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**REMARKS ON THE UNIQUENESS OF COMPARABLE  
RENORMALIZED SOLUTIONS OF ELLIPTIC EQUATIONS  
WITH MEASURE DATA**

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ABSTRACT. We give a partial uniqueness result concerning comparable renormalized solutions of the nonlinear elliptic problem  $-\operatorname{div}(\mathbf{a}(x, Du)) = \mu$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ , where  $\mu$  is a Radon measure with bounded variation on  $\Omega$ .

1. INTRODUCTION

Let us consider the nonlinear elliptic problem

$$\begin{aligned} (1) \quad & -\operatorname{div}(\mathbf{a}(x, Du)) = \mu \quad \text{in } \Omega, \\ (2) \quad & u = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where  $\Omega$  is a bounded open subset of  $\mathbb{R}^N$  with  $N \geq 2$ ,  $u \mapsto -\operatorname{div}(\mathbf{a}(x, Du))$  is a strictly monotone operator from  $W_0^{1,p}(\Omega)$  into  $W^{-1,p'}(\Omega)$  and  $\mu$  is a Radon measure with bounded variation on  $\Omega$ .

In the linear case G. Stampacchia has defined in [17] the notion of “solution by transposition” which insures existence and uniqueness of such a solution. If  $p = 2$  and for the nonlinear case, this notion is generalized in [15] and the existence and uniqueness of the solution obtained as limit of approximations is proved in [15] (see also [2] and, for a class of pseudo-monotone operator [9]).

If  $2 - 1/N < p \leq N$  the existence of a solution of (1)–(2) in the sense of distributions is proved by L. Boccardo and T. Gallouët in [3]. However, using the counter example of J. Serrin [16] it is well known that this solution is not unique in general, except in the case  $p = N$  for an appropriate choice of the space to which the solution belongs (see [8] and [11]).

When  $\mu$  is a function of  $L^1(\Omega)$  the notions of entropy solution [1], of solution obtained as limit of approximations [7] and of renormalized solution [13] (see also [14] and [15]) provide existence and uniqueness results (and these three notions are actually equivalent).

When  $\mu$  is a Radon measure with bounded variation on  $\Omega$ , G. Dal Maso, F. Murat, L. Orsina and A. Prignet have recently introduced in [5] and [6]

a notion of renormalized solution of (1)–(2) which generalizes the three (and equivalent) previous ones. The authors prove in [6] the existence of such a renormalized solution, a stability result and partial uniqueness results for “comparable” solutions. In particular, under some assumptions on  $\mathbf{a}$ , if  $u_1$  and  $u_2$  are two renormalized solutions of (1)–(2) such that  $u_1 - u_2$  belongs to  $L^\infty(\Omega)$  (this condition is here the precise meaning of the fact that the two solutions are comparable), then  $u_1 = u_2$ . The uniqueness of the renormalized solution of (1)–(2) remains an open problem in general and the present paper is devoted to weaken this condition. We prove that the condition of being comparable can be localized in a neighborhood  $\mathcal{U}$  of the set where  $\mu$  is singular and that it is sufficient to assume that  $(u_1 - u_2)^-$  (the negative part of  $u_1 - u_2$ ) belongs to  $L^\infty(\mathcal{U})$ .

The paper is organized as follows: Section 2 is devoted to give the assumptions on the data and to recall the definition of a renormalized solution of (1)–(2). In Section 3 (Theorems 5 and 7) we establish partial uniqueness results concerning comparable renormalized solutions of (1)–(2).

## 2. ASSUMPTIONS AND DEFINITIONS

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^N$  with  $N \geq 2$ ,  $p$  and  $p'$  two real numbers such that  $1 < p < N$  and  $1/p + 1/p' = 1$ . We assume that  $\mathbf{a} : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$  is a Carathéodory function (i.e. measurable with respect to  $x$  and continuous with respect to  $\xi$ ) such that

$$(3) \quad \mathbf{a}(x, \xi) \cdot \xi \geq \alpha |\xi|^p,$$

$$(4) \quad (\mathbf{a}(x, \xi) - \mathbf{a}(x, \xi')) \cdot (\xi - \xi') > 0,$$

$$(5) \quad |\mathbf{a}(x, \xi)| \leq \gamma(b(x) + |\xi|^{p-1})$$

for every  $\xi, \xi'$  ( $\xi \neq \xi'$ ) in  $\mathbb{R}^N$  and almost everywhere in  $\Omega$ , where  $\gamma > 0$ ,  $\alpha > 0$  and  $b$  is a nonnegative function lying in  $L^p(\Omega)$ .

We denote by  $\mathcal{M}_b(\Omega)$  the set of Radon measures on  $\Omega$  with total bounded variation on  $\Omega$  and by  $\mathcal{M}_0(\Omega)$  the set of measures of  $\mathcal{M}_b(\Omega)$  that are absolutely continuous with respect to the  $p$ -capacity (i.e.  $\mu \in \mathcal{M}_b(\Omega)$  and  $\mu(E) = 0$  for every Borel set  $E$  such that  $\text{cap}_p(E, \Omega) = 0$ ). For  $K > 0$  we define as  $T_K(r) = \max(-K, \min(K, r))$  the truncation function at height  $\pm K$ . If  $A$  is a measurable set we denote by  $\mathbb{1}_A$  the characteristic function of  $A$ .

We recall now a decomposition result of the Radon measures (see [4] and [10]) and the definition of the gradient of a function whose truncates belong to  $W_0^{1,p}(\Omega)$  (see [1] Lemma 2.1 and [13]) which are needed to define (following [6]) a renormalized solution of (1)–(2).

**Proposition 1.** ([4] and [10]) *Let  $\mu$  be an element of  $\mathcal{M}_b(\Omega)$ . There exists two functions  $f \in L^1(\Omega)$ ,  $g \in (L^{p'}(\Omega))^N$ , two nonnegative measures in  $\mathcal{M}_b(\Omega)$ ,  $\lambda^+$  and  $\lambda^-$ , which are concentrated respectively on two disjoint Borel sets  $E^+$  and  $E^-$  of zero  $p$ -capacity such that*

$$\mu = f - \operatorname{div}(g) + \lambda^+ - \lambda^-.$$

Moreover, if  $\mu_0$  denotes  $f - \operatorname{div}(g)$  then  $\mu_0 \in \mathcal{M}_b(\Omega)$  and the decomposition  $\mu = \mu_0 + \lambda^+ - \lambda^-$  is unique.

**Definition 2.** ([1] and [13]) *Let  $u$  be a measurable function defined from  $\Omega$  into  $\overline{\mathbb{R}}$  which is finite almost everywhere in  $\Omega$ . Assume that  $T_K(u) \in W_0^{1,p}(\Omega) \forall K > 0$ . Then there exists a unique measurable function  $v : \Omega \mapsto \mathbb{R}^N$  such that*

$$\forall K > 0, \quad DT_K(u) = \mathbb{1}_{\{|u| < K\}} v \quad \text{a.e. in } \Omega.$$

This function  $v$  is called the gradient of  $u$  and is denoted by  $Du$ .

Following [6] we are now in a position to recall the definition of renormalized solution.

**Definition 3.** ([6]) *Let  $\mu$  be an element of  $\mathcal{M}_b(\Omega)$  and  $\mu = f - \operatorname{div}(g) + \lambda^+ - \lambda^-$  the decomposition given by Proposition 1. A function  $u$  defined from  $\Omega$  into  $\overline{\mathbb{R}}$  is a renormalized solution of (1)–(2) if*

(6)  *$u$  is measurable and finite almost everywhere in  $\Omega$  and  $T_K(u) \in W_0^{1,p}(\Omega) \forall K > 0$ ;*

(7)  $|Du|^{p-1} \in L^q(\Omega) \quad \forall q < \frac{N}{N-1}$ ;

(8)  $\forall w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$  such that  $\exists K > 0$  and two functions  $w^{+\infty}$  and  $w^{-\infty}$  lying in  $W^{1,r}(\Omega) \cap L^\infty(\Omega)$  with  $r > N$  and

$$\begin{cases} w = w^{+\infty} & \text{on } \{x; u(x) > K\}, \\ w = w^{-\infty} & \text{on } \{x; u(x) < -K\}, \end{cases}$$

we have

$$(9) \quad \int_{\Omega} \mathbf{a}(x, Du) \cdot Dw \, dx = \int_{\Omega} f w \, dx + \int_{\Omega} g \cdot Dw \, dx + \int_{\Omega} w^{+\infty} \, d\lambda^+ - \int_{\Omega} w^{-\infty} \, d\lambda^-.$$

It is proved in [6] that if  $\mathbf{a}$  verifies (3), (4) and (5) then for any element  $\mu$  belonging to  $\mathcal{M}_b(\Omega)$  there exists at least a renormalized solution of (1)–(2).

**Remark 4.** Every function  $w \in \mathcal{C}_c^\infty(\Omega)$  is an admissible test function in (9) and then any renormalized solution of (1)–(2) is also solution in the sense of distributions.

Furthermore if  $\varphi \in W^{1,r}(\Omega) \cap L^\infty(\Omega)$  with  $r > N$  then we have

$$(10) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\{|u| < n\}} \mathbf{a}(x, Du) \cdot Du \varphi \, dx = \int_{\Omega} \varphi \, d\lambda^+ + \int_{\Omega} \varphi \, d\lambda^-.$$

This property (see [6] for more details on the properties of renormalized solutions) is obtained by using the admissible test function  $w = \frac{1}{n} T_n(u) \varphi$  in (9) and by passing to the limit as  $n$  goes to infinity.

### 3. UNIQUENESS OF COMPARABLE SOLUTIONS

In [6] the authors prove under assumptions (3) and (4), the strong monotonicity of  $\mathbf{a}$  and the local Lipschitz continuity, or the Hölder continuity, with respect to  $\xi$ , i.e.  $\mathbf{a}$  verifies

$$(11) \quad \begin{cases} (\mathbf{a}(x, \xi) - \mathbf{a}(x, \xi')) \cdot (\xi - \xi') \geq \alpha |\xi - \xi'|^p & \text{if } p \geq 2 \\ (\mathbf{a}(x, \xi) - \mathbf{a}(x, \xi')) \cdot (\xi - \xi') \geq \alpha \frac{|\xi - \xi'|^2}{(|\xi| + |\xi'|)^{2-p}} & \text{if } p < 2, \end{cases}$$

$$(12) \quad \begin{cases} |\mathbf{a}(x, \xi) - \mathbf{a}(x, \xi')| \leq \gamma (b(x) + |\xi| + |\xi'|)^{p-2} |\xi - \xi'| & \text{if } p \geq 2, \\ |\mathbf{a}(x, \xi) - \mathbf{a}(x, \xi')| \leq \gamma |\xi - \xi'|^{p-1} & \text{if } p < 2, \end{cases}$$

for every  $\xi, \xi' \in \mathbb{R}^N$  and almost everywhere in  $\Omega$ , where  $\gamma > 0$  and  $b$  is a nonnegative function in  $L^p(\Omega)$ , that if two renormalized solutions  $u_1$  and  $u_2$  of (1)–(2) (relative to the same element  $\mu \in \mathcal{M}_b(\Omega)$ ) satisfy the condition of being comparable, in the sense that  $u_1 - u_2 \in L^\infty(\Omega)$ , then  $u_1 = u_2$ . In Theorem 7 below we weaken this condition; if there exists an open neighborhood  $\mathcal{U}$  of  $E = E^+ \cup E^-$  where  $E^+$  and  $E^-$  are given by Proposition 1 such that  $(u_1 - u_2)^- \in L^\infty(\mathcal{U})$ , then  $u_1 = u_2$ . This result is a consequence of the following theorem.

**Theorem 5.** *Assume that (3), (4), (5) and (12) hold true. Let  $\mu$  be an element of  $\mathcal{M}_b(\Omega)$  and let  $E = E^+ \cup E^-$  where  $E^+$  and  $E^-$  are the two disjoint Borel sets of zero  $p$ -capacity given by Proposition 1. Let  $u_1$  and  $u_2$  be two renormalized solutions of (1)–(2) with  $\mu$  as right-hand side. If moreover there exists an open set  $\mathcal{U}$  such that*

$$(13) \quad E \subset \mathcal{U} \subset \Omega,$$

$$(14) \quad \forall K > 0 \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\substack{\mathcal{U} \cap \{u_1 - u_2 < K\} \\ \cap \{|u_1| < n, |u_2| < n\}}} |Du_1 - Du_2|^p \, dx = 0,$$

then  $u_1 = u_2$ .

**Remark 6.** Using the following property for every  $m \in \mathbb{N}^*$

$$\begin{aligned} \{|u_1| < 2^m, |u_2| < 2^m\} \subset & \{|u_1| < 1, |u_2| < 1\} \\ & \cup \bigcup_{k=0}^{m-1} \{2^k \leq |u_1| < 2^{k+1}, |u_2| < 2^{k+1}\} \\ & \cup \bigcup_{k=0}^{m-1} \{2^k \leq |u_2| < 2^{k+1}, |u_1| < 2^{k+1}\}, \end{aligned}$$

a Cesaro argument and the fact that  $T_1(u_1)$  and  $T_1(u_2)$  belong to  $W_0^{1,p}(\Omega)$ , the condition (14) is equivalent to

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{n} \left( \int_{\substack{\mathcal{U} \cap \{u_1 - u_2 < K\} \\ \cap \{n \leq |u_1| < 2n, |u_2| < 2n\}}} |Du_1 - Du_2|^p dx \right. \\ \left. + \int_{\substack{\mathcal{U} \cap \{u_1 - u_2 < K\} \\ \cap \{|u_1| < 2n, n \leq |u_2| < 2n\}}} |Du_1 - Du_2|^p dx \right) = 0, \end{aligned}$$

for all  $K > 0$ . Notice that the condition above with  $\mathcal{U} = \Omega$  and  $K = +\infty$  (so that  $\mathcal{U} \cap \{u_1 - u_2 < K\} = \Omega$ ) is the one given in [6] (Theorem 10.3).

**Theorem 7.** Assume that (3), (4), (5), (11) and (12) hold true. Let  $\mu$  be an element of  $\mathcal{M}_b(\Omega)$  and let  $E = E^+ \cup E^-$  where  $E^+$  and  $E^-$  are the two disjoint Borel sets of zero  $p$ -capacity given by Proposition 1. Let  $u_1$  and  $u_2$  be two renormalized solutions of (1)–(2) with  $\mu$  as right-hand side. If moreover there exists an open set  $\mathcal{U}$  such that  $E \subset \mathcal{U}$  and  $(u_1 - u_2)^- \in L^\infty(\mathcal{U})$  (or  $(u_1 - u_2)^+ \in L^\infty(\mathcal{U})$ ), then  $u_1 = u_2$ .

*Proof of Theorem 5.* Using Proposition 1, let  $f \in L^1(\Omega)$ ,  $g \in (L^{p'}(\Omega))^N$ ,  $\lambda^+$  and  $\lambda^-$  two nonnegative measures of  $\mathcal{M}_b(\Omega)$  which are concentrated on two disjoint subsets  $E^+$  and  $E^-$  of zero  $p$ -capacity such that  $\mu = f - \operatorname{div}(g) + \lambda^+ - \lambda^-$ . Since  $\operatorname{cap}_p(E^+, \Omega) = 0$  and  $E^+ \subset \mathcal{U} \subset \Omega$  we have (see [12])  $\operatorname{cap}_p(E^+, \mathcal{U}) = 0$  (and also  $\operatorname{cap}_p(E^-, \mathcal{U}) = 0$ ). Thus, following the construction of the cut-off functions in [6], we define for all  $\delta > 0$  two functions,  $\psi_\delta^+$  and  $\psi_\delta^-$ , lying in  $\mathcal{C}_c^\infty(\mathcal{U})$  such that

$$(15) \quad 0 \leq \psi_\delta^+ \leq 1, \quad 0 \leq \psi_\delta^- \leq 1 \quad \text{on } \mathcal{U},$$

$$(16) \quad \operatorname{supp}(\psi_\delta^+) \cap \operatorname{supp}(\psi_\delta^-) = \emptyset,$$

$$(17) \quad \int_\Omega \psi_\delta^- d\lambda^+ < \delta, \quad \int_\Omega \psi_\delta^+ d\lambda^- < \delta,$$

$$(18) \quad \int_\Omega (1 - \psi_\delta^+) d\lambda^+ < \delta, \quad \int_\Omega (1 - \psi_\delta^-) d\lambda^- < \delta.$$

Since  $\mathcal{U} \subset \Omega$ , we define  $\psi_\delta^+ \equiv \psi_\delta^- \equiv 0$  on  $\Omega \setminus \mathcal{U}$  so that we have  $\psi_\delta^+, \psi_\delta^- \in \mathcal{C}_c^\infty(\Omega)$ .

For any  $n \in \mathbb{N}^*$  let  $h_n$  be the function defined by  $h_n(r) = (n - T_n^+(|r| - n))/n$   $\forall r \in \mathbb{R}$ .

Let  $K > 0$  be fixed,  $n \in \mathbb{N}^*$  and  $\delta > 0$ . Since the function  $h_n$  belongs to  $W^{1,\infty}(\mathbb{R})$  while  $\text{supp}(h_n) = [-2n, 2n]$  is compact, from the regularity of  $u_1$  and  $u_2$  we obtain that the function  $h_n(u_1)h_n(u_2)(T_K(u_1 - u_2) - K(\psi_\delta^+ + \psi_\delta^-))$  lies in  $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$  and is equal to zero on the set  $\{x; |u_i(x)| > 2n\}$  for  $i = 1, 2$ . Therefore setting  $W_K = T_K(u_1 - u_2)$  the function  $h_n(u_1)h_n(u_2)(W_K - K(\psi_\delta^+ + \psi_\delta^-))$  is an admissible test function on both equations (1) written for  $u_1$  and  $u_2$ , relative to (9) of Definition 3. Subtracting the resulting equalities gives

$$\begin{aligned} \text{(A)} \quad & \int_{\Omega} h_n(u_1)h_n(u_2)(\mathbf{a}(Du_1) - \mathbf{a}(Du_2)) \cdot (DW_K - KD(\psi_\delta^+ - \psi_\delta^-)) dx \\ \text{(B)} \quad & + \int_{\Omega} h_n'(u_1)h_n(u_2)(\mathbf{a}(Du_1) - \mathbf{a}(Du_2)) \cdot Du_1(W_K - K(\psi_\delta^+ + \psi_\delta^-)) dx \\ \text{(C)} \quad & + \int_{\Omega} h_n'(u_2)h_n(u_1)(\mathbf{a}(Du_1) - \mathbf{a}(Du_2)) \cdot Du_2(W_K - K(\psi_\delta^+ + \psi_\delta^-)) dx \\ & = 0 \end{aligned}$$

In order to study the behavior of the terms above as  $n$  goes to infinity and  $\delta$  goes to zero, A and B are split into  $A_1 + A_2$  and  $B_1 + B_2$  respectively, where

$$\begin{aligned} A_1 &= \int_{\Omega} h_n(u_1)h_n(u_2)(\mathbf{a}(x, Du_1) - \mathbf{a}(x, Du_2)) \cdot DT_K(u_1 - u_2) dx, \\ A_2 &= -K \int_{\Omega} h_n(u_1)h_n(u_2)(\mathbf{a}(x, Du_1) - \mathbf{a}(x, Du_2)) \cdot (D\psi_\delta^+ + D\psi_\delta^-) dx, \\ B_1 &= \int_{\Omega} h_n'(u_1)h_n(u_2)(\mathbf{a}(x, Du_1) - \mathbf{a}(x, Du_2)) \cdot Du_1 W_K (1 - \psi_\delta^+ - \psi_\delta^-) dx, \\ B_2 &= \int_{\Omega} h_n'(u_1)h_n(u_2)(\mathbf{a}(x, Du_1) - \mathbf{a}(x, Du_2)) \cdot Du_1 (W_K - K)(\psi_\delta^+ + \psi_\delta^-) dx. \end{aligned}$$

From (5) and (7) it follows that  $\mathbf{a}(x, Du_i)$  belongs in particular to  $L^1(\Omega)$  for  $i = 1, 2$  and then  $(\mathbf{a}(x, Du_1) - \mathbf{a}(x, Du_2)) \cdot (D\psi_\delta^+ + D\psi_\delta^-)$  belongs to  $L^1(\Omega)$ . Since  $h_n(u_1)h_n(u_2)$  converges to 1 almost everywhere as  $n$  tends to infinity and is uniformly bounded, Lebesgue Theorem leads to

$$\lim_{n \rightarrow +\infty} A_2 = -K \int_{\Omega} (\mathbf{a}(x, Du_1) - \mathbf{a}(x, Du_2)) \cdot (D\psi_\delta^+ + D\psi_\delta^-) dx.$$

Recalling that  $u_1$  and  $u_2$  are also solution of (1)–(2) in the sense of distributions and since  $\psi_\delta^+, \psi_\delta^- \in C_c^\infty(\Omega)$ , we obtain that

$$(19) \quad \lim_{n \rightarrow +\infty} A_2 = 0.$$

Due to the definition of  $\psi_\delta^+$  and  $\psi_\delta^-$  we have  $1 \geq 1 - \psi_\delta^+ - \psi_\delta^- \geq 0$ . Thus Assumption (5) and Young's inequality lead to

$$|\mathbf{B}_1| \leq \frac{C}{n} \left( \int_{\{|u_1| < 2n\}} |Du_1|^p (1 - \psi_\delta^+ - \psi_\delta^-) dx + \int_{\{|u_2| < 2n\}} |Du_2|^p (1 - \psi_\delta^+ - \psi_\delta^-) dx + \int_{\Omega} b^p dx \right)$$

and (3) gives

$$(20) \quad |\mathbf{B}_1| \leq \frac{C}{n} \left( \int_{\{|u_1| < 2n\}} \mathbf{a}(x, Du_1) \cdot Du_1 (1 - \psi_\delta^+ - \psi_\delta^-) dx + \int_{\{|u_2| < 2n\}} \mathbf{a}(x, Du_2) \cdot Du_2 (1 - \psi_\delta^+ - \psi_\delta^-) dx + \int_{\Omega} b^p dx \right),$$

where  $C$  is a generic constant independent of  $n$  and  $\delta$ . Since  $1 - \psi_\delta^+ - \psi_\delta^- \in C^\infty(\overline{\Omega})$ , using the property (10) of renormalized solutions we get for  $i = 1, 2$

$$\lim_{n \rightarrow +\infty} \frac{1}{2n} \int_{\{|u_i| < 2n\}} \mathbf{a}(x, Du_i) \cdot Du_i (1 - \psi_\delta^+ - \psi_\delta^-) dx = \int_{\Omega} (1 - \psi_\delta^+ - \psi_\delta^-) d\lambda^+ + \int_{\Omega} (1 - \psi_\delta^+ - \psi_\delta^-) d\lambda^-,$$

from which it follows, using (17), (18) and (20) and since  $b \in L^p(\Omega)$ ,

$$\overline{\lim}_{n \rightarrow +\infty} |\mathbf{B}_1| \leq C\delta,$$

and then

$$(21) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow +\infty} |\mathbf{B}_1| = 0.$$

Let  $\mathcal{U}_{n,K}$  be the set defined by

$$(22) \quad \mathcal{U}_{n,K} = \mathcal{U} \cap \{|u_1| < 2n\} \cap \{|u_2| < 2n\} \cap \{u_1 - u_2 < K\}.$$

Because  $0 \leq K - T_K(u_1 - u_2) \leq 2K \mathbb{1}_{\{u_1 - u_2 < K\}}$  (recall that  $W_K = T_K(u_1 - u_2)$ ), from the definition of the cut-off functions  $\psi_\delta^+$  and  $\psi_\delta^-$  we obtain

$$|\mathbf{B}_2| \leq \frac{2K}{n} \int_{\mathcal{U}_{n,K}} |\mathbf{a}(x, Du_1) - \mathbf{a}(x, Du_2)| |Du_1| dx.$$



Using Hölder inequalities together with (5) permits us to deduce that if  $p \geq 2$  then

$$|B_2| \leq CK \left( \frac{1}{n} \int_{\mathcal{U}_{n,K}} |Du_1 - Du_2|^p dx \right)^{1/p} \\ \times \left( \frac{1}{n} \int_{\substack{\{|u_1| < 2n\} \\ \cap \{|u_2| < 2n\}}} (b(x) + |Du_1| + |Du_2|)^p dx \right)^{1/p'}$$

and if  $p < 2$  then

$$|B_2| \leq CK \left( \frac{1}{n} \int_{\mathcal{U}_{n,K}} |Du_1 - Du_2|^p dx \right)^{1/p'} \left( \frac{1}{n} \int_{\substack{\{|u_1| < 2n\} \\ \cap \{|u_2| < 2n\}}} |Du_1|^p dx \right)^{1/p}.$$

In both cases, property (10) (with  $\varphi \equiv 1$ ) and (14) lead to

$$\forall \delta > 0 \quad \lim_{n \rightarrow +\infty} |B_2| = 0.$$

From (21) it follows that

$$(23) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow +\infty} |B| = 0 \quad \text{and} \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow +\infty} |C| = 0 \quad (\text{by symmetry}).$$

From (19) and (23) we then have  $\lim_{\delta \rightarrow 0} \lim_{n \rightarrow +\infty} A_1 = 0$ . Since  $h_n(u_1)h_n(u_2)$  is nonnegative and converges to 1 almost everywhere in  $\Omega$ , the monotone character of the operator  $\mathbf{a}$  and Fatou lemma imply that for all  $K > 0$

$$\int_{\{|u_1 - u_2| < K\}} (\mathbf{a}(x, Du_1) - \mathbf{a}(x, Du_2)) \cdot (Du_1 - Du_2) dx = 0,$$

and from (4) we can conclude that  $u_1 = u_2$ .  $\square$

*Proof of Theorem 7.* It is sufficient to show that (14) holds true and to use Theorem 5. We assume that  $(u_1 - u_2)^-$  belongs to  $L^\infty(\mathcal{U})$ .

According to the properties of the difference of two renormalized solutions (see [6]) we have for all  $K > 0$

$$(24) \quad \int_{\{|u_1 - u_2| < K\}} (\mathbf{a}(x, Du_1) - \mathbf{a}(x, Du_2)) \cdot (Du_1 - Du_2) dx \leq CK,$$

where  $C$  is a constant independent of  $K$ .

Let  $M$  be a real number such that  $M > \|(u_1 - u_2)^-\|_{L^\infty(\mathcal{U})}$  and let  $K > 0$ ,  $n \in \mathbb{N}^*$  and  $\mathcal{U}_{n,K}$  the set defined by (22). Since  $\mathcal{U} \subset \{-M < u_1 - u_2\}$  we get  $\mathcal{U}_{n,K} \subset \{|u_1| < 2n\} \cap \{|u_2| < 2n\} \cap \{|u_1 - u_2| < \max(M, K)\}$  and therefore

$$\frac{1}{n} \int_{\mathcal{U}_{n,K}} |Du_1 - Du_2|^p dx \leq \frac{1}{n} \int_{\substack{\{|u_1| < n, |u_2| < n\} \\ \cap \{|u_1 - u_2| < \max(M, K)\}}} |Du_1 - Du_2|^p dx.$$

In both cases ( $p < 2$  and  $p \geq 2$ ), the strong monotonicity of the operator  $\mathbf{a}$ , Hölder inequalities together with (10) (with  $\varphi \equiv 1$ ) and (24) allow us to prove that for all  $K > 0$

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\substack{\{|u_1| < n, |u_2| < n\} \\ \cap \{|u_1 - u_2| < \max(M, K)\}}} |Du_1 - Du_2|^p dx = 0.$$

It follows that the conditions of Theorem 5 are satisfied and then  $u_1 = u_2$ .  $\square$

**Remark 8.** In Theorem 5, assuming  $\mathbf{a}$  to be strongly monotone, if condition (14) is satisfied for  $K = 0$  only (and not for every  $K > 0$ ), then  $u_1 = u_2$ . Indeed, in this case (10), (11) and (24) imply that  $\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\{|u_1 - u_2| < K\}} |Du_1 - Du_2|^p dx = 0 \forall K > 0$  and then (14) is satisfied for all  $K > 0$ .

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