fact says that under $\mathbb{P}^{\bar{W}^+}$, $(X_n, n \in \mathbb{N})$ is a positive recurrent Markov chain, therefore for all $\epsilon > 0$, $\mathbb{P}^{\bar{W}^+} \left[\sum_{k \in I_r(0)} \frac{\mathcal{L}(k, n)}{n} \geq \sum_{k \in I_r(0)} \mu^+(x) + \epsilon \right] \leq \exp(-C(\epsilon)n)$, where $C(\epsilon)$ is strictly positive constant depending only on ϵ . We finally get that

$$\limsup_n \sup_{x \in \mathbb{Z}} \sum_{k \in I_r(x)} \frac{\mathcal{L}(k, n)}{n} = g(r), \quad Q^+ \otimes \mathbb{P}^\alpha .a.s. \quad (2.8)$$

If we do the same computations with respect to Q^- , the above result only differ by the fact that g is different, let us call it g' , it takes the following value:

$$g'(r) = 1 - \frac{(1 - 2\bar{\alpha})}{2(\bar{A} - \bar{\alpha})} \bar{A}^r - \frac{(2\bar{A} - 1)}{2(\bar{A} - \bar{\alpha})} \left(\bar{\alpha} \mathbb{I}_{\{\bar{A} < \bar{\alpha}\}} + \frac{1}{\bar{A}} \mathbb{I}_{\{\bar{A} \geq \bar{\alpha}\}} \right) \bar{\alpha}^r. \quad (2.9)$$

In order to get Theorem 1.2 from 2.8 it suffices to notice that $Y_{n, g(r)}$ is in some sense the dual of $\sum_{k \in I_r(x)} \frac{\mathcal{L}(k, n)}{n}$, indeed, let $0 < \beta < 1$, $\gamma > 0$ we easily get the following assertions,

$$\limsup_n \sup_{x \in \mathbb{Z}} \mathcal{L}(I_{f(\beta)}(x), n) \leq \beta \quad \Rightarrow \quad \liminf_n Y_{n, \beta} \geq f(\beta) - \gamma, \quad (2.10)$$

$$\limsup_n \sup_{x \in \mathbb{Z}} \mathcal{L}(I_{f(\beta)}(x), n) \geq \beta \quad \Rightarrow \quad \liminf_n Y_{n, \beta} \leq f(\beta) + \gamma, \quad (2.11)$$

where $I_{f(\beta)}(x) = \{x - f(\beta), x - f(\beta) + 1, \dots, x + f(\beta)\}$, and $f(\beta)$ is the smallest integer that satisfies the following inequality:

$$\beta \leq \sup \left\{ z, z \in \text{supp} \left(\sup_{x \in \mathbb{Z}} \sum_{k \in I_{f(\beta)}(x)} \mu(x) \right) \right\}. \quad (2.12)$$

By collecting the last fifth equations and by letting γ goes to 0, we finally get that $\liminf_n Y_n(\beta) = f(\beta) \mathbb{P}.a.s.$, and $f(\beta)$ is the smallest interger that satisfies $(1 - g(f(\beta))) \vee (1 - g'(f(\beta))) \geq \beta$. To end the proof is suffices to compare $(1 - g(f(\beta)))$ and $(1 - g'(f(\beta)))$ when $\bar{\alpha} < 1 - \bar{A}$ and $\bar{\alpha} \geq 1 - \bar{A}$ and choose the largest one.

2.2 Proof of Theorem 1.4

First we notice that for all fixed δ , $Z_{n,\delta}$ is bounded because $\mathcal{L}(k,n) \leq n$, in fact it is clear that $Z_{n,\delta} \leq 1/\delta$, so we can apply the 3 steps of the proof of Lemma 3 of [GS02] for $Z_{n,\delta}$ instead of $\mathcal{L}^*(n)/n$ to show that $\limsup_n Z_{n,\delta} = \text{const} \in [0, +\infty)$, $\mathbb{P}.a.s.$, with this result and 2.2 we easily get that

$$\limsup_n \sum_{x \in \mathbb{Z}} \mathbb{I}_{\mathcal{L}(k,n)/n \geq \delta} \geq m(\delta) = m(\delta, \mu) := \sup \left\{ z, z \in \text{supp} \left(\sum_{x \in \mathbb{Z}} \mathbb{I}_{\{\mu(x) \geq \delta\}} \right) \right\}, \mathbb{P}.a.s. \quad (2.13)$$

As before we need to extract a random environment that maximise $\sum_{x \in \mathbb{Z}} \mathbb{I}_{\{\mu(x) \geq \delta\}}$, this time it is not exactly the same as the preceding one, indeed we need to build an infinitely deep valley such that the asymptotic of the normalized local time reach the level δ and also flat enough such that there is a large number of points where the normalized local time reach this level δ , we prove that such an environment is given by

$$\exp(-\bar{W}^+(x)) = \begin{cases} \tilde{\alpha}^{x-g(\delta)} & \text{if } x > g(\delta), \\ 1 & \text{if } 0 \leq x \leq g(\delta), \\ \tilde{A}^{-x} & \text{if } x < 0. \end{cases} \quad (\text{under } Q^+)$$

or

$$\exp(-\bar{W}^-(x)) = \begin{cases} \tilde{\alpha}^x & \text{if } x > 0, \\ 1 & \text{if } -g(\delta) \leq x \leq 0, \\ \tilde{A}^{-x-g(\delta)} & \text{if } x < -g(\delta). \end{cases} \quad (\text{under } Q^-)$$

With the first choice we get the following expression for $\bar{\mu}^+(x) := \mu^+(x, \bar{W}^+)$

$$\bar{\mu}^+(x) = \frac{\exp(-\bar{W}^+(x-1)) + \exp(-\bar{W}^+(x))}{2(\tilde{\alpha}/(1-\tilde{\alpha}) + g(\delta) + 1 + \tilde{A}^{g(\delta)+1}/(1-\tilde{A}))} \quad (2.14)$$

and a similar expression for $\bar{\mu}^- := \mu^-(x, \bar{W}^-)$. Notice that for all $x \in G(\delta) := \{1, \dots, g(\delta)\}$, $\bar{\mu}^+(x)$ reaches its maximum which is equal to $1/(\tilde{\alpha}/(1-\tilde{\alpha}) + g(\delta) + 1 + \tilde{A}^{g(\delta)+1}/(1-\tilde{A}))$.

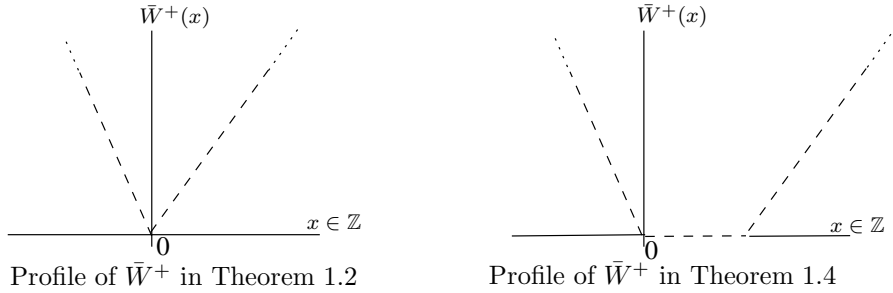


Figure 1: Infinite valleys, $(1 - \tilde{\alpha} < \tilde{A})$

We decompose $\bar{m}(\delta) := m(\delta, \bar{\mu}^+)$ in the following way

$$\bar{m}(\delta) := \bar{m}_S(\delta) + \bar{m}_L(\delta), \quad (2.15)$$

$$\bar{m}_S(\delta) := \sum_{x \in \mathbb{Z} \setminus G(\delta)} \mathbb{I}_{\{\bar{\mu}^+(x) \geq \delta\}}, \quad (2.16)$$

$$\bar{m}_L(\delta) := \sum_{x \in G(\delta)} \mathbb{I}_{\{\bar{\mu}^+(x) \geq \delta\}}. \quad (2.17)$$

Thanks to the above remark we know that

$$\begin{aligned}\bar{m}_L(\delta) = 0 &\Rightarrow \bar{m}_S(\delta) = 0, \\ \bar{m}_L(\delta) > 0 &\Rightarrow \bar{m}_L(\delta) = g(\delta),\end{aligned}$$

therefore $\bar{m}(\delta)\mathbb{I}_{\{\bar{m}(\delta)>0\}} = g(\delta) + \bar{m}_S(\delta)$. Moreover to get $\bar{m}(\delta)$ as large as possible, $g(\delta)$ must be the largest integer that satisfies the following

$$g(\delta) \leq \frac{1}{\delta} - 1 - \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} - \frac{\tilde{A}^{g(\delta)+1}}{1 - \tilde{A}}.$$

To get an upper bound for $\bar{m}_S(\delta)$, we solve the inequality $\bar{\mu}^+(x) \geq \delta$ for all $x \in \mathbb{Z} \setminus G(\delta)$ this leads for δ small enough to $|x| \leq \log(1/(1 + 1/\tilde{A})) / (\log(\tilde{A}))$. Finally we get

$$g(\delta) \leq \bar{m}(\delta) \leq g(\delta) \left(1 + O\left(\frac{1}{g(\delta)}\right)\right) \Rightarrow \bar{m}(\delta) = \frac{1}{\delta} - 1 - \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} + O(\delta) \quad (2.18)$$

A same expression for \bar{m} can be found under Q^- replacing $\tilde{\alpha}$ by $1 - \tilde{A}$.

To finish the proof we have to show that $\limsup_n \sum_{x \in \mathbb{Z}} \mathbb{I}_{\{\mathcal{L}(k,n) \geq \delta n\}} \leq m(\delta)$, *P.a.s.* for that we prove the following Lemma.

Lemma 2.1. *For n large enough, the distribution of $Z_{n,\delta}$ with respect to \mathbb{P}^α is stochastically dominated by the distribution of $Z_{n,\delta}$ with respect to $\mathbb{P}_{\bar{W}^+}$.*

Proof.

Let a be a strictly positive interger, first define the following auxiliary elementary processes, let $(\bar{X}_n, n \in \mathbb{N})$ be a Markov chain with state space $\mathcal{S} = \{0, \dots, a + 1\}$ such that for all $x \in \mathcal{S} \setminus \{0, a + 1\}$ \bar{X} is a simple random walk ($\forall x \in \mathcal{S} \setminus \{0, a + 1\}$, $p(\bar{X}_{n+1} = x + 1 | \bar{X}_n = x) = 1 - p(\bar{X}_{n+1} = x - 1 | \bar{X}_n = x) = 1/2$) and 0 (resp. $m - 1$) is a point of reflexion to the right (resp. to the left). In the same way define $(\bar{X}'_n, n \in \mathbb{N})$ a Markov chain with the same state space as \bar{X} and distributed as \mathbb{P}^α in $\mathcal{S} = \{0, \dots, a + 1\} \setminus \{0, a + 1\}$ and reflected at 0 and $a + 1$ like \bar{X} . It is easy to show that for n large enough the distribution of $Z_{n,1/a}$ under $\mathbb{P}_{\bar{X}'}$ is stochastically dominated by the distribution of $Z_{n,1/a}$ under $\mathbb{P}_{\bar{X}}$. It is also clear that if 0 and $m + 1$ are no more reflection state $Z_{n,1/a}$ will start to decrease under both $P_{\bar{X}}$ and $P_{\bar{X}'}$, moreover for all $i \leq 0$ $\alpha_i \leq \bar{A}$, and for all $i \geq a$, $\alpha_i \leq 1 - \bar{\alpha}$ so we get the Lemma. ■

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