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# Fractional mean curvature flows

Cyril Imbert<sup>1</sup>

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ABSTRACT. This paper is concerned with the study of a geometric flow whose law involves a singular integral operator. Such an operator is used to define a non-local mean curvature of a set and for this reason, the flow is referred to as fractional. Such a flow appears in two important applications: dislocation dynamics and phasefield theory for fractional reaction-diffusion equations. It is first defined by using the level-set method. It is proved that it can be also defined in terms of generalized flows (Barles, Souganidis, 1998) so that phasefield theory for fractional reaction-diffusion can be treated (see the working paper of the author and Souganidis).

**Keywords:** fractional mean curvature flows, singular dislocation dynamics, level-set approach, stability results, comparison principles

**Mathematics Subject Classification:** 35F25, 35K55, 35K65, 49L25, 35A05

## 1 Introduction

In this paper, we define a geometric flow whose law is non-local. A general geometric flow of a set  $\Omega$  is defined by imposing the velocity of a point  $x$  along its outer normal  $n(x)$ . In our case, this velocity does not only depend on  $x$  and  $n$  but also on a *fractional mean curvature* at  $x$ . Our motivation comes from two different problems: dislocation dynamics and phasefield theory for fractional reaction diffusion equations.

Such non-local moving fronts first appeared in the study of dislocation dynamics [3]. Dislocations are linear defaults in crystals and the study of their motion gives rise to the study of the non-local geometric flow. In [3], the level-set formulation of such a problem yields to the study of the following eikonal equation

$$\partial_t u = \mu(\widehat{Du}) \left[ c_1(x) + \kappa[x, u] \right] |Du| \quad \text{in } (0, +\infty) \times \mathbb{R}^N \quad (1)$$

supplemented with the following initial condition

$$u(0, x) = u_0(x) \quad \text{in } \mathbb{R}^N \quad (2)$$

where  $\hat{p}$  denotes  $p/|p|$  if  $p \neq 0$ ,  $\mu$  denotes the mobility vector field,  $c_1(x)$  is a driving force and  $\kappa[x, u]$  is a force (called the Peach-Koehler force) applied to the curve ( $N = 2$  in this application). As the notation suggests it, this force depends on the curve itself.

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Precisely, it depends on it through the resolution of an elliptic equation (from linear elasticity). Hence, this integral operator is a Dirichlet-to-Neumann operator associated with an elliptic equation and is thus singular. In order to define solutions for small times, the authors of [3] consider a physically relevant regularized problem and  $\kappa[x, u]$  reduces to

$$\int_{\{z:u(z)\geq 0\}} c_0(z)dz$$

with  $c_0 \in W^{1,1}(\mathbb{R}^N)$ . The major technical difficulty is that  $c_0$  does not have a constant sign and inclusion of initial sets is not preserved through the flow. In particular, this is one of the reasons why solutions are constructed for small times. If  $c_1$  is assumed to be large enough, Alvarez, Cardaliaguet and Monneau [2] managed to prove the existence and uniqueness for large times. In order to get homogenization results, it is assumed in [11] that the negative part of  $c_0$  is concentrated at the origin. Hence,  $\kappa[x, u]$  is first defined as

$$\int E(u(x+z) - u(x))c_0(z)dz$$

where  $E$  is the integer part (up to a constant). After an approximation procedure, the problem can be reduced to the study of

$$\partial_t U = \left[ c_1(x) + \int (U(x+z) - U(x))c_0(z)dz \right] |DU|$$

where  $c_0$  is smooth, non-negative and of finite mass. We used the letter  $U$  instead of  $u$  in order to emphasize the fact that a change of unknown function is needed in order to reduce the original equation to this new one. Finally, still assuming that the negative part of  $c_0$  is concentrated at the origin, a good geometric definition of the flow is obtained in [9] by regularizing the Green function of the Dirichlet-to-Neumann operator. This is achieved by considering a formulation “à la Slepčev” of the geometric flow. The equation now writes

$$\partial_t u = \left[ c_1(x) + \int_{\{z:u(t,x+z)>u(t,x)\}} c_0(z)dz \right] |Du|. \quad (3)$$

Notice that in this paper, several fronts move in interaction and the motion of a single front is a special case. Eventually, [4] gave existence results of very weak solutions in a very general setting; in particular, uniqueness is generally lost.

Our second main motivation comes from phasefield theory of fractional reaction-diffusion equations [13]. If one considers for instance stochastic Ising models with Kacs potentials with very slow decay at infinity (like a power law with proper exponent), then the study of the resulting mean field equation (after proper rescaling) is closely related to phasefield theory for fractional reaction-diffusion equations such as

$$\partial_t u^\varepsilon + (-\Delta)^{\alpha/2} u^\varepsilon + \frac{1}{\varepsilon^{1+\alpha}} f(u^\varepsilon) = 0$$

where  $(-\Delta)^{\alpha/2}$  denotes the fractional Laplacian with  $\alpha \in (0, 1)$  (in the case presented here) and  $f$  is a bistable non-linearity. In particular, it is *essential* in the analysis to deal with singular potentials. Indeed, we have to be able to treat the case where

$$c_0(z) = \frac{1}{|z|^{N+\alpha}}$$

with  $\alpha \in (0, 1)$ . It is also convenient to use the notion of generalized flows introduced by Barles and Souganidis [7] in order to develop a phasefield theory for such reaction-diffusion equations. See [13] for further details and [10] for analogous problems.

In [8], it is proved that if  $c_0(z)$  is smooth and regular near the origin and exactly behaves like  $|z|^{-N-1}$  at infinity, then a proper rescaling of (3) converges towards the mean curvature motion.

We are going to use ideas introduced by Slepčev [16] as we did in [9] in order, on the one hand, to define properly the geometric motion (by using the level-set approach) and, on the other hand, to describe this flow in terms of generalized ones. Because  $\nu(dz) = c_0(z)dz$  is singular, we cannot define  $\kappa[x, u]$  as in (3). Instead, we must *compensate* the singularity in a geometrical way. This is achieved by defining the integral operator as follows

$$\kappa[x, u] = \int_{\{z:u(x+z)>u(x),Du(x)\cdot z<0\}} \nu(dz) - \int_{\{z:u(x+z)\leq u(x),Du(x)\cdot z\geq 0\}} \nu(dz).$$

Because the first part (resp. the second one) measures how concave (resp. convex) is set  $\Omega = \{z : u(x+z) > u(x)\}$  “near  $x$ ” in a non-local way and because  $\nu$  is singular, we refer to  $\kappa[x, u]$  in this setting as the *fractional mean curvature* of  $\Gamma$  at  $x$ . It is explained in details below (see Lemma 2) the rigorous links between the different formulations we considered up to now.

Equation (1) is a non-linear non-local Hamilton-Jacobi equation and a lot of papers are dedicated to the study of such equations. The main issues are the definition of viscosity solutions, their stability and strong uniqueness results. We somehow use the ideas from [16] and combine them with the ones from [5], even if the results of these two papers do not apply to our equation.

In order to understand better the properties of the fractional mean curvature flow, a deterministic zero-sum repeated game is constructed in [12] in the spirit of [15, 14].

From a physical point of view and as far as dislocation dynamics are concerned, the measure  $\nu(dz) = c_0(z)dz$  should be  $\nu(dz) = g(z/|z|)|z|^{-N-1}dz$  but in this case, the fractional mean curvature is not well defined (see Remark 1). It is also physically relevant to say that close the dislocation line, in the *core* of the dislocation, the potential should be regularized. On the other hand, it is important to assume that  $\nu(dz) \sim g(z/|z|)|z|^{-N-1}dz$ , since this prescribes the long range interaction between dislocation lines. Another way to understand this difficulty is to say that in the core of the dislocation, the potential is very singular and the singularity should be compensated at a higher (second) order. This

can explain the loss of inclusion principles for such flows (if one can define them for large times). Or one can think that in this case, the first term in such an expansion should be a mean curvature term; but in this case, an inclusion principle still holds true. Curvature terms are commonly used to describe dislocation dynamics and this can be relevant to add a curvature term in (1). We choose not to do so in order to avoid technicalities and keep clear important points in the proof of the stability result and the comparison principle.

The paper is organized as follows. In Section 2, we first give the precise assumptions we make on data. We next give the definition(s) of the fractional mean curvature  $\kappa[x, \cdot]$ . In Section 3, we first give the definition of viscosity solutions for (1), we then state and prove stability results. We next obtain strong uniqueness results by establishing comparison principles and we finally construct solutions of (1) by Perron's method. In Section 4, we verify that the 0-level set of the solution  $u$  we constructed in the previous section only depends on the 0-level set of the initial condition. This provides a level-set formulation of the geometric flow. In the last section, we give an alternative geometric definition of the flow in terms of generalized flows in the sense of [7].

**Notation.**  $\mathbb{S}^{N-1}$  denotes the unit sphere of  $\mathbb{R}^N$ . The ball of radius  $\delta$  centered at  $x$  is denoted by  $B_\delta(x)$ . If  $x = 0$ , we simply write  $B_\delta$  and if moreover  $\delta = 1$ , we write  $B$ . If  $p \in \mathbb{R}^N \setminus \{0\}$ ,  $\hat{p} = p/|p|$ . If  $A$  is a subset of  $\mathbb{R}^d$  with  $d = N, N + 1$  for instance, then  $A^c$  denotes its complementary. For two subsets  $A$  and  $B$ ,  $A \sqcup B$  denotes  $A \cup B$  and means that  $A \cap B = \emptyset$ .

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## 2 Preliminaries

In this section, we make precise the assumptions we need on data and we give several definitions of the fractional mean curvature.

### 2.1 Assumptions

Here are the assumptions we make throughout the paper.

ASSUMPTIONS.

- The mobility function  $\mu : \mathbb{S}^{N-1} \rightarrow (0, +\infty)$  is continuous.
- The driving force  $c_1 : \mathbb{R}^N \rightarrow \mathbb{R}$  is Lipschitz continuous.

- The singular measure  $\nu$  is a non-negative Radon measure satisfying

$$\left\{ \begin{array}{l} \text{for all } \delta > 0, \quad \nu(\mathbb{R}^N \setminus B_\delta) < +\infty, \\ \text{for all } r > 0, e \in \mathbb{S}^{N-1}, \quad \nu\{z \in B : r|z \cdot e| \leq |z - (z \cdot e)e|^2\} < +\infty, \\ \text{for all } e \in \mathbb{S}^{N-1}, \quad \delta \nu(\mathbb{R}^N \setminus B_\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0, \\ \text{for all } e \in \mathbb{S}^{N-1}, \quad r \nu\{z \in B : r|z \cdot e| \leq |z - (z \cdot e)e|^2\} \rightarrow 0 \text{ as } r \rightarrow 0 \end{array} \right. \quad (4)$$

( $B_\delta$  denotes the ball of radius  $\delta$  centered at the origin and  $B = B_1$ ) and the last limit is uniform with respect to unit vectors  $e \in \mathbb{S}^{N-1}$ .

- The initial datum  $u_0 : \mathbb{R}^N \rightarrow \mathbb{R}$  is bounded and Lipschitz continuous.

*Example 1.* The Standing Example for the singular measure is

$$\nu(dz) = g(z/|z|)|z|^{-N-\alpha}$$

with  $g : \mathbb{S}^{N-1} \rightarrow (0, +\infty)$  continuous and  $\alpha \in (0, 1)$ .

## 2.2 Fractional mean curvature

In this subsection, we make precise the definition of fractional mean curvature. Our definition extends the ones given in [8, 9] to the case of singular measures  $\nu(dz) = c_0(z)dz$ .

Let us define the fractional curvature of a smooth curve  $\Gamma = \{x \in \mathbb{R}^N : u(x) = 0\} = \partial\{x \in \mathbb{R}^N : u(x) > 0\}$  associated with  $\nu$ . If  $u$  is  $C^{1,1}$  and  $Du(x) \neq 0$ , then the following quantity is well defined (see Lemma 1 below)

$$\begin{aligned} \kappa_*^*[x, \Gamma] &= \kappa_*^*[x, u] = \kappa_+^*[x, u] - \kappa_-^*[x, u] \\ \kappa_*^-[x, \Gamma] &= \kappa_*^-[x, u] = \kappa_*^+[x, u] - \kappa_*^-[x, u] \end{aligned} \quad (5)$$

where

$$\begin{aligned} \kappa_*^+[x, u] &= \nu(z : u(x+z) > u(x), Du(x) \cdot z < 0) \\ \kappa_*^-[x, u] &= \nu(z : u(x+z) < u(x), Du(x) \cdot z > 0) \end{aligned} \quad (6)$$

and

$$\begin{aligned} \kappa_+^*[x, u] &= \nu(z : u(x+z) \geq u(x), Du(x) \cdot z \leq 0) \\ \kappa_-^*[x, u] &= \nu(z : u(x+z) \leq u(x), Du(x) \cdot z \geq 0) . \end{aligned}$$

We will see later (see Lemma 3 below) that these functions are semi-continuous and this explains the choice of notation we made. In order to understand the way these quantities are related to the geometry of the curve  $\{u = u(x)\}$ , it is convenient to write for instance

$$\kappa_*^+[x, u] = \nu(z : 0 < -Du(x) \cdot z < u(x+z) - u(x) - Du(x) \cdot z) .$$

As shown on Figure 1,  $\kappa^+[x, u]$  measures how concave is the curve at  $x$  and  $\kappa^-[x, u]$  how convex it is.

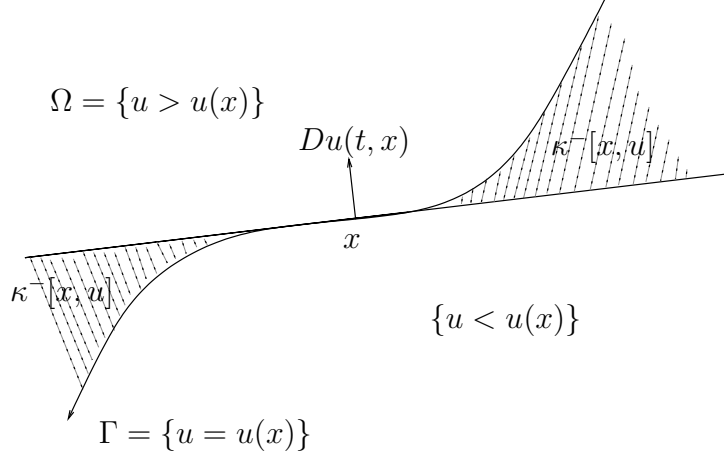


Figure 1: Fractional mean curvature of a curve

**Lemma 1 (Well-posedness of the fractional mean curvature).** *If  $u$  is  $C^2$  and its gradient  $Du(x) \neq 0$ , then  $\kappa_+^*[x, u]$  and  $\kappa_-^*[x, u]$  are finite.*

*Remark 1.* One can check that this lemma is false if  $\alpha = 1$  in the Standing Example 1.

*Proof.* Since  $\nu$  is bounded on  $\mathbb{R}^N \setminus B_\delta$  for all  $\delta > 0$ , it is enough to consider

$$\begin{aligned} (\kappa_+^*)^{1,\delta}[x, u] &= \nu(z \in B_\delta : u(x+z) \geq u(x), Du(x) \cdot z \leq 0) \\ &= \nu(z \in B_\delta : 0 \leq re \cdot z \leq u(x+z) - u(x) + re \cdot z) \end{aligned}$$

where  $r = |Du(x)| \neq 0$  and  $e = r^{-1}Du(x)$ . If now  $z_N$  denotes  $e \cdot z$  and  $z' = z - z_N e$ , and if we choose  $\delta$  such that  $r - C\delta > 0$  with  $C = \|D^2u\|_{L^\infty(B_1(x))}$ , we can write

$$\begin{aligned} (\kappa_+^*)^{1,\delta}[x, u] &\leq \nu(z \in B_\delta : 0 \leq rz_N \leq Cz_N^2 + C|z'|^2) \\ &\leq \nu(z \in B_\delta : 0 \leq C^{-1}(r - C\delta)z_N \leq |z'|^2) \end{aligned}$$

and the result now follows from Condition (4).  $\square$

The following lemma explains rigourously the link between (1) and (3) and the link with the formulation used in [9] in the case where  $\nu$  is a bounded measure.

**Lemma 2 (Link with regular dislocation dynamics).** *Consider  $c_0 \in L^1(\mathbb{R}^N)$  such that  $c_0(x) = c_0(-x)$ . Then*

$$\begin{aligned} \int_{\{z:u(t,x+z)>u(t,x)\}} c_0(z)dz &= \frac{1}{2} \int c_0 + \kappa_*[x, u] \\ \int E^*(u(x+z) - u(x))c_0(z)dz &= \frac{1}{2}\kappa^*[x, u] \\ \int E_*(u(x+z) - u(x))c_0(z)dz &= \frac{1}{2}\kappa_*[x, u] \end{aligned}$$

with  $\nu(dz) = c_0(z)dz$ .

Since the proof is elementary, we omit it.

### 3 Viscosity solutions for (1)

#### 3.1 Definitions

The viscosity solution theory introduced in [16] suggests that the good notion of solution for the fractional equation (1) is the following one.

**Definition 1 (Viscosity solutions for (1)).** 1. An upper semi-continuous function  $u : [0, T] \times \mathbb{R}^N$  is a viscosity subsolution of (1) if for every smooth test-function  $\phi$  such that  $u - \phi$  admits a global zero maximum at  $(t, x)$ , we have

$$\partial_t \phi(t, x) \leq \mu(\widehat{D\phi(t, x)}) \left[ c_1(x) + \kappa^*[x, \phi(t, \cdot)] \right] |D\phi|(t, x) \quad (7)$$

if  $D\phi(t, x) \neq 0$  and  $\partial_t \phi(t, x) \leq 0$  if not.

2. A lower semi-continuous function  $u$  is a viscosity supersolution of (1) if for every smooth test-function  $\phi$  such that  $u - \phi$  admits a global minimum 0 at  $(t, x)$ , we have

$$\partial_t \phi(t, x) \geq \mu(\widehat{D\phi(t, x)}) \left[ c_1(x) + \kappa_*[x, \phi(t, \cdot)] \right] |D\phi|(x_0, t_0) \quad (8)$$

if  $D\phi(t, x) \neq 0$  and  $\partial_t \phi(t, x) \geq 0$  if not.

3. A locally bounded function  $u$  is a viscosity solution of (1) if  $u^*$  (resp.  $u_*$ ) is a subsolution (resp. supersolution).

*Remark 2.* Given  $\delta > 0$ , the global extrema in Definition 1 can be assumed to be strict in a ball of radius  $\delta$  centered at  $(t, x)$ . Such a result is classically expected and the reader can have a look, for instance, at the proof of the stability result in [5].

If one uses the notation introduced in [16], the equation reads

$$\partial_t u + F(x, Du, \{z : u(x+z) \geq u(x)\}) = 0$$

with, for  $x, p \in \mathbb{R}^N$  and  $K \subset \mathbb{R}^N$ ,

$$F(x, p, K) = \begin{cases} -\mu(\hat{p}) \left[ c_1(x) + \nu(K \cap \{p \cdot z \leq 0\}) - \nu(K^c \cap \{p \cdot z > 0\}) \right] |p| & \text{if } p \neq 0, \\ 0 & \text{if not} \end{cases}$$

where  $K^c$  is the complementary set of  $K$ . With this notation in hand, one can check that this non-linearity does not satisfy Assumption (F5) of [16]. The idea is to check that, somehow, Assumption (NLT) in [5] is satisfied and stability results thus hold true.

Let us be more precise. We previously associated with  $\kappa[\cdot, \cdot]$  the following non-local operators (see the proof of Lemma 1)

$$\begin{aligned} (\kappa_*^+)^{1, \delta}[x, \phi] &= \nu(z \in B_\delta : \phi(x+z) > \phi(x), z \cdot D\phi(x) < 0), \\ (\kappa_*^+)^{2, \delta}[x, p, \phi] &= \nu(z \notin B_\delta : \phi(x+z) > \phi(x), z \cdot p < 0). \end{aligned} \quad (9)$$

In the same way, we can define



- the negative non-local curvature operators  $(\kappa_*^-)^{i,\delta}$ ,  $i = 1, 2$ ,
- upper semi-continuous envelopes of these four integral operators  $(\kappa_{\pm}^*)^{i,\delta}$ ,  $i = 1, 2$ ,
- and lower/upper semi-continuous total non-local curvature operators  $(\kappa_*)^{i,\delta}$ ,  $(\kappa^*)^{i,\delta}$ ,  $i = 1, 2$ .

By using the idea of Lemma 2, it is easy to see that

$$\begin{cases} (\kappa^*)^{2,\delta}[x, p, u] &= \nu(z \notin B_\delta : u(t, x+z) \geq u(t, x)) - \frac{1}{2}\nu(B_\delta^c) =: (\kappa^*)^{2,\delta}[x, u], \\ (\kappa_*)^{2,\delta}[x, p, u] &= \nu(z \notin B_\delta : u(t, x+z) > u(t, x)) - \frac{1}{2}\nu(B_\delta^c) =: (\kappa_*)^{2,\delta}[x, u]. \end{cases} \quad (10)$$

We can now state an equivalent definition of viscosity solutions of (1).

**Definition 2 (Equivalent definition).** 1. *An upper semi-continuous function  $u : [0, T] \times \mathbb{R}^N$  is a viscosity subsolution of (1) if for every smooth test-function  $\phi$  such that  $u - \phi$  admits a maximum 0 at  $(t, x)$  on  $B_\delta(t, x)$ , we have*

$$\partial_t \phi(t, x) \leq \mu(\widehat{D\phi(t, x)}) \left[ c_1(x) + (\kappa^*)^{1,\delta}[x, \phi(t, \cdot)] + (\kappa^*)^{2,\delta}[x, u(t, \cdot)] \right] |D\phi|(t, x) \quad (11)$$

*if  $D\phi(t, x) \neq 0$  and  $\partial_t \phi(t, x) \leq 0$  if not.*

2. *A lower semi-continuous function  $u$  is a viscosity supersolution of (1) if for every smooth test-function  $\phi$  such that  $u - \phi$  admits a global minimum 0 at  $(t, x)$ , we have*

$$\partial_t \phi(t, x) \geq \mu(\widehat{D\phi(t, x)}) \left[ c_1(x) + (\kappa_*)^{1,\delta}[x, \phi(t, \cdot)] + (\kappa_*)^{2,\delta}[x, u(t, \cdot)] \right] |D\phi|(x_0, t_0) \quad (12)$$

*if  $D\phi(t, x) \neq 0$  and  $\partial_t \phi(t, x) \geq 0$  if not.*

3. *A continuous function  $u$  is a viscosity solution of (1) if it is both a sub and supersolution.*

*Remark 3.* Remark 2 applies to the equivalent definition too.

*Remark 4.* Definition 2 seems to depend on  $\delta$ . But since all these definitions are equivalent to Definition 1, it does not depend on it. Hence, when proving that a function is a solution of (1), it is enough to do it for a fixed (or not)  $\delta > 0$ .

## 3.2 Stability results

**Theorem 1 (Discontinuous stability).** • *Let  $(u_n)_{n \geq 1}$  be a family of subsolutions of (1) that is locally bounded, uniformly with respect to  $n$ . Then its relaxed upper limit  $u^*$  is a subsolution of (1).*

- *If moreover,  $u_n(0, x) = u_0^n(x)$ , then for all  $x \in \mathbb{R}^N$*

$$\bar{u}(0, x) \leq \bar{u}_0(x)$$

*where  $\bar{u}_0$  is the upper semicontinuous envelope of  $u_0^n$ .*

- Let  $(u_\alpha)_{\alpha \in \mathcal{A}}$  be a family of subsolutions of (1) that is locally bounded, uniformly with respect to  $\alpha \in \mathcal{A}$ . Then  $\bar{u}$ , the upper semicontinuous envelope of  $\sup_\alpha u_\alpha$  is a subsolution of (1).

Even if this result follows from ideas introduced in [5] together with classical ones, we give a detailed proof for the sake of completeness.

*Proof.* We only prove the first part of the theorem since it is easy to adapt it to get a proof of the third part. The second one is very classical and can be adapted from [1] for instance.

Consider a test function  $\varphi$  such that  $u^* - \varphi$  attains a global maximum at  $(t, x)$ . We can assume (see Remark 2) that  $u^* - \varphi$  attains a strict maximum at  $(t, x)$  on  $B_\delta(t, x)$ . Consider a subsequence  $p = p(n)$  and  $x_p$  such that

$$u^*(t, x) = \lim_{n \rightarrow +\infty} u_{p(n)}(t_n, x_n).$$

Classical arguments show that  $u_p - \varphi$  attains a maximum on  $B_\delta(t, x)$  at  $(s_p, y_p) \in B_\delta(t, x)$  and that

$$(s_p, y_p) \rightarrow (t, x) \quad \text{and} \quad u_p(s_p, y_p) \rightarrow u^*(t, x).$$

Since  $u_p$  is a subsolution of (1), we have

$$\partial_t \varphi(s_p, y_p) \leq \mu(\widehat{D\varphi}(s_p, y_p)) \left[ c_1(y_p) + (\kappa^*)^{1,\delta}[y_p, \varphi(s_p, \cdot)] + (\kappa^*)^{2,\delta}[y_p, D_x \varphi(s_p, y_p), u(s_p, \cdot)] \right] |D\varphi|(s_p, y_p)$$

if  $D\varphi(t_p, x_p) \neq 0$  and  $\partial_t \varphi(t_p, x_p) \leq 0$  if not. If there exists a subsequence  $q$  of  $p$  such that  $D\varphi(s_q, y_q) = 0$ , then it is easy to conclude. We thus now assume that  $D\varphi(s_p, y_p) \neq 0$  for  $p$  large enough. In view of the continuity of  $\mu$  and  $c_1$ , the following technical lemma (whose proof is given in Appendix) permits to conclude.

**Lemma 3.** *Assume that  $D\varphi(s_p, y_p) \neq 0$  for  $p$  large enough.*

- Assume moreover that  $D\varphi(t, x) \neq 0$ . Then

$$(s, y) \mapsto (\kappa^*)^{1,\delta}[y, \varphi(s, \cdot)] \quad \text{and} \quad (s, y) \mapsto (\kappa^*)^{2,\delta}[y, D_x \varphi(s, y), u_p(s, \cdot)]$$

are well defined for  $i = 1, 2$  in a neighbourhood of  $(t, x)$  and

$$\limsup_p \left\{ (\kappa^*)^{1,\delta}[y_p, \varphi(s_p, \cdot)] \right\} \leq (\kappa^*)^{1,\delta}[x, \varphi(t, \cdot)]$$

$$\limsup_p \left\{ (\kappa^*)^{2,\delta}[y_p, D_x \varphi(s_p, y_p), u(s_p, \cdot)] \right\} \leq (\kappa^*)^{2,\delta}[x, D_x \varphi(t, x), \varphi(t, \cdot)]$$

as soon as  $u_p(s_p, y_p) \rightarrow u(t, x)$  as  $p \rightarrow +\infty$ .

- Assume now that  $D\varphi(t, x) = 0$ . Then, for  $i = 1, 2$ ,

$$\left[ (\kappa^*)^{1,\delta} [y_p, \varphi(s_p, \cdot)] + (\kappa^*)^{2,\delta} [y_p, D\varphi(s_p, y_p), u(s_p, \cdot)] \right] |D\varphi|(s_p, y_p) \rightarrow 0 \quad \text{as } p \rightarrow +\infty.$$

As we shall see, this lemma is a consequence of the following one.

**Lemma 4 ([16]).** Consider  $f_p$  and  $g_p$  two sequences of measurable functions on a set  $U$  and  $f \geq \limsup^* f_p$ ,  $g \geq \limsup^* g_p$ , and  $a_p, b_p$  two sequences of real numbers converging to 0. Then

$$\nu(\{f_p \geq a_p, g_p \geq b_p\} \setminus \{f \geq 0, g \geq 0\}) \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

We mention that in [16], the measure is not singular and there is only one sequence of measurable functions but the reader can check that the slightly more general version we gave here can be proven exactly with the same arguments. An immediate consequence of the lemma is the following inequality

$$\limsup_p \nu(\{f_p \geq a_p, g_p \geq b_p\}) \leq \nu(\{f \geq 0, g \geq 0\}).$$

*Proof of Lemma 3.* Let us first assume that  $D\varphi(t, x) \neq 0$ . In this case, for  $(s, y)$  close to  $(t, x)$ ,  $D\varphi(s, y) \neq 0$  and all the integral operators we consider here are well defined (see Lemma 1). Recall next that, for  $i = 1, 2$ ,  $(\kappa^*)^{i,\delta} = (\kappa_+^*)^{i,\delta} - (\kappa_*^-)^{i,\delta}$ . Hence, it is enough to prove that

$$\begin{aligned} \limsup_p \left\{ (\kappa_+^*)^{1,\delta} [y_p, \varphi(s_p, \cdot)] \right\} &\leq (\kappa_+^*)^{1,\delta} [x, \varphi(t, \cdot)], \\ \liminf_p \left\{ (\kappa_*^-)^{1,\delta} [y_p, \varphi(s_p, \cdot)] \right\} &\geq (\kappa_*^-)^{1,\delta} [x, \varphi(t, \cdot)], \\ \limsup_p \left\{ (\kappa_+^*)^{2,\delta} [y_p, D_x \varphi(s_p, y_p), u_p(s_p, \cdot)] \right\} &\leq (\kappa_+^*)^{2,\delta} [x, D_x \varphi(t, x), u^*(t, \cdot)], \\ \liminf_p \left\{ (\kappa_*^-)^{2,\delta} [y_p, D_x \varphi(s_p, y_p), u_p(s_p, \cdot)] \right\} &\leq (\kappa_*^-)^{2,\delta} [x, D_x \varphi(t, x), u^*(t, \cdot)]. \end{aligned}$$

In order to prove the the first inequality above for instance, choose  $f_p(z) = \varphi(s_p, y_p + z) - \varphi(t, x)$ ,  $a_p = \varphi(s_p, y_p) - \varphi(t, x)$ ,  $g_p(z) = -D\varphi(s_p, y_p) \cdot z$ ,  $b_p = 0$  in Lemma 4.

We now turn to the case  $D\varphi(t, x) = 0$ . We look for  $\delta = \delta_p$  that goes to 0 as  $p \rightarrow +\infty$  such that

$$(\kappa_+^*)^{1,\delta_p} [y_p, \varphi(s_p, \cdot)] |D\varphi(s_p, y_p)| \rightarrow 0 \quad \text{and} \quad (\kappa_*^-)^{1,\delta_p} [y_p, \varphi(s_p, \cdot)] |D\varphi(s_p, y_p)| \rightarrow 0$$

as  $p \rightarrow +\infty$ . This is enough to conclude since Condition (4) implies that

$$\begin{aligned} (\kappa_+^*)^{2,\delta_p} [y_p, D\varphi(s_p, y_p), u(s_p, \cdot)] |D\varphi(s_p, y_p)| &\rightarrow 0 \\ (\kappa_*^-)^{2,\delta_p} [y_p, D\varphi(s_p, y_p), u(s_p, \cdot)] |D\varphi(s_p, y_p)| &\rightarrow 0. \end{aligned}$$

We only prove that the first limit equals zero since the argument is similar for the second one. If  $r_p$  denotes  $|D\varphi(s_p, y_p)|$  and  $e_p$  denotes  $-r_p^{-1}D\varphi(s_p, y_p)$ , and  $z_N = e_p \cdot z$  and  $z' = z - z_N e_p$ , then

$$\begin{aligned} & (\kappa_+^*)^{1,\delta}[y_p, \varphi(s_p, \cdot)]|D\varphi(s_p, y_p)| \\ &= r_p \nu(z \in B_{\delta_p} : 0 \leq r_p e_p \cdot z \leq \varphi(s_p, y_p + z) - \varphi(s_p, y_p) + r_p e_p \cdot z) \\ &\leq r_p \nu(z \in B_{\delta_p} : 0 \leq r_p z_N \leq C|z'|^2 + Cz_N^2) \\ &\leq r_p \nu(z \in B_{\delta_p} : 0 \leq r_p z_N \leq C|z'|^2 + C\delta_p z_N) \end{aligned}$$

where  $C$  is a bound for second derivatives of  $\varphi$  around  $(t, x)$ . Now if we choose  $\delta_p = r_p/(2C)$ , we get

$$\begin{aligned} (\kappa_+^*)^{1,\delta}[y_p, \varphi(s_p, \cdot)]|D\varphi(s_p, y_p)| &\leq r_p \nu(z \in B_{\delta_p} : 0 \leq (r_p/2C)z_N \leq |z'|^2) \\ &\leq r_p \nu(z \in B : 0 \leq (r_p/2C)z_N \leq |z'|^2) \end{aligned}$$

and the last limit in (4) permits now to conclude. □

□

### 3.3 Existence and uniqueness results

Let us first state a strong uniqueness result.

**Theorem 2 (Comparison principle).** *Assume that  $\nu$  satisfies (4). Assume moreover that for all  $e \in \mathbb{S}^{N-1}$  and  $r \in (0, 1)$*

$$r \nu\{z \in B_\delta : r|z \cdot e| \leq |z - (z \cdot e)e|^2\} \rightarrow 0 \quad \text{as } \delta \rightarrow 0 \quad (13)$$

*uniformly in  $e$  and  $r \in (0, 1)$ . Consider a bounded and Lipschitz continuous function  $u_0$ . Let  $u$  (resp.  $v$ ) be a bounded subsolution (resp. bounded supersolution) of (1). If  $u(0, x) \leq u_0(x) \leq v(0, x)$ , then  $u \leq v$  on  $(0, +\infty) \times \mathbb{R}^N$ .*

The proof is quite classical. The main difficulty is to deal with the singularity of the measure. This is the reason why the argument used in [8, 9] for instance does not apply. The argument we give below somehow simplify the old argument in the regular case. See in particular Lemma 5 below.

*Proof of Theorem 2.* We classically consider  $M = \sup_{t,x} \{u(t, x) - v(t, x)\}$  and argue by contradiction by assuming  $M > 0$ . We next consider the following approximation of  $M$

$$\tilde{M}_{\varepsilon, \alpha} = \sup_{t,s>0,x,y \in \mathbb{R}^N} \left\{ u(t, x) - v(s, y) - \frac{(t-s)^2}{2\gamma} - \frac{|x-y|^2}{2\varepsilon} - \eta t - \alpha|x|^2 \right\}.$$

Since  $u$  and  $v$  are bounded, this supremum is attained at a point  $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y})$ . We first observe that  $\tilde{M}_{\varepsilon, \alpha} \geq M/2 \geq 0$  for  $\eta$  and  $\alpha$  small enough. Since  $u$  and  $v$  are bounded, this implies in particular

$$\frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon} + \alpha|\tilde{x}|^2 \leq C_0 \quad (14)$$

where  $C_0 = \|u\|_\infty + \|v\|_\infty$ .

Classical results about penalization imply that  $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y}) \rightarrow (\bar{t}, \bar{t}, \bar{x}, \bar{y})$  as  $\gamma \rightarrow 0$  and  $(\bar{t}, \bar{t}, \bar{x}, \bar{y})$  realizes the following supremum

$$M_{\varepsilon, \alpha} = \sup_{t>0, x, y \in \mathbb{R}} \left\{ u(t, x) - v(t, y) - \frac{|x - y|^2}{2\varepsilon} - \eta t - \alpha |x|^2 \right\}.$$

We claim next that this supremum cannot be achieved at  $t = 0$  if  $\varepsilon, \alpha, \eta$  are small enough. To see this, remark first that Recall that  $M_{\varepsilon, \alpha} \geq M/2 \geq 0$  for  $\eta$  and  $\alpha$  small enough and, if  $\bar{t} = 0$ , use the fact that  $u_0$  is Lipschitz continuous and get

$$0 < \frac{M}{2} \leq \sup_{x, y \in \mathbb{R}^N} \left\{ u_0(x) - u_0(y) - \frac{|x - y|^2}{2\varepsilon} \right\} \leq \sup_{r>0} \left\{ C_0 r - \frac{r^2}{2\varepsilon} \right\} = \frac{1}{2} C_0^2 \varepsilon$$

and this is obviously false if  $\varepsilon$  is small enough. We conclude that, if the four parameters are small enough,  $\tilde{t} > 0$  and  $\tilde{s} > 0$ . Hence, we can write two viscosity inequalities, for all  $\delta > 0$ ,

$$\begin{aligned} \eta + \frac{\tilde{t} - \tilde{s}}{\gamma} &\leq (c(\tilde{x}) + (\kappa^*)^{1, \delta} [\tilde{x}, \phi_u(\tilde{t}, \cdot)] + (\kappa^*)^{2, \delta} [\tilde{x}, u(\tilde{t}, \cdot)]) |\tilde{p} + 2\alpha \tilde{x}| \\ \frac{\tilde{t} - \tilde{s}}{\gamma} &\geq (c(\tilde{y}) + (\kappa_*)^{1, \delta} [\tilde{y}, \phi_v(\tilde{s}, \cdot)] + (\kappa_*)^{2, \delta} [\tilde{y}, v(\tilde{s}, \cdot)]) |\tilde{p}| \end{aligned}$$

where  $\tilde{p} = \frac{\tilde{x} - \tilde{y}}{\varepsilon}$  and

$$\phi_u(t, x) = \frac{(t - \tilde{s})^2}{2\gamma} + \frac{|x - \tilde{y}|^2}{2\varepsilon} + \eta t + \alpha |x|^2, \quad \phi_v(s, y) = -\frac{(s - \tilde{t})^2}{2\gamma} - \frac{|y - \tilde{x}|^2}{2\varepsilon} - \eta \tilde{t} - \alpha |\tilde{x}|^2.$$

Subtracting these inequalities yield

$$\eta \leq \|Dc\|_\infty |\tilde{x} - \tilde{y}| |\tilde{p}| + 2\|c\|_\infty \alpha |\tilde{x}| + T_{nl}$$

where

$$\begin{aligned} T_{nl} &= \left( (\kappa^*)^{1, \delta} [\tilde{x}, \phi_u(\tilde{t}, \cdot)] + (\kappa^*)^{2, \delta} [\tilde{x}, u(\tilde{t}, \cdot)] \right) |\tilde{p} + 2\alpha \tilde{x}| \\ &\quad - \left( (\kappa_*)^{1, \delta} [\tilde{y}, \phi_v(\tilde{s}, \cdot)] + (\kappa_*)^{2, \delta} [\tilde{y}, v(\tilde{s}, \cdot)] \right) |\tilde{p}|. \end{aligned}$$

Our task is now to find  $\delta = \delta(\alpha, \varepsilon)$  so that the right hand side of this inequality is small when the four parameters are small. Precisely, we have

**Lemma 5.** *By using (13), we have*

$$T_{nl} \leq \frac{1}{\varepsilon} o_\delta(1) + \frac{1}{\varepsilon} o_\alpha(1) + C \frac{\sqrt{\alpha}}{\delta}$$

where  $C$  only depends on  $\nu$  and  $\|u\|_\infty + \|v\|_\infty$ .

The proof of this lemma is postponed. We thus get (recall that  $\tilde{p} = (\tilde{x} - \tilde{y})/\varepsilon$ )

$$\eta \leq C\left(\frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon} + \sqrt{\alpha} + \frac{\sqrt{\alpha}}{\delta}\right) + \frac{1}{\varepsilon}(o_\alpha(1) + o_\delta(1)).$$

where  $C$  only depends on  $c, \nu$  and  $\|u\|_\infty + \|v\|_\infty$ . By letting successively  $\alpha, \delta$  and  $\varepsilon$  go to 0, we thus get a contradiction. This achieves the proof of the comparison principle.  $\square$

*Proof of Lemma 5.* We first write

$$\begin{aligned} T_{nl} &\leq |(\kappa^*)^{1,\delta}|[\tilde{x}, \phi_u(\tilde{t}, \cdot)]|\tilde{p} + 2\alpha\tilde{x}| + |(\kappa_*)^{1,\delta}|[\tilde{y}, v(\tilde{s}, \cdot)]|\tilde{p}| \\ &\quad + 2|(\kappa^*)^{2,\delta}|[\tilde{x}, u(\tilde{t}, \cdot)]|\alpha\tilde{x}| + \left( (\kappa^*)^{2,\delta}[\tilde{x}, u(\tilde{t}, \cdot)] - (\kappa_*)^{2,\delta}[\tilde{y}, v(\tilde{s}, \cdot)] \right)|\tilde{p}|. \end{aligned}$$

We thus estimate the right hand side of the previous inequality. We start with the first integral terms.

$$\begin{aligned} |(\kappa^*)^{1,\delta}|[\tilde{x}, \phi_u(\tilde{t}, \cdot)] &\leq (\kappa_+^*)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t}, \cdot)] + (\kappa_*^-)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t}, \cdot)] \\ &\leq \nu(z \in B_\delta : 0 \leq -(\tilde{p} + 2\alpha\tilde{x}) \cdot z \leq (\alpha + 1/(2\varepsilon))|z|^2) \\ &\quad + \nu(z \in B_\delta : 0 > -(\tilde{p} + 2\alpha\tilde{x}) \cdot z > (\alpha + 1/(2\varepsilon))|z|^2) \\ &\leq \nu(z \in B_\delta : |\varepsilon\tilde{p} + 2\varepsilon\alpha\tilde{x}| |e \cdot z| \leq |z|^2). \end{aligned}$$

If now  $r_{\alpha,\varepsilon}$  denotes  $|\varepsilon\tilde{p} + 2\varepsilon\alpha\tilde{x}|$  and we choose  $\delta \leq \frac{1}{2}r_{\alpha,\varepsilon}$ , (13) implies we get

$$|(\kappa^*)^{1,\delta}|[\tilde{x}, \phi_u(\tilde{t}, \cdot)]|\tilde{p} + 2\alpha\tilde{x}| \leq \left\{ \sup_{e \in \mathbb{S}^{N-1}} \nu(z \in B_\delta : r_{\alpha,\varepsilon}|e \cdot z| \leq |z - (e \cdot z)e|^2) \right\} r_{\alpha,\varepsilon} = \frac{1}{\varepsilon}o_\delta(1).$$

Arguing similarly, we get for  $\delta < \frac{1}{2}r'_\varepsilon = \frac{1}{2}\varepsilon|\tilde{p}|$ ,

$$|(\kappa_*)^{1,\delta}|[\tilde{y}, \phi_v(\tilde{s}, \cdot)]|\tilde{p}| \leq \frac{1}{\varepsilon} \left\{ \sup_{e \in \mathbb{S}^{N-1}} \nu(z \in B_\delta : r'_{\alpha,\varepsilon}|e \cdot z| \leq |z - (e \cdot z)e|^2) \right\} r'_\varepsilon = \frac{1}{\varepsilon}o_\delta(1).$$

As far as the third integral term is concerned, we simply write

$$2|(\kappa^*)^{2,\delta}|[\tilde{x}, u(\tilde{t}, \cdot)]|\alpha\tilde{x}| \leq 2\nu(B_\delta^c)|\alpha\tilde{x}| \leq \frac{\tilde{C}_0}{\delta}\sqrt{\alpha}$$

(we used (14)). We now turn to the last integral terms. In view of (10), we can write

$$\begin{aligned} (\kappa^*)^{2,\delta}[\tilde{x}, u(\tilde{t}, \cdot)] - (\kappa_*)^{2,\delta}[\tilde{y}, v(\tilde{s}, \cdot)] &= \nu(z \notin B_\delta : u(\tilde{t}, \tilde{x} + z) \geq u(\tilde{t}, \tilde{x})) \\ &\quad - \nu(z \notin B_\delta : v(\tilde{s}, \tilde{y} + z) < v(\tilde{s}, \tilde{y})). \end{aligned}$$

Remark next that the definition of  $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y})$  implies the following inequality: for all  $z \in \mathbb{R}^N$ ,

$$u(\tilde{t}, \tilde{x} + z) - u(\tilde{t}, \tilde{x}) \leq v(\tilde{s}, \tilde{y} + z) - v(\tilde{s}, \tilde{y}) + \alpha\psi(z)$$

where  $\psi(z) = |z|^2 + 2\tilde{x} \cdot z$ . Hence,

$$\begin{aligned} & (\kappa^*)^{2,\delta}[\tilde{x}, u(\tilde{t}, \cdot)] - (\kappa_*)^{2,\delta}[\tilde{y}, v(\tilde{s}, \cdot)] \\ & \leq \nu(z \notin B_\delta : v(\tilde{s}, \tilde{y} + z) \geq v(\tilde{s}, \tilde{y}) - \alpha\psi(z)) - \nu(z \notin B_\delta : v(\tilde{s}, \tilde{y} + z) > v(\tilde{s}, \tilde{y})). \end{aligned}$$

If now  $\bar{V}(z)$  (resp.  $\underline{V}$ ) denotes the upper (resp. lower) relaxed semi-limit of the family of bounded functions  $v(\tilde{s}, \tilde{y} + z) - v(\tilde{s}, \tilde{y})$  indexed by  $\alpha > 0$ , Lemma 4 implies

$$\begin{aligned} \limsup_{\alpha \rightarrow 0} \nu \left( z \notin B_\delta : v(\tilde{s}, \tilde{y} + z) \geq v(\tilde{s}, \tilde{y}) + \alpha\psi(z) \right) & \leq \nu(z \notin B_\delta : \bar{V}(z) \geq 0) \\ \liminf_{\alpha \rightarrow 0} \nu \left( z \notin B_\delta : (v(\tilde{s}, \tilde{y} + z) - v(\tilde{s}, \tilde{y})) > 0 \right) & \geq \nu(z \notin B_\delta : \underline{V}(z) > 0) \end{aligned}$$

and we conclude that

$$\limsup_{\alpha \rightarrow 0} (\kappa^*)^{2,\delta}[\tilde{x}, u(\tilde{t}, \cdot)] - (\kappa_*)^{2,\delta}[\tilde{y}, v(\tilde{s}, \cdot)] \leq 0.$$

Estimate (14) permits now to conclude.  $\square$

We now turn to the existence result.

**Theorem 3 (Existence).** *Let  $u_0$  be Lipschitz continuous and bounded. There then exists a unique bounded uniformly continuous viscosity solution  $u$  of (1).*

*Proof.* We first construct a solution for regular initial data. Precisely, we first assume that  $u_0 \in C_b^2(\mathbb{R}^N)$  (the function and its first and second derivatives are bounded).

Because we can apply Perron's method, it is enough to construct a sub- and a supersolution  $u^\pm$  to (1) such that  $(u^+)_*(0, x) = (u^-)^*(0, x) = u_0(x)$ . We assert that  $u^\pm(t, x) = u_0(x) \pm Ct$  are respectively a super- and a subsolution of (1) for  $C$  large enough. To see this, we first prove that there exists  $C_0 = C_0(\|D^2u_0\|_\infty)$  such that for all  $x \in \mathbb{R}^N$  such that  $Du_0(x) \neq 0$ , we have

$$(|\kappa_*|[x, u_0] + |\kappa^*|[x, u_0])|Du_0(x)| \leq C_0. \quad (15)$$

In order to prove this estimate, we simply write for  $x$  such that  $Du_0(x) \neq 0$

$$\begin{aligned} ((\kappa_+^*)^{1,\delta}[x, u_0] + (\kappa_-^*)^{1,\delta}[x, u_0])|Du_0(x)| & \leq 2\nu(z \in B_\delta : r|e \cdot z| \leq \frac{1}{2}\|D^2u_0\|_\infty|z|^2)r \\ & \leq 2\nu(z \in B_\delta : r|e \cdot z| \leq C|z - (e \cdot z)e|^2)r \leq C_\nu \end{aligned}$$

where  $r = |Du_0(x)|$ ,  $C = \max(\|Du_0\|_\infty, 1)$  and  $e = Du_0(x)/r$  and  $\delta = r/(2C)$  and  $C_\nu$  is given by (4). On the other hand

$$((\kappa_+^*)^{2,\delta}[x, u_0] + (\kappa_-^*)^{2,\delta}[x, u_0])|Du_0(x)| \leq \frac{C_\nu}{\delta}r = 2C_\nu C.$$

We thus get Estimate (15).

If now  $u_0$  is not regular, we approximate it with  $u_0^n \in C_b^2(\mathbb{R}^N)$  and can prove that the corresponding sequence of solutions  $u_n$  converges locally uniformly towards a solution  $u$ . Since this is very classical, we omit details (see for instance [1]).  $\square$

## 4 The level-set approach

In the previous section, we proved that we are able to solve (1) in the case of singular measures satisfying (4) and (13). In the present section, we prove that it permits to define a geometric flow. Precisely, we first prove (Theorem 4) that if  $u$  and  $v$  are solutions of (1) associated with two different initial data  $u_0$  and  $v_0$  that have the same 0-level sets, then so have  $u$  and  $v$ . Hence, the geometric flows is obtained by considering the 0-level sets of the solution  $u$  of (1) for any (Lipschitz continuous) initial datum. We also describe (Theorem 5) the maximal and minimal discontinuous solutions of (1) associated with an important class of discontinuous initial data.

**Theorem 4 (Consistency of the definition).** *Let  $u_0$  and  $v_0$  be two bounded and Lipschitz continuous functions and consider the viscosity solutions  $u, v$  associated with these initial conditions. If*

$$\begin{aligned} \{x \in \mathbb{R}^N : u_0(x) > 0\} &= \{x \in \mathbb{R}^N : v_0(x) > 0\} \\ \{x \in \mathbb{R}^N : u_0(x) < 0\} &= \{x \in \mathbb{R}^N : v_0(x) < 0\} \end{aligned}$$

then, for all time  $t > 0$ ,

$$\begin{aligned} \{x \in \mathbb{R}^N : u(t, x) > 0\} &= \{x \in \mathbb{R}^N : v(t, x) > 0\} \\ \{x \in \mathbb{R}^N : u(t, x) < 0\} &= \{x \in \mathbb{R}^N : v(t, x) < 0\} \end{aligned}$$

In view of the techniques used to prove the consistency of the definition of local geometric fronts (see for instance [6]), it is clear that this result is a straightforward consequence of the following proposition.

**Proposition 1 (Equation (1) is geometric).** *Consider  $u : [0, +\infty) \times \mathbb{R}^N$  a bounded subsolution of (1) and  $\theta : \mathbb{R} \rightarrow \mathbb{R}$  a upper semi-continuous non-decreasing function. Then  $\theta(u)$  is also a subsolution of (1).*

Such a proposition is classical by now. It is proved by regularizing  $\theta$  (in a proper way) with a strictly increasing function  $\theta^n$ , by remarking that  $\kappa^*[x, \theta^n(u)] = \kappa^*[x, u]$  in this case, and by using discontinuous stability. Details are left to the reader.

Thanks to Theorem 4, we can define a geometric flow in the following way. Given  $(\Gamma_0, D_0^+, D_0^-)$  such that  $\Gamma_0$  is closed,  $D_0^\pm$  are open and  $\mathbb{R}^N = \Gamma_0 \sqcup D_0^+ \sqcup D_0^-$ , we can write

$$D_0^+ = \{x \in \mathbb{R}^N : u_0(x) > 0\}, D_0^- = \{x \in \mathbb{R}^N : u_0(x) < 0\}, \Gamma_0 = \{x \in \mathbb{R}^N : u_0(x) = 0\}$$

for some bounded Lipschitz continuous function  $u_0$  (for instance the signed distance function). If  $u$  is the solution of (1) submitted to the initial condition  $u(0, x) = u_0(x)$  for  $x \in \mathbb{R}^N$ , then Theorem 4 precisely says that the sets

$$D_t^+ = \{x \in \mathbb{R}^N : u(t, x) > 0\}, D_t^- = \{x \in \mathbb{R}^N : u(t, x) < 0\}, \Gamma_t = \{x \in \mathbb{R}^N : u(t, x) = 0\}$$

does not depend on the choice of  $u_0$ .

The next theorem claims that there exists a maximal subsolution minimal supersolution of (1) associated with the appropriate discontinuous initial data.



**Theorem 5 (Maximal subsolution and minimal supersolution).** *Assume that (4) and (13) hold true. Then the function  $\mathbf{1}_{D_t^+ \cup \Gamma_t} - \mathbf{1}_{D_t^-}$  (resp.  $\mathbf{1}_{D_t^+} - \mathbf{1}_{D_t^- \cup \Gamma_t}$ ) is the maximal subsolution (resp. minimal supersolution) of (1) submitted to the initial datum  $\mathbf{1}_{D_0^+ \cup \Gamma_0} - \mathbf{1}_{D_0^-}$  (resp.  $\mathbf{1}_{D_0^+} - \mathbf{1}_{D_0^- \cup \Gamma_0}$ ).*

This result is a consequence of Proposition 1 together with discontinuous stability and the comparison principle. See [6, p. 445] for details.

We conclude this section by showing that a bounded front propagates with finite speed.

**Proposition 2 (Evolution of bounded sets).** *Let  $\Omega_0$  be a bounded open set of  $\mathbb{R}^N$ : there exists  $R > 0$  such that  $\Omega_0 \subset B_R$ . Then the level-set evolution  $(\Gamma_t, D_t^+, D_t^-)$  of  $(\partial\Omega_0, \Omega_0, (\bar{\Omega}_0)^c)$  satisfies  $D_t^+ \cup \Gamma_t \subset \bar{B}_{R+Ct}$  with*

$$C = \|c_1\|_\infty - \inf_{e \in \mathbb{S}^{N-1}} \nu(z \in \mathbb{R}^N : 0 \leq e \cdot z \leq |z|^2)$$

as long as  $R + Ct > 0$ .

*Remark 5.* Another consequence of this proposition is that, if there are no driving force ( $c_1 = 0$ ), then the set shrinks till it disappears.

*Proof.* The proof consists in constructing a supersolution of (1), (2). It is easy to check that  $C$  is chosen such that

$$u(t, x) = Ct + \sqrt{\varepsilon^2 + R^2} - \sqrt{\varepsilon^2 + |x|^2}$$

is a supersolution of (1). Since  $\bar{B}_R = \{x \in \mathbb{R}^N : u(0, x) \geq 0\}$ , we conclude that  $D_t^+ \cup \Gamma_t \subset \{x \in \mathbb{R}^N : u(t, x) \geq 0\} = \bar{B}_{R^\varepsilon(t)}$  with  $R^\varepsilon(t) = \sqrt{(Ct + \sqrt{\varepsilon^2 + R^2})^2 - \varepsilon^2}$ . Hence,  $D_t^+ \cup \Gamma_t \subset \cap_{\varepsilon > 0} \bar{B}_{R^\varepsilon(t)} = \bar{B}_{R+Ct}$ .  $\square$

## 5 Generalized flows

In this section, we follow [7] and give an equivalent definition of the flow by, freely speaking, replacing smooth test functions with smooth test fronts.

In order to give this equivalent definition, we use the geometrical non-linearities we partially introduced in Section 2 above. For all  $x, p \in \mathbb{R}^N$  and all closed set  $\mathcal{F} \subset \mathbb{R}^N$  and open set  $\mathcal{O} \subset \mathbb{R}^N$

$$F_*(x, p, \mathcal{F}) = \begin{cases} -\mu(\hat{p}) \left[ c_1(x) + \nu(\mathcal{F} \cap \{p \cdot z \leq 0\}) - \nu(\mathcal{F}^c \cap \{p \cdot z > 0\}) \right] |p| & \text{if } p \neq 0, \\ 0 & \text{if not,} \end{cases}$$

$$F^*(x, p, \mathcal{O}) = \begin{cases} -\mu(\hat{p}) \left[ c_1(x) + \nu(\mathcal{O} \cap \{p \cdot z < 0\}) - \nu(\mathcal{O}^c \cap \{p \cdot z \geq 0\}) \right] |p| & \text{if } p \neq 0, \\ 0 & \text{if not.} \end{cases}$$

We can now give the definition of a generalized flow.

**Definition 3 (Generalized flows).** *The family  $(\mathcal{O}_t)_{t \in (0, T)}$  of open subsets of  $\mathbb{R}^N$  (resp.  $(\mathcal{F}_t)_{t \in (0, T)}$  of closed subsets of  $\mathbb{R}^N$ ) is a generalized super-flow (resp. sub-flow) of (1) if for all  $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}^N$ ,  $r > 0$ ,  $h > 0$ , and for all smooth function  $\phi : (0; +\infty) \times \mathbb{R}^N \rightarrow \mathbb{R}$  such that*

1.  $\partial_t \phi + F^*(x, D\phi, \{z : \phi(t, x+z) > \phi(t, x)\}) \leq -\delta_\phi$  in  $[t_0, t_0+h] \times \bar{B}(x_0, r)$   
(resp.  $\partial_t \phi + F_*(x, D\phi, \{z : \phi(t, x+z) \geq \phi(t, x)\}) \geq -\delta_\phi$  in  $[t_0, t_0+h] \times \bar{B}(x_0, r)$ )
2.  $D\phi \neq 0$  in  $\{(s, y) \in [t_0, t_0+h] \times \bar{B}(x_0, r) : \phi(s, y) = 0\}$ ,
3.  $\{y \in \mathbb{R}^N : \phi(t_0, y) \geq 0\} \subset \mathcal{O}_{t_0}^1$ ,  
(resp.  $\{y \in \mathbb{R}^N : \phi(t_0, y) \leq 0\} \subset \mathbb{R}^N \setminus \mathcal{F}_{t_0}$ ),
4.  $\{y \notin \bar{B}(x_0, r) : \phi(s, y) \geq 0\} \subset \mathcal{O}_s^1$  for all  $s \in [t_0, t_0+h]$ ,  
(resp.  $\{y \notin \bar{B}(x_0, r) : \phi(s, y) \leq 0\} \subset \mathbb{R}^N \setminus \mathcal{F}_s$  for all  $s \in [t_0, t_0+h]$ ),

then  $\{y \in \bar{B}(x_0, r) : \phi(t_0+h, y) > 0\} \subset \mathcal{O}_{t_0+h}^1$  (resp.  $\{y \in \bar{B}(x_0, r) : \phi(t_0+h, y) < 0\} \subset \mathbb{R}^N \setminus \mathcal{F}_{t_0+h}$ ).

Freely speaking about generalized super-flows, Condition 1 says that in a prescribed neighbourhood  $\mathcal{V}$  around  $(t_0, x_0)$ , the normal velocity of the test front  $\{\phi > 0\}$  is strictly smaller than the one of the front  $\mathcal{O}$ ; Condition 2 asserts that the front  $\{\phi = 0\}$  is smooth in  $\mathcal{V}$ ; Conditions 3 and 4 assert that the test front is inside the front  $\mathcal{O}$  outside  $\mathcal{V}$ . The conclusion is that the test front is inside the neighbourhood  $\mathcal{O}$  at time  $t+h$ .

*Remark 6.* As far as local geometric fronts are considered, Conditions 3 and 4 impose that the test front is inside  $\mathcal{O}$  on the parabolic boundary of the neighbourhood. Here, because the front is not local, the test front has to be inside  $\mathcal{O}$  everywhere outside the neighbourhood.

The next theorem asserts that Definition 3 of the flow coincides with the level-set formulation of Section 4.

**Theorem 6 (Generalized flows and level-set approach).** *Let  $(\mathcal{O}_t)_{t \in (0, T)}$  be a family of open subsets of  $\mathbb{R}^N$  (resp.  $(\mathcal{F}_t)_{t \in (0, T)}$  of closed subsets of  $\mathbb{R}^N$ ) such that the set  $\cup_{t \in (0, T)} \{t\} \times \mathcal{O}_t$  is open in  $[0, T] \times \mathbb{R}^N$  (resp.  $\cup_{t \in (0, T)} \{t\} \times \mathcal{F}_t$  is closed in  $[0, T] \times \mathbb{R}^N$ ).*

*Then  $(\mathcal{O}_t)_{t \in (0, T)}$  (resp.  $(\mathcal{F}_t)_{t \in (0, T)}$ ) is a generalized super-flow (resp. sub-flow) of (1) if and only if  $\chi(t, x) = \mathbf{1}_{\mathcal{O}_t}(x) - \mathbf{1}_{\mathbb{R}^N \setminus \mathcal{O}_t}(x)$  (resp.  $\chi(t, x) = \mathbf{1}_{\mathcal{F}_t}(x) - \mathbf{1}_{\mathbb{R}^N \setminus \mathcal{F}_t}(x)$ ) is a viscosity supersolution (resp. subsolution) of (1), (2).*

Since the proof of [7] can be readily adapted, we omit it. We give a straightforward corollary of Theorems 5 and 6 that is used in [13].

**Corollary 1 (Abstract method).** *Assume that  $(\mathcal{O}_t)_t$  and  $(\mathcal{F}_t)_t$  are respectively a generalized super-flow and generalized sub-flow and suppose there exists two open sets  $D_0^+, D_0^-$  such that  $\mathbb{R}^N = \partial \mathcal{O}_0 \sqcup D_0^+ \sqcup D_0^-$  and such that  $D_0^+ \subset \mathcal{O}_0$  and  $D_0^- \subset \mathcal{F}_0^c$ . Then if  $(\Gamma_t, D_t^+, D_t^-)$  denotes the level-set evolution of  $(\partial \mathcal{O}_0, D_0^+, D_0^-)$ , we have for all time  $t > 0$*

$$D_t^+ \subset \mathcal{O}_t \subset D_t^+ \cup \Gamma_t, \quad D_t^- \subset \mathcal{F}_t^c \subset D_t^- \subset \Gamma_t.$$

*Remark 7.* One can check that under the assumptions of the previous corollary, we have in fact  $D_0^+ = \mathcal{O}_0$  and  $D_0^- = \mathcal{F}_0^c$ .

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