On group theory for quantum gates and quantum coherence

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Abstract. Finite group extensions offer a natural language to quantum computing. In a nutshell, one roughly describes the action of a quantum computer as consisting of two finite groups of gates: error gates from the general Pauli group $\mathcal{P}$ and stabilizing gates within an extension group $\mathcal{C}$. In this paper one explores the nice adequacy between group theoretical concepts such as commutators, normal subgroups, group of automorphisms, short exact sequences, wreath products... and the coherent quantum computational primitives. The structure of the single qubit and two-qubit Clifford groups is analyzed in detail. As a byproduct, one discovers that M_{20}, the smallest perfect group for which the commutator subgroup departs from the set of commutators, underlies quantum coherence of the two-qubit system. One recovers similar results by looking at the automorphisms of a complete set of mutually unbiased bases.

PACS numbers: 03.67.Pp, 03.67.Lx, 03.67.-a, 02.20.-a, 03.65.Fd, 03.65.Vf, 02.40.Dr

1. Introduction

Currently quantum computing is a very active and respectable area of research at the interface of the three pillars: quantum physics, mathematics and computer science. If large-scale quantum computers can be built, they will be able to solve certain problems, such as quantum factoring, quantum search or the graph isomorphism problem, in a very efficient way when compared to classical computing. However, one of the main drawbacks of quantum computing is its extreme sensitivity to the classical environment, which induces the decoherence of quantum preparations. To overcome this limitation, many designs have been proposed for correcting the unavoidable errors, or for preventing them to occur. Since the inception of the field, fault-tolerant procedures such as universal bases of gates [1], quantum codes [2] or quantum teleportation based protocols [3] have been proposed. Other approaches relate to topological quantum computation [4], decoherence free subspaces [5] or are based on sequences of measurements [6].

Despite the number of seemingly different proposals some of them are related: there is a close relation between the "oldfashioned" quantum gate circuitry, fault tolerant quantum codes and measurements, already apparent in the stabilizer formalism [7, 8]. It was shown that a few building block gates are enough to simulate any unitary evolution [2] and a few minimal resources are required for measurement-only quantum computation [9]. This paper explores the fresh view that the geometry of commutation relations [10]-[12] between the error operators, their corresponding group of symmetries (i.e. the automorphisms), and the typology of the stabilizer group in terms of maximal normal subgroups [13], sustain the explanation of quantum (de)coherence. Although the approach is performed for a reduced number of qubits, novel pieces of the puzzle appear such as perfect groups with special group theoretical or geometrical properties, and new links are established, such as the relevance of mutually unbiased bases to quantum coherence, or the embedding of quantum topological concepts within the Clifford group. Several recent papers concern closely related topics, see for example Refs [14]-[17].

Following an outline of useful group theoretical concepts in Sec 2, the structure of one and two-qubit Clifford groups is unraveled in Sec 3 in terms of split short exact sequences, which makes use of permutation groups acting on five or six letters. Calculations are performed using GAP [18] and MAGMA [19].

2. An outline of group commutators, group extensions and groups of automorphisms

For an introduction to group theory one may use the on-line Ref [20]. A subgroup N of a group G is called a normal subgroup if it is invariant under conjugation: that is, for each n in N and each g in G, the conjugate element gng^{-1} still belongs to N. In particular, the center $Z(G)$ of a group G (the set of all elements in G which commute with each

element of G) is a normal subgroup of G. The group $\tilde{G} = G/Z(G)$ is called the central quotient of G. A second important example of a normal subgroup of G is provided by the subgroup G' of commutators (also called the derived subgroup of G). It is defined as the subgroup generated by all the commutators $[g,h] = ghg^{-1}h^{-1}$ of elements of G. The quotient group $H^{\rm ab} = G/G'$ is an abelian group called the abelianization of G and corresponds to its first homology group. The set $K(G)$ of all commutators of a group G may depart from G' [21].

Our third example relates to group extensions. Let $\mathcal{P}$ and $\mathcal{C}$ be two groups such that $\mathcal{P}$ is normal subgroup of $\mathcal{C}$. The group $\mathcal{C}$ is an extension of $\mathcal{P}$ by H if there exists a short exact sequence of groups

$$1 \to \mathcal{P} \stackrel{f_1}{\to} \mathcal{C} \stackrel{f_2}{\to} H \to 1, \tag{1}$$

in which 1 is the trivial (single element) group.

The above definition can be reformulated as follows

(i) $\mathcal{P}$ is isomorphic to a normal subgroup N of $\mathcal{C}$,

(ii) H is isomorphic to the quotient group $\mathcal{C}/N$.

Because in an exact sequence the image of f_1 is equal to the kernel of f_2, then the map f_1 is injective and f_2 is surjective.

* Given any groups $\mathcal{P}$ and H the direct product of $\mathcal{P}$ and H is an extension of $\mathcal{P}$ by H.

* The semidirect product $\mathcal{P} \rtimes H$ of $\mathcal{P}$ and H is defined as follows. The group $\mathcal{C}$ is an extension of $\mathcal{P}$ by H (one identifies $\mathcal{P}$ with a normal subgroup of $\mathcal{C}$) and

(i) H is isomorphic to a subgroup of $\mathcal{C}$,

(ii) $\mathcal{C}$=$\mathcal{P}H$ and

(iii) $\mathcal{P} \cap H = \langle 1 \rangle$.

One says that the short exact sequence splits.

The wreath product $M \wr H$ of a group M with a permutation group H acting on n points is the semidirect product of the normal subgroup M^n with the group H, which acts on M^n by permuting its components.

* Let $G = \mathcal{Z}_2 \wr A_5$, in which A_5 is the five letters alternating group, then G' is a perfect group with order 960 and one has $G' \neq K(G)$. Let $H = Z_2^5 \rtimes A_5$, one can think of A_5 having a wreath action on Z_2^5. The group $G' = \tilde{H} = M_{20}$ [26] is the smallest perfect group having its commutator subgroup distinct from the set of the commutators [21]. One easily checks that M_{20} also corresponds to the derived subgroup W' of the Weyl group (also called hyperoctahedral group) $W = \mathcal{Z}_2 \wr S_5$ for the Lie algebra of type B_5. For a quantum version, see [22].

Group of automorphisms

Given the group operation $*$ of a group G, a group endomorphism is a function ϕ from G to itself such that $\phi(g_1 * g_2) = \phi(g_1) * \phi(g_2)$, for all $g_1, g_2 \in G$. If it is bijective, it is called an automorphism. An automorphism of G that is induced by conjugation of some $g \in G$ is called inner. Otherwise it is called an outer automorphism. Under composition

the set of all automorphisms defines a group denoted $\mathrm{Aut}(G)$. The inner automorphisms form a normal subgroup $\mathrm{Inn}(G)$ of $\mathrm{Aut}(G)$, that is isomorphic to the cental quotient of G. The quotient $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the outer automorphism group.

The decoherence problem is related to peculiar outer automorphisms occuring inside the Clifford group. It turns out that, among symmetric permutation groups, only S_6 has a nontrivial automorphism group $\mathrm{Out}(S_6) = \mathcal{Z}_2$. Later it is shown that S_6 governs the symmetries of commutation relations of the two-qubit system and one has to pass to S_5 for restoring quantum coherence.

3. Quantum computing and the Clifford group

Compared to group theory, the science of quantum computing is in its infancy [8]. In quantum codes and in quantum computing, one is interested in preventing or correcting errors that may affect one or many physical qubits [23]-[25]. A frequently used error group is the general Pauli group $\mathcal{P}_n$. It consists of tensor products of the Pauli matrices [10]

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y = i\sigma_x\sigma_z, \tag{2}$$

and the unity matrix σ_0. Pauli matrices generate the single qubit Pauli group $\mathcal{P}_1$ of order 16 and center $Z(\mathcal{P}_1) = \{\pm 1, \pm i\}$.

Let us assume a quantum computer in a state $|\psi\rangle$, and apply to it an error g belonging to the Pauli group $\mathcal{P}$ so that the new state of the computer is $g|\psi\rangle$. One allows unitary evolutions U so that the new state evolves as $Ug|\psi\rangle = UgU^\dagger U|\psi\rangle$. For stabilizing the error within the Pauli group $\mathcal{P}$, one requires that $UgU^\dagger \in \mathcal{P}$. The set of operators leaving $\mathcal{P}$ invariant under conjugation is the normalizer $\mathcal{C}$ in the unitary group U, also known as the Clifford group [7]. Within a unitary group one has the equality $U^\dagger = U^{-1}$. As a result, the group $\mathcal{P}$ is a normal subgroup of $\mathcal{C}$ and one may use the powerful formalism of group extensions to report on it. Additionnaly some subgroups of $\mathcal{C}$, which have the error group $\mathcal{P}$ as a normal subgroup, will play a role for displaying the quantum coherence.

For a system of n qubits, one denotes the Pauli group as $\mathcal{P}_n$ and the Clifford group as $\mathcal{C}_n$. One learned from Gottesman-Knill theorem that the Hadamard gate $H = 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and the phase gate $P = \mathrm{Diag}(1, i)$ are in the one-qubit Clifford group $\mathcal{C}_1$, and that the controlled-Z gate $CZ = \mathrm{Diag}(1, 1, 1, -1)$ is in the two-qubit Clifford group $\mathcal{C}_2$. Any gate in $\mathcal{C}_n$ may be generated from the application of gates from $\mathcal{C}_1$ and $\mathcal{C}_2$ [7, 14]. Clifford group $\mathcal{C}_n$ on n-qubits has order $|\mathcal{C}_n| = 2^{n^2+2n+3}\prod_{j=1}^{n} 4^j - 1$ [23].

Below we will concentrate on the properties of the Clifford group related to one and two qubits, using the group theoretical package GAP4 [18]. Generation of the gates will be ensured by the use of cyclotomic numbers, as described in Sec 18 of the GAP4

reference manual. For example the elements 1, -1, i and $2^{1/2}$ will be modelled as the roots of unity $E(1)$, $E(2)$, $E(4)$ and as $ER(2)$, respectively.

3.1. The Clifford group on a single qubit

The one-qubit Clifford group is defined as $\mathcal{C}_1 = \langle H, P\rangle$. It has order $|\mathcal{C}_1| = 192$, its center is $Z(\mathcal{C}_1) = \mathcal{Z}_8$ and the derived subgroup $\mathcal{C}_1'$ equals the special linear group $SL(2,3)$. The central quotient is $\tilde{\mathcal{C}_1} = S_4$ and one obtains the abelianization as the direct product $\mathcal{C}_1^{\rm ab} = \mathcal{Z}_4 \times \mathcal{Z}_2$.

Using the method described in Sec 2 two split extensions follow. The first one is attached to $\mathcal{C}_1' = SL(2,3)$ as follows

$$1 \to SL(2,3) \to \mathcal{C}_1 \to \mathcal{Z}_2 \times \mathcal{Z}_3 \to 1. \tag{3}$$

The second one is attached to the Pauli group

$$1 \to \mathcal{P}_1 \to \mathcal{C}_1 \to D_{12} \to 1, \tag{4}$$

in which $D_{12} = \mathcal{Z}_2 \times S_3$ is the dihedral symmetry group of a regular hexagon.

3.2. The Clifford group on two qubits

The two-qubit Pauli group may be generated as $\mathcal{P}_2 = \langle \sigma_x \otimes \sigma_x, \sigma_z \otimes \sigma_z, \sigma_x \otimes \sigma_y, \sigma_y \otimes \sigma_z, \sigma_z \otimes \sigma_x\rangle$. It is of order 64 and has center $\mathcal{Z}(\mathcal{P}_2) = \mathcal{Z}(\mathcal{P}_1)$. The two-qubit Clifford group, of order 92160, may be generated from H, P and CZ as $\mathcal{C}_2 = \langle H \otimes H, H \otimes P, CZ\rangle$. Its center is $Z(\mathcal{C}_2) = Z(\mathcal{C}_1)$ and the central quotient $\tilde{\mathcal{C}_2}$ is found to satisfy the exact sequence

$$1 \to U_6 \to \tilde{\mathcal{C}_2} \to \mathcal{Z}_2 \to 1, \tag{5}$$

in which we introduced the notation $U_6 = \tilde{\mathcal{C}_2'} = \mathcal{Z}_2^{\times 4} \rtimes A_6$. Another important relationship is $U_6 = \mathrm{Aut}(\mathcal{P}_2)'$, i.e. U_6 encodes the commutators of the Pauli group automorphisms. It turns out that the group $\tilde{\mathcal{C}_2}$ only contains two normal subgroups $\mathcal{Z}_2^{\times 4}$ and U_6. The group U_6, of order 5760, is a perfect group. It can be seen as a parent of the six element alternating group A_6. Its outer automorphism group $\mathrm{Out}(U_6)$ is the same, equal to the Klein group $\mathcal{Z}_2 \times \mathcal{Z}_2$.

The group U_6 is an important maximal subgroup of several sporadic groups. The group of smallest size where it appears is the Mathieu group M_{22}. Mathieu groups are sporadic simple groups, so that U_6 is not normal in M_{22}. It appears in the context of a subgeometry of M_{22} known as an *hexad*. Let us recall the definition of Steiner systems. A Steiner system $S(a,b,c)$ with parameters a, b, c, is a c-element set together with a set of b-element subsets of S (called *blocks*) with the property that each a-element subset of S is contained in exactly one block. A finite projective plane of order q, with the lines as blocks, is an $S(2, q+1, q^2+q+1)$, because it has q^2+q+1 points, each line passes through $q+1$ points, and each pair of distinct points lies on exactly one line. Any large Mathieu group can be defined as the automorphism (symmetry) group of a Steiner system [27]. The group M_{22} stabilizes the Steiner system $S(3,6,22)$ comprising

22 points and 6 blocks, each set of 3 points being contained exactly in one block‡. Any block in $S(3,6,22)$ is a Mathieu hexad, i.e. it is stabilized by the *general* alternating group U_6.

There is a relationship between the two-qubit Clifford and Pauli groups

$$\mathcal{C}_2/\mathcal{P}_2 = \mathcal{Z}_2 \times S_6 \tag{6}$$

which features the important role of the six-letter symmetric group S_6. The latter governs the Pauli graph attached to the two-qubit system, being the automorphism group of generalized quadrangle of order two $W(2)$ [10]. The group S_6 is peculiar among the symmetric permutation groups as having an outer automorphism group $\mathcal{Z}_2$.

3.3. Quantum coherence within the two-qubit system

Topological quantum computing based on anyons has been proposed as way of encoding quantum bits in nonlocal observables that are immune of decoherence [4, 28]. The basic idea is to use pairs $|v, v^{-1}\rangle$ of "magnetic fluxes" playing the roles of the qubits and permuting them within some large enough nonabelian finite group G such as A_5. The "magnetic flux" carried by the (anyonic) quantum particle is labeled by an element of G, and "electric charges" are labeled by irreducible representation of G [29].

The exchange within G modifies the quantum numbers of the fluxons according to the fundamental logical operation

$$|v_1, v_2\rangle \rightarrow \left|v_2, v_2^{-1} v_1 v_2\right\rangle, \tag{7}$$

a form of Aharonov-Bohm interactions, which is nontrivial in a nonabelian group. This process can be shown to produce universal quantum computation. It is intimately related to topological entanglement, the braid group and unitary solutions of the Yang-Baxter equation [30]

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R), \tag{8}$$

in which I denotes the identity transformation and the operator R: $V \otimes V \rightarrow V \otimes V$ acts on the tensor product of the bidimensional vector space V. One elegant unitary solution of the Yang-Baxter equation is a universal quantum gate known as the Bell basis change matrix

$$R = 1/\sqrt{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

‡ There exists up to equivalence a unique S(5,8,24) Steiner system called a Witt geometry. The group $M(24)$ is the automorphism group of this Steiner system, that is, the set of permutations which map every block to some other block. The subgroups $M(23)$ and $M(22)$ are defined to be the stabilizers of a single point and two points respectively.

It is straightforward to see two-qubit topological quantum computing as another group extension of the Pauli group. One may introduce a subgroup of the Clifford group, of order 15360, that we denote the Bell group as follows

$$\mathcal{B}_2 = \langle H \otimes H, H \otimes P, R \rangle \, . \tag{10}$$

The Bell group has center $\mathcal{Z}_8$ and its central quotient only contains two normal subgroups $\mathcal{Z}_2^{\times 4}$ and $M_{20} = \mathcal{Z}_2^{\times 4} \rtimes A_5$. The group M_{20} was already quoted in Sec 2 as being the smallest perfect group having the set of commutators departing from the commutator subgroup. The relationship between the Bell and Pauli groups

$$\mathcal{B}_2/\mathcal{P}_2 = \mathcal{Z}_2 \times S_5 \tag{11}$$

displays the important role of the five letters symmetric group S_5. At this point, it may be useful to mention that A_5 is the automorphism group of the icosahedron. Icosahedral symmetry and quantum coherence seems to be related in recent fullerene experiments [31].

3.4. Quantum coherence within mutually unbiased bases

To our knowledge the relationship between mutually unbiased bases (MUBs) of the Pauli group and the Clifford group has not yet been established. Two orthonormal bases are said to be mutually unbiased if each common state of one basis gives rise to the same probability distribution when measured with respect to the other basis. For prime power dimensions p^m, complete sets of MUBs have cardinality $p^m + 1$ and can be determined using different techniques such as the additive characters over a Galois field [32] §. In composite dimensions, MUBs strongly rely on projective lines over finite rings [35]. In addition, the continuous variable case was addressed recently [36].

Commuting/non-commuting relations between the Pauli operators of the two-qubit system have been determined [10]. The Pauli graph admits several decompositions: one of them is based on its minimum vertex cover (the Petersen graph) and a maximal independant set (of size five). If one uses a geometrical representation, operators correspond to the points of the geometry, maximal sets of mutually commuting operators, i.e. MUBs, correspond to the lines of the geometry, and a complete set of MUBs corresponds to an ovoid (the maximum number of mutually disjoint lines). The geometry of the two-qubit system is the smallest non-trivial generalized quadrangle. Due to the perfect duality between the fifteen points and fifteen lines of the quadrangle, the cardinality of a maximal independant set and the one of an ovoid is the same.

These graph theoretical and geometrical features of MUBs have a group theoretical counterpart that one may find in the group of automorphisms attached to an independant set. Let us denote m_i $(i = 1..5)$ the elements of such a maximal set, one forms groups of increasing size $g_2 = \langle m_1, m_2 \rangle$, ... $g_4 = \langle m_1, m_2, m_3, m_4 \rangle$. ($g_1$ is the trivial group and $g_5 = g_4$). The groups g_i and the corresponding groups of

§ Power of prime dimensions also play a pivotal role in the number theoretical approach of $1/f$ noise developed by one of us [33, 34].

automorphisms $\mathrm{Aut}(g_i)$ are identified in Table 1. One readily observes that the group of automorphisms of an independant set/ovoid of the two-qubit system is isomorphic to the wreath product $\mathcal{Z}_2 \wr A_5$ encountered in the context of topological quantum computing. One concludes that the symmetries in a complete set of MUBs also provide a signature of quantum coherence. Let us mention that the hyperoctahedral group $\mathcal{Z}_2 \wr S_5$, of order 3840, corresponds to the automorphism group of the code $((5, 6, 2))$, the first instance of a non-additive quantum code [37].

g_i	g_2	g_3	g_4	g_5	g_6
G	$\mathcal{Z}_2^{\times 2}$	$(\mathcal{Z}_4 \times \mathcal{Z}_2) \rtimes \mathcal{Z}_2$	$(\mathcal{Z}_2 \times \mathcal{Q}_8) \rtimes \mathcal{Z}_2$	$\mathcal{Z}_2 \times ((\mathcal{Z}_2 \times \mathcal{Q}_8) \rtimes \mathcal{Z}_2)$	g_6
$\mathrm{Aut}(G)$	$\mathcal{D}_8$	$\mathcal{Z}_2 \times S_4$	$\mathcal{Z}_2 \wr A_5$	$\mathcal{Z}_2^{\times 2} \wr A_5$	$\mathcal{Z}_2^{\times 3} \wr A_5$
$\lvert\mathrm{Aut}(G)\rvert$	8	48	1920	61440	1966080

Table 1. Group structure of an independant set of the two-qubit (g_2 to g_4) and three-qubit systems (g_2 to g_6). G denotes the identified group and $\mathrm{Aut}(G)$ the corresponding automorphism group. $\mathcal{Q}_8$ and $\mathcal{D}_8$ are the eight-element quaternion and dihedral groups.

The same approach can be applied to the three-qubit system and higher-order qubit systems. For the three-qubit system, the size of a maximal independant set is seven (it is different from the size $9 = 2^3 + 1$ of a complete set of MUBs). The corresponding automorphism group encompasses the one of the two-qubit system as shown in Table 1. The group $\mathrm{Aut}(g_n)$ $(n > 4)$ is found to be isomorphic to the wreath product $\mathcal{Z}_2^{\times m} \wr A_5$, with $m = n - 3$. Its central quotient equals its derived subgroup and may be identified to the perfect group $(\mathcal{Z}_2^{\times 4})^{\rtimes m} \rtimes A_5$. All these perfect groups of order 960, 15360, 245760 are found to contain at least one element, which is not a commutator ‖.

4. Conclusion

Advanced group theoretical tools may be used to explore fault tolerance in quantum computing. We found some fingerprints of quantum (de)coherence in exceptional groups such as U_6 (the stabilizer of an hexad in M_{22}), in the group M_{20}, and in the automorphism groups of mutually unbiased bases. Using this approach, disparate concepts such as the stabilizer formalism, topological quantum computing [38] and the mathematical approach of quantum complementary, tend to merge. Future work will be devoted to arbitrary n-qudit systems and composite systems, and the link to quantum codes.

Acknowledgements

The authors acknowledge the support of the PEPS program (Projets Exploratoires Pluridisciplinaires) from the ST2I department at CNRS, France (Sciences et Technologies de l'Information et de la Communication).

‖ The calculation is performed using theorem 6.6 in [21].

Bibliography

[1] O. Boykin, T Mor, M Pulver, V Roychowdhury and F Vatan. On universal and fault-tolerant quantum computing: a novel basis and a new constructive proof of universality for Shor's basis. 40th annual symposium on the foundations of computer science, 486-494 (1999). Prpeint quant-ph/9906054.

[2] J P. Francoise, G L Naber and S T Tsou. Quantum error correction and fault tolerance. D Gottesman. Encyclopedia of Mathematical Physics, Oxford: Elsevier 2006, vol. 4, pp. 196-201 (Preprint quant-ph/0507174).

[3] D Gottesman and I L Chuang. Quantum teleportation is a universal computational primitive. Nature 402, 390-392 (1999).

[4] A. Yu. Kitaev. Fault-tolerant quantum computation with anyons. Preprint quant-ph/970702.

[5] L A Wu, P Zanardi and D A Lidar. Holonomic quantum computation in decoherence-free subspaces. Phys Rev Lett 95, 130501 (2005).

[6] P. Jorrand and S. Perdrix. Unifying quantum computation with projective measurements only and one way quantum computation. Preprint quant-ph/0404125.

[7] D Gottesman. The Heisenberg representation of quantum computers. Preprint quant-ph/9807006.

[8] M A Nielsen and I L Chuang. Quantum computation and quantum information. Cambridge University Press (2000).

[9] S Perdrix. Toward minimam resources of measurement-based quantum computation. New J Phys 9, 206 (2007).

[10] M Planat and M Saniga. On the Pauli graphs of N-qudits. Quant Inf Comp 8, 127–46 (2008).

[11] M Planat, M Saniga and M Kibler. Quantum entanglement and projective ring geometry. SIGMA 2, 66 (2006).

[12] M Saniga and M Planat. Multiple qubits as symplectic polar spaces of order two. Adv. Stud. in Theor. Phys. 1, 1-4 (2007).

[13] M Planat Clifford quantum computer and the Mathieu groups. Preprint 0711.1733 [quant-ph].

[14] S Clark, R Jozsa and N Linden. Generalized Clifford groups and simulation of associated quantum circuits. Quant Inf Comp 8, 106–26 (2008).

[15] H Bombin and M A Martin-Delgado. Topological Quantum Distillation. Phys Rev Lett 97, 180501 (2006).

[16] D Aerts and M Czachor. Cartoon computation: Quantum-like algorithms without quantum mechanics. J Phys A: Math Theor 40, F259-F266 (2007).

[17] R Zeier, M Grassl and T Beth. Gate simulation and lower bounds on the simulation time. Phys Rev A 70, 032319 (2004).

[18] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.4; 2004. (http://www.gap-system.org).

[19] W Bosma, J Cannon and C Playoust. The Magma algebra system. I. The user language. J Symbolic Comput, 24, 235-265, 1997.

[20] J S Milne. Group theory. available on line at http://www.jmilne.org/math/

[21] L C Kappe and R F Morse. On commutators in groups. available on line at http://faculty.evansville.edu/rm43/publications/commutatorsurvey.pdf

[22] T Banica, J Bichon and B Collins. The hyperoctahedral quantum group. Preprint math/0701859 [math.RT]

[23] A R Calderbank, E M Rains, P W Schor and N J A Sloane. Quantum error correction via codes over $GF(4)$. IEEE Trans Inform Theory 44, 1369–87 (1998).

[24] A Klappenecker and M. Rötteler. Beyond stabilizer codes I: nice error bases. IEEE Trans Inform Theory 48, 2392–95 (2002).

[25] A Klappenecker and M. Rötteler. Beyond stabilizer codes II: Clifford codes. IEEE Trans Inform Theory 48, 2396–99 (2002).

[26] ATLAS of Finite Group Representations, http://brauer.maths.qmul.ac.uk/Atlas/v3/misc/M20/
[27] R A Wilson. The finite simple groups. available at http://www.maths.qmul.ac.uk/˜raw/fsgs.html
[28] J Preskill. Fault tolerant quantum computation. in Introduction to Quantum Computation and Information. H K Lo, T Spiller, S Popescu eds (Singapore, World Scientific, 1998). Preprint quant-ph/9712048.
[29] R W Ogburn and J Preskill. Topological quantum computation. Lecture Notes in Computer Science 1509, 341-356 (1999).
[30] L H Kauffman and S J Lomonaco. Braiding operators are universal quantum gates. New J Phys 6, 134 (2004).
[31] S Benjamin et al. Toward a fullerene-based quantum computer. J. Phys.: Condens. Matter 18, S867S883 (2006).
[32] M Planat, H C Rosu and S Perrine. A survey of finite algebraic geometrical structures underlying mutually unbiased measurements. Found of Phys 36, 1662–80 (2006).
[33] M Planat. $1/f$ noise, the measurement of time and number theory. Fluc Noise Lett 1, R65-R79 (2001).
[34] M Planat. $1/f$ frequency noise in a communication receiver and the Riemann hypothesis. Lect Notes Phys 550, 265-287 (2000).
[35] M Planat and A C Baboin. Qudits of composite dimension, mutually unbiased bases and projective ring geometry. J. Phys A Math and Theor 40, F1-F8 (2007).
[36] S. Weigert and M. Wilkinson. Mutually unbiased bases for continuous variables. Preprint 0802.0394 [quant-ph].
[37] E M Rains , R H Hardin, P W Schor and N J A Sloane. Phys Rev Lett 79, 953-54 (1997).
[38] H Bombin and M A Martin-Delgado. A family of non-abelian Kitaev models on a lattice: topological confinement and condensation. Preprint 0712.0190 [cond-mat.str-el].