



HAL
open science

Fractional mean curvature flows

Cyril Imbert

► **To cite this version:**

| Cyril Imbert. Fractional mean curvature flows. 2008. hal-00262386v2

HAL Id: hal-00262386

<https://hal.science/hal-00262386v2>

Preprint submitted on 1 Apr 2008 (v2), last revised 12 Mar 2009 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Fractional mean curvature flows

Cyril Imbert¹

April 1, 2008

ABSTRACT. This paper is concerned with the study of a geometric flow whose law involves a singular integral operator. Such an operator is used to define a non-local mean curvature of a set and for this reason, the flow is referred to as fractional. Such a flow appears in two important applications: dislocation dynamics and phasefield theory for fractional reaction-diffusion equations. It is first defined by using the level-set method. It is proved that it can be also defined in terms of generalized flows (Barles, Souganidis, 1998) so that phasefield theory for fractional reaction-diffusion can be treated (see the working paper of the author and Souganidis).

Keywords: fractional mean curvature flows, singular dislocation dynamics, level-set approach, stability results, comparison principles

Mathematics Subject Classification: 35F25, 35K55, 35K65, 49L25, 35A05

1 Introduction

In this paper, we define a geometric flow whose law is non-local. A general geometric flow of a set Ω is defined by imposing the velocity of a point x along its outer normal $n(x)$. In our case, this velocity does not only depend on x and n but also on a *fractional mean curvature* at x . Our motivation comes from two different problems: dislocation dynamics and phasefield theory for fractional reaction diffusion equations.

1.1 Motivation and existing results

Mathematical study of non-local moving fronts recently attracted a lot of attention (see in particular [8] and references therein). An important application is the study of dislocation dynamics [3].

Dislocation dynamics

Dislocations are linear defects in crystals and the study of their motion gives rise to the study of the non-local geometric flow. In recent years, several papers were dedicated to this problem; we make next a short review.

A dislocation creates in the whole space \mathbb{R}^3 an elastic field and this field creates a force (called the Peach-Koehler force) that acts not only on the dislocation that created

¹Université Paris-Dauphine, CEREMADE, place de Lattre de Tassigny, 75775 Paris cedex 16, France, imberty@ceremade.dauphine.fr

it (self-force) but also on all dislocations in the material. We restrict ourselves here to the case of a single curve moving in a plane (called the slip plane).

In [3], the level-set approach is used to describe the dynamics of a dislocation Γ_t , $t > 0$. If Γ_t is the 0-level set of a function $u(t, \cdot)$, the following eikonal equation is obtained

$$\partial_t u = \kappa[x, u] |Du|$$

$\kappa[x, u]$ is the Peach-Koehler force applied to the curve ($N = 2$ in this application). The Peach-Koehler force depends on the curve through the resolution of an elliptic equation (from linear elasticity). Hence, this integral operator is a Dirichlet-to-Neumann operator associated with an elliptic equation and is thus singular. In order to define solutions for small times, the authors of [3] consider a physically relevant regularized problem and $\kappa[x, u]$ reduces to

$$\int_{\{z:u(z)\geq 0\}} c_0(z) dz$$

with $c_0 \in W^{1,1}(\mathbb{R}^N)$.

The major technical difficulty of this first paper is that c_0 does not have a constant sign and consequently, solutions corresponding to ordered initial data are not ordered; in other words, comparison principle does not hold true. In particular, this is one of the reasons why solutions are constructed for small times. If c_1 is assumed to be large enough, Alvarez, Cardaliaguet and Monneau [2] managed to prove the existence and uniqueness for large times.

A second important remark is that solving such parabolic integro-differential equations does not permit to construct properly a geometric flow. More precisely, if the initial front Γ_0 is described with two different initial functions u_0 and v_0 , it is not sure that the 0-level sets of the corresponding solutions u and v coincide.

The difficulty related to comparison principle is circumvented in [12] by assuming that the negative part of c_0 is concentrated at the origin. We point out that this is reasonable if one keeps in mind that the singular operator is a Dirichlet-to-Neuman operator. The Peach-Koehler force $\kappa[x, u]$ (in the case of a curve moving alone) is defined in [12] as

$$\int \text{sign}(u(x+z) - u(x)) c_0(z) dz$$

where $\text{sign}(\alpha)$ equals 1 if $\alpha \geq 0$ and -1 if $\alpha < 0$. After an approximation procedure, the problem can be reduced to the study of

$$\partial_t U = \left[c_1(x) + \int (U(x+z) - U(x)) c_0(z) dz \right] |DU|$$

where c_0 is smooth, non-negative and of finite mass. We used the letter U instead of u in order to emphasize the fact that a change of unknown function is needed in order to reduce the original equation to this new one.

Finally, still assuming that the negative part of c_0 is concentrated at the origin, a good geometric definition of the flow is obtained in [10] by regularizing the Green function of

the Dirichlet-to-Neumann operator. This is achieved by considering a formulation “à la Slepčev” of the geometric flow. The equation now reads

$$\partial_t u = \left[c_1(x) + \int_{\{z: u(t, x+z) > u(t, x)\}} c_0(z) dz \right] |Du|. \quad (1)$$

In this approach, we cannot deal with singular potentials c_0 .

Notice that in that paper, several fronts move, and they are interacting. The motion of a single front is a special case. Eventually, [4] gave existence results of very weak solutions in a very general setting; in particular, uniqueness is generally lost.

In [9], it is proved that if $c_0(z)$ is smooth and regular near the origin and behaves exactly like $|z|^{-N-1}$ at infinity, then a proper rescaling of (1) converges towards the mean curvature motion.

Phasefield theory for fractional reaction-diffusion equations

Our second main motivation comes from phasefield theory of fractional reaction-diffusion equations [14]. If one considers for instance stochastic Ising models with Kacs potentials with very slow decay at infinity (like a power law with proper exponent), then the study of the resulting mean field equation (after proper rescaling) is closely related to phasefield theory for fractional reaction-diffusion equations such as

$$\partial_t u^\varepsilon + (-\Delta)^{\alpha/2} u^\varepsilon + \frac{1}{\varepsilon^{1+\alpha}} f(u^\varepsilon) = 0$$

where $(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian with $\alpha \in (0, 1)$ (in the case presented here) and f is a bistable non-linearity. In particular, it is *essential* in the analysis to deal with singular potentials. Indeed, we have to be able to treat the case where

$$c_0(z) = \frac{1}{|z|^{N+\alpha}}$$

with $\alpha \in (0, 1)$. It is also convenient to use the notion of generalized flows introduced by Barles and Souganidis [7] in order to develop a phasefield theory for such reaction-diffusion equations. See [14] for further details and [11] for analogous problems.

1.2 A new formulation

The main contributions of this paper are the following:

- to give a proper level-set formulation of dislocation dynamics for singular interaction potentials: in particular, sufficient conditions on the singularity to get stability results and comparison principles are exhibited;
- to shed light on the fact that the integral operator measures in a non-local way the curvature of the interface;

- to study the geometric flow in details: consistency of the definition, equivalent definition in terms of generalized flows, motion of bounded sets *etc.*

Because $\nu(dz) = c_0(z)dz$ is singular, we cannot define $\kappa[x, u]$ as in (1). Indeed, we must *compensate* the singularity as it is commonly done in order to get a proper integral representation of the fractional Laplacian. Here, we must do this in a geometrical way. This is achieved by defining the integral operator as follows

$$\kappa[x, u] = \int_{\{z:u(x+z)\geq u(x), Du(x)\cdot z\leq 0\}} \nu(dz) - \int_{\{z:u(x+z)<u(x), Du(x)\cdot z>0\}} \nu(dz). \quad (2)$$

An important example of a singular measure is the following one

$$\nu_{SE}(dz) = \frac{dz}{|z|^{N+\alpha}}$$

with $\alpha \in (0, 1)$.

We can say that this singular integral operator measures in a non-local way the curvature of the “curve”. Indeed, in Formula (2), the first part (resp. the second one) measures how concave (resp. convex) is the set $\Omega = \{z : u(x+z) > u(x)\}$ “near x ”. Because ν is singular, we refer to $\kappa[x, u]$ in this setting as the *fractional mean curvature* of Γ at x . We explain below in details (see Lemma 2) the rigorous links between the different formulations we considered up to now.

The variational case

When the singular measure $\nu(dz)$ has the form

$$\nu(dz) = -\left(\nabla \cdot G(z)\right)dz$$

for a vectorfield G , the previous singular integral operator can be written as follows

$$\kappa[x, u] = \int_{\{z:u(x+z)=u(x)\}} \left(G(z) \cdot \nabla u(x+z)\right) \sigma(dz) - b_G \cdot \nabla u(x) \quad (3)$$

where σ denotes the surface measure on the “curve” $\{z : u(x+z) = u(x)\}$ and where $b_G = \int_{\{z:\nabla u(x)\cdot z=0\}} G(z)\sigma(dz)$ is a fixed vector of \mathbb{R}^N .

Remark that the example we gave above is of this form. Indeed

$$\frac{dz}{|z|^{N+\alpha}} = -\frac{1}{\alpha} \left(\nabla \cdot \frac{z}{|z|^{N+\alpha}}\right) dz.$$

It is quite clear on this new formula that the singular integral operator is geometric (in the sense that it only depends on the curve and not and its parametrization u) and “fractional”.

Comments and related works

We gave two different formulations in the case of singular potentials. *We think that Formulation (2) is the proper one in order to get a complete level-set formulation of the geometric flow* even if Formulation (3) is somehow more intuitive since it only involves the curve itself. In particular, the approach proposed by Slepčev [18] can be adapted (see (11) below).

The level-set equation we study has the following form

$$\partial_t u = \mu(\widehat{Du}) \left[c_1(x) + \kappa[x, u] \right] |Du| \quad \text{in } (0, +\infty) \times \mathbb{R}^N \quad (4)$$

supplemented with the following initial condition

$$u(0, x) = u_0(x) \quad \text{in } \mathbb{R}^N \quad (5)$$

where \hat{p} denotes $p/|p|$ if $p \neq 0$, μ denotes the mobility vector field, $c_1(x)$ is a driving force and

Equation (4) is a non-linear non-local Hamilton-Jacobi equation and a lot of papers are dedicated to the study of such equations. The main technical issues are the definition of viscosity solutions, the proof of their stability and of a strong uniqueness result. We somehow use ideas from [18] and combine them with the ones from [5], even if the results of these two papers do not apply to our equation.

From a physical point of view and as far as dislocation dynamics are concerned, the measure $\nu(dz) = c_0(z)dz$ should be $\nu(dz) = g(z/|z|)|z|^{-N-1}dz$ but in this case, the fractional mean curvature is not well defined (see Remark 1). It is also physically relevant to say that close to the dislocation line, in the *core* of the dislocation, the potential should be regularized. On the other hand, it is important to assume that $\nu(dz) \sim g(z/|z|)|z|^{-N-1}dz$, since this prescribes the long range interaction between dislocation lines. Another way to understand this difficulty is to say that in the core of the dislocation, the potential is very singular and the singularity should be compensated at a higher (second) order. This can explain the loss of inclusion principles for such flows (if one can define them for large times). Or one can think that in this case, the first term in such an expansion should be a mean curvature term; but in this case, an inclusion principle still holds true. Curvature terms are commonly used to describe dislocation dynamics and it can be relevant to add a curvature term in (4). We choose not to do so in order to avoid technicalities and keep clear some important points in the proof of the stability result and the comparison principle.

In order to understand better the properties of the fractional mean curvature flow, a deterministic zero-sum repeated game is constructed in [13] in the spirit of [16, 15].

Organization of the article. In Section 2, we first give the precise assumptions we make on the data. We next give the definition(s) of the fractional mean curvature $\kappa[x, \cdot]$. In Section 3, we first give the definition of viscosity solutions for (4), we then state and prove

stability results. We next obtain strong uniqueness results by establishing comparison principles and we finally construct solutions of (4) by Perron’s method. In Section 4, we verify that the 0-level set of the solution u we constructed in the previous section only depends on the 0-level set of the initial condition. This provides a level-set formulation of the geometric flow. In the last section, we give an alternative geometric definition of the flow in terms of generalized flows in the sense of [7].

Notation. \mathbb{S}^{N-1} denotes the unit sphere of \mathbb{R}^N . The ball of radius δ centered at x is denoted by $B_\delta(x)$. If $x = 0$, we simply write B_δ and if moreover $\delta = 1$, we write B . If $p \in \mathbb{R}^N \setminus \{0\}$, $\hat{p} = p/|p|$. If A is a subset of \mathbb{R}^d with $d = N, N + 1$ for instance, then A^c denotes its complementary. For two subsets A and B , $A \sqcup B$ denotes $A \cup B$ and means that $A \cap B = \emptyset$. The function $\mathbf{1}_A(z)$ equals 1 if $z \in A$ and 0 if not.

Acknowledgments. This paper is partially supported by the ANR grant “MICA”. The author thanks R. Monneau and P. E. Souganidis for the fruitful discussions they had together. He also thanks G. Barles for its attentive reading of a preliminary version of this paper.

2 Preliminaries

In this section, we make precise the assumptions we need on data and we give several definitions of the fractional mean curvature.

2.1 Assumptions

Here are the assumptions we make on the singular measure throughout the paper.

ASSUMPTIONS.

- The mobility function $\mu : \mathbb{S}^{N-1} \rightarrow (0, +\infty)$ is continuous.
- The driving force $c_1 : \mathbb{R}^N \rightarrow \mathbb{R}$ is Lipschitz continuous.
- The singular measure ν is a non-negative Radon measure satisfying

$$\left\{ \begin{array}{l} \text{for all } \delta > 0, \quad \nu(\mathbb{R}^N \setminus B_\delta) < +\infty, \\ \text{for all } r > 0, e \in \mathbb{S}^{N-1}, \quad \nu\{z \in B : r|z \cdot e| \leq |z - (z \cdot e)e|^2\} < +\infty, \\ \delta \nu(\mathbb{R}^N \setminus B_\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0, \\ \text{for all } e \in \mathbb{S}^{N-1}, \quad r \nu\{z \in B : r|z \cdot e| \leq |z - (z \cdot e)e|^2\} \rightarrow 0 \text{ as } r \rightarrow 0 \end{array} \right. \quad (6)$$

(B_δ denotes the ball of radius δ centered at the origin and $B = B_1$) and the last limit is uniform with respect to unit vectors $e \in \mathbb{S}^{N-1}$.

- The initial datum $u_0 : \mathbb{R}^N \rightarrow \mathbb{R}$ is bounded and Lipschitz continuous.

Even if the assumptions on the singular measure look technical at first glance, they are quite natural in the sense that they ensure

- that the measure is bounded away from the origin;
- that the measure is not too singular in order that the fractional mean curvature of regular curves be well defined;
- model regular curves $\{z : rz_N = |z'|^2\}$ can be handled, even when they degenerate ($r \rightarrow 0$).

In particular, the fourth line of (6) is required to prove the discontinuous stability result.

Example 1. The Standing Example for the singular measure is

$$\nu(dz) = g\left(\frac{z}{|z|}\right) \frac{dz}{|z|^{N+\alpha}}$$

with $g : \mathbb{S}^{N-1} \rightarrow (0, +\infty)$ continuous and $\alpha \in (0, 1)$.

2.2 Fractional mean curvature

In this subsection, we make precise the definition of fractional mean curvature. Our definition extends the ones given in [9, 10] where $\nu(dz) = c_0(z)dz$ to the case of singular measures.

Let us define the fractional curvature of a smooth curve $\Gamma = \{x \in \mathbb{R}^N : u(x) = 0\} = \partial\{x \in \mathbb{R}^N : u(x) > 0\}$ associated with ν . If u is $C^{1,1}$ and $Du(x) \neq 0$, then the following quantity is well defined (see Lemma 1 below)

$$\begin{aligned} \kappa^*[x, \Gamma] &= \kappa^*[x, u] = \kappa_+^*[x, u] - \kappa_*^-[x, u] \\ \kappa_*[x, \Gamma] &= \kappa_*[x, u] = \kappa_*^+[x, u] - \kappa_-^*[x, u] \end{aligned} \tag{7}$$

where

$$\begin{aligned} \kappa_*^+[x, u] &= \nu(z : u(x+z) > u(x), Du(x) \cdot z < 0) \\ \kappa_*^-[x, u] &= \nu(z : u(x+z) < u(x), Du(x) \cdot z > 0) \end{aligned} \tag{8}$$

and

$$\begin{aligned} \kappa_+^*[x, u] &= \nu(z : u(x+z) \geq u(x), Du(x) \cdot z \leq 0) \\ \kappa_-^*[x, u] &= \nu(z : u(x+z) \leq u(x), Du(x) \cdot z \geq 0) . \end{aligned}$$

We will see later (see Lemma 3 below) that these functions are semi-continuous and this explains the choice of notation we made. In order to understand the way these quantities are related to the geometry of the curve $\{u = u(x)\}$, it is convenient to write for instance

$$\kappa_*^+[x, u] = \nu(z : 0 < -Du(x) \cdot z < u(x+z) - u(x) - Du(x) \cdot z) .$$

As shown on Figure 1, $\kappa^+[x, u]$ measures how concave the curve is at x and $\kappa^-[x, u]$ how convex it is.

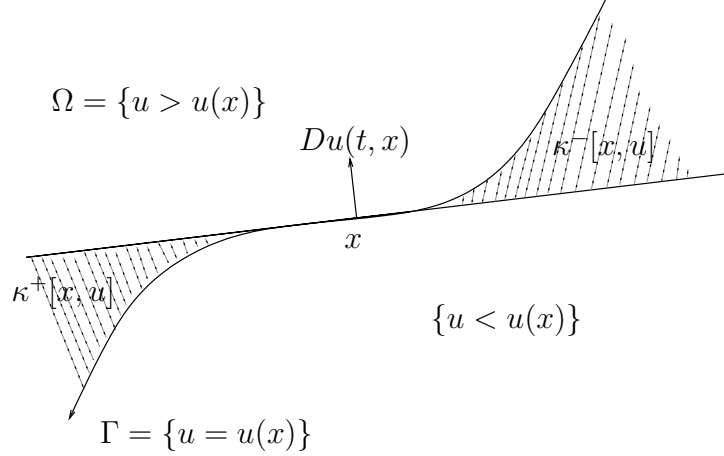


Figure 1: Fractional mean curvature of a curve

Lemma 1 (Fractional mean curvature is finite). *If u is C^2 and its gradient $Du(x) \neq 0$, then $\kappa_+^*[x, u]$ and $\kappa_-^*[x, u]$ are finite.*

Remark 1. One can check that this lemma is false if $\alpha = 1$ in the Standing Example 1.

Proof. Since ν is bounded on $\mathbb{R}^N \setminus B_\delta$ for all $\delta > 0$, it is enough to consider

$$\begin{aligned} (\kappa_+^*)^{1,\delta}[x, u] &= \nu(z \in B_\delta : u(x+z) \geq u(x), Du(x) \cdot z \leq 0) \\ &= \nu(z \in B_\delta : 0 \leq re \cdot z \leq u(x+z) - u(x) + re \cdot z) \end{aligned}$$

where $r = |Du(x)| \neq 0$ and $e = r^{-1}Du(x)$. If now z_N denotes $e \cdot z$ and $z' = z - z_N e$, and if we choose δ such that $r - C\delta > 0$ with $C = \|D^2u\|_{L^\infty(B_1(x))}$, we can write

$$\begin{aligned} (\kappa_+^*)^{1,\delta}[x, u] &\leq \nu(z \in B_\delta : 0 \leq rz_N \leq Cz_N^2 + C|z'|^2) \\ &\leq \nu(z \in B_\delta : 0 \leq C^{-1}(r - C\delta)z_N \leq |z'|^2) \end{aligned}$$

and the result now follows from Condition (6). \square

The following lemma explains rigourously the link between (4) and (1) and the link with the formulation used in [10] in the case where ν is a bounded measure.

Lemma 2 (Link with regular dislocation dynamics). *Consider $c_0 \in L^1(\mathbb{R}^N)$ such that $c_0(x) = c_0(-x)$. Then*

$$\begin{aligned} \int_{\{z:u(t,x+z)>u(t,x)\}} c_0(z)dz &= \frac{1}{2} \int c_0 + \kappa_*[x, u] \\ \int \text{sign}^*(u(x+z) - u(x))c_0(z)dz &= \frac{1}{2}\kappa_*^*[x, u] \\ \int \text{sign}_*(u(x+z) - u(x))c_0(z)dz &= \frac{1}{2}\kappa_*[x, u] \end{aligned}$$

with $\text{sign}^*(r) = 1$ (resp. $\text{sign}_*(r) = 1$) if $r \geq 0$ (resp. $r > 0$) and 0 if not and with $\nu(dz) = c_0(z)dz$.

Since the proof is elementary, we omit it.

3 Viscosity solutions for (4)

3.1 Definitions

The viscosity solution theory introduced in [18] suggests that the good notion of solution for the fractional equation (4) is the following one.

Definition 1 (Viscosity solutions for (4)). 1. An upper semi-continuous function $u : [0, T] \times \mathbb{R}^N$ is a viscosity subsolution of (4) if for every smooth test-function ϕ such that $u - \phi$ admits a global zero maximum at (t, x) , we have

$$\partial_t \phi(t, x) \leq \mu(\widehat{D\phi(t, x)}) \left[c_1(x) + \kappa^*[x, \phi(t, \cdot)] \right] |D\phi|(t, x) \quad (9)$$

if $D\phi(t, x) \neq 0$ and $\partial_t \phi(t, x) \leq 0$ if not.

2. A lower semi-continuous function u is a viscosity supersolution of (4) if for every smooth test-function ϕ such that $u - \phi$ admits a global minimum 0 at (t, x) , we have

$$\partial_t \phi(t, x) \geq \mu(\widehat{D\phi(t, x)}) \left[c_1(x) + \kappa_*[x, \phi(t, \cdot)] \right] |D\phi|(x_0, t_0) \quad (10)$$

if $D\phi(t, x) \neq 0$ and $\partial_t \phi(t, x) \geq 0$ if not.

3. A locally bounded function u is a viscosity solution of (4) if u^* (resp. u_*) is a subsolution (resp. supersolution).

Remark 2. Given $\delta > 0$, the global extrema in Definition 1 can be assumed to be strict in a ball of radius δ centered at (t, x) . Such a result is classically expected and the reader can have a look, for instance, at the proof of the stability result in [5].

If one uses the notation introduced in [18], the equation reads

$$\partial_t u + F(x, Du, \{z : u(x+z) \geq u(x)\}) = 0 \quad (11)$$

with, for $x, p \in \mathbb{R}^N$ and $K \subset \mathbb{R}^N$,

$$F(x, p, K) = \begin{cases} -\mu(\hat{p}) \left[c_1(x) + \nu(K \cap \{p \cdot z \leq 0\}) - \nu(K^c \cap \{p \cdot z > 0\}) \right] |p| & \text{if } p \neq 0, \\ 0 & \text{if not} \end{cases}$$

where K^c is the complementary set of K . With this notation in hand, one can check that this non-linearity does not satisfy Assumption (F5) of [18]. The idea is to check that, somehow, Assumption (NLT) in [5] is satisfied and stability results thus hold true.

Let us be more precise. We previously associated with $\kappa[\cdot, \cdot]$ the following non-local operators (see the proof of Lemma 1)

$$\begin{aligned} (\kappa_*^+)^{1,\delta}[x, \phi] &= \nu(z \in B_\delta : \phi(x+z) > \phi(x), z \cdot D\phi(x) < 0), \\ (\kappa_*^+)^{2,\delta}[x, p, \phi] &= \nu(z \notin B_\delta : \phi(x+z) > \phi(x), z \cdot p < 0). \end{aligned} \quad (12)$$

In the same way, we can define

- the negative non-local curvature operators $(\kappa_*^-)^{i,\delta}$, $i = 1, 2$,
- upper semi-continuous envelopes of these four integral operators $(\kappa_\pm^*)^{i,\delta}$, $i = 1, 2$,
- and lower/upper semi-continuous total non-local curvature operators $(\kappa_*)^{i,\delta}$, $(\kappa^*)^{i,\delta}$, $i = 1, 2$.

By using the idea of Lemma 2, it is easy to see that

$$\begin{cases} (\kappa^*)^{2,\delta}[x, p, u] &= \nu(z \notin B_\delta : u(t, x+z) \geq u(t, x)) - \frac{1}{2}\nu(B_\delta^c) =: (\kappa^*)^{2,\delta}[x, u], \\ (\kappa_*)^{2,\delta}[x, p, u] &= \nu(z \notin B_\delta : u(t, x+z) > u(t, x)) - \frac{1}{2}\nu(B_\delta^c) =: (\kappa_*)^{2,\delta}[x, u]. \end{cases} \quad (13)$$

We can now state an equivalent definition of viscosity solutions of (4).

Definition 2 (Equivalent definition). 1. An upper semi-continuous function $u : [0, T] \times \mathbb{R}^N$ is a viscosity subsolution of (4) if for every smooth test-function ϕ such that $u - \phi$ admits a maximum 0 at (t, x) on $B_\delta(t, x)$, we have

$$\partial_t \phi(t, x) \leq \mu(\widehat{D\phi(t, x)}) \left[c_1(x) + (\kappa^*)^{1,\delta}[x, \phi(t, \cdot)] + (\kappa^*)^{2,\delta}[x, u(t, \cdot)] \right] |D\phi|(t, x) \quad (14)$$

if $D\phi(t, x) \neq 0$ and $\partial_t \phi(t, x) \leq 0$ if not.

2. A lower semi-continuous function u is a viscosity supersolution of (4) if for every smooth test-function ϕ such that $u - \phi$ admits a global minimum 0 at (t, x) , we have

$$\partial_t \phi(t, x) \geq \mu(\widehat{D\phi(t, x)}) \left[c_1(x) + (\kappa_*)^{1,\delta}[x, \phi(t, \cdot)] + (\kappa_*)^{2,\delta}[x, u(t, \cdot)] \right] |D\phi|(x_0, t_0) \quad (15)$$

if $D\phi(t, x) \neq 0$ and $\partial_t \phi(t, x) \geq 0$ if not.

3. A continuous function u is a viscosity solution of (4) if it is both a sub and supersolution.

Remark 3. Equivalent definitions of this type first appeared in [17] and since the proof is the same, we omit it.

Remark 4. Remark 2 applies to the equivalent definition too.

Remark 5. Definition 2 seems to depend on δ . But since all these definitions are equivalent to Definition 1, it does not depend on it. Hence, when proving that a function is a solution of (4), it is enough to do it for a fixed (or not) $\delta > 0$.

3.2 Stability results

Theorem 1 (Discontinuous stability). • *Let $(u_n)_{n \geq 1}$ be a family of subsolutions of (4) that is locally bounded, uniformly with respect to n . Then its relaxed upper limit u^* is a subsolution of (4).*

- *If moreover, $u_n(0, x) = u_0^n(x)$, then for all $x \in \mathbb{R}^N$*

$$u^*(0, x) \leq u_0^*(x)$$

where u_0^* is the relaxed upper limit of u_0^n .

- *Let $(u_\alpha)_{\alpha \in \mathcal{A}}$ be a family of subsolutions of (4) that is locally bounded, uniformly with respect to $\alpha \in \mathcal{A}$. Then \bar{u} , the upper semicontinuous envelope of $\sup_\alpha u_\alpha$ is a subsolution of (4).*

Even if this result follows from ideas introduced in [5] together with classical ones, we give a detailed proof for the sake of completeness.

Proof. We only prove the first part of the theorem since it is easy to adapt it to get a proof of the third part. The second one is very classical and can be adapted from [1] for instance.

Consider a test function φ such that $u^* - \varphi$ attains a global maximum at (t, x) . We can assume (see Remark 2) that $u^* - \varphi$ attains a strict maximum at (t, x) on $B_\delta(t, x)$. Consider a subsequence $p = p(n)$ and x_p such that

$$u^*(t, x) = \lim_{n \rightarrow +\infty} u_{p(n)}(t_n, x_n).$$

Classical arguments show that $u_p - \varphi$ attains a maximum on $B_\delta(t, x)$ at $(s_p, y_p) \in B_\delta(t, x)$ and that

$$(s_p, y_p) \rightarrow (t, x) \quad \text{and} \quad u_p(s_p, y_p) \rightarrow u^*(t, x).$$

Since u_p is a subsolution of (4), we have

$$\begin{aligned} \partial_t \varphi(s_p, y_p) \leq \\ \mu(\widehat{D\varphi}(s_p, y_p)) \left[c_1(y_p) + (\kappa^*)^{1, \delta} [y_p, \varphi(s_p, \cdot)] + (\kappa^*)^{2, \delta} [y_p, D_x \varphi(s_p, y_p), u(s_p, \cdot)] \right] |D\varphi|(s_p, y_p) \end{aligned}$$

if $D\varphi(t_p, x_p) \neq 0$ and $\partial_t \varphi(t_p, x_p) \leq 0$ if not. If there exists a subsequence q of p such that $D\varphi(s_q, y_q) = 0$, then it is easy to conclude. We thus now assume that $D\varphi(s_p, y_p) \neq 0$ for p large enough. In view of the continuity of μ and c_1 , the following technical lemma (whose proof is given in Appendix) permits to conclude.

Lemma 3. *Assume that $D\varphi(s_p, y_p) \neq 0$ for p large enough.*

- Assume moreover that $D\varphi(t, x) \neq 0$. Then

$$(s, y) \mapsto (\kappa^*)^{1,\delta}[y, \varphi(s, \cdot)] \quad \text{and} \quad (s, y) \mapsto (\kappa^*)^{2,\delta}[y, D_x\varphi(s, y), u_p(s, \cdot)]$$

are well defined for $i = 1, 2$ in a neighbourhood of (t, x) and

$$\limsup_p \left\{ (\kappa^*)^{1,\delta}[y_p, \varphi(s_p, \cdot)] \right\} \leq (\kappa^*)^{1,\delta}[x, \varphi(t, \cdot)]$$

$$\limsup_p \left\{ (\kappa^*)^{2,\delta}[y_p, D_x\varphi(s_p, y_p), u(s_p, \cdot)] \right\} \leq (\kappa^*)^{2,\delta}[x, D_x\varphi(t, x), \varphi(t, \cdot)]$$

as soon as $u_p(s_p, y_p) \rightarrow u(t, x)$ as $p \rightarrow +\infty$.

- Assume now that $D\varphi(t, x) = 0$. Then, for $i = 1, 2$,

$$\left[(\kappa^*)^{1,\delta}[y_p, \varphi(s_p, \cdot)] + (\kappa^*)^{2,\delta}[y_p, D\varphi(s_p, y_p), u(s_p, \cdot)] \right] |D\varphi|(s_p, y_p) \rightarrow 0 \quad \text{as } p \rightarrow +\infty.$$

As we shall see, this lemma is a consequence of the following one.

Lemma 4 ([18]). Consider f_p and g_p two sequences of measurable functions on a set U and $f \geq \limsup^* f_p$, $g \geq \limsup^* g_p$, and a_p, b_p two sequences of real numbers converging to 0. Then

$$\nu(\{f_p \geq a_p, g_p \geq b_p\} \setminus \{f \geq 0, g \geq 0\}) \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

We mention that in [18], the measure is not singular and there is only one sequence of measurable functions but the reader can check that the slightly more general version we gave here can be proven with exactly the same arguments. An immediate consequence of the lemma is the following inequality

$$\limsup_p \nu(\{f_p \geq a_p, g_p \geq b_p\}) \leq \nu(\{f \geq 0, g \geq 0\}).$$

Proof of Lemma 3. Let us first assume that $D\varphi(t, x) \neq 0$. In this case, for (s, y) close to (t, x) , $D\varphi(s, y) \neq 0$ and all the integral operators we consider here are well defined (see Lemma 1). Recall next that, for $i = 1, 2$, $(\kappa^*)^{i,\delta} = (\kappa_+^*)^{i,\delta} - (\kappa_*^-)^{i,\delta}$. Hence, it is enough to prove that

$$\limsup_p \left\{ (\kappa_+^*)^{1,\delta}[y_p, \varphi(s_p, \cdot)] \right\} \leq (\kappa_+^*)^{1,\delta}[x, \varphi(t, \cdot)],$$

$$\liminf_p \left\{ (\kappa_*^-)^{1,\delta}[y_p, \varphi(s_p, \cdot)] \right\} \geq (\kappa_*^-)^{1,\delta}[x, \varphi(t, \cdot)],$$

$$\limsup_p \left\{ (\kappa_+^*)^{2,\delta}[y_p, D_x\varphi(s_p, y_p), u_p(s_p, \cdot)] \right\} \leq (\kappa_+^*)^{2,\delta}[x, D_x\varphi(t, x), u^*(t, \cdot)],$$

$$\liminf_p \left\{ (\kappa_*^-)^{2,\delta}[y_p, D_x\varphi(s_p, y_p), u_p(s_p, \cdot)] \right\} \leq (\kappa_*^-)^{2,\delta}[x, D_x\varphi(t, x), u^*(t, \cdot)].$$

In order to prove the first inequality above for instance, choose $f_p(z) = \varphi(s_p, y_p + z) - \varphi(t, x)$, $a_p = \varphi(s_p, y_p) - \varphi(t, x)$, $g_p(z) = -D\varphi(s_p, y_p) \cdot z$, $b_p = 0$ in Lemma 4.

We now turn to the case $D\varphi(t, x) = 0$. We look for $\delta = \delta_p$ that goes to 0 as $p \rightarrow +\infty$ such that

$$(\kappa_+^*)^{1, \delta_p} [y_p, \varphi(s_p, \cdot)] |D\varphi(s_p, y_p)| \rightarrow 0 \quad \text{and} \quad (\kappa_*^-)^{1, \delta_p} [y_p, \varphi(s_p, \cdot)] |D\varphi(s_p, y_p)| \rightarrow 0$$

as $p \rightarrow +\infty$. This is enough to conclude since Condition (6) implies that

$$\begin{aligned} (\kappa_+^*)^{2, \delta_p} [y_p, D\varphi(s_p, y_p), u(s_p, \cdot)] |D\varphi(s_p, y_p)| &\rightarrow 0 \\ (\kappa_*^-)^{2, \delta_p} [y_p, D\varphi(s_p, y_p), u(s_p, \cdot)] |D\varphi(s_p, y_p)| &\rightarrow 0. \end{aligned}$$

We only prove that the first limit equals zero since the argument is similar for the second one. If r_p denotes $|D\varphi(s_p, y_p)|$ and e_p denotes $-r_p^{-1}D\varphi(s_p, y_p)$, and $z_N = e_p \cdot z$ and $z' = z - z_N e_p$, then

$$\begin{aligned} (\kappa_+^*)^{1, \delta} [y_p, \varphi(s_p, \cdot)] |D\varphi(s_p, y_p)| \\ &= r_p \nu(z \in B_{\delta_p} : 0 \leq r_p e_p \cdot z \leq \varphi(s_p, y_p + z) - \varphi(s_p, y_p) + r_p e_p \cdot z) \\ &\leq r_p \nu(z \in B_{\delta_p} : 0 \leq r_p z_N \leq C|z'|^2 + C z_N^2) \\ &\leq r_p \nu(z \in B_{\delta_p} : 0 \leq r_p z_N \leq C|z'|^2 + C\delta_p z_N) \end{aligned}$$

where C is a bound for second derivatives of φ around (t, x) . Now if we choose $\delta_p = r_p/(2C)$, we get

$$\begin{aligned} (\kappa_+^*)^{1, \delta} [y_p, \varphi(s_p, \cdot)] |D\varphi(s_p, y_p)| &\leq r_p \nu(z \in B_{\delta_p} : 0 \leq (r_p/2C)z_N \leq |z'|^2) \\ &\leq r_p \nu(z \in B : 0 \leq (r_p/2C)z_N \leq |z'|^2) \end{aligned}$$

and the last limit in (6) permits now to conclude. □

□

3.3 Existence and uniqueness results

Let us first state a strong uniqueness result.

Theorem 2 (Comparison principle). *Assume that ν satisfies (6). Assume moreover that for all $e \in \mathbb{S}^{N-1}$ and $r \in (0, 1)$*

$$r \nu\{z \in B_\delta : r|z \cdot e| \leq |z - (z \cdot e)e|^2\} \rightarrow 0 \quad \text{as } \delta \rightarrow 0 \tag{16}$$

uniformly in e and $r \in (0, 1)$ and

$$\nu(dz) = J(z)dz \quad \text{with } J \in W^{1,1}(\mathbb{R}^N \setminus B_\delta) \text{ for all } \delta > 0. \tag{17}$$

Consider a bounded and Lipschitz continuous function u_0 . Let u (resp. v) be a bounded subsolution (resp. bounded supersolution) of (4). If $u(0, x) \leq u_0(x) \leq v(0, x)$, then $u \leq v$ on $(0, +\infty) \times \mathbb{R}^N$.

The proof is quite classical. The main difficulty is to deal with the singularity of the measure.

Proof of Theorem 2. We classically consider $M = \sup_{t,x} \{u(t, x) - v(t, x)\}$ and argue by contradiction by assuming $M > 0$. We next consider the following approximation of M

$$\tilde{M}_{\varepsilon, \alpha} = \sup_{t, s > 0, x, y \in \mathbb{R}^N} \left\{ u(t, x) - v(s, y) - \frac{(t-s)^2}{2\gamma} - e^{Kt} \frac{|x-y|^2}{2\varepsilon} - \eta t - \alpha |x|^2 \right\}.$$

Since u and v are bounded, this supremum is attained at a point $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y})$. We first observe that $\tilde{M}_{\varepsilon, \alpha} \geq M/2 \geq 0$ for η and α small enough. Since u and v are bounded, this implies in particular

$$e^{K\tilde{t}} \frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon} + \alpha |\tilde{x}|^2 \leq C_0 \quad (18)$$

where $C_0 = \|u\|_\infty + \|v\|_\infty$.

Classical results about penalization imply that $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y}) \rightarrow (\bar{t}, \bar{t}, \bar{x}, \bar{y})$ as $\gamma \rightarrow 0$ and $(\bar{t}, \bar{t}, \bar{x}, \bar{y})$ realizes the following supremum

$$M_{\varepsilon, \alpha} = \sup_{t > 0, x, y \in \mathbb{R}^N} \left\{ u(t, x) - v(t, y) - e^{Kt} \frac{|x-y|^2}{2\varepsilon} - \eta t - \alpha |x|^2 \right\}.$$

We claim next that this supremum cannot be achieved at $t = 0$ if $\varepsilon, \alpha, \eta$ are small enough. To see this, remark first that Recall that $M_{\varepsilon, \alpha} \geq M/2 \geq 0$ for η and α small enough and, if $\bar{t} = 0$, use the fact that u_0 is Lipschitz continuous and get

$$0 < \frac{M}{2} \leq \sup_{x, y \in \mathbb{R}^N} \left\{ u_0(x) - u_0(y) - \frac{|x-y|^2}{2\varepsilon} \right\} \leq \sup_{r > 0} \left\{ C_0 r - \frac{r^2}{2\varepsilon} \right\} = \frac{1}{2} C_0^2 \varepsilon$$

and this is obviously false if ε is small enough. We conclude that, if the four parameters are small enough, $\tilde{t} > 0$ and $\tilde{s} > 0$. Hence, we can write two viscosity inequalities, for all $\delta > 0$,

$$\begin{aligned} \eta + \frac{\tilde{t} - \tilde{s}}{\gamma} + K e^{L\tilde{t}} \frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon} &\leq (c(\tilde{x}) + (\kappa^*)^{1, \delta} [\tilde{x}, \phi_u(\tilde{t}, \cdot)] + (\kappa^*)^{2, \delta} [\tilde{x}, u(\tilde{t}, \cdot)]) |\tilde{p} + 2\alpha \tilde{x}| \\ \frac{\tilde{t} - \tilde{s}}{\gamma} &\geq (c(\tilde{y}) + (\kappa_*)^{1, \delta} [\tilde{y}, \phi_v(\tilde{s}, \cdot)] + (\kappa_*)^{2, \delta} [\tilde{y}, v(\tilde{s}, \cdot)]) |\tilde{p}| \end{aligned}$$

where $\tilde{p} = e^{K\tilde{t}} \frac{\tilde{x} - \tilde{y}}{\varepsilon}$ and

$$\begin{aligned} \phi_u(t, x) &= \frac{(t - \tilde{s})^2}{2\gamma} + e^{Kt} \frac{|x - \tilde{y}|^2}{2\varepsilon} + \eta t + \alpha |x|^2, \\ \phi_v(s, y) &= -\frac{(s - \tilde{t})^2}{2\gamma} - e^{K\tilde{t}} \frac{|y - \tilde{x}|^2}{2\varepsilon} - \eta \tilde{t} - \alpha |\tilde{x}|^2. \end{aligned}$$

Subtracting these inequalities yield

$$\eta + Ke^{K\tilde{t}}\frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon} \leq \|Dc\|_\infty e^{K\tilde{t}}\frac{|\tilde{x} - \tilde{y}|^2}{\varepsilon} + 2\|c\|_\infty\alpha|\tilde{x}| + T_{nl} \quad (19)$$

where

$$\begin{aligned} T_{nl} = & \left((\kappa^*)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t}, \cdot)] + (\kappa^*)^{2,\delta}[\tilde{x}, u(\tilde{t}, \cdot)] \right) |\tilde{p} + 2\alpha\tilde{x}| \\ & - \left((\kappa_*)^{1,\delta}[\tilde{y}, \phi_v(\tilde{s}, \cdot)] + (\kappa_*)^{2,\delta}[\tilde{y}, v(\tilde{s}, \cdot)] \right) |\tilde{p}|. \end{aligned}$$

Our task is now to find $\delta = \delta(\alpha, \varepsilon)$ so that the right hand side of this inequality is small when the four parameters are small. We distinguish two cases.

Assume first that there exists a sequence $\alpha_n \rightarrow 0$ and $\varepsilon_n \rightarrow 0$ such that

$$\tilde{p} = \tilde{p}_n \rightarrow 0.$$

In this case, we simply choose $\delta = 1$ and we pass to the limit as $n \rightarrow +\infty$ in (19) and we get the desired contradiction: $\eta \leq 0$.

Assume now that for α and ε small enough, we have a constant C_ε independent of α such that

$$|\tilde{p}| \geq C_\varepsilon > 0. \quad (20)$$

In this case, the following technical lemma holds true.

Lemma 5. *By using (16), we have*

$$T_{nl} \leq \frac{1}{\varepsilon}o_\delta(1) + C\frac{\sqrt{\alpha}}{\delta} + o_\alpha(1)[\varepsilon] + C_\delta e^{K\tilde{t}}\frac{|\tilde{x} - \tilde{y}|^2}{\varepsilon}$$

where C only depends on ν and $\|u\|_\infty + \|v\|_\infty$ and C_δ only depends on J and δ (we emphasize that the third term goes to 0 as $\alpha \rightarrow 0$ for fixed ε).

The proof of this lemma is postponed. We thus get (recall that $\tilde{p} = (\tilde{x} - \tilde{y})/\varepsilon$)

$$\eta + Ke^{K\tilde{t}}\frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon} \leq C \left(e^{K\tilde{t}}\frac{|\tilde{x} - \tilde{y}|^2}{\varepsilon} + \sqrt{\alpha} + \frac{\sqrt{\alpha}}{\delta} \right) + \frac{1}{\varepsilon}o_\delta(1) + o_\alpha(1)[\varepsilon] + C_\delta e^{K\tilde{t}}\frac{|\tilde{x} - \tilde{y}|^2}{\varepsilon}$$

where C only depends on c , ν and $\|u\|_\infty + \|v\|_\infty$ and C_δ is given by the lemma. By choosing $K = 2(C + C_\delta)$, we get

$$\eta \leq C(\sqrt{\alpha} + \frac{\sqrt{\alpha}}{\delta}) + \frac{1}{\varepsilon}o_\delta(1) + o_\alpha(1)[\varepsilon].$$

By letting successively α and δ go to 0, we thus get a contradiction. This achieves the proof of the comparison principle. \square

Proof of Lemma 5. We first write

$$\begin{aligned} T_{nl} \leq & |(\kappa^*)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t}, \cdot)]\tilde{p} + 2\alpha\tilde{x}| + |(\kappa_*)^{1,\delta}[\tilde{y}, v(\tilde{s}, \cdot)]\tilde{p}| \\ & + 2|(\kappa^*)^{2,\delta}[\tilde{x}, u(\tilde{t}, \cdot)]\alpha\tilde{x}| + \left((\kappa^*)^{2,\delta}[\tilde{x}, u(\tilde{t}, \cdot)] - (\kappa_*)^{2,\delta}[\tilde{y}, v(\tilde{s}, \cdot)] \right) |\tilde{p}|. \end{aligned}$$

We thus estimate the right hand side of the previous inequality. We start with the first integral terms.

$$\begin{aligned} |(\kappa^*)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t}, \cdot)]| & \leq (\kappa_+^*)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t}, \cdot)] + (\kappa_*^-)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t}, \cdot)] \\ & \leq \nu(z \in B_\delta : 0 \leq -(\tilde{p} + 2\alpha\tilde{x}) \cdot z \leq (\alpha + 1/(2\varepsilon))|z|^2) \\ & \quad + \nu(z \in B_\delta : 0 > -(\tilde{p} + 2\alpha\tilde{x}) \cdot z > (\alpha + 1/(2\varepsilon))|z|^2) \\ & \leq \nu(z \in B_\delta : |\varepsilon\tilde{p} + 2\varepsilon\alpha\tilde{x}| |e \cdot z| \leq |z|^2). \end{aligned}$$

If now $r_{\alpha,\varepsilon}$ denotes $|\varepsilon\tilde{p} + 2\varepsilon\alpha\tilde{x}|$ and we choose $\delta \leq \frac{1}{2}r_{\alpha,\varepsilon}$, (16) implies we get

$$|(\kappa^*)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t}, \cdot)]\tilde{p} + 2\alpha\tilde{x}| \leq \frac{1}{\varepsilon} \left\{ \sup_{e \in \mathbb{S}^{N-1}} \nu(z \in B_\delta : r_{\alpha,\varepsilon}|e \cdot z| \leq |z - (e \cdot z)e|^2) \right\} r_{\alpha,\varepsilon} = \frac{1}{\varepsilon} o_\delta(1).$$

Since $\alpha\tilde{x} \rightarrow 0$, we can choose

$$\delta \leq \frac{1}{4}\varepsilon|\tilde{p}|.$$

Arguing similarly, we get for $\delta < \frac{1}{2}r'_\varepsilon = \frac{1}{2}\varepsilon|\tilde{p}'|$,

$$|(\kappa_*)^{1,\delta}[\tilde{y}, \phi_v(\tilde{s}, \cdot)]\tilde{p}| \leq \frac{1}{\varepsilon} \left\{ \sup_{e \in \mathbb{S}^{N-1}} \nu(z \in B_\delta : r'_{\alpha,\varepsilon}|e \cdot z| \leq |z - (e \cdot z)e|^2) \right\} r'_\varepsilon = \frac{1}{\varepsilon} o_\delta(1).$$

As far as the third integral term is concerned, we simply write

$$2|(\kappa^*)^{2,\delta}[\tilde{x}, u(\tilde{t}, \cdot)]\alpha\tilde{x}| \leq 2\nu(B_\delta^c)|\alpha\tilde{x}| \leq \frac{\tilde{C}_0}{\delta}\sqrt{\alpha}$$

(we used (18)). We now turn to the last integral terms. In view of (13), we can write

$$\begin{aligned} \tilde{T}_{nl} = (\kappa^*)^{2,\delta}[\tilde{x}, u(\tilde{t}, \cdot)] - (\kappa_*)^{2,\delta}[\tilde{y}, v(\tilde{s}, \cdot)] & = \nu(z \notin B_\delta : u(\tilde{t}, \tilde{x} + z) \geq u(\tilde{t}, \tilde{x})) \\ & \quad - \nu(z \notin B_\delta : v(\tilde{s}, \tilde{y} + z) > v(\tilde{s}, \tilde{y})). \end{aligned}$$

Here, we have to use (17):

$$\tilde{T}_{nl} = \int_{B_\delta^c} J(z - \tilde{x}) \mathbf{1}_{\{u(\tilde{t}, \cdot) > u(\tilde{t}, \tilde{x})\}}(z) dz - \int_{B_\delta^c} J(z - \tilde{y}) \mathbf{1}_{\{v(\tilde{s}, \cdot) > v(\tilde{s}, \tilde{y})\}}(z) dz$$

Remark next that the definition of $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y})$ implies the following inequality: for all $z \in \mathbb{R}^N$,

$$u(\tilde{t}, z) - u(\tilde{t}, \tilde{x}) \leq v(\tilde{s}, z) - v(\tilde{s}, \tilde{y}) + \alpha(|z|^2 - |\tilde{x}|^2) - e^{K\tilde{t}} \frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon}.$$

This implies that for $|z| \leq R_{\alpha, \varepsilon}$, we have

$$\mathbf{1}_{\{u(\tilde{t}, \cdot) > u(\tilde{t}, \tilde{x})\}}(z) \leq \mathbf{1}_{\{v(\tilde{s}, \cdot) > u(\tilde{s}, \tilde{y})\}}(z)$$

where

$$R_{\alpha, \varepsilon}^2 = \frac{1}{\alpha} \left(\alpha |\tilde{x}|^2 + e^{K\tilde{t}} \frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon} \right) \geq \frac{\varepsilon C_\varepsilon^2}{2\alpha}$$

where C_ε appears in (20) (we used that $\alpha |\tilde{x}|^2 \rightarrow 0$ as $\alpha \rightarrow 0$). Hence, we have

$$\begin{aligned} \tilde{T}_{nl} &\leq \int_{|z| \geq R_{\alpha, \varepsilon}} J(z - \tilde{x}) dz + \int_{\delta \leq |z| \leq R_{\alpha, \varepsilon}} |J(z - \tilde{x}) - J(z - \tilde{y})| dz \\ &\leq \int_{|z| \geq \frac{\varepsilon C_\varepsilon^2}{4\alpha}} J(z) dz + C_\delta |\tilde{x} - \tilde{y}| = o_\alpha(1)[\varepsilon] + C_\delta |\tilde{x} - \tilde{y}| \end{aligned}$$

where we used that $\alpha |\tilde{x}| \rightarrow 0$ as $\alpha \rightarrow 0$. □

We now turn to the existence result.

Theorem 3 (Existence). *Let u_0 be Lipschitz continuous and bounded. There then exists a unique bounded uniformly continuous viscosity solution u of (4).*

Proof. We first construct a solution for regular initial data. Precisely, we first assume that $u_0 \in C_b^2(\mathbb{R}^N)$ (the function and its first and second derivatives are bounded).

Because we can apply Perron's method, it is enough to construct a sub- and a supersolution u^\pm to (4) such that $(u^+)_*(0, x) = (u^-)^*(0, x) = u_0(x)$. We assert that $u^\pm(t, x) = u_0(x) \pm Ct$ are respectively a super- and a subsolution of (4) for C large enough. To see this, we first prove that there exists $C_0 = C_0(\|D^2 u_0\|_\infty)$ such that for all $x \in \mathbb{R}^N$ such that $Du_0(x) \neq 0$, we have

$$(|\kappa_*|[x, u_0] + |\kappa^*|[x, u_0])|Du_0(x)| \leq C_0. \quad (21)$$

In order to prove this estimate, we simply write for x such that $Du_0(x) \neq 0$

$$\begin{aligned} ((\kappa_+^*)^{1, \delta}[x, u_0] + (\kappa_-^*)^{1, \delta}[x, u_0])|Du_0(x)| &\leq 2\nu(z \in B_\delta : r|e \cdot z| \leq \frac{1}{2}\|D^2 u_0\|_\infty |z|^2) r \\ &\leq 2\nu(z \in B_\delta : r|e \cdot z| \leq C|z - (e \cdot z)e|^2) r \leq C_\nu \end{aligned}$$

where $r = |Du_0(x)|$, $C = \max(\|Du_0\|_\infty, 1)$ and $e = Du_0(x)/r$ and $\delta = r/(2C)$ and C_ν is given by (6). On the other hand

$$((\kappa_+^*)^{2, \delta}[x, u_0] + (\kappa_-^*)^{2, \delta}[x, u_0])|Du_0(x)| \leq \frac{C_\nu}{\delta} r = 2C_\nu C.$$

We thus get Estimate (21).

If now u_0 is not regular, we approximate it with $u_0^n \in C_b^2(\mathbb{R}^N)$ and can prove that the corresponding sequence of solutions u_n converges locally uniformly towards a solution u . Since this is very classical, we omit details (see for instance [1]). □

4 The level-set approach

In the previous section, we proved that we are able to solve (4) in the case of singular measures satisfying (6), (16) and (17). In the present section, we prove that it permits to define a geometric flow. Precisely, we first prove (Theorem 4) that if u and v are solutions of (4) associated with two different initial data u_0 and v_0 that have the same 0-level sets, then so have u and v . Hence, the geometric flows is obtained by considering the 0-level sets of the solution u of (4) for any (Lipschitz continuous) initial datum. We also describe (Theorem 5) the maximal and minimal discontinuous solutions of (4) associated with an important class of discontinuous initial data.

Theorem 4 (Consistency of the definition). *Let u_0 and v_0 be two bounded and Lipschitz continuous functions and consider the viscosity solutions u, v associated with these initial conditions. If*

$$\begin{aligned} \{x \in \mathbb{R}^N : u_0(x) > 0\} &= \{x \in \mathbb{R}^N : v_0(x) > 0\} \\ \{x \in \mathbb{R}^N : u_0(x) < 0\} &= \{x \in \mathbb{R}^N : v_0(x) < 0\} \end{aligned}$$

then, for all time $t > 0$,

$$\begin{aligned} \{x \in \mathbb{R}^N : u(t, x) > 0\} &= \{x \in \mathbb{R}^N : v(t, x) > 0\} \\ \{x \in \mathbb{R}^N : u(t, x) < 0\} &= \{x \in \mathbb{R}^N : v(t, x) < 0\} \end{aligned}$$

In view of the techniques used to prove the consistency of the definition of local geometric fronts (see for instance [6]), it is clear that this result is a straightforward consequence of the following proposition.

Proposition 1 (Equation (4) is geometric). *Consider $u : [0, +\infty) \times \mathbb{R}^N$ a bounded subsolution of (4) and $\theta : \mathbb{R} \rightarrow \mathbb{R}$ a upper semi-continuous non-decreasing function. Then $\theta(u)$ is also a subsolution of (4).*

Such a proposition is classical by now. It is proved by regularizing θ (in a proper way) with a strictly increasing function θ^n , by remarking that $\kappa^*[x, \theta^n(u)] = \kappa^*[x, u]$ in this case, and by using discontinuous stability. Details are left to the reader.

Thanks to Theorem 4, we can define a geometric flow in the following way. Given (Γ_0, D_0^+, D_0^-) such that Γ_0 is closed, D_0^\pm are open and $\mathbb{R}^N = \Gamma_0 \sqcup D_0^+ \sqcup D_0^-$, we can write

$$D_0^+ = \{x \in \mathbb{R}^N : u_0(x) > 0\}, D_0^- = \{x \in \mathbb{R}^N : u_0(x) < 0\}, \Gamma_0 = \{x \in \mathbb{R}^N : u_0(x) = 0\}$$

for some bounded Lipschitz continuous function u_0 (for instance the signed distance function). If u is the solution of (4) submitted to the initial condition $u(0, x) = u_0(x)$ for $x \in \mathbb{R}^N$, then Theorem 4 precisely says that the sets

$$D_t^+ = \{x \in \mathbb{R}^N : u(t, x) > 0\}, D_t^- = \{x \in \mathbb{R}^N : u(t, x) < 0\}, \Gamma_t = \{x \in \mathbb{R}^N : u(t, x) = 0\}$$

does not depend on the choice of u_0 .

The next theorem claims that there exists a maximal subsolution minimal supersolution of (4) associated with the appropriate discontinuous initial data.

Theorem 5 (Maximal subsolution and minimal supersolution). *Assume that (6) and (16) hold true. Then the function $\mathbf{1}_{D_t^+ \cup \Gamma_t} - \mathbf{1}_{D_t^-}$ (resp. $\mathbf{1}_{D_t^+} - \mathbf{1}_{D_t^- \cup \Gamma_t}$) is the maximal subsolution (resp. minimal supersolution) of (4) submitted to the initial datum $\mathbf{1}_{D_0^+ \cup \Gamma_0} - \mathbf{1}_{D_0^-}$ (resp. $\mathbf{1}_{D_0^+} - \mathbf{1}_{D_0^- \cup \Gamma_0}$).*

This result is a consequence of Proposition 1 together with discontinuous stability and the comparison principle. See [6, p. 445] for details.

We conclude this section by showing that a bounded front propagates with finite speed.

Proposition 2 (Evolution of bounded sets). *Let Ω_0 be a bounded open set of \mathbb{R}^N : there exists $R > 0$ such that $\Omega_0 \subset B_R$. Then the level-set evolution (Γ_t, D_t^+, D_t^-) of $(\partial\Omega_0, \Omega_0, (\bar{\Omega}_0)^c)$ satisfies $D_t^+ \cup \Gamma_t \subset \bar{B}_{R+Ct}$ with*

$$C = \|c_1\|_\infty - \inf_{e \in \mathbb{S}^{N-1}} \nu(z \in \mathbb{R}^N : 0 \leq e \cdot z \leq |z|^2)$$

as long as $R + Ct > 0$.

Remark 6. Another consequence of this proposition is that, if there are no driving force ($c_1 = 0$), then the set shrinks till it disappears.

Proof. The proof consists in constructing a supersolution of (4), (5). It is easy to check that C is chosen such that

$$u(t, x) = Ct + \sqrt{\varepsilon^2 + R^2} - \sqrt{\varepsilon^2 + |x|^2}$$

is a supersolution of (4). Since $\bar{B}_R = \{x \in \mathbb{R}^N : u(0, x) \geq 0\}$, we conclude that $D_t^+ \cup \Gamma_t \subset \{x \in \mathbb{R}^N : u(t, x) \geq 0\} = \bar{B}_{R^\varepsilon(t)}$ with $R^\varepsilon(t) = \sqrt{(Ct + \sqrt{\varepsilon^2 + R^2})^2 - \varepsilon^2}$. Hence, $D_t^+ \cup \Gamma_t \subset \cap_{\varepsilon > 0} \bar{B}_{R^\varepsilon(t)} = \bar{B}_{R+Ct}$. \square

5 Generalized flows

In this section, we follow [7] and give an equivalent definition of the flow by, freely speaking, replacing smooth test functions with smooth test fronts.

In order to give this equivalent definition, we use the geometrical non-linearities we partially introduced in Section 2 above. For all $x, p \in \mathbb{R}^N$ and all closed set $\mathcal{F} \subset \mathbb{R}^N$ and open set $\mathcal{O} \subset \mathbb{R}^N$

$$F_*(x, p, \mathcal{F}) = \begin{cases} -\mu(\hat{p}) \left[c_1(x) + \nu(\mathcal{F} \cap \{p \cdot z \leq 0\}) - \nu(\mathcal{F}^c \cap \{p \cdot z > 0\}) \right] |p| & \text{if } p \neq 0, \\ 0 & \text{if not,} \end{cases}$$

$$F^*(x, p, \mathcal{O}) = \begin{cases} -\mu(\hat{p}) \left[c_1(x) + \nu(\mathcal{O} \cap \{p \cdot z < 0\}) - \nu(\mathcal{O}^c \cap \{p \cdot z \geq 0\}) \right] |p| & \text{if } p \neq 0, \\ 0 & \text{if not.} \end{cases}$$

We can now give the definition of a generalized flow.

Definition 3 (Generalized flows). *The family $(\mathcal{O}_t)_{t \in (0, T)}$ of open subsets of \mathbb{R}^N (resp. $(\mathcal{F}_t)_{t \in (0, T)}$ of closed subsets of \mathbb{R}^N) is a generalized super-flow (resp. sub-flow) of (4) if for all $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}^N$, $r > 0$, $h > 0$, and for all smooth function $\phi : (0; +\infty) \times \mathbb{R}^N \rightarrow \mathbb{R}$ such that*

1. $\partial_t \phi + F^*(x, D\phi, \{z : \phi(t, x+z) > \phi(t, x)\}) \leq -\delta_\phi$ in $[t_0, t_0+h] \times \bar{B}(x_0, r)$
(resp. $\partial_t \phi + F_*(x, D\phi, \{z : \phi(t, x+z) \geq \phi(t, x)\}) \geq -\delta_\phi$ in $[t_0, t_0+h] \times \bar{B}(x_0, r)$)
2. $D\phi \neq 0$ in $\{(s, y) \in [t_0, t_0+h] \times \bar{B}(x_0, r) : \phi(s, y) = 0\}$,
3. $\{y \in \mathbb{R}^N : \phi(t_0, y) \geq 0\} \subset \mathcal{O}_{t_0}^1$,
(resp. $\{y \in \mathbb{R}^N : \phi(t_0, y) \leq 0\} \subset \mathbb{R}^N \setminus \mathcal{F}_{t_0}$),
4. $\{y \notin \bar{B}(x_0, r) : \phi(s, y) \geq 0\} \subset \mathcal{O}_s^1$ for all $s \in [t_0, t_0+h]$,
(resp. $\{y \notin \bar{B}(x_0, r) : \phi(s, y) \leq 0\} \subset \mathbb{R}^N \setminus \mathcal{F}_s$ for all $s \in [t_0, t_0+h]$),

then $\{y \in \bar{B}(x_0, r) : \phi(t_0+h, y) > 0\} \subset \mathcal{O}_{t_0+h}^1$ (resp. $\{y \in \bar{B}(x_0, r) : \phi(t_0+h, y) < 0\} \subset \mathbb{R}^N \setminus \mathcal{F}_{t_0+h}$).

Loosely speaking about generalized super-flows, Condition 1 says that in a prescribed neighbourhood \mathcal{V} around (t_0, x_0) , the normal velocity of the test front $\{\phi > 0\}$ is strictly smaller than the one of the front \mathcal{O} ; Condition 2 asserts that the front $\{\phi = 0\}$ is smooth in \mathcal{V} ; Conditions 3 and 4 assert that the test front is inside the front \mathcal{O} outside \mathcal{V} . The conclusion is that the test front is inside the neighbourhood \mathcal{O} at time $t+h$.

Remark 7. As far as local geometric fronts are considered, Conditions 3 and 4 impose that the test front is inside \mathcal{O} on the parabolic boundary of the neighbourhood. Here, because the front is not local, the test front has to be inside \mathcal{O} everywhere outside the neighbourhood.

The next theorem asserts that Definition 3 of the flow coincides with the level-set formulation of Section 4.

Theorem 6 (Generalized flows and level-set approach). *Let $(\mathcal{O}_t)_{t \in (0, T)}$ be a family of open subsets of \mathbb{R}^N (resp. $(\mathcal{F}_t)_{t \in (0, T)}$ of closed subsets of \mathbb{R}^N) such that the set $\cup_{t \in (0, T)} \{t\} \times \mathcal{O}_t$ is open in $[0, T] \times \mathbb{R}^N$ (resp. $\cup_{t \in (0, T)} \{t\} \times \mathcal{F}_t$ is closed in $[0, T] \times \mathbb{R}^N$).*

Then $(\mathcal{O}_t)_{t \in (0, T)}$ (resp. $(\mathcal{F}_t)_{t \in (0, T)}$) is a generalized super-flow (resp. sub-flow) of (4) if and only if $\chi(t, x) = \mathbf{1}_{\mathcal{O}_t}(x) - \mathbf{1}_{\mathbb{R}^N \setminus \mathcal{O}_t}(x)$ (resp. $\chi(t, x) = \mathbf{1}_{\mathcal{F}_t}(x) - \mathbf{1}_{\mathbb{R}^N \setminus \mathcal{F}_t}(x)$) is a viscosity supersolution (resp. subsolution) of (4), (5).

Since the proof of [7] can be readily adapted, we omit it. We give a straightforward corollary of Theorems 5 and 6 that is used in [14].

Corollary 1 (Abstract method). *Assume that $(\mathcal{O}_t)_t$ and $(\mathcal{F}_t)_t$ are respectively a generalized super-flow and generalized sub-flow and suppose there exists two open sets D_0^+, D_0^- such that $\mathbb{R}^N = \partial \mathcal{O}_0 \sqcup D_0^+ \sqcup D_0^-$ and such that $D_0^+ \subset \mathcal{O}_0$ and $D_0^- \subset \mathcal{F}_0^c$. Then if (Γ_t, D_t^+, D_t^-) denotes the level-set evolution of $(\partial \mathcal{O}_0, D_0^+, D_0^-)$, we have for all time $t > 0$*

$$D_t^+ \subset \mathcal{O}_t \subset D_t^+ \cup \Gamma_t, \quad D_t^- \subset \mathcal{F}_t^c \subset D_t^- \subset \Gamma_t.$$

Remark 8. One can check that under the assumptions of the previous corollary, we have in fact $D_0^+ = \mathcal{O}_0$ and $D_0^- = \mathcal{F}_0^c$.

References

- [1] N. ALIBAUD AND C. IMBERT, *Fractional semi-linear parabolic equations with unbounded data*, Trans. of the AMS, (2008). to appear.
- [2] O. ALVAREZ, P. CARDALIAGUET, AND R. MONNEAU, *Existence and uniqueness for dislocation dynamics with nonnegative velocity*, Interfaces Free Bound., 7 (2005), pp. 415–434.
- [3] O. ALVAREZ, P. HOCH, Y. LE BOUAR, AND R. MONNEAU, *Dislocation dynamics: short-time existence and uniqueness of the solution*, Arch. Ration. Mech. Anal., 181 (2006), pp. 449–504.
- [4] G. BARLES, P. CARDALIAGUET, O. LEY, AND R. MONNEAU, *Global existence results and uniqueness for dislocation equations*. to appear.
- [5] G. BARLES AND C. IMBERT, *Second-order elliptic integro-differential equations: Viscosity solutions’ theory revisited*, Annales de l’Institut Henri Poincaré, Analyse non linéaire, (2007). To appear.
- [6] G. BARLES, H. M. SONER, AND P. E. SOUGANIDIS, *Front propagation and phase field theory*, SIAM J. Control Optim., 31 (1993), pp. 439–469.
- [7] G. BARLES AND P. E. SOUGANIDIS, *A new approach to front propagation problems: theory and applications*, Arch. Rational Mech. Anal., 141 (1998), pp. 237–296.
- [8] P. CARDALIAGUET, *On front propagation problems with nonlocal terms*, Adv. Differential Equations, 5 (2000), pp. 213–268.
- [9] F. DA LIO, N. FORCADEL, AND R. MONNEAU, *Convergence of a non-local eikonal equation to anisotropic mean curvature motion. Application to dislocations dynamics*, J. Eur. Math. Soc, (2006). To appear.
- [10] N. FORCADEL, C. IMBERT, AND R. MONNEAU, *Homogenization of some particle systems with two-body interactions and of the dislocation dynamics*. Submitted, 2007.
- [11] N. FORCADEL AND R. MONNEAU, *Existence of solutions for a model describing the dynamics of junctions between dislocations*, 2008. Submitted.
- [12] C. IMBERT, R. MONNEAU, AND E. ROUY, *Homogenization of first order equations with u/ϵ -periodic hamiltonians. part II: application to dislocation dynamics*, Comm. in PDEs, (2007). accepted for publication.

- [13] C. IMBERT AND S. SERFATY, *Repeated games for eikonal equations, dislocation dynamics and parabolic integro-differential equations*, 2008. In preparation.
- [14] C. IMBERT AND P. E. SOUGANIDIS, *Phasefield theory for fractional reaction-diffusion equations and applications*, 2008. In preparation.
- [15] R. KOHN AND S. SERFATY, *A deterministic-control-based approach to fully nonlinear parabolic and elliptic equations*, tech. rep., Courant Institute, New York University, 2007.
- [16] R. V. KOHN AND S. SERFATY, *A deterministic-control-based approach to motion by curvature*, *Comm. Pure Appl. Math.*, 59 (2006), pp. 344–407.
- [17] A. SAYAH, *Équations d’Hamilton-Jacobi du premier ordre avec termes intégrodifférentiels. I. Unicité des solutions de viscosité*, *Comm. Partial Differential Equations*, 16 (1991), pp. 1057–1074.
- [18] D. SLEPČEV, *Approximation schemes for propagation of fronts with nonlocal velocities and Neumann boundary conditions*, *Nonlinear Anal.*, 52 (2003), pp. 79–115.