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POINTWISE EXISTENCE OF THE LYAPUNOV EXPONENT FOR A QUASI-PERIODIC EQUATION

ALEXANDER FEDOTOV AND FRÉDÉRIC KLOPP

0. INTRODUCTION

0.1. Quasi-periodic finite difference equations. Consider the finite difference Schrödinger equation

$$(1) \quad (H_\theta \psi)(n) = \psi(n+1) + \psi(n-1) + v(n\omega + \theta)\psi(n) = E\psi(n)$$

where $v : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and periodic, $v(x+1) = v(x)$, $0 < \omega < 1$ and $0 \leq \theta < 1$.

When $\omega \notin \mathbb{Q}$, the mapping $n \mapsto v(n\omega + \theta)$ is quasi-periodic.

The spectral theory of such quasi-periodic equations is very rich, and the study has generated a vast literature; among the authors are A. Avila, Y. Avron, J. Bellissard, J. Bourgain, V. Buslaev, V. Chulaevsky, D. Damanik, E. Dinaburg, H. Eliasson, A. F., B. Helffer, M. Hermann, S. Jitomirskaya, F. K., R. Krikorian, Y. Last, L. Pastur, J. Puig, M. Shubin, B. Simon, Y. Sinai, J. Sjöstrand, S. Sorets, T. Spencer, M. Wilkinson and many others (see e.g. [9] for a recent survey).

Speaking about intriguing spectral phenomena, one can mention for example that

- for such equations, the spectral nature depends on the “number theoretical” properties of the frequency ω ;

and that one expects that

- typically such equations exhibit Cantorian spectrum;
- $\sigma_{pp}(H_\theta)$, the singular continuous spectrum, is topologically typical.

This has been well understood only for a few models, most prominently, for the almost Mathieu equation when $v(x) = 2\lambda \cos(x)$.

0.2. Lyapunov exponent. One of the central objects of the spectral study of the quasi-periodic equations is the Lyapunov exponent. Recall its definition.

Equation (1) can be rewritten as

$$(2) \quad \begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = M(n\omega + \theta) \begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix}, \quad M(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}.$$

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So the large n behavior of solutions to (1) can be characterized by the limits (when they exist) :

$$(3) \quad \gamma^+(E, \theta) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|M((n-1)\omega + \theta) \cdots M(\theta + \omega) M(\theta)\|$$

$$(4) \quad \gamma^-(E, \theta) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|M^{-1}(\theta - n\omega) \cdots M^{-1}(\theta - 2\omega) M^{-1}(\theta - \omega)\|$$

Furstenberg and Kesten have proved

Theorem 1 ([2]). *Fix E . For almost every θ , these limits exist, coincide and do not depend on θ .*

For energies E such that the limits exist, coincide and do not depend on θ , their common value is called the *Lyapunov exponent*; we denote it by $\gamma(E)$.

We are interested in the pointwise (in both E and θ) existence of the limits $\gamma^+(E, \theta)$ and $\gamma^-(E, \theta)$. We call them the *right* and *left* Lyapunov exponents. Speaking about the pointwise existence of the Lyapunov exponent itself, we say that it does not exist for a pair (E, θ) when either at least one of $\gamma^\pm(E, \theta)$ does not exist or both of them exist, but at least one of them differs from $\gamma(E)$.

0.3. Lyapunov exponents and the spectrum. For $\omega \notin \mathbb{Q}$, one has the following theorem by Ishii - Pastur - Kotani

Theorem 2 ([2]). *The absolutely continuous spectrum, $\sigma_{ac}(H_\theta)$, is the essential closure of the set of energies where the Lyapunov exponent vanishes.*

This theorem immediately implies

Corollary 1 ([2]). *If $\gamma(E)$ is positive on I , an interval, then the spectrum in I (if any) is singular, $\sigma \cap I \subset \sigma_s$.*

As, in general, singular continuous spectrum can be present, in this statement, one cannot replace σ_s , the singular spectrum, with σ_{pp} , the pure point spectrum. One may ask if it is possible to characterize the singular continuous spectrum in terms of the Lyapunov exponent.

Consider equation (1) on the interval $E \in I$ where $\gamma(E) > 0$.

Almost surely, for a given θ , the Lyapunov exponents exist a priori only almost everywhere in E . Denote by I_{Lyapunov} the subset of I where $\gamma^+(E, \theta)$ and $\gamma^-(E, \theta)$ both exist and are positive. For $E \in I_{\text{Lyapunov}}$, the solutions to (1) have to increase or decrease exponentially (see, e.g., [2]).

This implies that the singular continuous component of the spectral measure vanishes on I_{Lyapunov} . So, it can be positive only on $I \setminus I_{\text{Lyapunov}}$. And, the latter must happen if the spectrum on I is singular continuous.

0.4. **B. Simon's example:** We now recall an example by B.Simon showing that, for quasi-periodic operators, one can find singular continuous spectrum on an interval where the Lyapunov exponent is positive.

Consider the Almost Mathieu equation, i.e., equation (1) with $v(\theta) = 2\lambda \cos \theta$.

For this equation, by Herman's theorem ([7]), $\gamma(E) \geq \log \lambda$. We assume that $\lambda > 1$. Then, $\gamma(E)$ is positive for all E , and the spectrum is singular.

Let the frequency ω be such that, for some infinite sequence $(p_m, q_m) \in \mathbb{N} \times \mathbb{N}^*$,

$$\left| \omega - \frac{p_m}{q_m} \right| \leq m^{-q_m}.$$

Such Liouvillean frequencies are topologically typical but of zero measure.

One has

Theorem 3 ([2]). *Under the above conditions, there are no eigenvalues and the spectrum is purely singular continuous.*

Note that this result is a consequence of a theorem by A. Gordon (see [5, 4]) which roughly says that when the quasi-periodic potential can be super-exponentially well approximated by periodic potentials, the equation (1) does not admit any decreasing solutions.

Note that actually, in the case of the almost Mathieu equation, Gordon's result implies that any of its solution ϕ satisfies the inequality

$$\overline{\lim}_{m \rightarrow \infty} \max(\phi(\pm q_m), \phi(\pm 2q_m)) \geq \frac{1}{2}\phi(0),$$

$$\phi(n) = (|\psi(n+1)|^2 + |\psi(n)|^2)^{1/2}.$$

This means that the corresponding generalized eigenfunctions have to have infinitely many humps located at some of the points $\pm q_m, \pm 2q_m, m \in \mathbb{N}$. These humps prevent the solutions from being square summable.

0.5. **Non-trivial model problem.** In the present note, we concentrate on the model equation

$$(5) \quad \psi(n+1) + \psi(n-1) = \lambda v_0(n\omega + \theta) \psi(n), \quad n \in \mathbb{Z},$$

$$(6) \quad v_0(\theta) = 2 e^{i\pi\omega/2} \sin(\pi\theta).$$

where $0 < \omega < 1$ is an irrational frequency, $1 < \lambda$ is a coupling constant, and θ is the ergodic parameter. Actually, up to a shift in θ , this is an Almost Mathieu equation with the spectral parameter equal to zero.

We study this equation for the following reasons:

- (1) a large part of analysis is quite simple whereas (we believe that) to carry it out one has to use a non trivial renormalization procedure;
- (2) the techniques developed in this study can be generalized to the case of real analytic potentials v ;

(3) this model is related to various self-adjoint models via a cocycle representation (see [3]), e.g., it comes up naturally when studying the spectral properties of the equation

$$-\psi''(t) + \alpha \sum_{l \geq 0} \delta(l(l-1)/2 + l\phi_1 + \phi_2 - t) \psi(t) = E\psi(t).$$

For the model equation (5), our ultimate goal is to describe the set of θ for which the Lyapunov exponent exists or does not exist and to describe the solutions both when the Lyapunov exponent exists and does not exist.

We concentrate on the case of frequencies complementary to the frequencies occurring in Simon's example. And, in the case when the Lyapunov exponent does not exist, this leads to a new scenario for the behavior of solutions of (1).

Our main tool is the *monodromization* renormalization method introduced by V. Buslaev - A. Fedotov originally for the semi-classical study of the geometry of the spectrum of one dimensional finite difference almost periodic equations, see [1]. The idea was to construct Weyl solutions outside the spectrum but, at each step of the renormalization, closer to spectrum so as to uncover smaller and smaller gaps in the spectrum. Now, essentially, we use it to study the solutions of the model equation on the spectrum.

1. EXISTENCE OF THE LYAPUNOV EXPONENT FOR THE MODEL EQUATION

We now formulate our results on the pointwise existence of the right Lyapunov exponent $\gamma^+(\theta)$ for the model equation (5); as we have set the energy parameter, to a fixed value, we omit it in the Lyapunov exponents. The right Lyapunov exponent is defined by the formula (3) with

$$(7) \quad M(x) = \begin{pmatrix} \lambda v_0(\theta) & -1 \\ 1 & 0 \end{pmatrix},$$

where v_0 is given by (6).

Note that for $\gamma^-(\theta)$, the left Lyapunov exponent, one has similar results.

1.1. Main result. Here, we formulate a sufficient condition for the existence of the Lyapunov exponent. Therefore, we need to introduce some notations.

For $L = 0, 1, 2, \dots$, define

$$\omega_{L+1} = \left\{ \frac{1}{\omega_L} \right\}, \quad \omega_0 = \omega.$$

where $\{a\}$ is the fractional part of $a \in \mathbb{R}$, and

$$\lambda_{L+1} = \lambda_L^{\frac{1}{\omega_L}}, \quad \lambda_0 = \lambda$$

Remark 1. The numbers $\{\omega_l\}_{l=1}^\infty$ are related to the continued fraction expansion of ω :

$$\omega = \frac{1}{a_1 + \omega_1} = \frac{1}{a_1 + \frac{1}{a_2 + \omega_2}} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \omega_3}}} = \dots = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}}$$

where $a_1, a_2, a_3 \dots \in \mathbb{N}$ are the elements of the continuous fraction for ω . It is well known that, for any $l \in \mathbb{N}$, one has $\omega_l \omega_{l+1} \leq 1/2$. This implies that the numbers λ_l increase super-exponentially.

Furthermore, for a given $\omega \in (0, 1) \setminus \mathbb{Q}$ and $s \in (0, 1)$, define the following sequence

$$(8) \quad s_L = \left\{ \frac{s_{L-1}}{\omega_{L-1}} \right\}, \quad s_0 = s.$$

One has

Lemma 1. If $s = k_0 + \omega_0 l_0$, where $k_0, l_0 \in \mathbb{Z}$, then

- for all L , one has $s_L = k_L + \omega_L l_L$ with $k_L, l_L \in \mathbb{Z}$;
- if $k_0 > 0$, then the sequence $(k_{2L})_{L \geq 0}$ is monotonically decreasing until it vanishes and then it stays constant equal to 0;
- let $k_0 > 0$ and L be the first number for which $k_{2L} = 0$, then

$$k_0 \omega_0 \omega_1 \dots \omega_{2L-1} \leq 2.$$

For a given $L \in \mathbb{N}$, define $K(2L, \omega)$ being the maximal k_0 such that $k_{2L} = 0$ and set $K(2L-1, \omega) = K(2L, \omega)$.

Now, we are ready to discuss the Lyapunov exponent. We have

Theorem 4. Pick $\lambda > 1$ and $\omega \in (0, 1)$ irrational. Assume that there exists a function $M : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(L) < L$ and that, for $L \rightarrow \infty$,

$$(9) \quad \omega_{M(L)} \omega_{M(L)+1} \dots \omega_{L-1} \rightarrow 0, \quad \text{and} \quad \lambda_{M(L)} \omega_{M(L)} \omega_{M(L)+1} \dots \omega_L \rightarrow \infty.$$

For a given $0 \leq \theta \leq 1$, the Lyapunov exponent $\gamma^+(\theta)$ for equation (5) exists if, for all sufficiently large L , one has:

$$(10) \quad |\theta - k - l\omega_0| \geq \omega_0 \omega_1 \dots \omega_{M(L)-1} e^{-\frac{1}{\omega_0 \omega_1 \dots \omega_{M(L)-1}}},$$

for all $k, l \in \mathbb{Z}$ such that $0 \leq k + l\omega_0 \leq 1$ and

$$(11) \quad K(M(L), \omega) < k \leq K(\tilde{L}, \omega), \quad \tilde{L} = \begin{cases} L & \text{if } L \text{ is even,} \\ L+1 & \text{otherwise.} \end{cases}$$

Furthermore, when $\gamma^+(\theta)$ exists, it is equal to $\log \lambda$.

One also has a similar statement on the pointwise existence of the left Lyapunov exponent γ^- . Note that for γ^- to exist, θ has to avoid neighborhoods of the points $k + l\omega_0$ with negative k .

Now, turn to a discussion of the results given in Theorem 4.

1.2. **Admissible frequencies.** Denote by Ω the set of $\omega \in (0, 1)$ satisfying the conditions of Theorem 4

1.2.1. *The measure of Ω .* Khinchin's famous result (see e.g. [8]) on the geometric means of the products of the elements of the continued fractions implies

Lemma 2. $\text{mes } \Omega = 1$.

Proof. Let $\{a_l\}$ be the elements of the continued fraction for ω . By Khinchin, for almost all ω , one has $\lim_{L \rightarrow \infty} (a_1 a_2 \dots a_L)^{\frac{1}{L}} = C$, where $C = 2, 6 \dots$ is a universal constant. Pick $l \in \mathbb{N}$. One has $\frac{1}{2a_l} < \omega_{l-1} < \frac{1}{a_l}$. Therefore, for almost all frequencies ω ,

$$\overline{\lim}_{L \rightarrow \infty} (\omega_0 \omega_1 \dots \omega_{L-1})^{\frac{1}{L}} \leq \frac{1}{C}, \quad \underline{\lim}_{L \rightarrow \infty} (\omega_0 \omega_1 \dots \omega_{L-1})^{\frac{1}{L}} \geq \frac{1}{2C}.$$

Such ω belong to Ω : in (9) one can take $M(L) = [L/2]$. \square

1.2.2. *Liouvillean numbers in Ω .* Recall that an irrational number ω is called Liouvillean if, for any $n \in \mathbb{N}$, there are infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that

$$\left| \omega - \frac{p}{q} \right| \leq \frac{1}{q^n}.$$

(see e.g. [8]).

One has

Lemma 3. *The set Ω contains Liouvillean numbers satisfying*

$$(12) \quad \left| \omega - \frac{p}{q} \right| \leq \frac{1}{q \lambda^{c q}}, \quad c = c(\omega) > 0,$$

for infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$.

Proof. We construct a Liouvillean $\omega \in \Omega$ by choosing inductively $(a_l)_{l \geq 1}$, the elements of its continued fraction. Therefore, we pick $a_1 \geq 1$ large and, for all $L \geq 1$, we choose a_{L+1} so that

$$(13) \quad \frac{1}{2} a_{L+1} \leq (a_1 a_2 \dots a_L)^{-1} \lambda^{a_1 a_2 \dots a_L} \leq a_{L+1}.$$

We now check that such an ω belongs to Ω . Therefore, we check that one has (9) for $M(l) = l - 1$. As $\lambda > 1$, the sequence $(a_l)_l$ is quickly increasing, and so

$$(14) \quad \omega_{l-1} \rightarrow 0, \quad l \rightarrow \infty.$$

Furthermore, as, for all $l \geq 0$, one has $\omega_l = (a_{l+1} + \omega_{l+1})^{-1}$, we get

$$\omega_{l-1} \omega_l \lambda_{l-1} > \frac{1}{4 a_l a_{l+1}} \lambda^{a_1 a_2 \dots a_l} \geq \frac{a_1 a_2 \dots a_{l-1}}{8}.$$

This implies that

$$\lambda_{l-1} \omega_{l-1} \omega_l \rightarrow \infty,$$

and so $\omega \in \Omega$.

Now, let us check that ω satisfies (12) (and, thus, is a Liouville number). Consider $\left(\frac{p_l}{q_l}\right)$, the sequence of the best approximates for ω . Recall that (see e.g. [8]), for all $l \in \mathbb{N}$,

$$(15) \quad \left| \omega - \frac{p_l}{q_l} \right| \leq \frac{1}{a_{l+1}q_l^2},$$

$$(16) \quad q_{l+1} = a_{l+1}q_l + q_{l-1}, \quad q_1 = a_1, \quad q_0 = 1.$$

The relations (16) imply that

$$(17) \quad a_l a_l \dots a_2 a_1 < q_l < P a_l \dots a_2 a_1, \quad P = \prod_{l=1}^{\infty} \left(1 + \frac{1}{a_l a_{l+1}} \right);$$

the product P converges as the sequence $(a_l)_l$ is quickly increasing. Relations (17) and (13) imply that $a_{l+1} \geq q_l^{-1} \lambda^{q_l/P}$. This and (15) imply (12). \square

1.3. The set of “bad” phases. For given $\lambda > 1$ and $\omega \in \Omega$, denote by Θ the set of phases θ not satisfying (10) for infinitely many L . One has

Lemma 4. *The set Θ is topologically typical (countable intersection of dense open sets) and, under the condition*

$$(18) \quad \sum_{L>0}^{\infty} (\lambda_{M(L)} \omega_{M(L)} \omega_{M(L)+1} \dots \omega_L)^{-1} < \infty$$

(which is stronger than (9)), it has zero Lebesgue measure.

Proof. For a given $L > 0$, denote the set of θ not verifying (10) by Θ_L . Then

$$(19) \quad \Theta = \bigcap_{N \geq 0} \bigcup_{L \geq N} \Theta_L.$$

Thus, Θ is a countable intersection of open sets.

As ω is irrational, the points $\theta_{k,l} = k + \omega_0 l$ ($k, l \in \mathbb{Z}$, $k \geq 0$) are dense in the interval $(0, 1)$. So, to complete the proof of the first property of Θ , it suffices to show that the set $\bigcup_{L \leq N} \Theta_L$ contains all the points $\theta_{k,l}$ with k sufficiently large. But, this follows from (11) and the inequality $M(L) < l$.

Finally, note that, by (10),

$$\text{mes } \Theta_L \leq \frac{1}{\omega} K(L) \omega_0 \omega_1 \dots \omega_{M(L)-1} \lambda_{M(L)}^{-1} \leq \frac{2}{\omega} (\lambda_{M(L)} \omega_{M(L)} \dots \omega_{L-1} \omega_L)^{-1}.$$

Under the condition (18), this implies that the Lebesgue measure of Θ is zero. \square

1.4. **Heuristics and the statement of Theorem 4.** Let us now describe some heuristics “explaining” Theorem 4.

Consider a continuous version of equation (5) :

$$(20) \quad \phi(s + \omega) + \phi(s - \omega) = \lambda v_0(s) \phi(s), \quad s \in \mathbb{R}.$$

If ϕ satisfies this equation, then the formula $\psi(n) = \phi(n\omega + \theta)$, $n \in \mathbb{Z}$, defines a solution to (5).

If $\lambda \gg 1$, then one can expect that, on a fixed compact interval, equation (20) has an exponentially increasing solution ϕ^+ with the leading term ϕ_0^+ satisfying the equation

$$(21) \quad \phi_0^+(s + \omega) = \lambda v_0(s) \phi_0^+(s), \quad s \in \mathbb{R}.$$

For the last equation, one can easily construct a solution ϕ_0^+ that is analytic and has no zeros in the band $0 < \operatorname{re} s < 1 + \omega$. One can extend this solution analytically to the left of this band using equation (21). As v_0 vanishes at integers, ϕ_0^+ has zeros at all the points of the form $s_{k,l} = k + l\omega$ where $k, l > 0$ are integers.

If there is a true solution to (20) with the leading term ϕ_0^+ , then (5) has a solution ψ^+ with the leading term $\phi_0^+(n\omega + \theta)$. Furthermore, if $\theta \in (0, 1)$ admits the representation $\theta = k_0 - l_0\omega$ with some positive integers k_0 and l_0 , then, at least for sufficiently large λ , the leading term of ψ^+ increases exponentially on the “interval” where $-\frac{k_0}{\omega} + l_0 < n < l_0 + 1$ and then vanishes at the points $n = l_0 + 1, l_0 + 2, \dots$. The equality $\theta = k_0 - l_0\omega$ can be interpreted as a quantization condition: when this condition is satisfied, the solution ψ^+ that is exponentially growing up to the point $n = l_0$, at this point, changes to the exponential decay.

So, it is natural to expect that the solution ψ^+ keeps growing up to the infinity if θ is “far enough” from all the points of the form $k_0 - l_0\omega$ with positive integers k_0 and l_0 . Hence, the right Lyapunov exponent should exist.

2. NON-EXISTENCE OF THE LYAPUNOV EXPONENT

Theorem 4 is rather rough in the sense that the sizes of the “secure intervals” that θ has to avoid for the Lyapunov exponent to exist (see (10)) are too big. This is actually due to the fact that, under the conditions of Theorem 4, one has much more than the existence of Lyapunov exponent. Roughly, under these conditions, for each L large enough, equation (5) has solutions that, locally, on intervals of length of order $(\omega_0\omega_1 \dots \omega_{M(L)-1})^{-1}$, can have complicated behavior whereas globally, on the interval $0 < k < \mathcal{K}$ of length of order $(\omega_0\omega_1 \dots \omega_L)^{-1}$, they are nicely exponentially increasing.

Our method also allows a precise description of the set of θ where the Lyapunov exponent does not exist. The structure of this set is quite complicated; in the

present note, we only describe it for frequencies in $\Omega_1 \subset \Omega$, the set of ω satisfying the conditions

$$(22) \quad \omega_L \rightarrow 0, \quad \text{and} \quad \lambda_{L-1} \omega_{L-1} \omega_L \rightarrow \infty.$$

instead of (9).

One has the following two statements:

Theorem 5. *Pick $\lambda > 1$. Let $\omega \in \Omega_1$. For a $0 < \theta < 1$, define the sequence $\{s_L\}$ by (8) with $s_0 = \theta$. Assume that there is a positive constant c such that for infinitely many even positive integers L one has*

$$\text{dist}(s_{L-1}, \omega_{L-1} \cdot \mathbb{N}) \leq \omega_{L-1} \lambda_L^{-c} \quad \text{and} \quad s_{L-1} \geq c.$$

Then, the right Lyapunov exponent $\gamma^+(\theta)$ does not exist.

and

Theorem 6. *Pick $\lambda > 1$. Let $\omega \in \Omega_1$. For a $0 < \theta < 1$, define the sequence $\{s_L\}$ by (8) with $s_0 = \theta$. Assume that there is a positive constants c and N such that for infinitely many odd positive integers L one has*

$$\text{dist}(s_L, \omega_L \cdot \mathbb{N}) \leq \lambda_L^{-c}, \quad s_{L-1} \leq 1 - c, \quad \text{and} \quad s_L \leq \omega_L N.$$

Then, the right Lyapunov exponent $\gamma^+(\theta)$ does not exist.

The above two theorems are sharp: in the case of $\omega \in \Omega_1$, if the Lyapunov exponent does not exist, then θ satisfies the conditions of one of the them.

As for the behavior of the solutions, in both cases, roughly, we find that, for infinitely many L , even if we forget of the complicated local behavior of the solutions on the intervals of the length of order $(\omega_0 \omega_1 \dots \omega_{M(L)-1})^{-1}$, one can see that globally, on the interval $0 < k < \mathcal{K}$ of the length of order $(\omega_0 \omega_1 \dots \omega_L)^{-1}$, the solutions change from exponential growth to the exponential decay. For example, in the case of Theorem 6, there exists solutions that, at first, are globally exponentially increasing then are globally exponentially decaying, the length of the interval of increase and the interval of decrease being of the same order. Here we use the word globally to refer to the fact that this exponential growth or decay happens at a large scale.

3. THE MAIN IDEAS OF THE PROOF

As we have mentioned in the introduction, the main tool of the proof is the *monodromization* renormalization method. The new idea is that one can consider the infinite sequence of the almost periodic equations arising in the course of the monodromization as a sequence of equations describing a given solution of the input equation on larger and larger intervals, the ratio of their length being

determined by the continued fraction of the frequency.

Now, the renormalization formulas can be written in the form

$$(23) \quad \begin{aligned} & M(\theta + (k-1)\omega) \dots M(\theta + \omega) M(\theta) \\ & \sim \Psi(\{k\omega + \theta\}) [M_1(\theta_1 - \omega_1) M_1(\theta - 2\omega_1) \dots M_1(\theta - k_1\omega_1)]^t \Psi^{-1}(\theta). \end{aligned}$$

Here, “ \sim ” means “equal up to a sign”,

- $M(\theta) = \begin{pmatrix} 2\lambda \sin(\pi\theta) & -e^{-i\pi\theta} \\ e^{i\pi\theta} & 0 \end{pmatrix}$; the second order difference equation (20), the continuous analog of (5), is equivalent to the first order matrix difference equation

$$(24) \quad \Psi(s + \omega) = M(s)\Psi(s), \quad s \in \mathbb{R};$$

- Ψ is a fundamental solution to (24), i.e., such that $\Psi(s) \in SL(2, \mathbb{C})$ for all s ;
- t denotes the transposition;
- M_1 is a monodromy matrix corresponding to this solution, i.e., the matrix defined by $\Psi(s + 1) = \Psi(s)M_1^t(s/\omega)$.

The new constants ω_1 , θ_1 and the number k_1 are defined by

$$\omega_1 = \{1/\omega\}, \quad \theta_1 = \{\theta/\omega\}, \quad k_1 = [\theta + k\omega].$$

And, as usual $\{a\}$ and $[a]$ denote the fractional and the integer part of $a \in \mathbb{R}$.

Formula (23) relates the study of the matrix product $M(\theta + (k-1)\omega) \dots M(\theta + \omega) M(\theta)$ to that of a similar product: the monodromy matrix M_1 is unimodular and, as the matrix M , it is 1 anti-periodic. One can apply the same renormalization formula for the new matrix product and so on. It is easy to check that after a finite number of renormalizations, one gets a matrix product containing at most ... one matrix. This feature recalls the renormalization of the quadratic exponential sums carried out by Hardy and Littlewood (see [6, 3]).

At each step of the monodromization, one has to study similar difference equations $\Psi_L(s + \omega_k) = M_L(s)\Psi_L(s)$, $L = 0, 1, 2, \dots$. One needs to have a good enough control of their solutions but only on one fixed compact interval namely $[0, 1]$.

For our model, one can choose the fundamental solutions so that all the matrices M_L have the same functional structure, and the numbers (λ_L) are the successive coupling constants in these equations.

For $\lambda = \lambda_0 > 1$, the sequence $(\lambda_L)_L$ tends to infinity very rapidly; this enables an effective asymptotic analysis of the successive equations. For general almost periodic equations, one finds an analogous effect at least when the coupling constant in the input equation is large enough.

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