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► **To cite this version:**

Philippe Blanchard, Daniel Gandolfo, Jean Ruiz, Marc Wouts. Thermodynamic versus Topological Phase Transitions: Cusp in the Kertész Line. 2008. hal-00263693v1

HAL Id: hal-00263693

<https://hal.science/hal-00263693v1>

Preprint submitted on 13 Mar 2008 (v1), last revised 2 Sep 2008 (v2)

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Thermodynamic versus Topological Phase Transitions: Cusp in the Kertész Line

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We present a study of phase transitions of the Curie–Weiss Potts model at (inverse) temperature β , in presence of an external field h . Both thermodynamic and topological aspects of these transitions are considered. For the first aspect we complement previous results and give an explicit equation of the thermodynamic transition line in the β – h plane as well as the magnitude of the jump of the magnetization (for $q \geq 3$). The signature of the latter aspect is characterized here by the presence or not of a giant component in the clusters of a Fortuin–Kasteleyn type representation of the model. We give the equation of the Kertész line separating (in the β – h plane) the two behaviours. As a result, we get that this line exhibits, as soon as $q \geq 3$, a very interesting cusp where it separates from the thermodynamic transition line.

PACS numbers: 05.50.+q, 05.70.Fh, 64.60.ah, 02.10.Ox

In [1], Kertész pointed out that a very interesting phenomenon arises in the Ising model subject to an external field. The so-called Coniglio–Klein droplets [2] associated to Fortuin–Kasteleyn clusters [3] have a whole percolation transition line extending from the Curie point to infinite fields. This seems, at first sight in contradiction with the fact that on the other hand, thermodynamic quantities do not have any singularities in any of their derivatives (with respect to the temperature or the field) as soon as the field is non-zero. But, as Kertész already mentioned:

However, we emphasize that the suggested picture is not in contradiction with the non existence of singularities in the thermodynamic quantities because the total free energy remains analytic.

Indeed in such a model, the criticality can be specified in two different ways: the thermodynamic criticality associated to the thermodynamic limit of the bulk free energy and the geometric criticality associated in the same limit to another lower order free energy.

Since then, the Kertész line has remained the subject of interests over the years for various models of statistical mechanics. In particular, it has been considered recently in [4] within the Potts' model on the regular lattice \mathbb{Z}^d . There it was found, that, in dimension $d = 2$, the whole thermodynamic first order transition line (when such first order behavior is present in the system) coincides with the Kertész line. The latter is also first order in the corresponding range of values of temperature and field.

The aim of this letter is to present a study of such a

model when the underlying lattice is the complete graph with n vertices. In this case the model is called mean-field or Curie–Weiss Potts model.

In this model, the spin variables σ_i attached to the sites $i = 1, \dots, n$ of the graph take values in the set $\{1, \dots, q\}$. The Curie–Weiss Potts model at temperature $T = 1/\beta$ and subject to an external field $H = h/\beta$ is defined by the Gibbs measure

$$\mu_{\text{Potts}}(\boldsymbol{\sigma}) = \frac{1}{Z_{\text{Potts}}} \prod_{i < j} e^{(\beta/n)(\delta_{\sigma_i, \sigma_j} - 1)} \prod_i e^{h\delta_{\sigma_i, 1}} \quad (1)$$

over spins configurations $\boldsymbol{\sigma}$. Here, Z_{Potts} denotes the partition function, the indices i, j runs over the set $\{1, \dots, n\}$, and δ is the Kronecker symbol. The critical (thermodynamic) behaviour of this model is well known [5], [6], [7], and mainly governed by the mean field equation

$$h = -\beta s + \ln \frac{1 + (q-1)s}{1-s} \quad (2)$$

Namely, when $h = 0$, there exists a threshold value

$$\beta_c^{(q)} = 2 \quad \text{for } q = 2 \quad (3)$$

$$\beta_c^{(q)} = 2 \frac{q-1}{q-2} \ln(q-1) \quad \text{for } q \geq 3 \quad (4)$$

such that the system exhibits at $\beta_c^{(q)}$ a continuous transition when $q = 2$, and a first order transition when $q \geq 3$. When $h > 0$, no transition occurs if $q = 2$, while as soon as $q \geq 3$ a first order transition line appears for which the microcanonical free energy of the model has two minima

associated with two two different solutions of the mean field equation [5], [7].

To study the behaviour of clusters previously mentioned, we turn to the Edwards–Sokal joint measure [8] given by

$$\mu_{\text{ES}}(\boldsymbol{\sigma}, \boldsymbol{\eta}) = \frac{1}{Z_{\text{ES}}} \prod_{i < j} [e^{-\frac{\beta}{n}}(1 - \eta_{ij}) + (1 - e^{-\frac{\beta}{n}})\eta_{ij}\delta_{\sigma_i, \sigma_j}] \times \prod_i e^{h\delta_{\sigma_i, 1}} \quad (5)$$

where the edges variables η_{ij} belong to $\{0, 1\}$.

Notice that when $h = \infty$, all $\sigma_i = 1$. This means that we open edges ($\eta_{ij} = 1$) with probability (w.p.) $p = 1 - e^{-\beta/n}$ and close edges ($\eta_{ij} = 0$) w.p. $e^{-\beta/n}$. This is nothing else than the well known Erdős–Rényi random graph $\mathcal{G}(n, p)$ [9]. This random graph is known to exhibit a (topological) transition at $\beta = 1$ such that with probability tending to 1 as $n \rightarrow \infty$:

- i. for $\beta < 1$ all components of open edges are at most of order $\ln n$.
- ii. at $\beta = 1$ a giant component of order $n^{2/3}$ appears.
- iii. for $\beta > 1$ this giant component becomes of order s^*n where s^* is the largest root of the mean field equation (2) with $q = 1$ and $h = 0$.

We refer the reader to [10], [11] for detailed discussions and proofs (see also [12] for a new approach of this transition).

Notice also that, on the other hand, when $h = 0$, the marginal of the ES measure over the edges variables is the random cluster model:

$$\mu_{\text{RC}}(\boldsymbol{\eta}) = \frac{1}{Z_{\text{RC}}} \prod_{i < j} e^{-(\beta/n)(1 - \eta_{ij})} (1 - e^{-\beta/n})\eta_{ij} q^{C(\boldsymbol{\eta})} \quad (6)$$

where $C(\boldsymbol{\eta})$ denotes the number of connected components (including isolated sites) of open edges of the configuration $\boldsymbol{\eta}$. For $q = 1$, this model again reduces to $\mathcal{G}(n, p)$ with p as before. A refined study of the random cluster model (6) is given in [13] (see also [14], [15]). There, it is shown that with the threshold value $\beta_c^{(q)}$, given by (3) for $0 < q \leq 2$ and by (4) for $q > 2$, then a.e.:

- a) if $\beta < \beta_c^{(q)}$, the largest component (of open edges) of μ_{RC} is of order $\ln n$.
- b) if $\beta > \beta_c^{(q)}$, the largest component of μ_{RC} is of order $s_0 n$, where $s_0 > 0$ is the largest root of the mean field equation (2) with $h = 0$.
- c) if $\beta = \beta_c^{(q)}$ and $0 < q \leq 2$, μ_{RC} has largest component of order $n^{2/3}$.
- d) if $\beta = \beta_c^{(q)}$ and $q > 2$, μ_{RC} is either as in a) or as in b).

According to these results, it is natural to expect for model μ_{ES} (and its marginal over the edges variables), a Kertész line $h_K(\beta)$, interpolating between $h_K(1) = \infty$ and $h_K(\beta_c^{(q)}) = 0$, that signs the appearance of a giant component in the open edges of the configuration $\boldsymbol{\eta}$.

To study this line, it is convenient to consider a colored version of the Edwards–Sokal formulation. Namely, whenever the endpoints ij of a given edge are occupied by a spin of color a , we label the variable η_{ij} with a superscript indicating the color. For a spin configuration $\boldsymbol{\sigma}$, let $n_a(\boldsymbol{\sigma})$ be the number of sites occupied by spins of color $a = 1, \dots, q$. We then relabel the variables η_{ij}^a by $\eta_{kl}^a : k < l \in \{1, \dots, n_a(\boldsymbol{\sigma})\}$. This means that, for any pairs $i < j \in \{1, \dots, n_a(\boldsymbol{\sigma})\}$, we open an edge $\eta_{ij}^a = 1$ w.p. $1 - e^{-\beta/n}$, and close this edge $\eta_{ij}^a = 0$ w.p. $e^{-\beta/n}$. All the other edges between two sites of different colors are closed. The resulting measure becomes

$$\mu_{\text{CES}}(\boldsymbol{\sigma}, \boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^q) = \frac{e^{hn_1(\boldsymbol{\sigma})}}{Z_{\text{CES}}} \prod_{a=1}^q e^{\beta n_a(\boldsymbol{\sigma})(n_a(\boldsymbol{\sigma})-1)/2n} \times \prod_{i < j \in \{1, \dots, n_a(\boldsymbol{\sigma})\}} [e^{-\beta/n}(1 - \eta_{ij}^a) + (1 - e^{-\beta/n})\eta_{ij}^a] \quad (7)$$

By summing over the spins variables, we get the following Colored–Random–Cluster model

$$\mu_{\text{CRC}}(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^q) = \frac{1}{Z_{\text{CRC}}} \sum_{n_1 + \dots + n_q = n} Z(n_1, \dots, n_q) \times \prod_{a=1}^q \prod_{i < j \in \{1, \dots, n_a\}} [e^{-\beta x_a/n_a}(1 - \eta_{ij}^a) + (1 - e^{-\beta x_a/n_a})\eta_{ij}^a] \quad (8)$$

where $x_a = n_a/n$ are the densities of the colors and

$$Z(n_1, \dots, n_q) = \frac{n!}{n_1! \dots n_q!} e^{hn_1} \prod_{a=1}^q e^{\beta n_a(n_a-1)/2n} \quad (9)$$

This colored random cluster model can be thought as follows. Given a partition n_1, \dots, n_q of n , we have q classical Erdős–Rényi random graphs where the edges are open w.p. $1 - e^{-\beta x_a/n_a}$. The asymptotic behavior of the partition as $n \rightarrow \infty$ will be determined by minimizing the free energy

$$f(x_1, \dots, x_q) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln Z(n_1, \dots, n_q) \quad (10)$$

where the limit is taken in such a way that $n_a/n \rightarrow x_a$.

Notice that, up to a normalizing factor, $Z(n_1, \dots, n_q)$ is the microcanonical partition function of the Potts model (1) restricted to configurations such that the number of spins of color a is fixed to n_a . This will give the thermodynamic behavior of the system and will determine the densities x_a . Once this is done the topological properties can be analysed by using the known properties

of classical random graphs. Due to the presence of the field or using the symmetry of colors for vanishing field, we will pick up the first color. Then, the threshold value of the (topological) transition will be given by

$$\beta_K x_1 = 1 \quad (11)$$

For $\lambda = \beta x_1 > 1$ a giant component will appear, with size of order $\Theta_{x_1} n \equiv \theta_{\beta x_1} x_1 n$ where θ_λ is the largest root of the mean field equation

$$\lambda \theta = -\ln(1 - \theta) \quad (12)$$

Let us briefly recall the minimization procedure for (10). By Stirling's formula, we have $n!/n_1! \cdots n_q! \sim \prod_{a=1}^q e^{x_a \ln x_a}$ at the leading order in n , giving

$$f(x_1, \dots, x_q) = \sum_{a=1}^q x_a \ln x_a - \frac{\beta}{2} \sum_{a=1}^q x_a^2 - h x_1 \quad (13)$$

The minima of this function can be parametrised by a real number s , $0 \leq s \leq 1$, such that the minimizing vector (x_1, \dots, x_q) have the components

$$x_1 = \frac{1 + (q-1)s}{q} \quad (14)$$

$$x_a = \frac{1-s}{q} \quad a = 2, \dots, q \quad (15)$$

(with arbitrary order for $h = 0$).

The free energy (13) then takes the form

$$\begin{aligned} f(s) = & \frac{1 + (q-1)s}{q} \ln(1 + (q-1)s) \\ & + \frac{q-1}{q} (1-s) \ln(1-s) - \beta \frac{(q-1)}{2q} s^2 \\ & - h \frac{1 + (q-1)s}{q} - \frac{\beta}{2q} - \ln q \quad (16) \end{aligned}$$

The minimizers have to be found among the solutions of the mean field equation (2) obtained by differentiating f with respect to s . Note that by taking into account (14) the condition (11) becomes

$$\beta_K \frac{1 + (q-1)s}{q} = 1 \quad (17)$$

We now turn to the results.

Let us first consider the case $q = 2$.

When $h = 0$, the mean field equation (2) has a unique solution $s_0 = 0$ for $\beta \leq 2$, and a unique solution $s_0 > 0$ for $\beta > 2$. Inserting s_0 into (17), we obtain that the giant component appears at $\beta_K^{(2)} = 2$, getting thus $\beta_K^{(2)} = \beta_c^{(2)}$. There, this component is of order $n^{2/3}$. For $\beta > 2$, $\beta x_1(s_0) > 1$, and we are in the supercritical regime with a giant component of order $\Theta_{x_1(s_0)} n$. For $\beta < 2$, $\beta x_1(s_0) < 1$, and we are in the subcritical regime with components of order at most $\ln n$.

When $h > 0$, the mean field equation has a unique solution $s_0 > 0$ for any β (no thermodynamic transition occurs). The transition line $\beta_K^{(2)}(h)$, or equivalently $h_K^{(2)}(\beta)$, is obtained by eliminating s between (2) and (17). This Kertész line is given by

$$h_K^{(2)}(\beta) = \beta - 2 - \ln(\beta - 1) \quad (18)$$

On this line, the largest component is of order $n^{2/3}$. For $h < h_K$, the largest component is of order $\ln n$, while for $h > h_K$, the largest component is of order $\Theta_{x_1(s_0)} n$.

Let us now consider the case $q \geq 3$. We have now a thermodynamic transition line

$$h_T(\beta) = -\beta \frac{q-2}{2(q-1)} + \ln(q-1), \quad \beta_0 \leq \beta \leq \beta_c^{(q)} \quad (19)$$

with endpoints

$$(\beta_0 = 4 \frac{q-1}{q}, h_0 = -2 \frac{q-2}{q} + \ln(q-1)), \quad (\beta_c^{(q)}, 0) \quad (20)$$

On this line the free energy has two minima s_- and s_+ for which $f(s_-) = f(s_+)$. They satisfy the mean field equation (2) and are given by the parametric equations

$$(\beta, h) = (\beta_{s_\pm}, -\frac{q-2}{2(q-1)} \beta_{s_\pm} + \ln(q-1)) \quad (21)$$

where

$$\beta_s = (s - \frac{q-2}{2(q-1)})^{-1} \ln \frac{1 + (q-1)s}{(q-1)(1-s)} \quad (22)$$

and $s_- \in [0, \frac{q-2}{2(q-1)}]$, $s_+ \in [\frac{q-2}{2(q-1)}, \frac{q-2}{(q-1)}]$.

For $\beta_0 < \beta \leq \beta_c^{(q)}$ (or equivalently $0 \leq h < h_0$) these two minima are distinct: $s_- < s_+$. At (β_0, h_0) , $s_- = s_+ = \frac{q-2}{2(q-1)}$, and at $(\beta_c^{(q)}, 0)$, $s_- = 0$ and $s_+ = \frac{q-2}{2(q-1)}$.

Outside of the segment (19), there is only one minimum s_0 . This minimum is an analytic function of β and h , and $s_0 \rightarrow s_-$ as $h \uparrow h_T$, $s_0 \rightarrow s_+$ as $h \downarrow h_T$. In addition, on h_T , as β increases from β_0 to $\beta_c^{(q)}$ (or equivalently as h decreases from h_0 to 0), $\beta x_1(s_-)$ strictly decreases from $\beta_0/2 > 1$ to $\beta_c^{(q)}/q < 1$, while $\beta x_1(s_+)$ strictly increases from $\beta_0/2$ to $\beta_c^{(q)}(q-1)/q$. See Appendix.

We now turn to the topological behavior.

Let (β_{cp}, h_{cp}) the point of h_T for which $\beta x_1(s_-) = 1$. This point is distinct from the two endpoints (20). It is given by the solution of the equation $(\beta_{cp} - 2)q = 2(q-1) \ln(\beta_{cp} - 1)$ which can be solved by using the Lambert W function.

As a consequence of the above remarks, one gets the following topological behaviour for the system. On the thermodynamic line h_T :

1. at (β_{cp}, h_{cp}) , the largest component of μ_{CRC} is either of order $n^{2/3}$ or of order $\Theta_{x_1(s_+)} n$.

2. if $h_{cp} < h \leq h_0$, the largest component of μ_{CRC} is either of order $\Theta_{x_1(s_-)}n$ or of order $\Theta_{x_1(s_+)}n$.
3. if $0 \leq h < h_{cp}$, the largest component of μ_{CRC} is either of order $\ln n$ or of order $\Theta_{x_1(s_+)}n$.

The item 3 extends to the values $0 \leq h < h_{cp}$ what happens in item d) of the random cluster model for vanishing field. It implies that the thermodynamic and topological lines coincide there. This holds also at (β_{cp}, h_{cp}) , with a new behaviour in the sense that the giant component is of order $n^{2/3}$ or of order $\Theta_{x_1(s_+)}n$.

The Kertész line is given in this range of temperature (and field) by

$$h_K^{(q)}(\beta) = h_T(\beta) \quad \text{for} \quad \beta_{cp} \leq \beta \leq \beta_c^{(q)} \quad \text{or} \quad 0 \leq h \leq h_{cp} \quad (23)$$

The item 2 shows that the giant component exhibits a jump, however, for given $h > h_{cp}$, this giant component already appeared for lower values of β . The topological transition line is determined, as in case $q = 2$, by eliminating s between the mean field equation and the condition (17). This gives

$$h_K^{(q)}(\beta) = \frac{\beta - q}{q - 1} - \ln \frac{\beta - 1}{q - 1} \quad \text{for} \quad \beta \leq \beta_{cp} \quad \text{or} \quad h \geq h_{cp} \quad (24)$$

see Fig. 1. Below the Kertész line $h_K^{(q)}$ the largest component is of order $\ln n$, while above $h_K^{(q)}$ and outside of the segment (19), the largest component is of order $\Theta_{x_1(s_0)}n$.

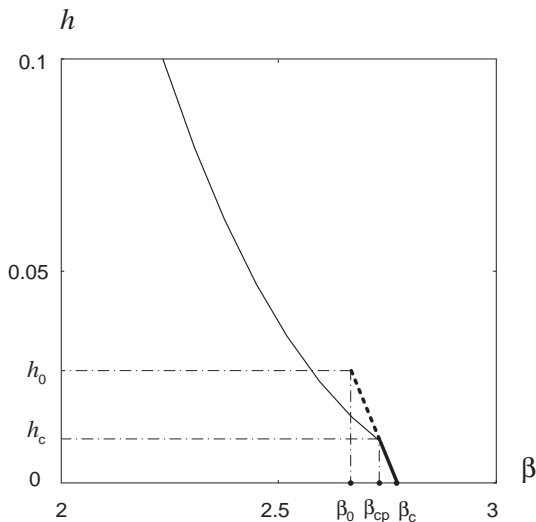


FIG. 1: The thermodynamic transition curve: bold (plain and dashed) line; the topological transition curve: plain (dashed and thin) line.

To summarize, we have shown that for low fields ($h \leq h_{cp}$) the Kertész line coincides with the thermodynamic transition line, and that for large fields ($h \geq h_0$) only the topological transition remains while the thermodynamic transition disappears. In addition, at intermediate fields ($h_{cp} < h < h_0$) the Kertész line separates

from the thermodynamic line. This means that decreasing the temperature one sees first the appearance of a giant component (on h_K) and, then on h_T , this component exhibits a jump. This behavior is new compared to what happens for the model on the 2-dimensional regular lattice. There, no intermediate regime appears [4]. We expect for sufficiently high lattice dimensions, a behaviour similar to the model on the complete graph.

Acknowledgments

Warm hospitality and Financial support from the Bi-BoS Research Center, University of Bielefeld and Centre de Physique Théorique, CNRS Marseille are gratefully acknowledged.

Appendix

Main part of the claims about the thermodynamic transition have been already proven in [7], for a thermodynamic curve with endpoints (20). We present below an alternative approach which allows to show that, in particular, the thermodynamic curve is the straight line (19). Indeed, as it is easy to observe, on the line of transition h_T given by (19), the free energy (16) has the symmetry $f(s) = f(\frac{q-2}{q-1} - s)$ with $s \in [0, \frac{q-2}{q-1}]$, see Fig. 2.

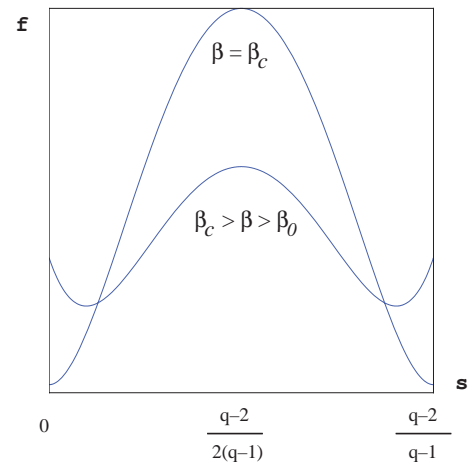


FIG. 2: The free energy on h_T .

This symmetry of the free energy is better seen if we parametrize the magnetization s by $z \in (\pm 1)$ as follows:

$$s = \frac{q-2}{2(q-1)} + \frac{q}{2(q-1)}z. \quad (25)$$

The corresponding densities are given by

$$x_1 = \frac{1+z}{2} \quad \text{and} \quad x_2 = \dots = x_q = \frac{1-z}{2(q-1)}$$

and the free energy decomposes immediately into its even and odd parts:

$$f(z) = \frac{1}{2}[(1+z)\ln(1+z) + (1-z)\ln(1-z) - \beta \frac{q(1+z^2)}{4(q-1)} - h - \ln(4(q-1))] + \frac{z}{2} \left[\ln(q-1) - \beta \frac{q-2}{2(q-1)} - h \right]. \quad (26)$$

The first derivative of f yields the mean field equation (2) while the second derivative is

$$f''(z) = \frac{1}{1-z^2} - \frac{\beta q}{4(q-1)}. \quad (27)$$

Hence f is convex when $\beta \leq \beta_0 = 4(q-1)/q$, whereas when $\beta > \beta_0$ it is convex on $(-1, -z_i)$, concave on $(-z_i, z_i)$ and convex again on $(z_i, 1)$, where $\pm z_i$ are the position of the inflexion points of f : $z_i = \sqrt{1 - \beta_0/\beta}$. In particular f has at most two local minima. In the following we denote z^- (resp. z^+) the position of the lowest (resp. largest) local minimizer. They correspond to the lowest (resp. largest) root of the mean field equation $f'(z) = 0$.

As a consequence, on the line h_T given by (19), the free energy f is an *even* function of z . If $\beta \leq \beta_0$, f is convex and attains its global minimum at $z_0 = z^+ = z^- = 0$. If $\beta > \beta_0$, the free energy attains twice its global minimum at z^+ and z^- , with $z^+ = -z^- > 0$ thanks to the symmetry of f . In the rest of the phase diagram, the decomposition of f at (26) shows that $z_0 = z^+$ above h_T , while $z_0 = z^-$ below h_T .

Now we prove the analyticity of z^- and z^+ in the parameters β, h and q . Consider for instance z^- and assume $(\beta, h) \in \mathcal{H} = \{\beta < \beta_0 \text{ or } f'(-z_i) > 0\}$. In this region, z^- can be characterized as the unique root of the mean field equation $f'(z) = 0$ in the interval

$$\begin{aligned} (-1, 1) & \quad \text{if } \beta < \beta_0 \\ (-1, -z_i) & \quad \text{if } \beta \geq \beta_0 \text{ and } f'(-z_i) > 0. \end{aligned}$$

There f' is an analytic function of β, h and q and satisfies $\partial f'/\partial z > 0$, yielding [16] the analyticity of z^- in β, h and q . Now we prove that \mathcal{H} covers the region strictly below h_T , together with a neighborhood of all points on h_T except (β_0, h_0) . This is trivial when $\beta < \beta_0$, so we consider now some (β, h) with $\beta = \beta_0$ that is strictly below h_T . z^- is the global minimizer of f and $z^- < 0 = z_i$. Because f is strictly convex, we have $0 = f'(z^-) < f'(-z_i)$, hence $(\beta, h) \in \mathcal{H}$. Consider at last (β, h) on h_T with $\beta > \beta_0$. Because $z^- < 0$ is in the domain of convexity of f we have $z^- \leq -z_i$. The equality $z^- = -z_i$ would imply that f is convex until $-z_i$, has an horizontal tangent at $-z_i$ (since $f'(z^-) = 0$) and is concave afterwards, which is in contradiction with z^- being a minimizer of f . Hence $z^- < -z_i$ and the strict convexity of f until

$-z_i$ shows that $f'(-z_i) > 0$, in other words $(\beta, h) \in \mathcal{H}$. This is still true for a neighborhood of (β, h) since \mathcal{H} is an open set.

Being analytic, the quantities z^- and z^+ are continuous until the line h_T . Hence the jump of z_0 , when crossing the line h_T at $\beta > \beta_0$ in the direction of higher β and h , equals $z^+ - z^- = 2z^+$.

Let us describe the *contour lines* of z_0 , or equivalently of the magnetization s (see (25)). The mean field equation (2) is satisfied at $z = z_0$, that is to say : the contour lines are straight lines of slope $-s$. Clearly (2) is verified at low $\beta \geq 0$. On the lines of (2) the magnetization is constant which means that the validity of (2) holds until the intersection with the line of phase transition h_T at $\beta > \beta_0$.

- When $s \in \left[0, \frac{q-2}{q-1}\right]$ i.e. $z_0 \in \left[\pm \frac{q-2}{q}\right]$, the contour line (2) is valid until the intersection (β_s, h_s) with the phase transition line, which has coordinates

$$\begin{aligned} \beta_s &= \left(s - \frac{q-2}{2(q-1)}\right)^{-1} \ln \frac{1+(q-1)s}{(q-1)(1-s)} \\ &= \frac{2(q-1)}{q} \frac{1}{z_0} \ln \frac{1+z_0}{1-z_0} \end{aligned} \quad (28)$$

$$\begin{aligned} h_s &= \log(q-1) - \frac{q-2}{2(q-1)} \beta_s \\ &= \log(q-1) - \frac{q-2}{q} \frac{1}{z_0} \ln \frac{1+z_0}{1-z_0} \end{aligned} \quad (29)$$

where the function

$$z \mapsto \frac{1}{z} \ln \frac{1+z}{1-z} = 2 \sum_{n \geq 0} \frac{z^{2n}}{2n+1}$$

is prolonged by 2 at $z = 0$. The coordinates β_s and h_s are even functions of z_0 . When s increases from 0 to $\frac{q-2}{2(q-1)}$, h_s increases from 0 to h_0 while β_s decreases from $\beta_c^{(q)}$ to β_0 . The reverse occurs when s goes from $\frac{q-2}{2(q-1)}$ to $\frac{q-2}{q-1}$.

- When $s \in \left(\frac{q-2}{q-1}, 1\right)$ i.e. $z_0 \in \left(\frac{q-2}{q}, 1\right)$ the contour line (2) is valid up to the intersection with the axis $h = 0$ at $(\beta_s, h_s = 0)$ determined by

$$\beta_s = \frac{1}{s} \ln \frac{1+(q-1)s}{1-s}$$

which is an increasing function of s in the interval $\left(\frac{q-2}{q-1}, 1\right)$ because the magnetization s increases with β (see below).

Now we prove that s increases with β and h . First we consider the region $s < \frac{q-2}{2(q-1)}$. It is situated below h_T since the contour line (2) ends at (β_s, h_s) given by (28) and (29), and its slope is s (in absolute value), lower than

that of h_T . As h_s is increasing, the contour lines move in the direction of higher β and h as s increases. The region $s > \frac{q-2}{2(q-1)}$ is situated above the critical line h_T because the contour lines (2) end again at (β_s, h_s) and have a greater slope than h_T . The fact that h_s given at (29) decreases proves that s increases with β and h , even in the region when $s > \frac{q-2}{q-1}$ where the intersection between h_T and (2) occurs at negative h_s .

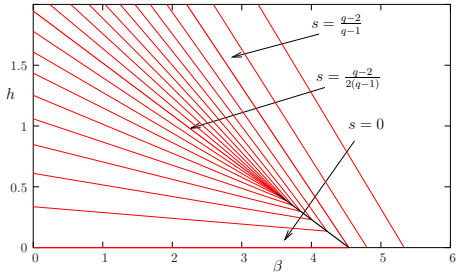


FIG. 3: Contour lines of the magnetization at $q = 8$.

Consider now the mean field equation (12). We fix $\lambda > 0$ and consider the contour lines of $\beta x_1 = \lambda$ (which give the contour lines for θ). The elimination of z between the necessary condition $f'(z) = 0$ and the equation $\beta x_1 = \lambda$ yields

$$h = \frac{\beta - q\lambda}{q-1} + \ln \frac{(q-1)\lambda}{\beta - \lambda} \quad (30)$$

with $\beta > \lambda$. Note that (30) is correct if and only z calculated from $\beta x_1 = \lambda$ coincides with the effective position z_0 of the global minima. It does at $\beta > \lambda$ close to λ by uniqueness of z_0 in the region of large h , as $f'(-z_i)$ is positive in that region. The value of z defined by $\beta x_1 = \lambda$ is continuous on (30) and this means that (30) is not correct after it crosses the line of phase transition h_T with $\beta \geq \beta_0$. Let us define $(h_\lambda, \beta_\lambda)$ as the point of intersection of (30) with the segment of phase transition, and if none, with the horizontal axis. Then (30) holds for any

$$\lambda < \beta < \beta_\lambda \quad \text{and} \quad h > h_\lambda.$$

The point $(\beta_\lambda, h_\lambda)$ is characterized as follows:

- If $\lambda \leq \beta_c/q$, then the intersection is on the horizontal axis where $s = 0$: $\beta_\lambda = q\lambda$ and $h_\lambda = 0$.
- If $\beta_c/q \leq \lambda \leq \beta_0/2$, then the intersection is on the lower side of the critical line h_T . Thanks to (28) and (29) one can parametrize λ in function of $s \in \left[0, \frac{q-2}{2(q-1)}\right]$ letting

$$\lambda_s = \beta_s \frac{1 + (q-1)s}{q} \quad (31)$$

Note that λ_s is strictly increasing with s because the intersection of (30) with the critical line occurs from below.

- If $\beta_0/2 \leq \lambda \leq \beta_c(q-1)/q$ then (30) intersects the critical line h_T from above and one can parametrise again λ according to (31), with $s \in \left[\frac{q-2}{2(q-1)}, \frac{q-2}{q-1}\right]$.
- When $\lambda \geq \beta_c^{(q)}(q-1)/q$ then (30) is valid until some β_λ determined by $h = 0$.

For any $q > 2$, the value $\lambda = 1$ corresponds to the second point and the line (30) intersects the line of phase transition h_T somewhere between its extremities. The region in which there is no giant component is hence delimited by (30) and h_T , whose extremities *do not* coincide.

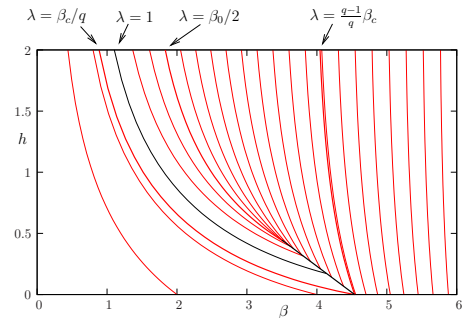


FIG. 4: Contour lines of λ at $q = 8$.

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