Level set approach for fractional mean curvature flows
Cyril Imbert

Level set approach for fractional mean curvature flows

Cyril Imbert[1]

March 12, 2009

ABSTRACT. This paper is concerned with the study of a geometric flow whose law involves a singular integral operator. This operator is used to define a non-local mean curvature of a set. Moreover the associated flow appears in two important applications: dislocation dynamics and phasefield theory for fractional reaction-diffusion equations. It is defined by using the level set method. The main results of this paper are: on one hand, the proper level set formulation of the geometric flow; on the other hand, stability and comparison results for the geometric equation associated with the flow.

1 Introduction

In this paper, we define a geometric flow whose law is non-local. We recall that a geometric flow of a set Ω is a family $\{\Omega_t\}_{t>0}$ such that the velocity of a point $x \in \partial\Omega_t$ along its outer normal $n(x)$ is a given function of x and $n(x)$ for instance. In our case, this velocity does not only depend on x and $n(x)$ but also on a *fractional mean curvature* at x. Our motivation comes from two different problems: dislocation dynamics and phasefield theory for fractional reaction-diffusion equations.

1.1 Motivation and existing results

Mathematical study of non-local moving fronts recently attracted a lot of attention (see in particular [12] and references therein). An important application is the study of dislocation dynamics [3].

Dislocation dynamics

Dislocations are linear defects in crystals and the study of their motion gives rise to the study of a non-local geometric flow. In recent years, several papers were dedicated to this problem. We next briefly recall the results contained in these papers.

A dislocation creates an elastic field in the whole space $\mathbb{R}^3$ and this field creates a force (called the Peach-Koehler force) that acts not only on the dislocation that created

---

[1] Université Paris-Dauphine, CEREMADE, place de Lattre de Tassigny, 75775 Paris cedex 16, France, `imbert@ceremade.dauphine.fr`

it (self-force) but also on all dislocations in the material. We restrict ourselves here to the case of a single curve. We also assume that this curve moves in a plane (called the slip plane).

The level set approach [24, 13, 16] is a general method for constructing moving interfaces. It consists in representing Ω_t as zero level sets of functions $u(t, \cdot)$. The geometric law satisfied by the interface $\partial \Omega_t$ is thus translated into an evolution equation satisfied by u. This approach is used in [3] to describe the dynamics of a dislocation line. If $\partial \Omega_t$ is the zero level set of a function $u(t, \cdot)$, the following non-local eikonal equation is obtained

$$\partial_t u = (c_1(x) + \kappa[x, u])|Du|$$

where c_1 is an external force and $\kappa[x, u]$ is the Peach-Koehler force applied to the curve ($N = 2$ in this application).

We briefly mentioned above that the Peach-Koehler force is created by the curve. Let us be a bit more specific. This force is computed through the resolution of an elliptic equation on a half space (corresponding to the law of linear elasticity). This equation is supplemented with Dirichlet boundary conditions. On one hand, the boundary datum equals the indicator function of the interior of the curve. On the other hand, loosely speaking, the force on the curve equals the normal derivative of the solution of the elliptic equation. Hence, the integral operator which defines the Peach-Koehler force is a Dirichet-to-Neumann operator associated with an elliptic equation. In particular, the operator is singular.

In order to define solutions for small times, the authors of [3] consider a physically relevant regularized problem and $\kappa[x, u]$ reduces to

$$\int_{\{z:u(z)\geq 0\}} c_0(z)dz$$

with $c_0 \in W^{1,1}(\mathbb{R}^N)$.

The major technical difficulty of this paper is that c_0 does not have a constant sign and consequently, solutions corresponding to ordered initial data are not ordered; in other words, comparison principle does not hold true. In particular, this is one of the reasons why solutions are constructed for small times. If c_1 is assumed to be large enough, Alvarez, Cardaliaguet and Monneau [2] managed to prove the existence and uniqueness for large times.

The difficulty related to comparison principle is circumvented in [19] by assuming that the negative part of c_0 is concentrated at the origin. The Peach-Koehler force $\kappa[x, u]$ (in the case of a single dislocation line) is defined in [19] as

$$\int \text{sign}(u(x + z) - u(x))c_0(z)dz = \int_{\{z:u(x+z)\geq u(x)\}} c_0(z)dz - \int_{\{z:u(x+z)<u(x)\}} c_0(z)dz \quad (1)$$

where $\text{sign}(r)$ equals 1 if $r \geq 0$ and -1 if $r < 0$. After an approximation procedure, the problem can be reduced to the study of

$$\partial_t U = \left[c_1(x) + \int (U(x + z) - U(x))c_0(z)dz \right] |DU|$$

where c_0 is smooth, non-negative and of finite mass. We used the letter U instead of u in order to emphasize the fact that a change of unknown function is needed in order to reduce the study of the original equation to the study of this new one.

A second important remark is that solving such non-local eikonal equations does not permit to construct properly a geometric flow. More precisely, if the initial front $\partial\Omega_0$ is described with two different initial functions u_0 and v_0, it is not sure that the zero level sets of the corresponding solutions u and v coincide. In other words, the invariance principle does not hold true.

Still assuming that the negative part of c_0 is concentrated at the origin, a good geometric definition of the flow is obtained in [17] by considering a formulation "à la Slepčev" of the geometric flow. The equation now becomes

$$\partial_t u = \left[c_1(x) + \int_{\{z:u(t,x+z)>u(t,x)\}} c_0(z)dz \right] |Du|. \tag{2}$$

We point out that, with such a formulation, we cannot deal with singular potentials c_0.

Notice that in [17], several fronts move, and they are interacting. The motion of a single front is a special case. Eventually, existence results of very weak solutions in a very general setting are obtained in [5] and uniqueness is studied in [6].

In [15], it is proved that if $c_0(z)$ is smooth and regular near the origin and behaves exactly like $|z|^{-N-1}$ at infinity, then a proper rescaling of (2) converges towards the mean curvature motion (see Proposition 1 and Theorem 4 below).

We finally mention that Caffarelli and Souganidis [11] consider a Bence-Merriman-Osher scheme with kernels associated with the fractional heat equation (that is to say the heat equation where the usual Laplacian is replaced with the fractional one). They prove that this scheme approximates the geometric flow at stake in this paper.

Phasefield theory for fractional reaction-diffusion equations

Our second main motivation comes from phasefield theory for fractional reaction-diffusion equations [21]. If one considers for instance stochastic Ising models with Kač potentials with very slow decay at infinity (like a power law with proper exponent), then the study of the resulting mean field equation (after proper rescaling) is closely related to phasefield theory for fractional reaction-diffusion equations such as

$$\partial_t u^\varepsilon + (-\Delta)^{\alpha/2} u^\varepsilon + \frac{1}{\varepsilon^{1+\alpha}} f(u^\varepsilon) = 0$$

where $(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian with $\alpha \in (0,1)$ (in the case presented here) and f is a bistable non-linearity. In particular, it is *essential* in the analysis to deal with singular potentials. Indeed, we have to be able to treat the case where

$$c_0(z) = \frac{1}{|z|^{N+\alpha}}$$

3

with $\alpha \in (0,1)$. It is also convenient to use the notion of generalized flows introduced by Barles and Souganidis [9] in order to develop a phasefield theory for such reaction-diffusion equations. See [21] for further details and [18] for analogous problems.

1.2 A new formulation

The main contributions of this paper are the following:

- to give a proper level set formulation of dislocation dynamics for singular interaction potentials; in particular, sufficient conditions on the singularity to get stability results and comparison principles are exhibited;

- to shead light on the fact that the integral operator measures in a non-local way the curvature of the interface;

- to study the geometric flow in detail: consistency of the definition, equivalent definition in terms of generalized flows, motion of bounded sets *etc.*

Because $\nu(dz) = c_0(z)dz$ is singular, we cannot define $\kappa[x,u]$ as in (2). Indeed, we must *compensate* the singularity as it is commonly done in order to get a proper integral representation of the fractional Laplacian. We recall that the fractional Laplacian can be defined as follows

$$(-\Delta)^{\alpha/2}u(x) = -c_N(\alpha) \int (u(x+z) - u(x)) \frac{dz}{|z|^{N+\alpha}}$$

where $c_N(\alpha)$ is a given positive constant depending on N, α. Notice that if $\alpha < 1$ and u is Lipschitz continuous at x and u is globally bounded, the integral is well defined. If $\alpha \geq 1$, the integral is not convergent in the neighbourhood of $z = 0$. In this case, the fractional Laplacian is defined either by considering the principal value of the previous singular integral or by writing

$$(-\Delta)^{\alpha/2}u(x) = -c_N(\alpha) \int (u(x+z) - u(x) - Du(x) \cdot z \mathbf{1}_B(z)) \frac{dz}{|z|^{N+\alpha}}$$

where $\mathbf{1}_B(z)$ denotes the indicator function of the unit ball B. Notice that we used the fact that the singular measure

$$\nu(dz) = \frac{dz}{|z|^{N+\alpha}} \tag{3}$$

(with $\alpha \in (0,2)$) is even in order to get (at least formally)

$$\int (Du(x) \cdot z \mathbf{1}_B(z)) \frac{dz}{|z|^{N+\alpha}} = 0 \,.$$

As far as the fractional mean curvature is concerned, we must compensate the singularity of the measure ν in a geometrical way. We explain how to do it when $\nu(dz) = c_0(z)dz$

4

with $c_0(z) = |z|^{-N-\alpha}$. Hence, we start from (1). We use the fact that c_0 is even in order to get (formally)

$$\nu(z \in \mathbb{R}^N : Du(x) \cdot z \geq 0) = \nu(z \in \mathbb{R}^N : Du(x) \cdot z < 0).$$

Straightforward computations yield

$$\int \text{sign}(u(x+z) - u(x))c_0(z)dz =$$
$$\nu\{z : u(x+z) \geq u(x), Du(x) \cdot z \leq 0\} - \nu\{z : u(x+z) < u(x), Du(x) \cdot z > 0\}.$$

We thus define an integral operator $\kappa[x, u]$ for a general singular non-negative measure ν as follows

$$\kappa[x, u] = \nu\{z : u(x+z) \geq u(x), Du(x) \cdot z \leq 0\} - \nu\{z : u(x+z) < u(x), Du(x) \cdot z > 0\}. \quad (4)$$

We explain below in detail (see Lemma 2) the rigourous links between the different formulations we considered up to now.

Notice that this definition makes sense even if ν is not even. We recall that the fractional Laplacian is a Lévy operator. Since Lévy operators [4] are defined for singular (Lévy) measure that are not necessarily even, this seems to be relevant to define the fractional mean curvature for singular measures that are not necessarily even.

We can say that this singular integral operator measures in a non-local way the curvature of the "curve" $\{u = u(x)\}$. Indeed, loosely speaking, we can say that in Formula (4) the first part (resp. the second one) measures how concave (resp. convex) is the set $\Omega = \{z : u(x+z) > u(x)\}$ "near x". Moreover, we prove (see Proposition 2 below) that, when ν is given by (3), the function $(1-\alpha)\kappa[x, u]$ converges as $\alpha \in (0, 1)$ goes to 1 towards the classical mean curvature of $\{u = u(x)\}$ at x. This is the reason why we refer to $\kappa[x, u]$ as the fractional mean curvature of the curve $\{u = u(x)\}$ at point x.

The variational case

When the singular measure $\nu(dz)$ has the form

$$\nu(dz) = -\left(\nabla \cdot G(z)\right)dz$$

for a vectorfield G, the previous singular integral operator can be written as follows

$$\kappa[x, u] = \int_{\{z:u(x+z)=u(x)\}} \left(G(z) \cdot \frac{\nabla u(x+z)}{|\nabla u(x+z)|}\right)\sigma(dz) - b_G\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right)\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right) \cdot \frac{\nabla u(x)}{|\nabla u(x)|} \quad (5)$$

where σ denotes the surface measure on the "curve" $\{z : u(x+z) = u(x)\}$ and where $b_G = \int_{\{z:\nabla u(x) \cdot z=0\}} G(z)\sigma(dz)$ is a vector field of $\mathbb{R}^N$.

Remark that the example we gave above is of this form. Indeed

$$\frac{dz}{|z|^{N+\alpha}} = -\frac{1}{\alpha}\left(\nabla \cdot \frac{z}{|z|^{N+\alpha}}\right)dz\,.$$

It is quite clear on this new formula that the singular integral operator is geometric (in the sense that it only depends on the curve and not and its parametrization u) and "fractional".

After this work was finished, we have been told that non-local minimal surfaces are being studied by Caffarelli, Roquejoffre and Savin [10]. Loosely speaking, they study sets whose indicator functions minimize a fractional Sobolev norm $\|\cdot\|_{H^\alpha}$, $\alpha \in (0,1)$. They prove in particular that local minimizers are viscosity solutions of $\kappa[x,u] = 0$.

Comments and related works

We gave two different formulations in the case of singular potentials. *We think that Formulation (4) is the proper one in order to get a complete level set formulation of the geometric flow* even if Formulation (5) is somehow more intuitive since it only involves the curve itself. In particular, the approach proposed by Slepčev [26] can be adapted (see (14) below).

The level set equation we study has the following form

$$\partial_t u = \mu(\widehat{Du})\left[c_1(x) + \kappa[x,u]\right]|Du| \quad \text{in } (0,+\infty) \times \mathbb{R}^N \tag{6}$$

supplemented with the following initial condition

$$u(0,x) = u_0(x) \quad \text{in } \mathbb{R}^N \tag{7}$$

where $\hat{p}$ denotes $p/|p|$ if $p \neq 0$, μ denotes the mobility vector field, $c_1(x)$ is a driving force.

Equation (6) is a non-linear non-local Hamilton-Jacobi equation. A lot of papers are dedicated to the study of such equations. In our case, the main technical issues are the definition of viscosity solutions, the proof of their stability and the proof of a strong uniqueness result. We somehow use ideas from [26] and combine them with the ones from [7], even if the results of these two papers do not apply to our equation.

From a physical point of view and as far as dislocation dynamics is concerned, the measure $\nu(dz) = c_0(z)dz$ should be $\nu(dz) = g(z/|z|)|z|^{-N-1}dz$ but in this case, the fractional mean curvature is not well defined (see Remark 1). It is also physically relevant to say that close to the dislocation line, in the *core* of the dislocation, the potential should be regularized. On the other hand, it is important to assume that $\nu(dz) \sim g(z/|z|)|z|^{-N-1}dz$ as $|z| \to +\infty$ since this prescribes the long range interaction between dislocation lines. Another way to understand this difficulty is to say that in the core of the dislocation, the potential is very singular and the singularity should be compensated at a higher (second) order. On one hand, this can explain the loss of inclusion principle for such flows (if one

can define them for large times). On the other hand, one can think that in this case, the first term in such an expansion should be a mean curvature term. This can make sense since curvature terms are commonly used to describe dislocation dynamics. It can be relevant to add one in (6). However, we choose not to do so in order to avoid technicalities and keep clear some important points in the proof of the stability result and the comparison principle.

In order to better understand properties of the fractional mean curvature flow, a deterministic zero-sum repeated game is constructed in [20] in the spirit of [23, 22].

Organization of the article. In Section 2, we first give the precise assumptions we make on data. We next give the definition(s) of the fractional mean curvature $\kappa[x, \cdot]$. In Section 3, we first give the definition of viscosity solutions for (6), we then state and prove stability results. We next obtain strong uniqueness results by establishing comparison principles. We also construct solutions of (6) by Perron's method. We finally give two convergence results which explain in which limit one recovers the classical mean curvature equation. In Section 4, we verify that the zero levet set of the solution u we constructed in the previous section only depends on the zero level set of the initial condition. This provides a level set formulation of the geometric flow. In the last section, we give an alternative geometric definition of the flow in terms of generalized flows in the sense of [9].

Notation. $\mathbb{S}^{N-1}$ denotes the unit sphere of $\mathbb{R}^N$. The ball of radius δ centered at x is denoted by $B_\delta(x)$. If $x = 0$, we simply write B_δ and if moreover $\delta = 1$, we write B. If $p \in \mathbb{R}^N \setminus \{0\}$, $\hat{p}$ denotes $p/|p|$. If A is a subset of $\mathbb{R}^d$ with $d = N, N + 1$ for instance, then A^c denotes $\mathbb{R}^d \setminus A$. For two subsets A and B, $A \sqcup B$ denotes $A \cup B$ and means that $A \cap B = \emptyset$. The function $\mathbf{1}_A(z)$ equals 1 if $z \in A$ and 0 if not.

2 Preliminaries

In this section, we make precise the assumptions we need on data and we give several definitions of the fractional mean curvature.

2.1 Assumptions

Here are the assumptions we make on the singular measure throughout the paper.

ASSUMPTIONS.

(A1) The mobility function $\mu : \mathbb{S}^{N-1} \to (0, +\infty)$ is continuous.

(A2) The driving force $c_1 : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous.

(A3) The singular measure ν is a non-negative Radon measure satisfying

$$
\begin{cases}
\text{for all } \delta > 0, & \nu(\mathbb{R}^N \setminus B_\delta) < +\infty \,, \\
\text{for all } r > 0, e \in \mathbb{S}^{N-1}, & \nu\{z \in B : r|z \cdot e| \le |z - (z \cdot e)e|^2\} < +\infty \,, \\
& \delta\nu(\mathbb{R}^N \setminus B_\delta) \to 0 \ \text{as } \delta \to 0 \,, \\
\text{for all } e \in \mathbb{S}^{N-1}, & r\,\nu\{z \in B : r|z \cdot e| \le |z - (z \cdot e)e|^2\} \to 0 \ \text{as } r \to 0
\end{cases}
\tag{8}
$$

(B_δ denotes the ball of radius δ centered at the origin and $B = B_1$) and the last limit is uniform with respect to unit vectors $e \in \mathbb{S}^{N-1}$.

(A4) The initial datum $u_0 : \mathbb{R}^N \to \mathbb{R}$ is bounded and Lipschitz continuous.

We point out that the set $\{z \in B : r|z \cdot e| \le |z - (z \cdot e)e|^2\}$ appearing in the second and the fourth lines of (8) is the region between an upward and downward (with respect to vector e) parabola.

Even if the assumptions on the singular measure look technical at first glance, they are quite natural in the sense that they imply several important properties:

- the measure is bounded away from the origin;

- the singularity at the origin (if any) is a weak singularity in the sense that the fractional mean curvature of regular curves can be defined; if the reader thinks of the example given in (3), this means that we choose $\alpha < 1$;

- Parabolas $\{z : rz_N = |z'|^2\}$ (which are the model regular curves for us) can be handled, even when they degenerate ($r \to 0$).

Example 1. The Standing Example for the singular measure is

$$
\nu_{SE}(dz) = g\left(\frac{z}{|z|}\right) \frac{dz}{|z|^{N+\alpha}}
$$

with $g : \mathbb{S}^{N-1} \to (0, +\infty)$ continuous and $\alpha \in (0,1)$. The measure in (3) corresponds to the isotropic case ($g \equiv 1$).

2.2 Fractional mean curvature

In this subsection, we make precise the definition of fractional mean curvature. Our definition extends the ones given in [15, 17] where $\nu(dz) = c_0(z)dz$ to the case of singular measures.

Let us define the fractional curvature of a smooth curve $\Gamma = \{x \in \mathbb{R}^N : u(x) = 0\} = \partial\{x \in \mathbb{R}^N : u(x) > 0\}$ associated with ν. If u is $C^{1,1}$ and $Du(x) \neq 0$, then the following quantity is well defined (see Lemma 1 below)

$$
\begin{aligned}
\kappa^*[x, \Gamma] &= \kappa^*[x, u] = \kappa_+^*[x, u] - \kappa_*^-[x, u] \\
\kappa_*[x, \Gamma] &= \kappa_*[x, u] = \kappa_*^+[x, u] - \kappa_-^*[x, u]
\end{aligned}
\tag{9}
$$

$$\Omega = \{u > u(x)\}$$

$$Du(t,x)$$

$$\kappa^-[x,u]$$

$$x$$

$$\kappa^+[x,u]$$

$$\{u < u(x)\}$$

$$\Gamma = \{u = u(x)\}$$

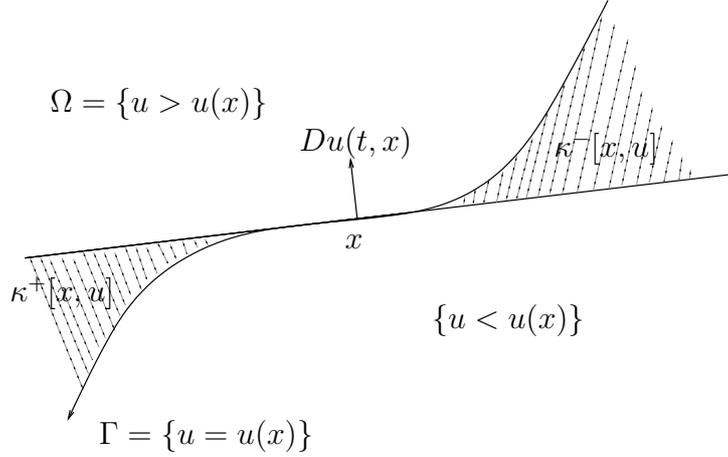Figure 1: Fractional mean curvature of a curve

where

$$
\begin{aligned}
\kappa_*^+[x,u] &= \nu\big(z : u(x+z) > u(x),\ Du(x) \cdot z < 0\big)\\
\kappa_*^-[x,u] &= \nu\big(z : u(x+z) < u(x),\ Du(x) \cdot z > 0\big)
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
\kappa_+^*[x,u] &= \nu\big(z : u(x+z) \geq u(x),\ Du(x) \cdot z \leq 0\big)\\
\kappa_-^*[x,u] &= \nu\big(z : u(x+z) \leq u(x),\ Du(x) \cdot z \geq 0\big)\,.
\end{aligned}
$$

We will see later (see Lemma 3 below) that these functions are semi-continuous and this explains the choice of notation we made. In order to understand the way these quantities are related to the geometry of the curve $\{u = u(x)\}$, it is convenient to write for instance

$$\kappa_*^+[x,u] = \nu\big(z : 0 < -Du(x) \cdot z < u(x+z) - u(x) - Du(x) \cdot z\big)\,.$$

As shown on Figure 1, $\kappa_*^+[x,u]$ measures how concave the curve is "near" x and $\kappa_*^-[x,u]$ how convex it is.

Lemma 1 (Fractional mean curvature is finite). *If u is $C^{1,1}$ at point x, i.e. there exists a constant $C = C(x) > 0$ such that for all $z \in \mathbb{R}^N$*

$$|u(x+z) - u(x) - Du(x) \cdot z| \leq C|z|^2$$

and its gradient $Du(x) \neq 0$, then $\kappa_\pm^*[x,u]$ are finite.
 If u is $C^{1,1}$ at x and $Du \neq 0$ everywhere on $\{y \in \mathbb{R}^N : u(y) = u(x)\}$ and ν is absolutely continuous with respect to the Lebesgue measure, then $\kappa_\pm^*[x,u]$ are finite and

$$\kappa^*[x,u] = \kappa_*[x,u]\,.$$

Remark 1. One can check that this lemma is false if $\alpha = 1$ in the Standing Example 1.

9

Proof. We only prove the first part of the Lemma since the second part is clear.

Since ν is bounded on $\mathbb{R}^N \setminus B_\delta$ for all $\delta > 0$, it is enough to consider

$$\begin{aligned}(\kappa_+^*)^{1,\delta}[x,u] &= \nu\big(z \in B_\delta : u(x+z) \geq u(x),\ Du(x) \cdot z \leq 0\big) \\ &= \nu\big(z \in B_\delta : 0 \leq re \cdot z \leq u(x+z) - u(x) + re \cdot z\big)\end{aligned}$$

where $r = |Du(x)| \neq 0$ and $e = r^{-1}Du(x)$. If now z_N denotes $e \cdot z$ and $z' = z - z_N e$, and if we choose δ such that $r - C\delta > 0$, we can write

$$\begin{aligned}(\kappa_+^*)^{1,\delta}[x,u] &\leq \nu\big(z \in B_\delta : 0 \leq rz_N \leq Cz_N^2 + C|z'|^2\big) \\ &\leq \nu\big(z \in B_\delta : 0 \leq C^{-1}(r - C\delta)z_N \leq |z'|^2\big)\end{aligned}$$

and the result now follows from Condition (8). $\qquad\square$

The following lemma explains rigourously the link between (6) and (2) and the link with the formulation used in [17] in the case where ν is a bounded measure.

Lemma 2 (Link with regular dislocation dynamics). *Consider $c_0 \in L^1(\mathbb{R}^N)$ such that $c_0(x) = c_0(-x)$. Then*

$$\begin{aligned}\int_{\{z : u(t,x+z) > u(t,x)\}} c_0(z)dz &= \frac{1}{2}\int c_0 + \kappa_*[x,u] \\ \int \mathrm{sign}^*(u(x+z) - u(x))c_0(z)dz &= \frac{1}{2}\kappa^*[x,u] \\ \int \mathrm{sign}_*(u(x+z) - u(x))c_0(z)dz &= \frac{1}{2}\kappa_*[x,u]\end{aligned}$$

with $\mathrm{sign}^*(r) = 1$ (resp. $\mathrm{sign}_*(r) = 1$) if $r \geq 0$ (resp. $r > 0$) and -1 if not and with $\nu(dz) = c_0(z)dz$.

Since the proof is elementary, we omit it.

We conclude this section by stating two results which explain the link between two special cases of fractional mean curvature operator and the classical mean curvature operator. The first one appears in [15] (see their Corollary 4.2). We state it in a special case in order to simplify the presentation.

Proposition 1 (From dislocation dynamics to mean curvature flow – [15]). *Assume that $\nu = \nu^\varepsilon$ has the following form*

$$\nu(dz) = \nu^\varepsilon(dz) = \frac{1}{\varepsilon^{N+1}|\ln \varepsilon|}c_0\left(\frac{z}{\varepsilon}\right)dz$$

with c_0 even, smooth, non-negative and such that $c_0(z) = |z|^{-N-1}$ for $|z| \geq 1$.

Assume that $u \in C^2(\mathbb{R}^N)$ and $Du(x) \neq 0$. Then

$$\kappa[x,u] = \kappa^\varepsilon[x,u] \to C\,\mathrm{div}\left(\frac{Du}{|Du|}\right)(x)$$

as $\varepsilon \to 0$ for some constant $C > 0$.

Remark 2. In [15], general anisotropic mean curvature operators can be obtained by considering anisotropic measures $\nu(dz)$.

This result can be compared with the following one.

Proposition 2 (From fractional mean curvature to mean curvature). *Assume that ν has the following form*

$$\nu(dz) = \nu^\alpha(dz) = (1 - \alpha)\frac{dz}{|z|^{N+\alpha}}$$

with $\alpha \in (0, 1)$.

Assume that $u \in C^2(\mathbb{R}^N)$ and $Du(x) \neq 0$. Then

$$\kappa[x, u] = \kappa^\alpha[x, u] \to C \operatorname{div}\left(\frac{Du}{|Du|}\right)$$

as $\alpha \to 1$, $\alpha < 1$, where C is some positive constant.

Remark 3. Anisotropic mean curvature can be obtained by considering

$$\nu^\alpha(dz) = (1 - \alpha)g\left(\frac{z}{|z|}\right)\frac{dz}{|z|^{N+\alpha}}.$$

Sketch of the proof of Proposition 2. For all η, we first choose δ such that

$$|u(x + z) - u(x) - Du(x) \cdot z - \frac{1}{2}D^2u(x)z \cdot z| \leq \eta|z|^2. \qquad (11)$$

If e denotes $-Du(x)$ and $W(z)$ denotes $u(x + z) - u(x) - Du(x) \cdot z$, we have

$$
\begin{aligned}
\kappa^\alpha[x, u] &= \nu^\alpha\{z \in \mathbb{R}^N : 0 \leq e \cdot z \leq W(z)\} \\
&\quad -\nu^\alpha\{z \in \mathbb{R}^N : W(z) \leq e \cdot z \leq 0\} \\
&= (1 - \alpha)\int_{\{z \in B_\delta : 0 \leq e \cdot z \leq W(z)\}} \frac{dz}{|z|^{N+\alpha}} - (1 - \alpha)\int_{\{z \in B_\delta : W(z) \leq e \cdot z \leq 0\}} \frac{dz}{|z|^{N+\alpha}} \\
&\quad +O(1 - \alpha)
\end{aligned}
$$

since $|z|^{-N-\alpha}$ is a bounded measure in B_δ^c.

In view of (11), it is enough to prove the result for $W(z) = Bz \cdot z$ where B is a symmetric $N \times N$ matrix. Hence we study the convergence of

$$K^\alpha = (1 - \alpha)\int_{\{z \in B_\delta : 0 \leq e \cdot z \leq Bz \cdot z\}} \frac{dz}{|z|^{N+\alpha}} - (1 - \alpha)\int_{\{z \in B_\delta : Bz \cdot z \leq e \cdot z \leq 0\}} \frac{dz}{|z|^{N+\alpha}}.$$

We next use the following system of coordinates: $z_1 = \hat{e} \cdot z$ and $z = (z_1, z')$. We now write

$$Bz \cdot z = b_1 z_1^2 + z_1(b_1' \cdot z') + B'z' \cdot z'$$

for some $b_1 \in \mathbb{R}$, $b_1' \in \mathbb{R}^{N-1}$ and a $(N-1) \times (N-1)$ symmetric matrix B'. We thus want to prove

$$K^\alpha \to |e|^{-1}\operatorname{tr}B'$$

11

as $\alpha \to 1$. We can assume without loss of generality that $|e| = 1$. For $z \in B_\delta$, we have

$$e \cdot z \leq Bz \cdot z \quad \Rightarrow \quad z_1 \leq (1 - C\delta)^{-1} B'z' \cdot z'$$
$$z_1 \geq (1 - C\delta)^{-1} B'z' \cdot z' \quad \Rightarrow \quad e \cdot z \geq Bz \cdot z \,.$$

Hence, it is enough to study the convergence of

$$\tilde{K}^\alpha = (1 - \alpha) \int_{\{(z_1, z') \in B_\delta : 0 \leq z_1 \leq B'z' \cdot z'\}} \frac{dz}{|z|^{N+\alpha}} - (1 - \alpha) \int_{\{(z_1, z') \in B_\delta : B'z' \cdot z' \leq z_1 \leq 0\}} \frac{dz}{|z|^{N+\alpha}} \,.$$

If $\sigma(d\theta)$ denotes the measure on the sphere $\mathbb{S}^{N-2}$, we can write

$$\begin{aligned}
\tilde{K}^\alpha &= (1 - \alpha) \int_{\{(z_1, z') : |z'| \leq \delta, 0 \leq z_1 \leq B'z' \cdot z'\}} \frac{dz}{|z|^{N+\alpha}} - (1 - \alpha) \int_{\{(z_1, z') : |z'| \leq \delta, B'z' \cdot z' \leq z_1 \leq 0\}} \frac{dz}{|z|^{N+\alpha}} \\
&= (1 - \alpha) \int_{\theta \in \mathbb{S}^{N-2} : B'\theta \cdot \theta \geq 0} \int_{r=0}^{\delta} \int_{z_1 = 0}^{r^2 B'\theta \cdot \theta} \frac{r^{N-2}}{(z_1^2 + r^2)^{(N+\alpha)/2}} \sigma(d\theta) dr dz_1 \\
&\quad - (1 - \alpha) \int_{\theta \in \mathbb{S}^{N-2} : B'\theta \cdot \theta \leq 0} \int_{r=0}^{\delta} \int_{z_1 = r^2 B'\theta \cdot \theta}^{0} \frac{r^{N-2}}{(z_1^2 + r^2)^{(N+\alpha)/2}} \sigma(d\theta) dr dz_1 \,.
\end{aligned}$$

We next make the change of variables $z_1 = r^2 \tau$ and we get

$$\tilde{K}^\alpha = \int_{\theta \in \mathbb{S}^{N-2} : B'\theta \cdot \theta \geq 0} (1 - \alpha) \int_{r=0}^{\delta} r^{-\alpha} \int_{\tau=0}^{B'\theta \cdot \theta} \frac{1}{(r^2 \tau^2 + 1)^{(N+\alpha)/2}} \sigma(d\theta) dr d\tau - (\dots) \,.$$

We finally remark that

$$\forall r \in (0, \delta), \quad \int_{\tau=0}^{B'\theta \cdot \theta} \frac{1}{(r^2 \tau^2 + 1)^{(N+\alpha)/2}} \to B'\theta \cdot \theta \text{ as } \delta \to 0 \,.$$

In particular, for δ small enough,

$$(1 - \eta) B'\theta \cdot \theta \leq \int_{\tau=0}^{B'\theta \cdot \theta} \frac{1}{(r^2 \tau^2 + 1)^{(N+\alpha)/2}} \leq (1 + \eta) B'\theta \cdot \theta \,.$$

It is now easy to conclude by remarking

$$(1 - \alpha) \int_0^\delta r^{-\alpha} = \delta^{1-\alpha}$$

$$\int_{\mathbb{S}^{N-2}} \theta \otimes \theta \sigma(d\theta) = C I_{N-1}$$

where I denotes the $(N-1) \times (N-1)$ identity matrix and C is a positive constant. $\qquad \square$

3 Viscosity solutions for (6)

3.1 Definitions

The viscosity solution theory introduced in [26] suggests that the good notion of solution for the fractional equation (6) is the following one.

Definition 1 (Viscosity solutions for (6)). *1. An upper semi-continuous function u : $[0, T] \times \mathbb{R}^N$ is a* viscosity subsolution *of* (6) *if for every smooth test-function ϕ such that $u - \phi$ admits a global zero maximum at (t, x), we have*

$$\partial_t \phi(t, x) \leq \mu(\widehat{D\phi(t, x)}) \left[c_1(x) + \kappa^*[x, \phi(t, \cdot)] \right] |D\phi|(t, x) \tag{12}$$

if $D\phi(t, x) \neq 0$ and $\partial_t \phi(t, x) \leq 0$ if not.

2. A lower semi-continuous function u is a viscosity supersolution *of* (6) *if for every smooth test-function ϕ such that $u - \phi$ admits a global minimum 0 at (t, x), we have*

$$\partial_t \phi(t, x) \geq \mu(\widehat{D\phi(t, x)}) \left[c_1(x) + \kappa_*[x, \phi(t, \cdot)] \right] |D\phi|(x_0, t_0) \tag{13}$$

if $D\phi(t, x) \neq 0$ and $\partial_t \phi(t, x) \geq 0$ if not.

3. A locally bounded function u is a viscosity solution *of* (6) *if u^* (resp. u_*) is a subsolution (resp. supersolution).*

Remark 4. Given $\delta > 0$, the global extrema in Definition 1 can be assumed to be strict in a ball of radius δ centered at (t, x). Such a result is classically expected and the reader can have a look, for instance, at the proof of the stability result in [7].

If one uses the notation introduced in [26], the equation reads

$$\partial_t u + F(x, Du, \{z : u(x + z) \geq u(x)\}) = 0 \tag{14}$$

with, for $x, p \in \mathbb{R}^N$ and $K \subset \mathbb{R}^N$,

$$F(x, p, K) = \begin{cases} -\mu(\hat{p}) \left[c_1(x) + \nu(K \cap \{p \cdot z \leq 0\}) - \nu(K^c \cap \{p \cdot z > 0\}) \right] |p| & \text{if } p \neq 0, \\ 0 & \text{if not} \end{cases}$$

where K^c is the complementary set of K. With this notation in hand, one can check that this non-linearity does not satisfy Assumption (F5) of [26]. The idea is to check that, somehow, Assumption (NLT) in [7] is satisfied and stability results thus hold true.

Let us be more precise. We previously associated with $\kappa[\cdot, \cdot]$ the following non-local operators (see the proof of Lemma 1)

$$\begin{aligned} (\kappa_*^+)^{1,\delta}[x, \phi] &= \nu\big(z \in B_\delta : \phi(x + z) > \phi(x), \; z \cdot D\phi(x) < 0\big), \\ (\kappa_*^+)^{2,\delta}[x, p, \phi] &= \nu\big(z \notin B_\delta : \phi(x + z) > \phi(x), \; z \cdot p < 0\big). \end{aligned} \tag{15}$$

In the same way, we can define

- the negative non-local curvature operators $(\kappa_*^-)^{i,\delta}$, $i = 1, 2$,

- upper semi-continuous envelopes of these four integral operators $(\kappa_\pm^*)^{i,\delta}$, $i = 1, 2$,

- and lower/upper semi-continuous total non-local curvature operators $(\kappa_*)^{i,\delta}$, $(\kappa^*)^{i,\delta}$, $i = 1, 2$.

By using the idea of Lemma 2, it is easy to see that

$$\begin{cases} (\kappa^*)^{2,\delta}[x, p, u] &= \nu\big(z \notin B_\delta : u(t, x+z) \geq u(t, x)\big) - \nu(z \notin B_\delta : p \cdot z > 0), \\ (\kappa_*)^{2,\delta}[x, p, u] &= \nu\big(z \notin B_\delta : u(t, x+z) > u(t, x)\big) - \nu(z \notin B_\delta : p \cdot z \geq 0). \end{cases} \quad (16)$$

We can now state an equivalent definition of viscosity solutions of (6).

Definition 2 (Equivalent definition). *1. An upper semi-continuous function $u : [0, T] \times \mathbb{R}^N$ is a viscosity subsolution of (6) if for every smooth test-function ϕ such that $u - \phi$ admits a maximum 0 at (t, x) on $B_\delta(t, x)$, we have*

$$\partial_t \phi(t, x) \leq \mu(\widehat{D\phi(t, x)}) \Big[c_1(x) + (\kappa^*)^{1,\delta}[x, \phi(t, \cdot)] + (\kappa^*)^{2,\delta}[x, u(t, \cdot)] \Big] |D\phi|(t, x) \quad (17)$$

if $D\phi(t, x) \neq 0$ and $\partial_t \phi(t, x) \leq 0$ if not.

2. A lower semi-continuous function u is a viscosity supersolution of (6) if for every smooth test-function ϕ such that $u - \phi$ admits a global minimum 0 at (t, x), we have

$$\partial_t \phi(t, x) \geq \mu(\widehat{D\phi(t, x)}) \Big[c_1(x) + (\kappa_*)^{1,\delta}[x, \phi(t, \cdot)] + (\kappa_*)^{2,\delta}[x, u(t, \cdot)] \Big] |D\phi|(x_0, t_0) \quad (18)$$

if $D\phi(t, x) \neq 0$ and $\partial_t \phi(t, x) \geq 0$ if not.

3. A continuous function u is a viscosity solution of (6) if it is both a sub and supersolution.

Remark 5. Equivalent definitions of this type first appeared in [25] and since the proof is the same, we omit it.

Remark 6. Remark 4 applies to the equivalent definition too.

Remark 7. Definition 2 seems to depend on δ. But since all these definitions are equivalent to Definition 1, it does not depend on it. Hence, when proving that a function is a solution of (6), it is enough to do it for a fixed (or not) $\delta > 0$.

3.2 Stability results

Theorem 1 (Discontinuous stability). *Assume (A1)-(A3).*

- *Let $(u_n)_{n \geq 1}$ be a family of subsolutions of (6) that is locally bounded, uniformly with respect to n. Then its relaxed upper limit u^* is a subsolution of (6).*

- *If moreover, $u_n(0, x) = u_0^n(x)$, then for all $x \in \mathbb{R}^N$*

$$u^*(0, x) \leq u_0^*(x)$$

  where u_0^* is the relaxed upper limit of u_0^n.

- *Let $(u_\alpha)_{\alpha \in \mathcal{A}}$ be a family of subsolutions of (6) that is locally bounded, uniformly with respect to $\alpha \in \mathcal{A}$. Then $\bar{u}$, the upper semicontinuous envelope of $\sup_\alpha u_\alpha$ is a subsolution of (6).*

Even if this result follows from ideas introduced in [7] together with classical ones, we give a detailed proof for the sake of completeness.

Proof. We only prove the first part of the theorem since it is easy to adapt it to get a proof of the third part. The second one is very classical and can be adapted from [1] for instance.

Consider a test function φ such that $u^* - \varphi$ attains a global maximum at (t, x). We can assume (see Remark 4) that $u^* - \varphi$ attains a strict maximum at (t, x) on $B_\delta(t, x)$. Consider a subsequence $p = p(n)$ and (t_p, x_p) such that

$$u^*(t, x) = \lim_{n \to +\infty} u_{p(n)}(t_p, x_p) \,.$$

Classical arguments show that $u_p - \varphi$ attains a maximum on $B_\delta(t, x)$ at $(s_p, y_p) \in B_\delta(t, x)$ and that

$$(s_p, y_p) \to (t, x) \quad \text{and} \quad u_p(s_p, y_p) \to u^*(t, x) \,.$$

Since u_p is a subsolution of (6), we have

$$\partial_t \varphi(s_p, y_p) \leq$$
$$\mu(\widehat{D\varphi(s_p, y_p)}) \left[c_1(y_p) + (\kappa^*)^{1,\delta}[y_p, \varphi(s_p, \cdot)] + (\kappa^*)^{2,\delta}[y_p, D_x\varphi(s_p, y_p), u(s_p, \cdot)] \right] |D\varphi|(s_p, y_p)$$

if $D\varphi(t_p, x_p) \neq 0$ and $\partial_t \varphi(t_p, x_p) \leq 0$ if not. If there exists a subsequence q of p such that $D\varphi(s_q, y_q) = 0$, then it is easy to conclude. We thus now assume that $D\varphi(s_p, y_p) \neq 0$ for p large enough. In view of the continuity of μ and c_1, the following technical lemma permits to conclude. $\square$

Lemma 3. *Assume that $D\varphi(s_p, y_p) \neq 0$ for p large enough.*

- *Assume moreover that $D\varphi(t, x) \neq 0$. Then*

$$(s, y) \mapsto (\kappa^*)^{1,\delta}[y, \varphi(s, \cdot)] \quad and \quad (s, y) \mapsto (\kappa^*)^{2,\delta}[y, D_x\varphi(s, y), u_p(s, \cdot)]$$

  are well defined for $i = 1, 2$ in a neighbourhood of (t, x) and

$$\limsup_p \left\{ (\kappa^*)^{1,\delta}[y_p, \varphi(s_p, \cdot)] \right\} \leq (\kappa^*)^{1,\delta}[x, \varphi(t, \cdot)]$$

$$\limsup_p \left\{ (\kappa^*)^{2,\delta}[y_p, , D_x\varphi(s_p, y_p), u(s_p, \cdot)] \right\} \leq (\kappa^*)^{2,\delta}[x, D_x\varphi(t, x), \varphi(t, \cdot)]$$

  as soon as $u_p(s_p, y_p) \to u(t, x)$ as $p \to +\infty$.

15

- *Assume now that $D\varphi(t,x) = 0$. Then, for $i = 1,2$,*

$$\left[(\kappa^*)^{1,\delta}[y_p, \varphi(s_p, \cdot)] + (\kappa^*)^{2,\delta}[y_p, D\varphi(s_p, y_p), u(s_p, \cdot)] \right] |D\varphi|(s_p, y_p) \to 0 \quad \text{as } p \to +\infty.$$

As we shall see, this lemma is a consequence of the following one.

Lemma 4 ([26]). *Consider f_p and g_p two sequences of measurable functions on a set U and $f \geq \limsup^* f_p$, $g \geq \limsup^* g_p$, and a_p, b_p two sequences of real numbers converging to 0. Then*

$$\nu(\{f_p \geq a_p, g_p \geq b_p\} \setminus \{f \geq 0, g \geq 0\}) \to 0 \quad \text{as } n \to +\infty.$$

We mention that in [26], the measure is not singular and there is only one sequence of measurable functions but the reader can check that the slightly more general version we gave here can be proven with exactly the same arguments. An immediate consequence of the lemma is the following inequality

$$\limsup_p \nu(\{f_p \geq a_p, g_p \geq b_p\}) \leq \nu(\{f \geq 0, g \geq 0\}).$$

Proof of Lemma 3. Let us first assume that $D\varphi(t,x) \neq 0$. In this case, for (s,y) close to (t,x), $D\varphi(s,y) \neq 0$ and all the integral operators we consider here are well defined (see Lemma 1). Recall next that, for $i = 1,2$, $(\kappa^*)^{i,\delta} = (\kappa^*_+)^{i,\delta} - (\kappa^-_*)^{i,\delta}$. Hence, it is enough to prove that

$$\limsup_p \left\{ (\kappa^*_+)^{1,\delta}[y_p, \varphi(s_p, \cdot)] \right\} \leq (\kappa^*_+)^{1,\delta}[x, \varphi(t, \cdot)],$$

$$\liminf_p \left\{ (\kappa^-_*)^{1,\delta}[y_p, \varphi(s_p, \cdot)] \right\} \geq (\kappa^-_*)^{1,\delta}[x, \varphi(t, \cdot)],$$

$$\limsup_p \left\{ (\kappa^*_+)^{2,\delta}[y_p, D_x\varphi(s_p, y_p), u_p(s_p, \cdot)] \right\} \leq (\kappa^*_+)^{2,\delta}[x, D_x\varphi(t, x), u^*(t, \cdot)],$$

$$\liminf_p \left\{ (\kappa^-_*)^{2,\delta}[y_p, D_x\varphi(s_p, y_p), u_p(s_p, \cdot)] \right\} \leq (\kappa^-_*)^{2,\delta}[x, D_x\varphi(t, x), u^*(t, \cdot)].$$

In order to prove the first inequality above for instance, choose $f_p(z) = \varphi(s_p, y_p + z) - \varphi(t, x)$, $a_p = \varphi(s_p, y_p) - \varphi(t, x)$, $g_p(z) = -D\varphi(s_p, y_p) \cdot z$, $b_p = 0$ in Lemma 4.

We now turn to the case $D\varphi(t,x) = 0$. We look for $\delta = \delta_p$ that goes to 0 as $p \to +\infty$ such that $|D\varphi(s_p, y_p)| \leq C\delta_p$ and

$$(\kappa^*_+)^{1,\delta_p}[y_p, \varphi(s_p, \cdot)]|D\varphi(s_p, y_p)| \to 0 \quad \text{and} \quad (\kappa^-_*)^{1,\delta_p}[y_p, \varphi(s_p, \cdot)]|D\varphi(s_p, y_p)| \to 0$$

as $p \to +\infty$. This is enough to conclude since Condition (8) implies that

$$(\kappa^*_+)^{2,\delta_p}[y_p, D\varphi(s_p, y_p), u(s_p, \cdot)]|D\varphi(s_p, y_p)| \to 0$$
$$(\kappa^-_*)^{2,\delta_p}[y_p, D\varphi(s_p, y_p), u(s_p, \cdot)]|D\varphi(s_p, y_p)| \to 0.$$

We only prove that the first limit equals zero since the argument is similar for the second one. If r_p denotes $|D\varphi(s_p, y_p)|$ and e_p denotes $-r_p^{-1}D\varphi(s_p, y_p)$, and $z_N = e_P \cdot z$ and $z' = z - z_N e_p$, then

$$
\begin{aligned}
(\kappa_+^*)^{1,\delta}[y_p, \varphi(s_p, \cdot)]&|D\varphi(s_p, y_p)| \\
&= r_p \nu(z \in B_{\delta_p} : 0 \le r_p e_p \cdot z \le \varphi(s_p, y_p + z) - \varphi(s_p, y_p) + r_p e_p \cdot z) \\
&\le r_p \nu(z \in B_{\delta_p} : 0 \le r_p z_N \le C|z'|^2 + C z_N^2) \\
&\le r_p \nu(z \in B_{\delta_p} : 0 \le r_p z_N \le C|z'|^2 + C \delta_p z_N)
\end{aligned}
$$

where C is a bound for second derivatives of φ around (t, x). Now if we choose $\delta_p = r_p/(2C)$, we get

$$
\begin{aligned}
(\kappa_+^*)^{1,\delta}[y_p, \varphi(s_p, \cdot)]|D\varphi(s_p, y_p)| &\le r_p \nu(z \in B_{\delta_p} : 0 \le (r_p/2C) z_N \le |z'|^2) \\
&\le r_p \nu(z \in B : 0 \le (r_p/2C) z_N \le |z'|^2)
\end{aligned}
$$

and the last limit in (8) permits now to conclude. $\qquad\square$

3.3 Existence and uniqueness results

Let us first state a strong uniqueness result.

Theorem 2 (Comparison principle)**.** *Assume (A1)-(A4). Assume moreover*

(A3') For all $e \in \mathbb{S}^{N-1}$ and $r \in (0, 1)$

$$
r \, \nu\{z \in B_\delta : r|z \cdot e| \le |z - (z \cdot e)e|^2\} \to 0 \quad \text{as } \delta \to 0 \tag{19}
$$

uniformly in e and $r \in (0, 1)$ and

$$
\nu(dz) = J(z)dz \text{ with } J \in W^{1,1}(\mathbb{R}^N \setminus B_\delta) \text{ for all } \delta > 0. \tag{20}
$$

Consider a bounded and Lipschitz continuous function u_0. Let u (resp. v) be a bounded subsolution (resp. bounded supersolution) of (6). If $u(0, x) \le u_0(x) \le v(0, x)$, then $u \le v$ on $(0, +\infty) \times \mathbb{R}^N$.

The proof is quite classical. The main difficulty is to deal with the singularity of the measure.

Proof of Theorem 2. We classically consider $M = \sup_{t,x}\{u(t, x) - v(t, x)\}$ and argue by contradiction by assuming $M > 0$. We next consider the following approximation of M

$$
\tilde{M}_{\varepsilon,\alpha} = \sup_{t,s>0, x,y \in \mathbb{R}^N} \left\{ u(t, x) - v(s, y) - \frac{(t-s)^2}{2\gamma} - e^{Kt}\frac{|x-y|^2}{2\varepsilon} - \eta t - \alpha|x|^2 \right\}.
$$

Since u and v are bounded, this supremum is attained at a point $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y})$. We first observe that $\tilde{M}_{\varepsilon,\alpha} \geq M/2 \geq 0$ for η and α small enough. Since u and v are bounded, this implies in particular

$$\eta \tilde{t} + e^{K\tilde{t}}\frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon} + \alpha|\tilde{x}|^2 \leq C_0 \tag{21}$$

where $C_0 = \|u\|_\infty + \|v\|_\infty$.

Classical results about penalization imply that $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y}) \to (\bar{t}, \bar{t}, \bar{x}, \bar{y})$ as $\gamma \to 0$ and $(\bar{t}, \bar{t}, \bar{x}, \bar{y})$ realizes the following supremum

$$M_{\varepsilon,\alpha} = \sup_{t>0, x,y\in\mathbb{R}} \{u(t,x) - v(t,y) - e^{Kt}\frac{|x-y|^2}{2\varepsilon} - \eta t - \alpha|x|^2\}.$$

It is also classical [14] to get, for ε and η fixed,

$$\alpha|\bar{x}|^2 \to 0 \text{ as } \alpha \to 0. \tag{22}$$

We claim next that this supremum cannot be achieved at $t = 0$ if $\varepsilon, \alpha, \eta$ are small enough. To see this, remark first that $M_{\varepsilon,\alpha} \geq M/2 \geq 0$ for η and α small enough and, if $\bar{t} = 0$, use the fact that u_0 is Lipschitz continuous and get

$$0 < \frac{M}{2} \leq \sup_{x,y\in\mathbb{R}^N} \{u_0(x) - u_0(y) - \frac{|x-y|^2}{2\varepsilon}\} \leq \sup_{r>0}\{C_0 r - \frac{r^2}{2\varepsilon}\} = \frac{1}{2}C_0^2\varepsilon$$

and this is obviously false if ε is small enough. We conclude that, if the four parameters are small enough, $\tilde{t} > 0$ and $\tilde{s} > 0$.

Hence, we can write two viscosity inequalities. In order to clarify computations, we introduce the function $M(p)$ defined as follows

$$M(p) = \begin{cases} \mu(\hat{p})|p| & \text{if } p \neq 0, \\ 0 & \text{if } p = 0. \end{cases}$$

It is easy to see that M is uniformly continuous and it trivially satisfies

$$|M(p)| \leq \|\mu\|_\infty|p|.$$

In the following, ω_M denotes the modulus of continuity of M.

We now write viscosity inequalities: for all $\delta > 0$,

$$\eta + \frac{\tilde{t} - \tilde{s}}{\gamma} + Ke^{K\tilde{t}}\frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon} \leq \left(c_1(\tilde{x}) + (\kappa^*)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t}, \cdot)] + (\kappa^*)^{2,\delta}[\tilde{x}, \tilde{p} + 2\alpha\tilde{x}, u(\tilde{t}, \cdot)]\right)M(\tilde{p} + 2\alpha\tilde{x})$$

$$\frac{\tilde{t} - \tilde{s}}{\gamma} \geq \left(c_1(\tilde{y}) + (\kappa_*)^{1,\delta}[\tilde{y}, \phi_v(\tilde{s}, \cdot)] + (\kappa_*)^{2,\delta}[\tilde{y}, \tilde{p}, v(\tilde{s}, \cdot)]\right)M(\tilde{p})$$

where $\tilde{p} = e^{K\tilde{t}}\frac{\tilde{x} - \tilde{y}}{\varepsilon}$ and

$$\phi_u(t, x) = v(\tilde{s}, \tilde{y}) + \frac{(t - \tilde{s})^2}{2\gamma} + e^{Kt}\frac{|x - \tilde{y}|^2}{2\varepsilon} + \eta t + \alpha|x|^2,$$

$$\phi_v(s, y) = u(\tilde{t}, \tilde{x}) - \frac{(s - \tilde{t})^2}{2\gamma} - e^{K\tilde{t}}\frac{|y - \tilde{x}|^2}{2\varepsilon} - \eta\tilde{t} - \alpha|\tilde{x}|^2.$$

18

Substracting these inequalities yield

$$\eta + Ke^{K\tilde{t}}\frac{|\tilde{x}-\tilde{y}|^2}{2\varepsilon} \le \|\mu\|_\infty\|Dc_1\|_\infty e^{K\tilde{t}}\frac{|\tilde{x}-\tilde{y}|^2}{\varepsilon} + \|c_1\|_\infty\omega_M(2\sqrt{C_0\alpha}) + T_{nl} \qquad (23)$$

(we used (21)) where

$$\begin{aligned}
T_{nl} &= \left((\kappa^*)^{1,\delta}[\tilde{x},\phi_u(\tilde{t},\cdot)] + (\kappa^*)^{2,\delta}[\tilde{x},\tilde{p}+2\alpha\tilde{x},u(\tilde{t},\cdot)]\right)M(\tilde{p}+2\alpha\tilde{x}) \\
&\quad -\left((\kappa_*)^{1,\delta}[\tilde{y},\phi_v(\tilde{s},\cdot)] + (\kappa_*)^{2,\delta}[\tilde{y},\tilde{p},v(\tilde{s},\cdot)]\right)M(\tilde{p}).
\end{aligned}$$

Our task is now to find $\delta = \delta(\alpha,\varepsilon)$ so that the right hand side of this inequality is small when the four parameters are small. We distinguish two cases.

Assume first that there exists a sequence $\alpha_n \to 0$ and $\varepsilon_n \to 0$ such that

$$\tilde{p} = \tilde{p}_n \to 0.$$

In this case, we simply choose $\delta = 1$, $K = 2\|\mu\|_\infty\|Dc_1\|_\infty$ and we pass to the limit as $n \to +\infty$ in (23) and we get the desired contradiction: $\eta \le 0$.

Assume now that for α and ε small enough, we have a constant C_ε independent of α such that

$$|\tilde{p}| \ge C_\varepsilon > 0. \qquad (24)$$

In this case, the following technical lemma holds true.

Lemma 5. *By using (19), we have*

$$T_{nl} \le \frac{1}{\varepsilon}o_\delta(1) + \frac{1}{\delta}\omega_M(2\sqrt{C_0\alpha}) + o_\alpha(1)[\varepsilon] + C_\delta e^{K\tilde{t}}\frac{|\tilde{x}-\tilde{y}|^2}{\varepsilon}$$

where C_0 appears in (21) and C_δ only depends on J, $\|\mu\|_\infty$ and δ (we emphasize that the third term goes to 0 as $\alpha \to 0$ for fixed ε).

The proof of this lemma is postponed. We thus get (recall that $\tilde{p} = e^{K\tilde{t}}(\tilde{x}-\tilde{y})/\varepsilon$)

$$\eta + Ke^{K\tilde{t}}\frac{|\tilde{x}-\tilde{y}|^2}{2\varepsilon} \le Ce^{K\tilde{t}}\frac{|\tilde{x}-\tilde{y}|^2}{\varepsilon} + C(1+\frac{1}{\delta})\omega_M(2\sqrt{C_0\alpha}) + \frac{1}{\varepsilon}o_\delta(1) + o_\alpha(1)[\varepsilon] + C_\delta e^{K\tilde{t}}\frac{|\tilde{x}-\tilde{y}|^2}{\varepsilon}$$

where C only depends on c_1, ν and $\|u\|_\infty + \|v\|_\infty$ and C_δ is given by the lemma. By choosing $K = 2(C + C_\delta)$, we get

$$\eta \le C(1+\frac{1}{\delta})\omega_M(2\sqrt{C_0\alpha}) + \frac{1}{\varepsilon}o_\delta(1) + o_\alpha(1)[\varepsilon].$$

By letting successively α and δ go to 0, we thus get a contradiction. This achieves the proof of the comparison principle. $\qquad\square$

Proof of Lemma 5. We first write

$$
\begin{aligned}
T_{nl} \;\leq\; & \|\mu\|_\infty |(\kappa^*)^{1,\delta}|[\tilde{x}, \phi_u(\tilde{t}, \cdot)]|\tilde{p} + 2\alpha\tilde{x}| + \|\mu\|_\infty |(\kappa_*)^{1,\delta}|[\tilde{y}, \phi_v(\tilde{s}, \cdot)]|\tilde{p}| \\
& + |(\kappa^*)^{2,\delta}|[\tilde{x}, \tilde{p} + 2\alpha\tilde{x}, u(\tilde{t}, \cdot)]\omega_M(|2\alpha\tilde{x}|) \\
& + \left((\kappa^*)^{2,\delta}[\tilde{x}, \tilde{p} + 2\alpha\tilde{x}, u(\tilde{t}, \cdot)] - (\kappa_*)^{2,\delta}[\tilde{y}, \tilde{p}, v(\tilde{s}, \cdot)] \right) M(\tilde{p}) \,.
\end{aligned}
$$

We thus estimate the right hand side of the previous inequality. We start with the first two integral terms.

$$
\begin{aligned}
|(\kappa^*)^{1,\delta}|[\tilde{x}, \phi_u(\tilde{t}, \cdot)] \;\leq\; & (\kappa_+^*)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t}, \cdot)] + (\kappa_*^-)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t}, \cdot)] \\
\;\leq\; & \nu(z \in B_\delta : 0 \leq -(\tilde{p} + 2\alpha\tilde{x}) \cdot z \leq (\alpha + e^{K\tilde{t}}/(2\varepsilon))|z|^2) \\
& + \nu(z \in B_\delta : 0 > -(\tilde{p} + 2\alpha\tilde{x}) \cdot z > (\alpha + e^{K\tilde{t}}/(2\varepsilon))|z|^2) \\
\;\leq\; & \nu(z \in B_\delta : |\varepsilon\tilde{p} + 2\varepsilon\alpha\tilde{x}||e \cdot z| \leq C(\eta)|z|^2)
\end{aligned}
$$

where we use (21) to ensure, for α, ε small enough,

$$
\frac{e^{K\tilde{t}}}{2} + \alpha\varepsilon \leq \frac{1}{2}e^{KC_0/\eta} + 1 := C(\eta)\,.
$$

If now $r_{\alpha,\varepsilon}$ denotes $|\varepsilon\tilde{p} + 2\varepsilon\alpha\tilde{x}|$ and we choose $\delta \leq r_{\alpha,\varepsilon}/(2C(\eta))$, we use (19) to write

$$
\begin{aligned}
|(\kappa^*)^{1,\delta}|[\tilde{x}, \phi_u(\tilde{t}, \cdot)]|\tilde{p} + 2\alpha\tilde{x}| \;=\; & \frac{1}{\varepsilon}r_{\alpha,\varepsilon}\nu(z \in B_\delta : r_{\alpha,\varepsilon}|e \cdot z| \leq C(\eta)|z|^2) \\
\;\leq\; & \frac{1}{\varepsilon}r_{\alpha,\varepsilon}\nu(z \in B_\delta : \frac{1}{2}r_{\alpha,\varepsilon}|e \cdot z| \leq C(\eta)|z - (e \cdot z)e|^2) \\
\;\leq\; & \frac{2C(\eta)}{\varepsilon}\left\{ \sup_{e \in \mathbb{S}^{N-1}, r \in (0,1)} r\nu(z \in B_\delta : r|e \cdot z| \leq |z - (e \cdot z)e|^2) \right\} \\
\;=\; & \frac{1}{\varepsilon}o_\delta(1)\,.
\end{aligned}
$$

Since $\alpha\tilde{x} \to 0$ (see (21)), we choose for instance

$$
\delta \leq \frac{\varepsilon|\tilde{p}|}{4C(\eta)}\,.
$$

Arguing similarly, we get for $\delta \leq \frac{\varepsilon|\tilde{p}|}{4C(\eta)}$,

$$
|(\kappa_*)^{1,\delta}|[\tilde{y}, \phi_v(\tilde{s}, \cdot)]|\tilde{p}| \leq \frac{1}{\varepsilon}o_\delta(1)\,.
$$

As far as the third integral term is concerned, we simply write

$$
|(\kappa^*)^{2,\delta}|[\tilde{x}, u(\tilde{t}, \cdot)]\omega_M(|2\alpha\tilde{x}|) \leq \nu(B_\delta^c)\omega_M(|2\alpha\tilde{x}|) \leq \frac{1}{\delta}\omega_M(2\sqrt{C_0}\alpha)
$$

20

(we used (21)). We now turn to the last two integral terms. In view of (16), we can write

$$
\begin{aligned}
\tilde{T}_{nl} &= (\kappa^*)^{2,\delta}[\tilde{x}, \tilde{p} + 2\alpha\tilde{x}, u(\tilde{t}, \cdot)] - (\kappa_*)^{2,\delta}[\tilde{y}, \tilde{p}, v(\tilde{s}, \cdot)] \\
&= \nu(z \notin B_\delta : u(\tilde{t}, \tilde{x} + z) \geq u(\tilde{t}, \tilde{x})) - \nu(z \notin B_\delta : v(\tilde{s}, \tilde{y} + z) > v(\tilde{s}, \tilde{y})) \\
&\quad -\nu(z \notin B_\delta : (\tilde{p} + 2\alpha\tilde{x}) \cdot z > 0) + \nu(z \notin B_\delta : \tilde{p} \cdot z \geq 0) \, .
\end{aligned}
$$

Now, we use (20) to get

$$
\tilde{T}_{nl} = \int_{B_\delta^c} J(z - \tilde{x}) \mathbf{1}_{\{u(\tilde{t}, \cdot) > u(\tilde{t}, \tilde{x})\}}(z) dz - \int_{B_\delta^c} J(z - \tilde{y}) \mathbf{1}_{\{v(\tilde{s}, \cdot) > u(\tilde{s}, \tilde{y})\}}(z) dz + o_\alpha(1)[\varepsilon] \, .
$$

Remark next that the definition of $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y})$ implies the following inequality: for all $z \in \mathbb{R}^N$,

$$
u(\tilde{t}, z) - u(\tilde{t}, \tilde{x}) \leq v(\tilde{s}, z) - v(\tilde{s}, \tilde{y}) + \alpha(|z|^2 - |\tilde{x}|^2) - e^{K\tilde{t}} \frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon} \, .
$$

This implies that for $|z| \leq R_{\alpha,\varepsilon}$, we have

$$
\mathbf{1}_{\{u(\tilde{t}, \cdot) > u(\tilde{t}, \tilde{x})\}}(z) \leq \mathbf{1}_{\{v(\tilde{s}, \cdot) > u(\tilde{s}, \tilde{y})\}}(z)
$$

where

$$
\begin{aligned}
R_{\alpha,\varepsilon}^2 &= \frac{1}{\alpha} \left(\alpha |\tilde{x}|^2 + e^{K\tilde{t}} \frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon} \right) \\
&= \frac{1}{\alpha} \left(o_\alpha(1) + \frac{\varepsilon C_\varepsilon^2}{4C(\eta)} \right) \\
&\geq \frac{\varepsilon C_\varepsilon^2}{8C(\eta)\alpha}
\end{aligned}
$$

where C_ε appears in (24). We used here (22). Hence, we have

$$
\begin{aligned}
\tilde{T}_{nl} &\leq \int_{|z| \geq R_{\alpha,\varepsilon}} J(z - \tilde{x}) dz + \int_{z \in B_\delta^c} |J(z - \tilde{x}) - J(z - \tilde{y})| dz \\
&\leq \int_{|\tilde{z}| \geq \frac{\sqrt{\varepsilon} C_\varepsilon}{2\sqrt{8C(\eta)\alpha}}} J(\tilde{z}) d\tilde{z} + C_\delta |\tilde{x} - \tilde{y}| = o_\alpha(1)[\varepsilon] + C_\delta |\tilde{x} - \tilde{y}|
\end{aligned}
$$

where we used once again (22). It is now easy to conclude. $\qquad \square$

We now turn to the existence result.

Theorem 3 (Existence). *Assume (A1)-(A4) and (A3'). There then exists a unique bounded uniformly continuous viscosity solution u of (6), (7).*

Proof. We first construct a solution for regular initial data. Precisely, we first assume that $u_0 \in C_b^2(\mathbb{R}^N)$ (the function and its first and second derivatives are bounded).

Because we can apply Perron's method, it is enough to construct a sub- and a supersolution $u^\pm$ to (6) such that $(u^+)_*(0,x) = (u^-)^*(0,x) = u_0(x)$. We assert that $u^\pm(t,x) = u_0(x) \pm Ct$ are respectively a super- and a subsolution of (6) for C large enough. To see this, we first prove that there exists $C_0 = C_0(\|D^2 u_0\|_\infty)$ such that for all $x \in \mathbb{R}^N$ such that $Du_0(x) \neq 0$, we have

$$(|\kappa_*|[x, u_0] + |\kappa^*|[x, u_0])|Du_0(x)| \leq C_0. \tag{25}$$

In order to prove this estimate, we simply write for x such that $Du_0(x) \neq 0$

$$((\kappa_+^*)^{1,\delta}[x, u_0] + (\kappa_-^*)^{1,\delta}[x, u_0])|Du_0(x)| \leq 2\nu(z \in B_\delta : r|e \cdot z| \leq \frac{1}{2}\|D^2 u_0\|_\infty |z|^2)r$$

$$\leq 2\nu(z \in B_\delta : r|e \cdot z| \leq C|z - (e \cdot z)e|^2)r \leq C_\nu$$

where $r = |Du_0(x)|$, $C = \max(\|Du_0\|_\infty, 1)$ and $e = Du_0(x)/r$ and $\delta = r/(2C)$ and C_ν is given by (8). On the other hand

$$((\kappa_+^*)^{2,\delta}[x, u_0] + (\kappa_-^*)^{2,\delta}[x, u_0])|Du_0(x)| \leq \frac{C_\nu}{\delta}r = 2C_\nu C.$$

We thus get Estimate (25).

If now u_0 is not regular, we approximate it with $u_0^n \in C_b^2(\mathbb{R}^N)$ and can prove that the corresponding sequence of solutions u_n converges locally uniformly towards a solution u. Since this is very classical, we omit details (see for instance [1]). $\square$

We now explain in which limit one recovers the mean curvature flow. To do so, we state two convergence results. Their proofs rely on Propositions 1 and 2. The first one (Theorem 4) appears in [15] and the second one can be proved by using Proposition 2.

Theorem 4 ([15]). *Assume that $\mu \equiv 1$, $c_1 \equiv 0$, u_0 is Lipschitz continuous and bounded and*

$$\nu(dz) = \nu^\varepsilon(dz) = \frac{1}{\varepsilon^{N+1}|\ln \varepsilon|} c_0\left(\frac{z}{\varepsilon}\right) dz$$

with c_0 even, smooth, non-negative and such that $c_0(z) = |z|^{-N-1}$ for $|z| \geq 1$. Then the viscosity solution u^ε of (6), (7) converges locally uniformly as $\varepsilon \to 0$ towards the viscosity solution u of

$$\partial_t u = C|Du| \operatorname{div}\left(\frac{Du}{|Du|}\right)$$

(C is a positive constant) supplemented with the initial condition (7).

Theorem 5. *Assume that $\mu \equiv 1$, $c_1 \equiv 0$, u_0 is Lipschitz continuous and bounded and*

$$\nu(dz) = \nu^\alpha(dz) = (1 - \alpha)\frac{dz}{|z|^{N+\alpha}}$$

with $\alpha \in (0,1)$. Then the viscosity solution u^α of (6), (7) converges locally uniformly as $\alpha \to 1$ towards the viscosity solution u of

$$\partial_t u = C|Du| \operatorname{div}\left(\frac{Du}{|Du|}\right)$$

(C is a positive constant) supplemented with the initial condition (7).

4   The level set approach

In the previous section, we constructed a unique solution of (6) in the case of singular measures satisfying (A3) and (A3') and for bounded Lipschitz continuous initial data (see (A4)). In the present section, we explain how to define a geometric flow by using these solutions of (6). Precisely, we first prove (Theorem 6) that if u and v are solutions of (6) associated with two different initial data u_0 and v_0 that have the same zero level sets, then so have u and v. Hence, the geometric flows is obtained by considering the zero level sets of the solution u of (6) for any (Lipschitz continuous) initial datum. We also describe (Theorem 7) the maximal and minimal discontinuous solutions of (6) associated with an important class of discontinuous initial data.

Theorem 6 (Consistency of the definition). *Assume (A1)-(A3) and (A3'). Let u_0 and v_0 be two bounded Lipschitz continuous functions and consider the viscosity solutions u, v associated with these initial conditions. If*

$$\begin{aligned}
\{x \in \mathbb{R}^N : u_0(x) > 0\} &= \{x \in \mathbb{R}^N : v_0(x) > 0\} \\
\{x \in \mathbb{R}^N : u_0(x) < 0\} &= \{x \in \mathbb{R}^N : v_0(x) < 0\}
\end{aligned}$$

then, for all time $t > 0$,

$$\begin{aligned}
\{x \in \mathbb{R}^N : u(t,x) > 0\} &= \{x \in \mathbb{R}^N : v(t,x) > 0\} \\
\{x \in \mathbb{R}^N : u(t,x) < 0\} &= \{x \in \mathbb{R}^N : v(t,x) < 0\}
\end{aligned}$$

In view of the techniques used to prove the consistency of the definition of local geometric fronts (see for instance [8]), it is clear that this result is a straightforward consequence of the following proposition.

Proposition 3 (Equation (6) is geometric). *Consider $u : [0, +\infty) \times \mathbb{R}^N$ a bounded subsolution of (6) and $\theta : \mathbb{R} \to \mathbb{R}$ a upper semi-continuous non-decreasing function. Then $\theta(u)$ is also a subsolution of (6).*

Such a proposition is classical by now. It is proved by regularizing θ (in a proper way) with a strictly increasing function θ^n, by remarking that $\kappa^*[x, \theta^n(u)] = \kappa^*[x, u]$ in this case, and by using discontinuous stability. Details are left to the reader.

Thanks to Theorem 6, we can define a geometric flow in the following way. Given (Γ_0, D_0^+, D_0^-) such that Γ_0 is closed, $D_0^\pm$ are open and $\mathbb{R}^N = \Gamma_0 \sqcup D_0^+ \sqcup D_0^-$, we can write

$$D_0^+ = \{x \in \mathbb{R}^N : u_0(x) > 0\}, D_0^- = \{x \in \mathbb{R}^N : u_0(x) < 0\}, \Gamma_0 = \{x \in \mathbb{R}^N : u_0(x) = 0\}$$

for some bounded Lipschitz continuous function u_0 (for instance the signed distance function). If u is the solution of (6) submitted to the initial condition $u(0, x) = u_0(x)$ for $x \in \mathbb{R}^N$, then Theorem 6 precisely says that the sets

$$D_t^+ = \{x \in \mathbb{R}^N : u(t, x) > 0\}, D_t^- = \{x \in \mathbb{R}^N : u(t, x) < 0\}, \Gamma_t = \{x \in \mathbb{R}^N : u(t, x) = 0\}$$

does not depend on the choice of u_0.

The next theorem claims that there exists a maximal subsolution minimal supersolution of (6) associated with the apropriate discontinuous initial data.

Theorem 7 (Maximal subsolution and minimal supersolution). *Assume (A1)-(A3) and (A3'). Then the function $\mathbf{1}_{D_t^+ \cup \Gamma_t} - \mathbf{1}_{D_t^-}$ (resp. $\mathbf{1}_{D_t^+} - \mathbf{1}_{D_t^- \cup \Gamma_t}$) is the maximal subsolution (resp. minimal supersolution) of (6) submitted to the initial datum $\mathbf{1}_{D_0^+ \cup \Gamma_0} - \mathbf{1}_{D_0^-}$ (resp. $\mathbf{1}_{D_0^+} - \mathbf{1}_{D_0^- \cup \Gamma_0}$).*

This result is a consequence of Proposition 3 together with discontinuous stability and the comparison principle. See [8, p. 445] for details.

We conclude this section by showing that a bounded front propagates with finite speed.

Proposition 4 (Evolution of bounded sets). *Assume (A1)-(A3) and (A3'). Let Ω_0 be a bounded open set of $\mathbb{R}^N$: there exists $R > 0$ such that $\Omega_0 \subset B_R$. Then the level set evolution (Γ_t, D_t^+, D_t^-) of $(\partial \Omega_0, \Omega_0, (\bar{\Omega}_0)^c)$ satisfies $D_t^+ \cup \Gamma_t \subset \bar{B}_{R+Ct}$ with*

$$C = \|c_1\|_\infty - \inf_{e \in \mathbb{S}^{N-1}} \nu(z \in \mathbb{R}^N : 0 \leq e \cdot z \leq |z|^2)$$

as long as $R + Ct > 0$.

Remark 8. Another consequence of this proposition is that, if there are no driving force $(c_1 = 0)$, then the set shrinks till it disappears.

Proof. The proof consists in constructing a supersolution of (6), (7). It is easy to check that C is chosen such that

$$u(t, x) = Ct + \sqrt{\varepsilon^2 + R^2} - \sqrt{\varepsilon^2 + |x|^2}$$

is a supersolution of (6). Since $\bar{B}_R = \{x \in \mathbb{R}^N : u(0, x) \geq 0\}$, we conclude that $D_t^+ \cup \Gamma_t \subset \{x \in \mathbb{R}^N : u(t, x) \geq 0\} = \bar{B}_{R^\varepsilon(t)}$ with $R^\varepsilon(t) = \sqrt{(Ct + \sqrt{\varepsilon^2 + R^2})^2 - \varepsilon^2}$. Hence, $D_t^+ \cup \Gamma_t \subset \cap_{\varepsilon > 0} \bar{B}_{R^\varepsilon(t)} = \bar{B}_{R+Ct}$. $\qquad\square$

5   Generalized flows

In this section, we follow [9] and give an equivalent definition of the flow by, freely speaking, replacing smooth test functions with smooth test fronts.

In order to give this equivalent definition, we use the geometrical non-linearities we partially introduced in Section 2. For all $x, p \in \mathbb{R}^N$ and all closed set $\mathcal{F} \subset \mathbb{R}^N$ and open set $\mathcal{O} \subset \mathbb{R}^N$

$$F_*(x, p, \mathcal{F}) = \begin{cases} -\mu(\hat{p})\Big[c_1(x) + \nu(\mathcal{F} \cap \{p \cdot z \le 0\}) - \nu(\mathcal{F}^c \cap \{p \cdot z > 0\})\Big]|p| & \text{if } p \ne 0, \\ 0 & \text{if not}, \end{cases}$$

$$F^*(x, p, \mathcal{O}) = \begin{cases} -\mu(\hat{p})\Big[c_1(x) + \nu(\mathcal{O} \cap \{p \cdot z < 0\}) - \nu(\mathcal{O}^c \cap \{p \cdot z \ge 0\})\Big]|p| & \text{if } p \ne 0, \\ 0 & \text{if not}. \end{cases}$$

We can now give the definition of a generalized flow.

Definition 3 (Generalized flows). *The family $(\mathcal{O}_t)_{t \in (0,T)}$ of open subsets of $\mathbb{R}^N$ (resp. $(\mathcal{F}_t)_{t \in (0,T)}$ of closed subsets of $\mathbb{R}^N$) is a* generalized super-flow *(resp.* sub-flow*) of (6) if for all $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}^N$, $r > 0$, $h > 0$, and for all smooth function $\phi : (0; +\infty) \times \mathbb{R}^N \to \mathbb{R}$ such that*

1. *$\partial_t \phi + F^*(x, D\phi, \{z : \phi(t, x+z) > \phi(t,x)\}) \le -\delta_\phi$ in $[t_0, t_0 + h] \times \bar{B}(x_0, r)$*
   (resp. $\partial_t \phi + F_*(x, D\phi, \{z : \phi(t, x+z) \ge \phi(t,x)\}) \ge -\delta_\phi$ in $[t_0, t_0 + h] \times \bar{B}(x_0, r)$)

2. *$D\phi \ne 0$ in $\{(s,y) \in [t_0, t_0 + h] \times \bar{B}(x_0, r) : \phi(s,y) = 0\}$,*

3. *$\{y \in \mathbb{R}^N : \phi(t_0, y) \ge 0\} \subset \mathcal{O}_{t_0}^1$,*
   (resp. $\{y \in \mathbb{R}^N : \phi(t_0, y) \le 0\} \subset \mathbb{R}^N \setminus \mathcal{F}_{t_0}$),

4. *$\{y \notin \bar{B}(x_0, r) : \phi(s,y) \ge 0\} \subset \mathcal{O}_s^1$ for all $s \in [t_0, t_0 + h]$,*
   (resp. $\{y \notin \bar{B}(x_0, r) : \phi(s,y) \le 0\} \subset \mathbb{R}^N \setminus \mathcal{F}_s$ for all $s \in [t_0, t_0 + h]$),

then $\{y \in \bar{B}(x_0, r) : \phi(t_0 + h, y) > 0\} \subset \mathcal{O}_{t_0+h}^1$ (resp. $\{y \in \bar{B}(x_0, r) : \phi(t_0 + h, y) < 0\} \subset \mathbb{R}^N \setminus \mathcal{F}_{t_0+h}$).

Loosely speaking about generalized super-flows, Condition 1 says that in a prescribed neighbourhood $\mathcal{V}$ around (t_0, x_0), the normal velocity of the test front $\{\phi > 0\}$ is strictly smaller than the one of the front $\mathcal{O}$; Condition 2 asserts that the front $\{\phi = 0\}$ is smooth in $\mathcal{V}$; Conditions 3 and 4 assert that the test front is inside the front $\mathcal{O}$ outside $\mathcal{V}$. The conclusion is that the test front is inside the neighbourhood $\mathcal{O}$ at time $t + h$.

Remark 9. As far as local geometric fronts are concerned, Conditions 3 and 4 impose that the test front is inside $\mathcal{O}$ on the parabolic boundary of the neighbourhood. Here, because the front is not local, the test front has to be inside $\mathcal{O}$ everywhere outside the neighbourhood.

The next theorem asserts that Definition 3 of the flow coincides with the level set formulation of Section 4.

Theorem 8 (Generalized flows and level set approach). *Assume (A1)-(A3) and (A3'). Let $(\mathcal{O}_t)_{t\in(0,T)}$ be a family of open subsets of $\mathbb{R}^N$ (resp. $(\mathcal{F}_t)_{t\in(0,T)}$ of closed subsets of $\mathbb{R}^N$) such that the set $\cup_{t\in(0,T)}\{t\}\times\mathcal{O}_t$ is open in $[0,T]\times\mathbb{R}^N$ (resp. $\cup_{t\in(0,T)}\{t\}\times\mathcal{F}_t$ is closed in $[0,T]\times\mathbb{R}^N$).*

Then $(\mathcal{O}_t)_{t\in(0,T)}$ (resp. $(\mathcal{F}_t)_{t\in(0,T)}$) is a generalized super-flow (resp. sub-flow) of (6) if and only if $\chi(t,x)=\mathbf{1}_{\mathcal{O}_t}(x)-\mathbf{1}_{\mathbb{R}^N\setminus\mathcal{O}_t}(x)$ (resp. $\chi(t,x)=\mathbf{1}_{\mathcal{F}_t}(x)-\mathbf{1}_{\mathbb{R}^N\setminus\mathcal{F}_t}(x)$) is a viscosity supersolution (resp. subsolution) of (6), (7).

Since the proof of [9] can be readily adapted, we omit it. We give a straightforward corollary of Theorems 7 and 8 that is used in [21].

Corollary 1 (Abstract method). *Assume (A1)-(A3) and (A3'). Assume that $(\mathcal{O}_t)_t$ and $(\mathcal{F}_t)_t$ are respectively a generalized super-flow and generalized sub-flow and suppose there exists two open sets D_0^+, D_0^- such that $\mathbb{R}^N = \partial\mathcal{O}_0 \sqcup D_0^+ \sqcup D_0^-$ and such that $D_0^+ \subset \mathcal{O}_0$ and $D_0^- \subset \mathcal{F}_0^c$. Then if (Γ_t, D_t^+, D_t^-) denotes the level set evolution of $(\partial\mathcal{O}_0, D_0^+, D_0^-)$, we have for all time $t>0$*

$$D_t^+ \subset \mathcal{O}_t \subset D_t^+ \cup \Gamma_t, \quad D_t^- \subset \mathcal{F}_t^c \subset D_t^- \subset \Gamma_t.$$

Remark 10. One can check that under the assumptions of the previous corollary, we have in fact $D_0^+ = \mathcal{O}_0$ and $D_0^- = \mathcal{F}_0^c$.

References

[1] N. ALIBAUD AND C. IMBERT, *Fractional semi-linear parabolic equations with unbounded data*, Trans. of the AMS, 361 (2009), pp. 2527–2566.

[2] O. ALVAREZ, P. CARDALIAGUET, AND R. MONNEAU, *Existence and uniqueness for dislocation dynamics with nonnegative velocity*, Interfaces Free Bound., 7 (2005), pp. 415–434.

[3] O. ALVAREZ, P. HOCH, Y. LE BOUAR, AND R. MONNEAU, *Dislocation dynamics: short-time existence and uniqueness of the solution*, Arch. Ration. Mech. Anal., 181 (2006), pp. 449–504.

[4] D. APPLEBAUM, *Lévy processes and stochastic calculus*, vol. 93 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2004.

[5] G. BARLES, P. CARDALIAGUET, O. LEY, AND R. MONNEAU, *Global existence results and uniqueness for dislocation equations*, SIAM J. Math. Anal., 40 (2008), pp. 44–69.

[6] G. Barles, P. Cardaliaguet, O. Ley, and A. Monteillet, *Uniqueness results for nonlocal hamilton-jacobi equations.* Submitted, 2008.

[7] G. Barles and C. Imbert, *Second-order elliptic integro-differential equations: Viscosity solutions' theory revisited*, Annales de l'Institut Henri Poincaré, Analyse non linéaire, 25 (2008), pp. 567–585.

[8] G. Barles, H. M. Soner, and P. E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control Optim., 31 (1993), pp. 439–469.

[9] G. Barles and P. E. Souganidis, *A new approach to front propagation problems: theory and applications*, Arch. Rational Mech. Anal., 141 (1998), pp. 237–296.

[10] L. Caffarelli, J.-M. Roquejoffre, and O. Savin, *Personal communication*, 2008.

[11] L. Caffarelli and P. E. Souganidis, *Convergence of nonlocal threshold dynamics approximations to front propagation*, Arch. Rational Mech. Anal., (2008). to appear.

[12] P. Cardaliaguet, *Front propagation problems with nonlocal terms. II*, J. Math. Anal. Appl., 260 (2001), pp. 572–601.

[13] Y. G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom., 33 (1991), pp. 749–786.

[14] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), 27 (1992), pp. 1–67.

[15] F. Da Lio, N. Forcadel, and R. Monneau, *Convergence of a non-local eikonal equation to anisotropic mean curvature motion. Application to dislocations dynamics*, J. Eur. Math. Soc, (2006). To appear.

[16] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature. I*, J. Differential Geom., 33 (1991), pp. 635–681.

[17] N. Forcadel, C. Imbert, and R. Monneau, *Homogenization of some particle systems with two-body interactions and of the dislocation dynamics*, Discrete Contin. Dyn. Syst., (2008). To appear.

[18] N. Forcadel and R. Monneau, *Existence of solutions for a model describing the dynamics of junctions between dislocations*, 2008. Submitted.

[19] C. Imbert, R. Monneau, and E. Rouy, *Homognization of first order equations with u/ε-periodic hamiltonians. part II: application to dislocation dynamics*, Comm. in PDEs, 33 (2008), pp. 479 – 516.

[20] C. IMBERT AND S. SERFATY, *Repeated games for eikonal equations, dislocation dynamics and parabolic integro-differential equations*, 2008. In preparation.

[21] C. IMBERT AND P. E. SOUGANIDIS, *Phasefield theory for fractional reaction-diffusion equations and applications*, 2008. In preparation.

[22] R. KOHN AND S. SERFATY, *A deterministic-control-based approach to fully non-linear parabolic and elliptic equations*, tech. rep., Courant Institute, New York University, 2007.

[23] R. V. KOHN AND S. SERFATY, *A deterministic-control-based approach to motion by curvature*, Comm. Pure Appl. Math., 59 (2006), pp. 344–407.

[24] S. OSHER AND J. A. SETHIAN, *Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations*, J. Comput. Phys., 79 (1988), pp. 12–49.

[25] A. SAYAH, *Équations d'Hamilton-Jacobi du premier ordre avec termes intégro-différentiels. I. Unicité des solutions de viscosité*, Comm. Partial Differential Equations, 16 (1991), pp. 1057–1074.

[26] D. SLEPČEV, *Approximation schemes for propagation of fronts with nonlocal velocities and Neumann boundary conditions*, Nonlinear Anal., 52 (2003), pp. 79–115.