Hodge modules on Shimura varieties and their higher direct images in the Baily–Borel compactification

by

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Abstract

We study the degeneration in the Baily–Borel compactification of variations of Hodge structure on Shimura varieties. Our main result Theorem 2.6 expresses the degeneration of variations given by algebraic representations in terms of Hochschild, and abstract group cohomology. It is the Hodge theoretic analogue of Pink's theorem on degeneration of étale and ℓ-adic sheaves [P2], and completes results by Harder and Looijenga–Rapoport [Hd, LR]. The induced formula on the level of singular cohomology is equivalent to the theorem of Harris–Zucker on the Hodge structure of deleted neighbourhood cohomology of strata in toroidal compactifications [HZ1].

Résumé : Ce travail concerne la dégénérescence des variations de structure de Hodge sur les variétés de Shimura. Le résultat principal Thm. 2.6 exprime cette dégénérescence en termes de cohomologie de Hochschild, et de cohomologie abstraite des groupes. Ce résultat est l'analogue, en théorie de Hodge, du théorème de Pink sur la dégénérescence des faisceaux étales et ℓ-adiques [P2], et complète des résultats obtenus par Harder et Looijenga–Rapoport [Hd, LR]. Il induit une formule au niveau de la cohomologie singulière, qui est équivalente au théorème de Harris–Zucker concernant la structure de Hodge sur la "deleted neighbourhood cohomology" des strates des compactifications toroïdales [HZ1].

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0 Introduction

In this paper, we consider the *Baily–Borel compactification* of a (pure) *Shimura variety*

$$j : M \longrightarrow M^* .$$

According to [P1], the *boundary* $M^* - M$ has a natural *stratification* into locally closed subsets, each of which is itself (a quotient by the action of a finite group of) a Shimura variety. Let

$$i : M_1 \longrightarrow M^*$$

be the inclusion of an individual such stratum. Saito's theory of *mixed algebraic Hodge modules* [Sa] comes equipped with the formalism of *Grothendieck's functors*. In particular, there is a functor

$$i^* j_*$$

from the bounded derived category of Hodge modules on M to that of Hodge modules on M_1. We shall refer to this functor as the *degeneration of Hodge modules on M along the stratum M_1.*

The objective of the present article is a formula for the effect of i^*j_* on those complexes of Hodge modules coming about via the *canonical construction*, denoted μ: the Shimura variety M is associated to a reductive group G over $\mathbb{Q}$, and any complex of algebraic representations $\mathbb{V}^\bullet$ of G gives rise to a complex of Hodge modules $\mu(\mathbb{V}^\bullet)$ on M. Let G_1 be the group belonging to M_1; it is the maximal reductive quotient of a certain subgroup P_1 of G:

$$W_1 \quad \trianglelefteq \quad P_1 \quad \subset \quad G$$
$$\downarrow$$
$$G_1$$

($W_1 :=$ the unipotent radical of P_1.) The topological inertia group of M_1 in M is an extension of a certain arithmetic group $\overline{H}_C$ by a lattice in $W_1(\mathbb{Q})$.

Our main result Theorem 2.6 expresses $i^*j_* \circ \mu$ as a composition of *Hochschild cohomology* of W_1, *abstract cohomology* of $\overline{H}_C$, and the canonical construction on M_1. It completes results of Harder and of Looijenga–Rapoport; in fact, the result on the level of local systems is proved in [Hd], while the result "modulo Hodge filtrations" is basically contained in [LR]. Our result induces a comparison statement on the level of singular cohomology of M_1, which is equivalent to one of the main results of [HZ1]. Theorem 2.6 is the analogue of the main result of [P2], which identifies the degeneration of étale and ℓ-adic sheaves.

As far as the proof of our main result is concerned, our *geometric* approach is very close to the one employed in [P2] and [HZ1, Sections 4 and 5]: as there, we use a *toroidal compactification*, to reduce a difficult local calculation to an easier local calculation, together with a global calculation on the fibres of the projection from the toroidal compactification. By contrast, the *homological* aspects differ drastically from [P2]. The reason for this lies in the behaviour of the formalism of Grothendieck's functors on the two sheaf categories with respect to the *t*-structures: roughly speaking, on the ℓ-adic side, the functors on the level of derived categories are obtained by right derivation of (at worst) left exact functors. Since the same is true for group cohomology, the formalism of equivariant ℓ-adic sheaves can be controlled via the standard techniques using injective resolutions [P2, Section 1]. Due to the perverse nature of Hodge modules, there are no exactness properties for Grothendieck's functors associated to arbitrary morphisms. Even when half exactness is known (e.g., right exactness for the inverse image of a closed immersion, left exactness for the (shift by $-d$ of the) direct image of a smooth morphism of constant relative dimension d), the corresponding functor on the level of derived categories is not a priori obtained by derivation. As a consequence, we found ourselves unable to establish the full formalism of Grothendieck's functors for equivariant Hodge modules, except for some almost obvious results when the action of the group is free (see Section 4). It turns out that these are sufficient for our purposes, once we observe that

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certain combinatorial aspects of the toroidal compactification can be translated into group cohomology.

Talking about group cohomology, we should mention that to find the correct conceptual context for the statement of Theorem 2.6 turned out to be a major challenge in itself: recall that we express $i^*j_* \circ \mu$ as a composition of Hochschild cohomology, abstract group cohomology, and the canonical construction on M_1. Due to the nature of the canonical construction, it is necessary for abstract cohomology to map algebraic representations to algebraic representations. We found it most natural to develop a formalism of group cohomology "in Abelian categories", which on the one hand applies to a sufficiently general situation, and on the other hand is compatible with usual group cohomology "in the category of $\mathbb{Z}$-modules". This is the content of Section 3.

In [HZ1], the authors study the *Hodge structure* on the boundary cohomology of the *Borel–Serre compactification* $\bar{M}$. Their main result states that the *nerve spectral sequence* associated to the natural stratification of $\bar{M}$ is a spectral sequence of mixed Hodge structures. Given the non-algebraic nature of $\bar{M}$ and its strata, one of the achievements of loc. cit. is to *define* the Hodge structures in question. It turns out that the E_1-terms are given by *deleted neighbourhood cohomology* of certain strata in the toroidal compactification. Its Hodge structure is identified in [HZ1, Thm. (5.6.10)]. We are able to recover this latter result, for maximal parabolic subgroups ($R = P$ in the notation of [loc. cit.]), by applying singular cohomology to the comparison isomorphism of Theorem 2.6. Although Theorem 2.6 is not a formal consequence of the main results of [HZ1], it is fair to say that an important part of the local information needed in our proof is already contained in loc. cit.; see also [HZ2, 4.3], where some of the statements of [HZ1, Section 5] are strengthened. Roughly speaking, the fundamental difference between the approach of loc. cit. and ours is the following: loc. cit. uses the explicit description of the objects in order to deduce a comparison result. We derive the comparison result from the abstract properties of the categories involved; this gives in particular an explicit description, which turns out to be compatible with that of [HZ1, HZ2].

Our article is structured as follows: we assemble the notations and results necessary for the statement of Theorem 2.6 in Sections 1 and 2. Sections 3–8 contain the material needed in its proof, which is given in Section 9. We refer to Overview 2.14 for a more detailed description. Let us note that because of the homological difficulties mentioned further above, one is forced to identify $i^*j_* \circ \mu$ with the composition of a certain number of functors, each of which is simultaneously (1) relatively easy to handle, and (2) at least half exact. This explains the central role played by the *specialization functor* in the context of Hodge modules (see Section 7). It also explains the use of Čech coverings

in the computation of the direct image associated to the projection from the toroidal to the Baily–Borel compactification (see Section 5).

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Notations and Conventions: All Shimura varieties are defined over the field of complex numbers $\mathbb{C}$. Throughout the whole article, we make consistent use of the language and the main results of [P1]. Algebraic representations of an algebraic group are finite dimensional by definition. If a group G acts on X, then we write $\mathrm{Cent}_G\, X$ for the kernel of the action. If Y is a sub-object of X, then $\mathrm{Stab}_G\, Y$ denotes the subgroup of G stabilizing Y. Finally, the ring of finite adèles over $\mathbb{Q}$ is denoted by $\mathbb{A}_f$.

1 Strata in the Baily–Borel compactification

This section is intended for reference. We recall and prove what is stated in [P2, (3.7)], but using group actions from the left (as in [P1, 6.3]). This can be seen as an adelic version of the description of these actions contained in [LR, (6.1)–(6.2)].

Let $(G, \mathfrak{H})$ be *mixed Shimura data* [P1, Def. 2.1]. The *Shimura varieties* associated to $(G, \mathfrak{H})$ are indexed by the open compact subgroups of $G(\mathbb{A}_f)$. If K is one such, then the analytic space of $\mathbb{C}$-valued points of the corresponding variety $M^K := M^K(G, \mathfrak{H})$ is given as

$$M^K(\mathbb{C}) := G(\mathbb{Q})\backslash(\mathfrak{H} \times G(\mathbb{A}_f)/K) \ .$$

We assume that G is reductive, and hence that $(G, \mathfrak{H})$ is *pure* in the sense of [P1]. In order to describe the *Baily–Borel compactification* $(M^K)^*$ of M^K [BaBo, AMRT], recall that for any *admissible parabolic subgroup* Q of G [P1, Def. 4.5], there is associated a canonical normal subgroup P_1 of Q [P1, 4.7]. There is a finite collection of *rational boundary components* $(P_1, \mathfrak{X}_1)$ associated to P_1, and indexed by the $P_1(\mathbb{R})$-orbits in $\pi_0(\mathfrak{H})$ [P1, 4.11]. The $(P_1, \mathfrak{X}_1)$ are themselves mixed Shimura data. Denote by W_1 the unipotent radical of P_1, and by $(G_1, \mathfrak{H}_1)$ the quotient of $(P_1, \mathfrak{X}_1)$ by W_1 [P1, Prop. 2.9]. From the proof of [P1, Lemma 4.8], it follows that W_1 equals the unipotent radical of Q.

One defines

$$\mathfrak{H}^* := \coprod_{(P_1, \mathfrak{X}_1)} \mathfrak{H}_1 \ ,$$

where the disjoint union is extended over all rational boundary components $(P_1, \mathfrak{X}_1)$. This set comes equipped with the *Satake topology* (see [AMRT,

p. 257], or [P1, 6.2]), as well as a natural action of the group $G(\mathbb{Q})$ (see [P1, 4.16]). One defines

$$M^K(G,\mathfrak{H})^*(\mathbb{C}) := G(\mathbb{Q})\backslash(\mathfrak{H}^* \times G(\mathbb{A}_f)/K) \ .$$

This object is endowed with the quotient topology. By [BaBo, 10.4, 10.11] (whose proof works equally well in the more general context considered by Pink; see [P1, 8.2]), it can be canonically identified with the space of $\mathbb{C}$-valued points of a normal projective complex variety $(M^K)^* = M^K(G,\mathfrak{H})^*$, containing M^K as a Zariski-open dense subset. The stratification of $\mathfrak{H}^*$ induces a stratification of $(M^K)^*$. Let us explicitly describe this stratification: fix an admissible parabolic subgroup Q of G, and let $(P_1,\mathfrak{X}_1)$, W_1, and

$$\pi : (P_1,\mathfrak{X}_1) \longrightarrow\!\!\!\!\rightarrow (G_1,\mathfrak{H}_1) = (P_1,\mathfrak{X}_1)/W_1$$

as above. Fix an open compact subgroup $K \subset G(\mathbb{A}_f)$, and an element $g \in G(\mathbb{A}_f)$. Define $K' := g \cdot K \cdot g^{-1}$, and $K_1 := P_1(\mathbb{A}_f) \cap K'$. We have the following natural morphisms (cmp. [P2, (3.7.1)]):

$$M^{\pi(K_1)}(G_1,\mathfrak{H}_1)(\mathbb{C})$$
$$\|$$
$$G_1(\mathbb{Q})\backslash(\mathfrak{H}_1 \times G_1(\mathbb{A}_f)/\pi(K_1)) \qquad\qquad [(x,\pi(p_1)]$$
$$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad \uparrow$$
$$P_1(\mathbb{Q})\backslash(\mathfrak{H}_1 \times P_1(\mathbb{A}_f)/K_1) \qquad\qquad [(x,p_1)]$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$
$$G(\mathbb{Q})\backslash(\mathfrak{H}^* \times G(\mathbb{A}_f)/K) \qquad\qquad [(x,p_1g)]$$
$$\|$$
$$M^K(G,\mathfrak{H})^*(\mathbb{C})$$

The map $[(x,p_1)] \mapsto [(x,\pi(p_1))]$ is an isomorphism of complex spaces. The composition of these morphisms comes from a unique morphism of algebraic varieties

$$i = i_{G_1,K,g} : M^{\pi(K_1)} := M^{\pi(K_1)}(G_1,\mathfrak{H}_1) \longrightarrow (M^K)^*$$

([Se, Prop. 15], applied to the Baily–Borel compactification of $M^{\pi(K_1)}$; see [P1, 7.6]). Define the following groups (cmp. [P2, (3.7.4)], where the same notations are used for the groups corresponding to the actions from the right):

$$\begin{aligned} H_Q &:= \operatorname{Stab}_{Q(\mathbb{Q})}(\mathfrak{H}_1) \cap P_1(\mathbb{A}_f) \cdot K' \ , \\ H_C &:= \operatorname{Cent}_{Q(\mathbb{Q})}(\mathfrak{H}_1) \cap W_1(\mathbb{A}_f) \cdot K' \ ; \end{aligned}$$

note that these are indeed groups since Q normalizes P_1 and W_1. The group H_Q acts by analytic automorphisms on $\mathfrak{H}_1 \times P_1(\mathbb{A}_f)/K_1$ (see Lemma 1.4 below for an explicit description of this action). Hence the group $\Delta_1 := H_Q/P_1(\mathbb{Q})$ acts naturally on

$$M^{\pi(K_1)}(\mathbb{C}) = P_1(\mathbb{Q})\backslash(\mathfrak{H}_1 \times P_1(\mathbb{A}_f)/K_1) \ .$$

This action is one by algebraic automorphisms [P1, Prop. 9.24]. By [P1, 6.3] (see also Proposition 1.1 below), it factors through a finite quotient of Δ_1, which we shall denote by Δ. The quotient by this action is precisely the image of i:

$$M^{\pi(K_1)} \longrightarrow\!\!\!\!\!\rightarrow M_1^K := \Delta \backslash M^{\pi(K_1)} \overset{i}{\lhook\joinrel\longrightarrow} (M^K)^*$$
$$\underbrace{\phantom{M^{\pi(K_1)} \longrightarrow\!\!\!\!\!\rightarrow M_1^K := \Delta \backslash M^{\pi(K_1)} \overset{i}{\lhook\joinrel\longrightarrow} (M^K)^*}}_{i}$$

By abuse of notation, we denote by the same letter i the inclusion of M_1^K into $(M^K)^*$. We need to identify the group Δ, and the nature of its action on $M^{\pi(K_1)}$. Let us introduce the following condition on $(G, \mathfrak{H})$:

(+) The neutral connected component $Z(G)^0$ of the center $Z(G)$ of G is, up to isogeny, a direct product of a $\mathbb{Q}$-split torus with a torus T of compact type (i.e., $T(\mathbb{R})$ is compact) defined over $\mathbb{Q}$.

The proof of [P1, Cor. 4.10] shows that $(G_1, \mathfrak{H}_1)$ satisfies (+) if $(G, \mathfrak{H})$ does.

Proposition 1.1. *(a) The subgroup $P_1(\mathbb{Q})H_C$ of H_Q is of finite index.*

(b) The group $H_C/W_1(\mathbb{Q})$ centralizes G_1, and H_C is the kernel of the action of H_Q on $\mathfrak{H}_1 \times G_1(\mathbb{A}_f)/\pi(K_1)$. The group $P_1(\mathbb{Q})H_C$ acts trivially on $M^{\pi(K_1)}$.

(c) Assume that $(G, \mathfrak{H})$ satisfies (+), and that K is neat *(see e.g. [P1, 0.6]). Then we have an equality*

$$P_1(\mathbb{Q}) \cap H_C = W_1(\mathbb{Q})$$

of subgroups of H_Q.

(d) Under the hypotheses of (c), the action of the finite group $H_Q/P_1(\mathbb{Q})H_C$ on $M^{\pi(K_1)}$ is free. In particular, we have $\Delta = H_Q/P_1(\mathbb{Q})H_C$.

For the proof of this result, we shall need three lemmata. By slight abuse of notation, we denote by the letter π the projection $Q \to Q/W_1$ as well:

$$
\begin{array}{ccc}
P_1 & \lhook\joinrel\longrightarrow & Q \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
G_1 & \lhook\joinrel\longrightarrow & Q/W_1
\end{array}
$$

Note that since Q/W_1 is reductive, it is possible to choose a complement of G_1 in Q/W_1, i.e., a normal connected reductive subgroup G_2 of Q/W_1 such that

$$Q/W_1 = G_1 \cdot G_2 ,$$

and such that the intersection $G_1 \cap G_2$ is finite. Let us mention that in the literature, the groups G_1 and G_2 are sometimes called G_h and G_ℓ, respectively.

Lemma 1.2. *Let $\gamma \in \mathrm{Cent}_{Q(\mathbb{Q})}(\mathfrak{H}_1)$. Then $\pi(\gamma) \in Q/W_1$ centralizes G_1.*

Proof. For any $g_1 \in G_1(\mathbb{Q})$, the element $g_1\pi(\gamma)g_1^{-1}$ centralizes $\mathfrak{H}_1$. It follows that $\pi(\gamma)$ centralizes the normal subgroup of G_1 generated by the images of the morphisms

$$h_x : \mathbb{S} \longrightarrow G_{1,\mathbb{R}} \quad , \quad x \in \mathfrak{H}_1 .$$

But by definition of P_1 (see [P1, 4.7]), this subgroup is G_1 itself. **q.e.d.**

Lemma 1.3. *Let $(G', \mathfrak{H}')$ be Shimura data satisfying condition $(+)$, and $\Gamma \subset G'(\mathbb{Q})$ an arithmetic subgroup (see e.g. [P1, 0.5]). Then Γ acts properly discontinuously on $\mathfrak{H}'$. In particular, the stabilizers of the action of Γ are finite.*

Proof. We cannot quote [P1, Prop. 3.3] directly because loc. cit. uses a more general notion of properly discontinuous actions (see [P1, 0.4]). However, the proof of [P1, Prop. 3.3] shows that the action of Γ is properly discontinuous in the usual sense if condition $(+)$ is satisfied. We refer to [W1, Prop. 1.2 b)] for the details. **q.e.d.**

Let us identify explicitly the action of H_Q on $\mathfrak{H}_1 \times P_1(\mathbb{A}_f)/K_1$:

Lemma 1.4. *Let $x \in \mathfrak{H}_1$, $p_1 \in P_1(\mathbb{A}_f)$, and $\gamma \in H_Q \subset \mathrm{Stab}_{Q(\mathbb{Q})}(\mathfrak{H}_1)$. Write*

$$\gamma = p_2 k ,$$

with $p_2 \in P_1(\mathbb{A}_f)$, and $k \in K'$. Since Q normalizes P_1, we have

$$p_3 := \gamma p_1 \gamma^{-1} \in P_1(\mathbb{A}_f) .$$

We then have

$$\gamma \cdot (x, [p_1]) = (\gamma(x), [p_3 p_2]) = (\gamma(x), [\gamma p_1 k^{-1}])$$

in $\mathfrak{H}_1 \times P_1(\mathbb{A}_f)/K_1$.

We leave the proof of this result to the reader.

Proof of Proposition 1.1. As for (a), observe that by Lemma 1.2, the images of both H_C and H_Q in $Q/P_1(\mathbb{Q})$ are arithmetic subgroups. Part (b) results directly from Lemmata 1.2 and 1.4. Let us turn to (c). By Lemma 1.2, the image of the group $P_1(\mathbb{Q}) \cap H_C$ under π is an arithmetic subgroup of the center of G_1. It is neat because K is. Because of $(+)$, it must be trivial. It remains to show (d). Fix $x \in \mathfrak{H}_1$, $p_1 \in P_1(\mathbb{A}_f)$, and $\gamma \in H_Q$ as well as $k \in K'$ as in Lemma 1.4. Suppose that

$$\gamma \cdot [(x, p_1)] = [(x, p_1)]$$

in $M^{\pi(K_1)}(\mathbb{C})$. There is thus an element γ' of $P_1(\mathbb{Q})$, such that

$$\gamma \cdot (x, [p_1]) = \gamma' \cdot (x, [p_1])$$

in $\mathfrak{H}_1 \times P_1(\mathbb{A}_f)/K_1$. In other words, we can find $k_1 \in K_1$ such that

(1) $\gamma'(x) = \gamma(x)$, i.e., $\gamma'' := \gamma^{-1}\gamma' \in Q(\mathbb{Q})$ stabilizes x.

(2) $\gamma'p_1 = \gamma p_1 k^{-1}k_1$. We thus have $\gamma'' \in p_1 K' p_1^{-1}$, which is a neat subgroup of $G(\mathbb{A}_f)$.

Choose a complement G_2 of G_1 in Q/W_1. The groups G_1 and G_2 centralism each other. Denote by $\Pi(\gamma'')$ the image of γ'' in Q/G_2. We identify this group with $G_1/G_1 \cap G_2$. Because of (1), the element $\Pi(\gamma'')$ stabilizes a point in the space $\mathfrak{H}_1/G_1 \cap G_2$ belonging to the quotient Shimura data

$$(Q/G_2, \mathfrak{H}_1/G_1 \cap G_2) := (G_1, \mathfrak{H}_1)/G_1 \cap G_2 \;.$$

By Lemma 1.3, the element $\Pi(\gamma'')$ is of finite order. Because of (2), it must be trivial. We conclude:

(3) $\pi(\gamma'')$ centralizes G_1. Hence $p_1 k^{-1}k_1 = \gamma''p_1 = p_1\gamma'' \mod W_1(\mathbb{A}_f)$, and we get:

(4) γ'' lies in $W_1(\mathbb{A}_f) \cdot K'$.

Since $G_1(\mathbb{R})$ acts transitively on $\mathfrak{H}_1$, (1) and (3) imply that γ'' acts trivially on $\mathfrak{H}_1$. Because of (4), we then have $\gamma'' \in H_C$, hence

$$\gamma = \gamma'(\gamma'')^{-1} \in P_1(\mathbb{Q})H_C \;,$$

as claimed. **q.e.d.**

For future reference, we note:

Corollary 1.5. *Assume that $(G, \mathfrak{H})$ satisfies $(+)$, and that K is neat. Then the kernel of the projection $\Delta_1 \to \Delta$ is canonically isomorphic to $\pi(H_C)$.*

Proof. This follows immediately from Proposition 1.1 (c). **q.e.d.**

2 Statement of the main result

Let $(M^K)^* = M^K(G, \mathfrak{H})^*$ be the Baily–Borel compactification of a Shimura variety $M^K = M^K(G, \mathfrak{H})$, and $M_1^K = \Delta\backslash M^{\pi(K_1)} = \Delta\backslash M^{\pi(K_1)}(G_1, \mathfrak{H}_1)$ a boundary stratum. Consider the situation

$$M^K \overset{j}{\hookrightarrow} (M^K)^* \overset{i}{\hookleftarrow} M_1^K \;.$$

Saito's formalism [Sa] gives a functor i^*j_* between the bounded derived categories of *algebraic mixed Hodge modules* on M^K and on M_1^K respectively. Our main result (Theorem 2.6) gives a formula for the restriction of i^*j_* onto the image of the natural functor μ_K associating to an algebraic representation of G a variation of Hodge structure on M^K. Its proof, which will rely

on the material developed in the next six sections, will be given in Section 9. In the present section, we shall restrict ourselves to a concise presentation of the ingredients necessary for the formulation of Theorem 2.6 (2.1–2.5), and we shall state the main corollaries (2.7–2.12). Let us mention that part of these results are already contained in work of Harder, Looijenga–Rapoport, and Harris–Zucker [Hd, LR, HZ1] (see Remark 2.13). 2.6–2.12 are the Hodge theoretic analogues of results obtained by Pink in the ℓ-adic context [P2].

Fix pure Shimura data $(G, \mathfrak{H})$ satisfying the hypothesis $(+)$, and an open compact neat subgroup K of $G(\mathbb{A}_f)$. Let F be a subfield of $\mathbb{R}$. By definition of Shimura data, there is a tensor functor associating to an algebraic F-representation $\mathbb{V}$ of G a variation of Hodge structure $\mu(\mathbb{V})$ on $\mathfrak{H}$ [P1, 1.18]. It descends to a variation $\mu_K(\mathbb{V})$ on $M^K(\mathbb{C})$. We refer to the tensor functor μ_K as the *canonical construction* of variations of Hodge structure from representations of G. Since the weight cocharacter associated to $(G, \mathfrak{H})$ is central [P1, Def. 2.1 (iii)], $\mu_K(\mathbb{V})$ is the direct sum of its weight graded objects. By Schmid's Nilpotent Orbit Theorem [Sch, Thm. (4.9)], the image of μ_K is contained in the category $\mathbf{Var}_F M^K$ of *admissible* variations, and hence [Sa, Thm. 3.27], in the category $\mathbf{MHM}_F M^K$ of algebraic mixed Hodge modules. Since the functor μ_K is exact, it descends to the level of derived categories:

$$\mu_K : D^b(\mathbf{Rep}_F G) \longrightarrow D^b(\mathbf{MHM}_F M^K) \ .$$

In order to state the main result, fix a rational boundary component $(P_1, \mathfrak{X}_1)$ of $(G, \mathfrak{H})$, and an element $g \in G(\mathbb{A}_f)$. We shall use the notation of Proposition 1.1. In particular, we have the following diagram of algebraic groups:

$$\begin{array}{ccccc} P_1 & \hookrightarrow & Q & \hookrightarrow & G \\ \pi \downarrow & & \downarrow \pi & & \\ G_1 & \hookrightarrow & Q/W_1 & & \end{array}$$

By Proposition 1.1 (c), we have a Cartesian diagram of subgroups of $Q(\mathbb{Q})$, all of which are normal in H_Q:

$$\begin{array}{ccc} & P_1(\mathbb{Q}) & \\ \nearrow & & \searrow \\ W_1(\mathbb{Q}) & & H_Q \\ \searrow & & \nearrow \\ & H_C & \end{array}$$

Writing $\overline{H}_Q$ for $\pi(H_Q)$, and $\overline{H}_C$ for $\pi(H_C)$, we thus have

$$G_1(\mathbb{Q}) \cap \overline{H}_C = \{1\} \ .$$

Definition 2.1. (a) The category $(\mathbf{Rep}_F G_1, \overline{H}_Q)$ consists of pairs

$$(\mathbb{V}_1, (\rho_\gamma)_{\gamma \in \overline{H}_Q}) \ ,$$

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where $\mathbb{V}_1 \in \mathbf{Rep}_F\, G_1$, and $(\rho_\gamma)_{\gamma \in \overline{H}_Q}$ is a family of isomorphisms

$$\rho_\gamma : (\operatorname{int} \gamma)^* \mathbb{V}_1 \xrightarrow{\ \sim\ } \mathbb{V}_1$$

in $\mathbf{Rep}_F\, G_1$ ($\operatorname{int} \gamma :=$ conjugation by γ on G_1) such that the following hold:

(i) ρ_γ is given by $v \mapsto \gamma^{-1}(v)$ if $\gamma \in G_1(\mathbb{Q})$,

(ii) the cocycle condition holds.

Morphisms in $(\mathbf{Rep}_F\, G_1, \overline{H}_Q)$ are defined in the obvious way.

(b) The category $(\mathbf{Rep}_F\, G_1, \overline{H}_Q\,/\,\overline{H}_C)$ is defined as the full sub-category of $(\mathbf{Rep}_F\, G_1, \overline{H}_Q)$ consisting of objects

$$(\mathbb{V}_1, (\rho_\gamma)_{\gamma \in \overline{H}_Q})$$

for which ρ_γ is the identity whenever γ lies in $\overline{H}_C$.

We also define variants of the above on the level of pro-categories, i.e., categories $(\operatorname{Pro}(\mathbf{Rep}_F\, G_1), \overline{H}_Q)$ and $(\operatorname{Pro}(\mathbf{Rep}_F\, G_1), \overline{H}_Q\,/\,\overline{H}_C)$. Note that by Proposition 1.1 (b), we have $(\operatorname{int} \gamma)^* \mathbb{V}_1 = \mathbb{V}_1$ for any $\mathbb{V}_1 \in \operatorname{Pro}(\mathbf{Rep}_F\, G_1)$ and $\gamma \in \overline{H}_C$.

The functor $\mu_{\pi(K_1)}$ extends to give an exact tensor functor from the category $(\mathbf{Rep}_F\, G_1, \overline{H}_Q\,/\,\overline{H}_C)$ to the category of objects of $\mathbf{Var}_F\, M^{\pi(K_1)}$ with an action of the finite group $\Delta = \overline{H}_Q\,/G_1(\mathbb{Q})\,\overline{H}_C$. From Proposition 1.1 (d), we get a canonical equivalence of this category and $\mathbf{Var}_F\, M_1^K$. Altogether, we have defined a tensor functor

$$(\mathbf{Rep}_F\, G_1, \overline{H}_Q\,/\,\overline{H}_C) \longrightarrow \mathbf{Var}_F\, M_1^K \subset \mathbf{MHM}_F\, M_1^K \ ,$$

equally referred to as $\mu_{\pi(K_1)}$. It is exact, and hence defines

$$\mu_{\pi(K_1)} : D^b(\mathbf{Rep}_F\, G_1, \overline{H}_Q\,/\,\overline{H}_C) \longrightarrow D^b(\mathbf{MHM}_F\, M_1^K) \ .$$

Definition 2.2. Denote by

$$\Gamma(\overline{H}_C, \bullet) : (\mathbf{Rep}_F\, G_1, \overline{H}_Q) \longrightarrow (\mathbf{Rep}_F\, G_1, \overline{H}_Q\,/\,\overline{H}_C)$$

the left exact functor associating to $\mathbb{V}_1 = (\mathbb{V}_1, (\rho_\gamma)_\gamma)$ the largest sub-object $\mathbb{V}_1'$ on which the ρ_γ act as the identity whenever $\gamma \in \overline{H}_C$.

Instead of $\Gamma(\overline{H}_C, (\mathbb{V}_1, (\rho_\gamma)_\gamma))$, we shall often write $(\mathbb{V}_1, (\rho_\gamma)_\gamma)^{\overline{H}_C}$, or simply $\mathbb{V}_1^{\overline{H}_C}$. Note that $\Gamma(\overline{H}_C, \bullet)$ extends to a functor

$$(\operatorname{Pro}(\mathbf{Rep}_F\, G_1), \overline{H}_Q) \longrightarrow (\operatorname{Pro}(\mathbf{Rep}_F\, G_1), \overline{H}_Q\,/\,\overline{H}_C) \ .$$

This functor will be studied in Section 3; for the time being, let us accept that the total right derived functor of $\Gamma(\overline{H}_C, \bullet)$ exists (Theorem 3.20 (a)):

$$R\Gamma(\overline{H}_C, \bullet) : D^+(\operatorname{Pro}(\mathbf{Rep}_F\, G_1), \overline{H}_Q) \longrightarrow D^+(\operatorname{Pro}(\mathbf{Rep}_F\, G_1), \overline{H}_Q\,/\,\overline{H}_C) \ ,$$

and that it respects the sub-categories $D^b(\mathbf{Rep}_F\, G_1, ?)$ (Theorem 3.20 (b)):

$$R\Gamma(\overline{H}_C, \bullet) : D^b(\mathbf{Rep}_F\, G_1, \overline{H}_Q) \longrightarrow D^b(\mathbf{Rep}_F\, G_1, \overline{H}_Q\,/\,\overline{H}_C)\ .$$

The cohomology functors associated to $R\Gamma(\overline{H}_C, \bullet)$ will be referred to by $H^p(\overline{H}_C, \bullet)$, for $p \in \mathbb{Z}$. Let us assemble the properties of these functors necessary for the understanding of our main result. For their proof, we refer to Section 3.

Proposition 2.3. *(a) The vector space underlying $H^p(\overline{H}_C, \bullet)$ is given by the cohomology of the abstract group $\overline{H}_C$. More precisely, there is a commutative diagram of functors*

$$
\begin{array}{ccc}
D^b(\mathbf{Rep}_F\, G_1, \overline{H}_Q) & \longrightarrow & D^+(\mathbf{Rep}_F\, \overline{H}_Q) \\
{\scriptstyle R\Gamma(\overline{H}_C,\bullet)}\downarrow & & \downarrow{\scriptstyle R\Gamma(\overline{H}_C,\bullet)} \\
D^b(\mathbf{Rep}_F\, G_1, \overline{H}_Q\,/\,\overline{H}_C) & \longrightarrow & D^+(\mathbf{Rep}_F\, \overline{H}_Q\,/\,\overline{H}_C)
\end{array}
$$

Here, the categories at the right hand side denote the derived categories of abstract representations, and the arrow $R\Gamma(\overline{H}_C, \bullet)$ between them denotes the total derived functor of the functor associating to an representation its $\overline{H}_C$-invariants. The horizontal arrows are the natural forgetful functors.

(b) Let $\mathbb{V}_1 \in (\mathbf{Rep}_F\, G_1, \overline{H}_Q)$, and $p \in \mathbb{Z}$. Consider the algebraic representations $\mathrm{Res}^{\overline{H}_Q}_{G_1}\mathbb{V}_1$ and $\mathrm{Res}^{\overline{H}_Q\,/\,\overline{H}_C}_{G_1}H^p(\overline{H}_C, \mathbb{V}_1)$ of G_1. Then any irreducible factor of $\mathrm{Res}^{\overline{H}_Q\,/\,\overline{H}_C}_{G_1}H^p(\overline{H}_C, \mathbb{V}_1)$ is an irreducible factor of $\mathrm{Res}^{\overline{H}_Q}_{G_1}\mathbb{V}_1$.

Observe that the weight cocharacter associated to the Shimura data $(G_1, \mathfrak{H}_1)$ maps to the center of G_1, and hence to the center of Q/W_1. It follows that any object of $(\mathbf{Rep}_F\, G_1, \overline{H}_Q)$ or of $(\mathbf{Rep}_F\, G_1, \overline{H}_Q\,/\,\overline{H}_C)$ is the direct sum of its weight-graded objects. Proposition 2.3 implies in particular:

Corollary 2.4. *The functors*

$$H^p(\overline{H}_C, \bullet) : (\mathbf{Rep}_F\, G_1, \overline{H}_Q) \longrightarrow (\mathbf{Rep}_F\, G_1, \overline{H}_Q\,/\,\overline{H}_C)$$

respect the sub-categories of pure objects. Hence they preserve the weight decompositions in both categories.

Definition 2.5. Denote by

$$\Gamma(W_1, \bullet) : \mathbf{Rep}_F\, Q \longrightarrow (\mathbf{Rep}_F\, G_1, \overline{H}_Q)$$

the left exact functor associating to a representation $\mathbb{X}$ of Q its invariants under W_1.

Instead of $\Gamma(W_1, \mathbb{X})$, we shall often write $\mathbb{X}^{W_1}$. The total right derived functor of $\Gamma(W_1, \bullet)$ in the sense of [D1, 1.2] exists (see e.g. [W3, Thm. 2.2, 2.3]):

$$R\Gamma(W_1, \bullet) : D^b(\mathbf{Rep}_F\, Q) \longrightarrow D^b(\mathbf{Rep}_F\, G_1, \overline{H}_Q)\ .$$

In fact, this is the composition of the functor

$$D^b(\mathbf{Rep}_F Q) \longrightarrow D^b(\mathbf{Rep}_F Q/W_1)$$

calculating algebraic (Hochschild) cohomology of W_1, and the forgetful functor from $D^b(\mathbf{Rep}_F Q/W_1)$ to $D^b(\mathbf{Rep}_F G_1, \overline{H}_Q)$. The cohomology functors associated to $R\Gamma(W_1, \bullet)$ will be referred to by $H^q(W_1, \bullet)$, for $q \in \mathbb{Z}$.

Denote by c the codimension of M_1^K in $(M^K)^*$. Our main result reads as follows:

Theorem 2.6. *There is a natural commutative diagram*

$$
\begin{array}{ccc}
D^b(\mathbf{Rep}_F G) & \xrightarrow{\ \mu_K\ } & D^b(\mathbf{MHM}_F M^K) \\
{\scriptstyle \mathrm{Res}_Q^G}\big\downarrow & & \big\downarrow{\scriptstyle i^* j_*[-c]} \\
D^b(\mathbf{Rep}_F Q) & & \\
{\scriptstyle R\Gamma(W_1, \bullet)}\big\downarrow & & \\
D^b(\mathbf{Rep}_F G_1, \overline{H}_Q) & & \\
{\scriptstyle R\Gamma(\overline{H}_C, \bullet)}\big\downarrow & & \\
D^b(\mathbf{Rep}_F G_1, \overline{H}_Q / \overline{H}_C) & \xrightarrow{\ \mu_{\pi(K_1)}\ } & D^b(\mathbf{MHM}_F M_1^K)
\end{array}
$$

In particular, the functor $i^* j_* \circ \mu_K$ takes values in the sub-category of objects of $D^b(\mathbf{MHM}_F M_1^K)$ that can be represented by complexes of direct sums of pure variations of Hodge structure.

The proof of this result will be given in Section 9. Theorem 2.6 is the Hodge theoretic analogue of the main result of [P2]. It expresses $i^* j_* \circ \mu_K$ as a composition of two derived functors.

Corollary 2.7. *For any $\mathbb{V}^\bullet \in D^b(\mathbf{Rep}_F G)$, there is a canonical and functorial spectral sequence*

$$E^{p,q} = \mu_{\pi(K_1)} \circ H^p(\overline{H}_C, H^q(W_1, \mathrm{Res}_Q^G \mathbb{V}^\bullet)) \Longrightarrow \mathcal{H}^{p+q-c} i^* j_* \circ \mu_K(\mathbb{V}^\bullet)$$

in the category of admissible variations on M_1^K.

The central ingredient for the analysis of this spectral sequence is the following well known fact (see e.g. [Hd, proof of 1.6.2, Satz 1]):

Proposition 2.8. *For any $\mathbb{X}^\bullet \in D^b(\mathbf{Rep}_F Q)$, there is a canonical and functorial isomorphism in $D^b(\mathbf{Rep}_F G_1, \overline{H}_Q)$*

$$R\Gamma(W_1, \mathbb{X}^\bullet) \xrightarrow{\ \sim\ } \bigoplus_{q \in \mathbb{Z}} H^q(W_1, \mathbb{X}^\bullet)[-q] \ .$$

Proof.   In fact, this isomorphism exists already on the level of the derived category $D^b(\mathbf{Rep}_F Q/W_1)$. Since Q/W_1 is reductive, the category $\mathbf{Rep}_F Q/W_1$ is semi-simple. Its derived category is therefore canonically equivalent to the category of graded objects in $\mathbf{Rep}_F Q/W_1$.   **q.e.d.**

Consequently, we have:

Theorem 2.9. *The spectral sequence of Corollary 2.7 degenerates and splits canonically. Therefore, there is a canonical and functorial isomorphism in $\mathbf{Var}_F M_1^K$ for any $\mathbb{V}^\bullet \in D^b(\mathbf{Rep}_F G)$*

$$\mathcal{H}^n i^* j_* \circ \mu_K(\mathbb{V}^\bullet) \overset{\sim}{\longrightarrow} \bigoplus_{p+q=n+c} \mu_{\pi(K_1)} \circ H^p(\overline{H}_C, H^q(W_1, \mathrm{Res}_Q^G \mathbb{V}^\bullet)),$$

for any $n \in \mathbb{Z}$.

This is the Hodge theoretic analogue of [P2, Thm. 5.3.1]. The isomorphism of local systems underlying the isomorphism of Theorem 2.9 is known; see e.g. [Hd, proof of 1.6.2, Satz 1].

Corollary 2.10. *Let $\mathbb{V}^\bullet \in D^b(\mathbf{Rep}_F G)$, and $n \in \mathbb{Z}$.*

(a) The admissible variation $\mathcal{H}^n i^* j_* \circ \mu_K(\mathbb{V}^\bullet)$ on M_1^K is the direct sum of its weight graded objects. In particular, it is semi-simple.

(b) For any $k \in \mathbb{Z}$, we have a canonical and functorial isomorphism in $\mathbf{Var}_F M_1^K$

$$\mathrm{Gr}_k^W \mathcal{H}^n i^* j_* \circ \mu_K(\mathbb{V}^\bullet) \overset{\sim}{\longrightarrow} \bigoplus_{p+q=n+c} \mu_{\pi(K_1)} \circ H^p(\overline{H}_C, \mathrm{Gr}_k^W H^q(W_1, \mathrm{Res}_Q^G \mathbb{V}^\bullet)).$$

(c) For any $k \in \mathbb{Z}$, any Hodge type occurring in $\mathrm{Gr}_k^W \mathcal{H}^n i^* j_* \circ \mu_K(\mathbb{V}^\bullet)$ occurs already in one of the $\mathrm{Gr}_k^W H^q(W_1, \mathrm{Res}_Q^G \mathbb{V}^\bullet)$, $q \in \mathbb{Z}$, $q \le n + c$.

Remark 2.11. Observe that the weights and the Hodge types of the objects $H^q(W_1, \mathrm{Res}_Q^G \mathbb{V}^\bullet)$ are computed in the category $\mathbf{Rep}_F G_1$. They thus depend on the restriction $\mathrm{Res}_{P_1}^G \mathbb{V}^\bullet$ of $\mathbb{V}^\bullet$ to the subgroup P_1 of G. We remind the reader that in general, the functor $\mathrm{Res}_{P_1}^G$ preserves neither the weights nor the Hodge types of a representation. However, it *does* preserve the Hodge filtration [P1, Prop. 4.12].

Proof of Corollary 2.10.   (a) follows from the existence of the weight decomposition in the category $(\mathbf{Rep}_F G_1, \overline{H}_Q / \overline{H}_C)$. The only point that needs to be explained in (b) is the equality

$$\mathrm{Gr}_k^W H^p(\overline{H}_C, H^q(W_1, \mathrm{Res}_Q^G \mathbb{V}^\bullet)) = H^p(\overline{H}_C, \mathrm{Gr}_k^W H^q(W_1, \mathrm{Res}_Q^G \mathbb{V}^\bullet)).$$

But this follows from Corollary 2.4. (c) results from (b), and from Proposition 2.3 (b).   **q.e.d.**

Passage to singular cohomology yields the following:

Corollary 2.12. *Denote by a_1 the structure morphism of M_1^K, and by* $\mathbf{MHS}_F$ *the category* $\mathbf{MHM}_F(\mathbf{Spec}\,\mathbb{C})$*, that is, the category of mixed graded-polarizable F-Hodge structures.*

(a) There is a natural commutative diagram

$$
\begin{array}{ccc}
D^b(\mathbf{Rep}_F G) & \xrightarrow{\ \mu_K\ } & D^b(\mathbf{MHM}_F M^K) \\[4pt]
\Big\downarrow {\scriptstyle \mathrm{Res}_Q^G} & & \Big\downarrow {\scriptstyle i^* j_*[-c]} \\[4pt]
D^b(\mathbf{Rep}_F Q) & & \\[4pt]
\Big\downarrow {\scriptstyle R\Gamma(W_1,\,\bullet\,)} & & \\[4pt]
D^b(\mathbf{Rep}_F G_1, \overline{H}_Q) & & \\[4pt]
\Big\downarrow {\scriptstyle R\Gamma(\overline{H}_C,\,\bullet\,)} & & \\[4pt]
D^b(\mathbf{Rep}_F G_1, \overline{H}_Q\,/\,\overline{H}_C) & & D^b(\mathbf{MHM}_F M_1^K) \\[4pt]
\Big\downarrow {\scriptstyle \mu_{\pi(K_1)}} & & \Big\downarrow {\scriptstyle a_{1*}} \\[4pt]
D^b(\mathbf{MHM}_F M_1^K) & \xrightarrow{\ a_{1*}\ } & D^b(\mathbf{MHS}_F)
\end{array}
$$

(b) For any $\mathbb{V}^\bullet \in D^b(\mathbf{Rep}_F G)$, there is a canonical and functorial spectral sequence

$$
E^{p,q} = H^p(M_1^K, \overline{H}_C, H^q(W_1, \mathrm{Res}_Q^G \mathbb{V}^\bullet)) \Longrightarrow H^{p+q}(M_1^K, i^* j_* \circ \mu_K(\mathbb{V}^\bullet))
$$

in the category of Hodge structures. Here, $H^p(M_1^K, \overline{H}_C, \bullet\,)$ denotes the cohomology functors associated to the composition

$$
a_{1*} \circ \mu_{\pi(K_1)} \circ R\Gamma(\overline{H}_C,\,\bullet\,) : D^b(\mathbf{Rep}_F G_1, \overline{H}_Q) \longrightarrow D^b(\mathbf{MHS}_F)\ .
$$

(c) The spectral sequence of (b) degenerates and splits canonically.

Proof. (a) follows from Theorem 2.6, (b) is clear, and (c) follows from Proposition 2.8. **q.e.d.**

Remark 2.13. (a) Part of the information contained in Corollary 2.10 is known. The splitting of the weight filtration on the variation $\mathcal{H}^n i^* j_* \circ \mu_K(\mathbb{V}^\bullet)$ is proved in [LR, Prop. (6.4)]. According to Corollary 2.10 (b), the local system underlying $\mathrm{Gr}_k^W \mathcal{H}^n i^* j_* \circ \mu_K(\mathbb{V}^\bullet)$ is the direct sum of the local systems underlying the $\mu_{\pi(K_1)} \circ H^p(\overline{H}_C, \mathrm{Gr}_k^W H^q(W_1, \mathrm{Res}_Q^G \mathbb{V}^\bullet))$, for $p+q = n+c$. This result can also be obtained by combining Proposition 2.8 and [LR, Cor. (6.6)].

(b) Note that we can do better than Corollary 2.10 (c). Let $\mathbb{V}$ be in $\mathbf{Rep}_F G$. Kostant's theorem [Vo, 3.2.3] allows us to identify the Q/W_1-representations

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$H^q(W_1, \operatorname{Res}_Q^G \mathbb{V})$, for $\mathbb{V} \in \mathbf{Rep}_F\, G$. Combining this with Corollary 2.10 (b) and Proposition 2.3 (b), we get an explicit a priori list of possible constituents of $\operatorname{Gr}_k^W \mathcal{H}^n i^* j_* \circ \mu_K(\mathbb{V})$.

(c) The cohomology $H^n(M_1^K, i^* j_* \circ \mu_K(\mathbb{V}^\bullet))$ coincides with what is called *deleted neighbourhood cohomology* (of certain strata in a toroidal compactification of M^K) in [HZ1], as can be seen from proper base change for the morphism from a toroidal to the Baily–Borel compactification. Thus, Corollary 2.12 is equivalent to [HZ1, Theorem (5.6.10)] for maximal parabolic subgroups ($R = P$ in the notation of [loc. cit.]).

Overview 2.14. Let us finish this section by an overview of the proof of Theorem 2.6, and of the remaining sections of this paper. We start by developing the basics of abstract group cohomology in Abelian categories (Section 3). We shall see in particular (Proposition 3.13) that as in the case of usual group cohomology, it can be calculated using free resolutions of the trivial module $\mathbb{Z}$. In Section 4, we state basic results on equivariant Hodge modules.
For simplicity, let us assume that the finite group Δ is trivial, and hence, that M_1^K is an actual Shimura variety. The proof of Theorem 2.6 involves the analysis of the degeneration in a toroidal compactification

$$j_{\mathfrak{S}} : M^K \longrightarrow M^K(\mathfrak{S})$$

of M^K, and the computation of its direct image under the proper morphism p from $M^K(\mathfrak{S})$ to $(M^K)^*$.
The pre-image $i_{\mathfrak{S}} : M_{1,\mathfrak{S}}^K \hookrightarrow M^K(\mathfrak{S})$ of M_1^K is itself stratified. We recall the description of this stratification in Section 8, following the presentation of [P2, (3.9)]. We recall in particular (Proposition 8.2) that the formal completion of $M^K(\mathfrak{S})$ along $M_{1,\mathfrak{S}}^K$ is isomorphic to the quotient by the action of Δ_1 of the formal completion of a certain torus embedding $M^{K_1}(\mathfrak{S}_1^0)$ along a union Z of strata. The action of Δ_1 is free, and so is the induced action on the set $\mathfrak{T}$ indexing the strata of Z. In particular, each individual stratum of $M^K(\mathfrak{S})$ can be seen as a stratum of the torus embedding. We are able to identify the composition

$$p_* i_{\sigma*} i_\sigma^* j_{\mathfrak{S}*} \circ \mu_K$$

for any stratum $i_\sigma : Z_\sigma \hookrightarrow Z$ of Z, by appealing to known results, which we recall in Section 6, on degeneration along strata, and on direct images to the base of torus compactifications: the value of $p_* i_{\sigma*} i_\sigma^* j_{\mathfrak{S}*}$ on $\mu_K(\mathbb{V}^\bullet)$ is equal to $\mu_{\pi(K_1)} \circ R\Gamma(W_1, \operatorname{Res}_{P_1}^G \mathbb{V}^\bullet)$. In particular, it does not depend on the stratum Z_σ. Since our isomorphisms are well behaved under the action of Δ_1, we get an object $\mathbb{X}^\bullet$ in the category Δ_1-$D^b((\mathbf{MHM}_F\, M_1^K)^{\mathfrak{T}})$, i.e., a class in $D^b((\mathbf{MHM}_F\, M_1^K)^{\mathfrak{T}})$ with an action of Δ_1. It is constant in the sense that its components, indexed by $\mathfrak{T}$, are all isomorphic.
The problem is now to put the information together, in order to compute

$$p_* i_{\mathfrak{S}}^* j_{\mathfrak{S}*} \circ \mu_K \ .$$

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The formal setting for this is provided by the theory of Čech complexes for Hodge modules, the basics of which are contained in Section 5. In particular (Corollary 5.8), we show how to compute the direct image p_* with the help of stratifications. The answer we get is perfectly adapted to the formalism of group cohomology: assume that $\mathbb{X}^\bullet$ is concentrated in a single degree. It is given by the same object for all strata of $\mathfrak{T}$. Since the action of Δ_1 is free on $\mathfrak{T}$, Proposition 3.13 and Corollary 5.8 tell us that the functor $p_* \, i_{\mathfrak{S}}^* \, j_{\mathfrak{S}*}$ maps $\mu_K(\mathbb{V}^\bullet)$ to $R\Gamma(\Delta_1, \mathbb{X}^\bullet)$.

Unfortunately, $\mathbb{X}^\bullet$ is always concentrated in more than one degree, unless $\mathbb{V}^\bullet = 0$. The formalism of Čech complexes does not in general allow to recover $p_* \, i_{\mathfrak{S}}^* \, j_{\mathfrak{S}*}$ just from the collection of the $p_* \, i_{\sigma *} \, i_\sigma^* \, j_{\mathfrak{S}*}$, viewed as functors on derived categories; we need to relate complexes on different strata *before* passing to the derived category. In other words, the information provided by the object

$$\mathbb{X}^\bullet \in \Delta_1\text{-}D^b((\mathbf{MHM}_F \, M_1^K)^{\mathfrak{T}})$$

is too weak; what is needed is an object in

$$D^b(\Delta_1\text{-}(\mathbf{MHM}_F \, M_1^K)^{\mathfrak{T}}) \, .$$

This explains the presence of Section 7, which provides the missing global information on the degeneration along Z. We work on the normal cone of $M^{K_1}(\mathfrak{S}_1^0)$ along Z, and identify the value of Saito's *specialization functor* Sp_Z on $j_{\mathfrak{S}*} \circ \mu_K(\mathbb{V}^\bullet)$. The most difficult ingredient is control of the *monodromy weight filtration*. We recall a number of results from the literature: first, the explicit description, due to Galligo–Granger–Maisonobe, of the category of perverse sheaves *of normal crossing type* on a product of unit disks; then, Saito's identification of the specialization functor in this description; next, the fundamental theorems, due to Schmid and Cattani–Kaplan, on *nilpotent orbits*; finally, Kashiwara's permanence result on nilpotent orbits under the *nearby cycle functor*. It then suffices to combine all these results in order to deduce the desired statement on the monodromy weight filtration of the composition $Sp_Z \, j_{\mathfrak{S}*} \circ \mu_K(\mathbb{V}^\bullet)$ (Theorem 7.2, Corollary 7.3).

Section 9 puts everything together, and concludes the proof of Theorem 2.6.

3   On the formalism of group cohomology

Let $\mathcal{A}$ be an Abelian category, and H an abstract group. We shall denote by $H\text{-}\mathcal{A}$ the category of objects of $\mathcal{A}$ provided with a left H-action and by $\mathrm{Pro}(\mathcal{A})$ the pro-category associated to $\mathcal{A}$ (see [D1, 0.5]). Hence $H\text{-}\mathrm{Pro}(\mathcal{A})$ is the category of pro-objects of $\mathcal{A}$ provided with a left H-action. All these categories are also Abelian. In this section all functors will be additive. If $\gamma \in H$ and $A \in \mathrm{Ob}(H\text{-}\mathcal{A})$, we denote by the same letter γ the corresponding automorphism of A. We denote by e the unit element of H.

Definition 3.1. The *fixed point functor* associated to H is the functor

$$\Gamma(H, \bullet\,) = (\,\bullet\,)^H : H\text{-}\mathrm{Pro}(\mathcal{A}) \longrightarrow \mathrm{Pro}(\mathcal{A})$$

given by

$$\Gamma(H, A) = (A)^H := \bigcap_{\gamma \in H} \mathrm{Ker}(e - \gamma) \ .$$

In general, the image of the category $H\text{-}\mathcal{A}$ under the functor $\Gamma(H, \bullet\,)$ is not contained in $\mathcal{A}$, unless certain conditions on H or $\mathcal{A}$ are satisfied. Examples for such conditions are: H is finitely generated, or $\mathcal{A}$ contains arbitrary products, or $\mathcal{A}$ is Artinian.

The main aim of this section is to show the existence of the right derived functor

$$R\Gamma(H, \bullet\,) : D^+(H\text{-}\mathrm{Pro}(\mathcal{A})) \longrightarrow D^+(\mathrm{Pro}(\mathcal{A}))$$

(Theorem 3.11). We shall also show that, under some finiteness conditions on H, the above derived functor can be lifted to define functors

$$R\Gamma(H, \bullet\,) : D^+(H\text{-}\mathcal{A}) \longrightarrow D^+(\mathcal{A})$$

or

$$R\Gamma(H, \bullet\,) : D^b(H\text{-}\mathcal{A}) \longrightarrow D^b(\mathcal{A})$$

(Theorem 3.14). As an application of these abstract principles, we establish in Theorem 3.20 the existence of the functor $R\Gamma(\overline{H}_C, \bullet\,)$, which occurs in the statement of our main result, Theorem 2.6. We end the section by giving a proof of Proposition 2.3.

The strategy for the construction of the derived functor is an abstract version of a well known theme. The main interest of this approach is that one does not need to suppose the existence of sufficiently many injective objects. We shall only treat the case of covariant left exact functors, the other cases being completely analogous.

Definition 3.2. Let $\mathcal{A}$ be an Abelian category and let Id be the identity functor. A *resolution functor* is an exact functor $C : \mathcal{A} \to \mathcal{A}$ provided with a morphism of functors Id $\to C$ such that, for every $A \in \mathrm{Ob}(\mathcal{A})$ the map $A \to C(A)$ is a monomorphism.

Definition 3.3. Let $\mathcal{A}$ be an Abelian category, and C a resolution functor. For any object A of $\mathcal{A}$, the *C-resolution* of A, denoted $C^*(A)$, is defined inductively as follows:

$$
\begin{aligned}
K_C^0(A) &:= A \ , \\
C^i(A) &:= C(K_C^i(A)) \ , & i \geq 0 \ , \\
K_C^{i+1}(A) &:= \mathrm{Coker}(K_C^i(A) \hookrightarrow C^i(A)) \ , & i \geq 0 \ .
\end{aligned}
$$

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The differential $d : C^i \to C^{i+1}$ is defined as the composition

$$C^i(A) \longrightarrow\!\!\!\!\rightarrow K_C^{i+1}(A) \lhook\joinrel\longrightarrow C^{i+1}(A) \ .$$

By definition, the sequence

$$0 \longrightarrow A \longrightarrow C^0(A) \longrightarrow C^1(A) \longrightarrow \dots$$

is exact.

Proposition 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact covariant additive functor. Let $C : \mathcal{A} \to \mathcal{A}$ be a resolution functor such that the composition $F \circ C$ is exact. Then the functor*

$$F \circ C^* : K^+(\mathcal{A}) \longrightarrow K^+(\mathcal{B})$$

descends to the level of derived categories. The resulting functor

$$D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}) \ ,$$

equally denoted by $F \circ C^*$, is the total right derived functor RF of F in the sense of [D1, 1.2].

Proof. We need to show that the functor

$$F \circ C^* : K^+(\mathcal{A}) \longrightarrow K^+(\mathcal{B}) \longrightarrow D^+(\mathcal{B})$$

transforms quasi-isomorphisms into isomorphisms; from the construction of the total derived functor [D1, 1.2], it is clear that this will imply the desired equality $RF = F \circ C^*$. Using the cone of such a quasi-isomorphism, we are reduced to showing that $F \circ C^*(K^\bullet)$ is acyclic for any acyclic complex $K^\bullet$ in $C^+(\mathcal{A})$. For this, it is enough to show that for each $i \geq 0$, the functor $F \circ C^i$ is exact, because in this case $F \circ C^*(K^\bullet)$ is the simple complex associated to a double complex with exact rows, hence acyclic.

The following lemma follows by induction from the exactness of the functor C.

Lemma 3.5. *The functors K_C^i and C^i are exact for all $i \geq 0$.*

In the situation of Proposition 3.4, we thus see that $F \circ C^i = (F \circ C) \circ K_C^i$ is the composition of two exact functors, hence exact. **q.e.d.**

Recall [D1, p. 23] that the derived functor RF in the sense of [D1, 1.2] satisfies the universal property of [V1, II.2.1.2].

Example 3.6. We recall how group homology can be defined using (the dual of) the above method. Let H be a group, We denote by $\mathcal{A}b$ the category of Abelian groups. Then $H\text{-}\mathcal{A}b$ is the category of left $\mathbb{Z}H$-modules. If A is an object of $H\text{-}\mathcal{A}b$, then the group of co-invariants is

$$A_H = \mathbb{Z} \underset{\mathbb{Z}H}{\otimes} A$$

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where $\mathbb{Z}$ has the trivial $\mathbb{Z}H$ action. This defines a right exact functor $H\text{-}\mathcal{A}b$ to $\mathcal{A}b$ that we want to derive. Let $\mathrm{Fo} : H\text{-}\mathcal{A}b \to \mathcal{A}b$ be the forgetful functor. We define the functor $C^H : H\text{-}\mathcal{A}b \to H\text{-}\mathcal{A}b$ by

$$C^H(A) := \mathbb{Z}H \underset{\mathbb{Z}}{\otimes} \mathrm{Fo}(A) = \mathrm{Ind}_{\{1\}}^H \mathrm{Res}_{\{1\}}^H(A) \ .$$

The functor C^H is exact and is equipped with a natural equivariant epimorphism $\epsilon : C^H(A) \to A$ given by

$$\epsilon\left(\sum n_i g_i \otimes a_i\right) = \sum n_i g_i a_i \ .$$

Applying the dual of the above construction we get a resolution

$$\ldots \longrightarrow C_2^H(A) \longrightarrow C_1^H(A) \longrightarrow C_0^H(A) \longrightarrow A \longrightarrow 0 \ .$$

Since the composition $(\ \bullet\ )_H \circ C^H$ is the forgetful functor, the total left derived functor of the co-invariant functor is given by $(C_*^H(A))_H$. Note that $C_*^H(\mathbb{Z})$ is a free resolution of $\mathbb{Z}$.

Next we use the general theory to define group cohomology in an arbitrary Abelian category. Let $\mathcal{A}$, $H\text{-}\mathcal{A}$, $\mathrm{Pro}(\mathcal{A})$ and $H\text{-}\mathrm{Pro}(\mathcal{A})$ be as in the beginning of the section. We denote by Fo the forgetful functor from $H\text{-}\mathcal{A}$ to $\mathcal{A}$, as well as the forgetful functor between the pro-categories.

Definition 3.7. The *resolution functor associated to H* is the functor C_H defined as follows. Given an object A of $H\text{-}\mathrm{Pro}(\mathcal{A})$, the underlying object of $C_H(A)$ is $\prod_{h \in H} A \in \mathrm{Ob}(\mathrm{Pro}(\mathcal{A}))$. Let $p_h : C_H(A) \to A$ be the projection over the factor h. The action of an element $\gamma \in H$ over $C_H(A)$ is defined by the family of morphisms

$$\gamma_h : C_H(A) \longrightarrow A \ , \quad h \in H \ ,$$

where $\gamma_h = \gamma \circ p_{\gamma^{-1}h}$.

By definition, we have

$$C_H(A) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}H, A) \ ,$$

with the diagonal action of H.

The following result is immediate from the definition of the action of H on $C_H(A)$. It is the basic ingredient to define morphisms to $C_H(A)$.

Lemma 3.8. *Let A and B be objects of $H\text{-}\mathrm{Pro}(\mathcal{A})$. Then there are canonical bijections between (a) the set of morphisms $f : B \to C_H(A)$ in the category $H\text{-}\mathrm{Pro}(\mathcal{A})$, (b) the set of families of morphisms $f_h : \mathrm{Fo}(B) \to \mathrm{Fo}(A)$ in $\mathrm{Pro}(\mathcal{A})$, $h \in H$, such that*

$$f_h \circ \gamma = \gamma \circ f_{\gamma^{-1}h} \ , \quad \gamma \in H \ ,$$

and (c) the set of morphisms $f_e : \mathrm{Fo}(B) \to \mathrm{Fo}(A)$ in $\mathrm{Pro}(\mathcal{A})$. In other words, the functor C_H represents the functor on $H\text{-}\mathrm{Pro}(\mathcal{A})$ given by

$$B \longmapsto Hom_{\mathrm{Pro}(\mathcal{A})}(\mathrm{Fo}(B), \mathrm{Fo}(A)) \ .$$

Definition 3.9. Denote by $\iota : A \to C_H(A)$ the canonical equivariant monomorphism determined by the family of morphisms $\iota_h = \mathrm{Id}$. We denote by $\tau : \mathrm{Fo}(A) \to \mathrm{Fo}(C_H(A))$ the morphism determined by the family of morphisms $\tau_h = h$.

The following result follows easily from the definitions.

Proposition 3.10. *(a) The functor C_H together with the morphism of functors ι is a resolution functor.*

(b) The morphism τ induces an isomorphism of functors between Fo and $(\bullet)^H \circ C_H$.

Since C_H is a resolution functor and $(\bullet)^H \circ C_H$ is exact (because the functor Fo is exact), we obtain the following result:

Theorem 3.11. *The functor $(\bullet)^H : H\text{-}\mathrm{Pro}(\mathcal{A}) \to \mathrm{Pro}(\mathcal{A})$ is right derivable, and the total right derived functor*

$$R\Gamma(H, \bullet) : D^+(H\text{-}\mathrm{Pro}(\mathcal{A})) \longrightarrow D^+(\mathrm{Pro}(\mathcal{A}))$$

is the functor induced by the exact functor $(\bullet)^H \circ C_H^*$.

The cohomology functors associated to $R\Gamma(H, \bullet)$ will be denoted by $H^p(H, \bullet)$, for $p \in \mathbb{Z}$. We shall see that the functor $R\Gamma(H, \bullet)$ can be computed using *any* right resolution of the trivial H-module $\mathbb{Z}$ by free $\mathbb{Z}H$-modules. For any Abelian category $\mathcal{A}$, we define a bifunctor

$$\mathrm{Hom} : \mathcal{A}b \times \mathrm{Pro}(\mathcal{A}) \longrightarrow \mathrm{Pro}(\mathcal{A}) \ :$$

Let $M \in \mathrm{Ob}(\mathcal{A}b)$ and $A \in \mathrm{Ob}(\mathrm{Pro}(\mathcal{A}))$, and consider the following contravariant functor $\mathcal{F}$ on $\mathrm{Pro}(\mathcal{A})$: by definition, $\mathcal{F}(B)$ is the set of all group homomorphisms

$$\alpha : M \longrightarrow Hom_{\mathrm{Pro}(\mathcal{A})}(B, A) \ .$$

To see that this functor is representable, we first treat the case of a free Abelian group M. Choose a basis $\{x_i\}_{i \in I}$ of M. For any object A of $\mathrm{Pro}(\mathcal{A})$, we see that

$$\prod_{i \in I} A$$

represents the functor $\mathcal{F}$. If M is any Abelian group, we choose any two step free resolution

$$F_2 \longrightarrow F_1 \longrightarrow M \longrightarrow 0 \ .$$

We then have

$$\mathrm{Hom}(M, A) = \mathrm{Ker}(\mathrm{Hom}(F_1, A) \longrightarrow \mathrm{Hom}(F_2, A)) \ .$$

We can take into account the action of H:

Definition 3.12. Let A be an object of $H\text{-}\mathrm{Pro}(\mathcal{A})$, and let M be a $\mathbb{Z}H$-module. The *diagonal action* of H over $\mathrm{Hom}(M, A)$ is defined as follows: for B in $\mathrm{Pro}(\mathcal{A})$ and $\alpha : M \to Hom_{\mathrm{Pro}(\mathcal{A})}(B, A)$ in $\mathrm{Hom}(M, A)(B)$, define $\gamma\alpha : M \to Hom_{\mathrm{Pro}(\mathcal{A})}(B, A)$ as

$$m \longmapsto \gamma \circ \alpha(\gamma^{-1}(m)) \ .$$

When M is a free $\mathbb{Z}H$-module, we can give an explicit description of this action: we choose a basis $\{x_i\}_{i \in I}$ of M as $\mathbb{Z}H$-module. Then

$$\mathrm{Hom}(M, A) = \prod_i \prod_{h \in H} A \ .$$

We write $p_{i,h}$ for the projection over the factor (i, h). Then the action of an element $\gamma \in H$ is determined by the family of morphisms

$$\gamma_{i,h} : \mathrm{Hom}(M, A) \longrightarrow A \ ,$$

with $\gamma_{i,h} = \gamma \circ p_{i,\gamma^{-1}h}$.

Proposition 3.13. *(a) There are canonical equivalences of functors between C_H and $\mathrm{Hom}(C^H(\mathbb{Z}), \bullet)$, and between C_H^* and $\mathrm{Hom}(C_*^H(\mathbb{Z}), \bullet)$.*

(b) Let $F_* \to \mathbb{Z}$ be any resolution of the trivial $\mathbb{Z}H$-module $\mathbb{Z}$ by free $\mathbb{Z}H$-modules. Then the functor $R\Gamma(H, \bullet)$ is induced by $(\mathrm{Hom}(F_*, \bullet))^H$.

Proof. The fact that $C_H = \mathrm{Hom}(C^H(\mathbb{Z}), \bullet)$ is a direct consequence of the definitions. Since the sequence

$$0 \longrightarrow K_1(\mathbb{Z}) \longrightarrow C_0^H(\mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

splits as a sequence of Abelian groups we obtain that the sequence

$$0 \longrightarrow \mathrm{Hom}(K_1(\mathbb{Z}), A) \longrightarrow \mathrm{Hom}(C_0^H(\mathbb{Z}), A) \longrightarrow \mathrm{Hom}(\mathbb{Z}, A) \longrightarrow 0$$

is exact in $\mathrm{Pro}(\mathcal{A})$. Since $\mathrm{Hom}(\mathbb{Z}, A) = A$ we can prove by induction that $K_i(\mathbb{Z})$ is projective as Abelian group, that $\mathrm{Hom}(K_i(\mathbb{Z}), A) = K^i(A)$ and that $\mathrm{Hom}(C_i^H(\mathbb{Z}), A) = C_H^i(A)$. This proves part (a).

For (b), use the fact that any $\mathbb{Z}H$-free resolution F_* of $\mathbb{Z}$ is homotopically equivalent to $C_*^H(\mathbb{Z})$. Therefore the complex $\mathrm{Hom}(F_*, A)^H$ is homotopically equivalent to $(C_H^*(A))^H$. **q.e.d.**

Next we shall put some finiteness conditions on the group H. Recall that a group is of type FL if the trivial $\mathbb{Z}H$-module $\mathbb{Z}$ admits a finite resolution

$$0 \longrightarrow F_n \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

by finitely generated free $\mathbb{Z}H$-modules. A group is called FP_∞ if $\mathbb{Z}$ admits a resolution by finitely generated free $\mathbb{Z}H$-modules.

Theorem 3.14. *(a) If the group H is of type FP_∞, then there exists a canonical functor, also denoted by*

$$R\Gamma(H, \bullet) : D^+(H\text{-}\mathcal{A}) \longrightarrow D^+(\mathcal{A}) \ ,$$

and a natural commutative diagram

$$
\begin{array}{ccc}
D^+(H\text{-}\mathcal{A}) & \xrightarrow{\ R\Gamma(H,\,\bullet\,)\ } & D^+(\mathcal{A}) \\
\downarrow & & \downarrow \\
D^+(H\text{-}\mathrm{Pro}(\mathcal{A})) & \xrightarrow{\ R\Gamma(H,\,\bullet\,)\ } & D^+(\mathrm{Pro}(\mathcal{A}))
\end{array}
$$

(b) If the group H is of type FL, then the functor in (a) respects the bounded derived categories. We thus get a canonical functor, still denoted by

$$
R\Gamma(H,\,\bullet\,) : D^b(H\text{-}\mathcal{A}) \longrightarrow D^b(\mathcal{A}) \ ,
$$

and a natural commutative diagram

$$
\begin{array}{ccc}
D^b(H\text{-}\mathcal{A}) & \xrightarrow{\ R\Gamma(H,\,\bullet\,)\ } & D^b(\mathcal{A}) \\
\downarrow & & \downarrow \\
D^+(H\text{-}\mathrm{Pro}(\mathcal{A})) & \xrightarrow{\ R\Gamma(H,\,\bullet\,)\ } & D^+(\mathrm{Pro}(\mathcal{A}))
\end{array}
$$

Proof. If the group is of type FP_∞, then there exists a resolution

$$
\ldots \xrightarrow{f_{n+1}} F_n \xrightarrow{f_n} \ldots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \longrightarrow \mathbb{Z} \longrightarrow 0 \ ,
$$

where every F_i is a finitely generated free $\mathbb{Z}H$-module. For each i, we choose a basis $(x_{i,j})_{j \in J_i}$ of F_i. Then the morphism f_i is determined by

$$
f_i(x_{i,j}) = \sum_{\substack{k \in J_{i-1} \\ h \in H}} n_j^{k,h} h(x_{i-1,k}) \ .
$$

For any object A of $H\text{-}\mathcal{A}$, we write

$$
S^i(A) = \prod_{j \in J_i} A,
$$

and let $d^{i-1} : S^{i-1}(A) \to S^i(A)$ be the morphism determined by the family of morphisms

$$
d_j^{i-1} : S^{i-1}(A) \longrightarrow A, \quad j \in J_i \ ,
$$

given by

$$
d_j^{i-1} = \sum_{\substack{k \in J_{i-1} \\ h \in H}} n_j^{k,h} h \circ p_{i-1,k} \ ,
$$

where $p_{i-1,k}$ is the projection of $S^{i-1}(A)$ onto the kth factor. Then there is a natural isomorphism of complexes $S^*(A) = \mathrm{Hom}(F_*, A)^H$. Since the complex $S^*(A)$ determines an element of $D^+(\mathcal{A})$, we have proved (a). The proof of (b) is analogous. Note that our construction is canonical, since it does not depend on the choice of the resolution F_*, any two such choices being homotopically equivalent. **q.e.d.**

Remark 3.15. (a) A natural question to ask is whether under the above finiteness conditions on H, the functors $R\Gamma(H, \bullet)$ of Theorem 3.14 (a) and (b) are the actual derived functors of the functor

$$\Gamma(H, \bullet) : H\text{-}\mathcal{A} \longrightarrow \mathcal{A} .$$

The answer in general is negative, and counterexamples occur right in the context of arithmetic groups: let $\mathcal{A}$ be the category of finite dimensional vector spaces over $\mathbb{C}$. Choose a connected, simply connected algebraic group P over $\mathbb{Q}$, which is simple over $\mathbb{Q}$, and of $\mathbb{Q}$-rank at least two. Let H be an arithmetic subgroup of $P(\mathbb{Q})$. Then $H\text{-}\mathcal{A}$, the category of abstract representations in finite dimensional $\mathbb{C}$-vector spaces, is semi-simple: indeed, for two objects $\mathbb{V}$ and $\mathbb{W}$ of $H\text{-}\mathcal{A}$, we have

$$\mathrm{Ext}^1_{H\text{-}\mathcal{A}}(\mathbb{V}, \mathbb{W}) = H^1(H, \mathbb{V}^* \otimes_{\mathbb{C}} \mathbb{W}) ,$$

and the latter group is zero by [Rg, Cor. 2 of Thm. 2]. Therefore, any additive functor on $H\text{-}\mathcal{A}$ is automatically exact. In particular, the derived functor of $\Gamma(H, \bullet)$ takes the value $\Gamma(H, \mathbb{V})[0]$ on any object $\mathbb{V}$ of $H\text{-}\mathcal{A}$. On the other hand, if H is neat in $P(\mathbb{Q})$, then it is of type FL by [BoS, 11.1 (c)], hence Theorem 3.14 (b) is applicable; but there exist examples of such H, and objects $\mathbb{V}$ of $H\text{-}\mathcal{A}$, for which

$$\{p \geq 1 \,|\, H^p(H, \mathbb{V}) \neq 0\}$$

is not empty (e.g., [BoS, Prop. 11.3 (b)]).

(b) If the category $\mathcal{A}$ is Artinian, then the natural functor

$$D^+(\mathcal{A}) \longrightarrow D^+(\mathrm{Pro}(\mathcal{A}))$$

is a full embedding. Its image consists of the complexes whose cohomology objects lie in $\mathcal{A}$, the sub-category of Artinian objects of $\mathrm{Pro}(\mathcal{A})$. Consequently, the conclusions of (a) and (b) of Theorem 3.14 are equivalent to the following: (a') for any object A of $H\text{-}\mathcal{A}$, the group cohomology objects $H^p(H, A)$ lie all in $\mathcal{A}$; (b') for any object A of $H\text{-}\mathcal{A}$, the group cohomology objects $H^p(H, A)$ lie all in $\mathcal{A}$, and are trivial for large p. If (a') or (b') is satisfied, then the respective functor $R\Gamma(H, \bullet)$ is uniquely determined by the commutative diagram in 3.14.

We note the following consequence of Theorem 3.14:

Corollary 3.16. *Assume that the group H is of type FP_∞, and that the Abelian category $\mathcal{A}$ is semi-simple. Let A be an object of $H\text{-}\mathcal{A}$, and $p \in \mathbb{Z}$. Then any irreducible factor of $H^p(H, A) \in \mathrm{Ob}(\mathcal{A})$ is an irreducible factor of $\mathrm{Fo}(A)$.*

Proof. By Theorem 3.14 (a), the object $H^p(H, A)$ is the cohomology object of a complex, all of whose components are finite products of copies of $\mathrm{Fo}(A)$. **q.e.d.**

Next we state the compatibility of group cohomology with respect to exact functors. Let $\mu : \mathcal{A} \to \mathcal{B}$ be an exact functor between Abelian categories. We denote by the same symbol μ the induced functor between the categories H-$\mathcal{A}$ (resp. $\mathrm{Pro}(\mathcal{A})$, H-$\mathrm{Pro}(\mathcal{A})$) and H-$\mathcal{B}$ (resp. $\mathrm{Pro}(\mathcal{B})$, H-$\mathrm{Pro}(\mathcal{B})$). The proof of the following result is immediate and left to the reader.

Proposition 3.17. *Let $\mu : \mathcal{A} \to \mathcal{B}$ be an exact functor between Abelian categories. Then there is a natural commutative diagram*

$$
\begin{array}{ccc}
D^+(H\text{-}\mathrm{Pro}(\mathcal{A})) & \xrightarrow{\;R\Gamma(H,\,\bullet\,)\;} & D^+(\mathrm{Pro}(\mathcal{A})) \\
\downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} \\
D^+(H\text{-}\mathrm{Pro}(\mathcal{B})) & \xrightarrow{\;R\Gamma(H,\,\bullet\,)\;} & D^+(\mathrm{Pro}(\mathcal{B}))
\end{array}
$$

If H is of type FL or FP_∞, then there are natural commutative diagrams

$$
\begin{array}{ccc}
D^?(H\text{-}\mathcal{A}) & \xrightarrow{\;R\Gamma(H,\,\bullet\,)\;} & D^?(\mathcal{A}) \\
\downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} \\
D^?(H\text{-}\mathcal{B}) & \xrightarrow{\;R\Gamma(H,\,\bullet\,)\;} & D^?(\mathcal{B})
\end{array}
$$

with $? = b$ if H is of type FL, and $? = +$ if H is of type FP_∞.

Variant 3.18. (a) There are obvious variants of 3.7–3.17 for Abelian categories $\mathcal{A}$ which are closed under arbitrary products. More precisely, in this case, the use of the pro-category $\mathrm{Pro}(\mathcal{A})$ is unnecessary, and the constructions and statements of 3.7–3.17 remain valid when the symbol $\mathrm{Pro}(\mathcal{A})$ is replaced by $\mathcal{A}$.

(b) Consider the case when the group H is normal in a larger group L. Then we may study the fixed point functor

$$
\Gamma(H,\,\bullet\,) = (\,\bullet\,)^H : L\text{-}\mathrm{Pro}(\mathcal{A}) \longrightarrow L/H\text{-}\mathrm{Pro}(\mathcal{A})
$$

defined in the same way as in 3.1. The resolution functor is the functor C_L of 3.7. The analogue of Proposition 3.10 (b) reads as follows: the functor $(\,\bullet\,)^H \circ C_L$ is isomorphic to $C_{L/H} \circ \mathrm{Fo}$. In particular, it is exact. Therefore, the analogue of Theorem 3.11 holds: the above fixed point functor is right derivable, and

$$
R\Gamma(H,\,\bullet\,) = (\,\bullet\,)^H \circ C_L^* \,.
$$

Furthermore, by Proposition 3.13 (b) (applied to the group H and the free $\mathbb{Z}H$-resolution $C_*^L(\mathbb{Z})$ of Example 3.6), we see that the diagram

$$
\begin{array}{ccc}
D^+(L\text{-}\mathrm{Pro}(\mathcal{A})) & \xrightarrow{\;R\Gamma(H,\,\bullet\,)\;} & D^+(L/H\text{-}\mathrm{Pro}(\mathcal{A})) \\
{\scriptstyle\mathrm{Res}^L_H}\downarrow & & \downarrow{\scriptstyle\mathrm{Res}^{L/H}_{\{e\}}} \\
D^+(H\text{-}\mathrm{Pro}(\mathcal{A})) & \xrightarrow{\;R\Gamma(H,\,\bullet\,)\;} & D^+(\mathrm{Pro}(\mathcal{A}))
\end{array}
$$

is commutative. Finally, the analogues of Theorem 3.14 (a), (b) hold if the group L is of type FP_∞, resp. of type FL.

(c) The construction of (b) continues to work, and the statements made in (b) continue to hold in a somewhat larger generality. Namely, let $\mathcal{A}$ be an Abelian category, on which the action of an abstract group L is given. This means that there are given contravariant functors γ^* on $\mathcal{A}$, for $\gamma \in L$, such that $(\gamma_1 \cdot \gamma_2)^* = \gamma_2^* \circ \gamma_1^*$ for all $\gamma_1, \gamma_2 \in L$, and such that $e^* = \mathrm{Id}$. We denote by L-$\mathcal{A}$ the category of pairs

$$(A, (\rho_\gamma)_{\gamma \in L}) \,,$$

where $A \in \mathrm{Ob}(\mathcal{A})$, and $(\rho_\gamma)_{\gamma \in L}$ is a family of isomorphisms

$$\rho_\gamma : \gamma^* A \xrightarrow{\ \sim\ } A$$

in $\mathcal{A}$ such that the cocycle condition holds. In the same way, define the category L-$\mathrm{Pro}(\mathcal{A})$. We assume that the action of a given normal subgroup H of L on $\mathcal{A}$ is trivial: $\gamma^* = \mathrm{Id}$ for all $\gamma \in H$. Therefore, the action of L on $\mathcal{A}$ is induced by an action of the quotient L/H. The fixed point functor

$$\Gamma(H, \ \bullet \ ) = (\ \bullet \ )^H : L\text{-}\mathrm{Pro}(\mathcal{A}) \longrightarrow L/H\text{-}\mathrm{Pro}(\mathcal{A})$$

is defined by the same formula as in 3.1. The resolution functor is the functor C_L of (b). The action of an element $\gamma \in L$,

$$\rho_\gamma : \gamma^* C_L(A) \xrightarrow{\ \sim\ } C_L A$$

is determined by the family of morphisms $\gamma_h = \rho_\gamma \circ \gamma^* p_{\gamma h} : \gamma^* C_L(A) \to A$.

It is clear that there is a variant of Proposition 3.13 (b) in the setting of Variant 3.18 (c). We quote the precise result for further reference:

Proposition 3.19. *Let $\mathcal{A}$ be an Abelian category with an action of a group L. Let H be a normal subgroup of L, which acts trivially on $\mathcal{A}$. Let F_* be a free $\mathbb{Z}L$-resolution of the trivial L-module $\mathbb{Z}$. Then the functor*

$$R\Gamma(H, \ \bullet \ ) : D^+(L\text{-}\mathrm{Pro}(\mathcal{A})) \longrightarrow D^+(L/H\text{-}\mathrm{Pro}(\mathcal{A}))$$

is represented by the functor $(\mathrm{Hom}(F_*, \ \bullet \ ))^H$.

It is possible to further enlarge the degree of generality by imposing conditions on the action of a second normal subgroup H' of the group L in 3.18 (c). This applies in particular to the situation considered in Definition 2.1, where $H' = G_1(\mathbb{Q})$, and $(\mathbf{Rep}_F G_1, \overline{H}_Q)$ is the full sub-category of $\overline{H}_Q$-$\mathbf{Rep}_F G_1$ of objects satisfying condition 2.1 (a) (i). We want to derive the functor $\Gamma(\overline{H}_C, \ \bullet \ )$ of Definition 2.2. In this case, we use the resolution functor

$$C_{\Delta_1} : (\mathrm{Pro}(\mathbf{Rep}_F G_1), \overline{H}_Q) \longrightarrow (\mathrm{Pro}(\mathbf{Rep}_F G_1), \overline{H}_Q)$$

given by

$$C_{\Delta_1}(\mathbb{V}_1) = \prod_{h \in \Delta_1} \mathbb{V}_1 \,.$$

Recall that by definition, we have $\Delta_1 = \overline{H}_Q / G_1(\mathbb{Q})$, and $\Delta = \overline{H}_Q / G_1(\mathbb{Q}) \overline{H}_C$. The composition $\Gamma(\overline{H}_C, \bullet) \circ C_{\Delta_1}$ maps $\mathbb{V}_1$ to $\prod_{\overline{h} \in \Delta} \mathbb{V}_1$, and hence is exact. Applying freely the results obtained so far, we get:

Theorem 3.20. *(a) The functor*

$$\Gamma(\overline{H}_C, \bullet) : (\mathrm{Pro}(\mathbf{Rep}_F\, G_1), \overline{H}_Q) \longrightarrow (\mathrm{Pro}(\mathbf{Rep}_F\, G_1), \overline{H}_Q / \overline{H}_C)$$

is right derivable:

$$R\Gamma(\overline{H}_C, \bullet) : D^+(\mathrm{Pro}(\mathbf{Rep}_F\, G_1), \overline{H}_Q) \longrightarrow D^+(\mathrm{Pro}(\mathbf{Rep}_F\, G_1), \overline{H}_Q / \overline{H}_C) .$$

(b) The functor $R\Gamma(\overline{H}_C, \bullet)$ respects the sub-categories $D^b((\mathbf{Rep}_F\, G_1), ?)$: there is a commutative diagram

$$
\begin{array}{ccc}
D^b(\mathbf{Rep}_F\, G_1, \overline{H}_Q) & \xrightarrow{\ R\Gamma(\overline{H}_C, \bullet)\ } & D^b(\mathbf{Rep}_F\, G_1, \overline{H}_Q / \overline{H}_C) \\
\downarrow & & \downarrow \\
D^+(\mathrm{Pro}(\mathbf{Rep}_F\, G_1), \overline{H}_Q) & \xrightarrow{\ R\Gamma(\overline{H}_C, \bullet)\ } & D^+(\mathrm{Pro}(\mathbf{Rep}_F\, G_1), \overline{H}_Q / \overline{H}_C)
\end{array}
$$

Proof. Part (a) follows from the general formalism developed above. For (b), we intend to apply the criterion of Theorem 3.14. Because of the form of our resolution functor C_{Δ_1}, we have to impose the finiteness condition on the group Δ_1 (see Variants 3.18 (b) and (c)). More precisely, we need to know that Δ_1 is of type FL. By definition, this group is an arithmetic subgroup of $Q/P_1(\mathbb{Q})$, which is neat since K is. By [BoS, 11.1 (c)], such a group is indeed of type FL. **q.e.d.**

We still need to prove what was left open in Section 2:

Proof of Proposition 2.3. Part (a) is a special case of Proposition 3.17, and (b) follows from semi-simplicity of the category $\mathbf{Rep}_F\, G_1$, and from Corollary 3.16. **q.e.d.**

4 Equivariant algebraic Hodge modules

The aim of this short section is to develop some elementary theory of equivariant algebraic Hodge modules.

Because of the local nature of Hodge modules, the category $\mathbf{MHM}_F\, X$ can be defined for reduced schemes X, which are only *locally* of finite type over $\mathbb{C}$. However, the formalism of Grothendieck's functors [Sa, Section 4] is constructed on the bounded derived categories of Hodge modules on reduced schemes which are (globally) of finite type over $\mathbb{C}$. It does not obviously extend to the derived categories of Hodge modules on schemes of a more general type.

Definition 4.1. Let X be a reduced scheme which is locally of finite type over $\mathbb{C}$, and H an abstract group acting on X by algebraic automorphisms. The category $H\text{-}\mathbf{MHM}_F\, X$ consists of pairs

$$\left(\mathbb{M}, (\rho_\gamma)_{\gamma \in H}\right) ,$$

where $\mathbb{M} \in \mathbf{MHM}_F\, X$, and $(\rho_\gamma)_{\gamma \in H}$ is a family of isomorphisms

$$\rho_\gamma : \gamma^*\mathbb{M} \overset{\sim}{\longrightarrow} \mathbb{M}$$

in $\mathbf{MHM}_F\, X$ such that the cocycle condition holds.

Note that this is a special case of what was done in Variant 3.18 (c). We shall repeatedly use the following principle:

Proposition 4.2. *In the situation of Definition 4.1, suppose that the action of H on X is free and proper in the sense of [P2, (1.7)], with quotient $H\backslash X$. Denote by Π the morphism from X to $H\backslash X$. Then the inverse image*

$$\Pi^* : \mathbf{MHM}_F(H\backslash X) \longrightarrow H\text{-}\mathbf{MHM}_F\, X$$

is an equivalence of categories, which possesses a canonical pseudo-inverse.

Proof. The pseudo-inverse is given by the direct image Π_*, followed by the H-invariants $\Gamma(H, \bullet)$. Since the direct image is not in general defined for morphisms which are only locally of finite type, this definition needs to be explained: choose an H-equivariant open covering $\mathfrak{V}$ of X, such that each open subset V in $\mathfrak{V}$ satisfies

$$\Pi^{-1}(\Pi(V)) = \coprod_{h \in H} h(V) .$$

This is possible because of our assumption on the action of H. It is then clear how to define the restriction of $\Gamma(H, \bullet) \circ \Pi_*$ to any open subset in the quotient $H\backslash \mathfrak{V}$. The resulting collection of Hodge modules glues to give a Hodge module on $H\backslash X$. **q.e.d.**

Corollary 4.3. *In the situation of Proposition 4.2, the inverse image*

$$\Pi^* : D^b(\mathbf{MHM}_F(H\backslash X)) \longrightarrow D^b(H\text{-}\mathbf{MHM}_F\, X)$$

is an equivalence of categories, which possesses a canonical pseudo-inverse.

Remark 4.4. Using the formalism developed in Section 3, we can give a more conceptual meaning of the canonical pseudo-inverse

$$D^b(H\text{-}\mathbf{MHM}_F\, X) \longrightarrow D^b(\mathbf{MHM}_F(H\backslash X))$$

of Corollary 4.3. As in the proof of [Sa, Thm. 4.3], it is possible, using a covering as in the proof of Proposition 4.2, to define the direct image

$$\begin{aligned}
\Pi_* : D^b(H\text{-}\mathbf{MHM}_F\, X) \;\longrightarrow\;& D^b(H\text{-}\mathrm{Pro}(\mathbf{MHM}_F(H\backslash X))) \\
\subset\;& D^+(H\text{-}\mathrm{Pro}(\mathbf{MHM}_F(H\backslash X))) \;\;.
\end{aligned}$$

Its image consists of $\Gamma(H, \bullet)$-acyclic complexes. The composition of Π_* and the functor

$$R\Gamma(H, \bullet) : D^+(H\text{-}\mathrm{Pro}(\mathbf{MHM}_F(H\backslash X))) \longrightarrow D^+(\mathrm{Pro}(\mathbf{MHM}_F(H\backslash X)))$$

of Theorem 3.11 factors through $D^b(\mathbf{MHM}_F(H\backslash X))$. Our quasi-inverse is the functor

$$D^b(H\text{-}\mathbf{MHM}_F X) \longrightarrow D^b(\mathbf{MHM}_F(H\backslash X))$$

induced by the composition $R\Gamma(H, \bullet) \circ \Pi_*$.

5 Čech complexes for Hodge modules

For later purposes, we need to develop the basics of the formalism of Čech complexes associated to closed coverings in the context of Hodge modules.

Fix a reduced scheme Z, which is separated and of finite type over $\mathbb{C}$. Let $\mathfrak{Z} = \{Z_\sigma\}_{\sigma \in \Sigma}$ be a finite covering of Z by reduced closed sub-schemes, not necessarily different from each other. We denote by $\mathfrak{Z}_\bullet$ the free simplicial set generated by the set of indices Σ. That is, $\mathfrak{Z}_p = \Sigma^{p+1}$ is the set of $p+1$-tuples $(\sigma_0, \ldots, \sigma_p)$. If

$$\tau : \{0, \ldots, q\} \longrightarrow \{0, \ldots, p\}$$

is an increasing map and $I = (\sigma_0, \ldots, \sigma_p) \in \mathfrak{Z}_p$, then

$$\mathfrak{Z}_\bullet(\tau)(I) = (\sigma_{\tau(0)}, \ldots, \sigma_{\tau(q)}) \in \mathfrak{Z}_q .$$

Definition 5.1. Define the Abelian category $(\mathbf{MHM}_F Z)^{\mathfrak{Z}}_\infty$ as the category of mixed Hodge modules over the simplicial scheme $Z \times \mathfrak{Z}_\bullet$.

Explicitly, an element of $(\mathbf{MHM}_F Z)^{\mathfrak{Z}}_\infty$ is a family $(\mathbb{M}_I)_I$ of objects of $\mathbf{MHM}_F Z$ indexed by $\mathfrak{Z}_\bullet$, and for every increasing map

$$\tau : \{0, \ldots, q\} \longrightarrow \{0, \ldots, p\}$$

and each $I \in \mathfrak{Z}_p$, a morphism

$$\tau_I : \mathbb{M}_{\mathfrak{Z}_\bullet(\tau)(I)} \longrightarrow \mathbb{M}_I ,$$

equal to the identity if $\tau = \mathrm{Id}_{\{0, \ldots, q\}}$, and such that

$$(\eta \circ \tau)_I = \eta_I \circ \tau_{\mathfrak{Z}_\bullet(\eta)(I)} .$$

Definition 5.2. Define $(\mathbf{MHM}_F Z)^{\mathfrak{Z}}$ as the full Abelian sub-category of $(\mathbf{MHM}_F Z)^{\mathfrak{Z}}_\infty$ consisting of objects

$$((\mathbb{M}_I)_I, (\tau_I)_{\tau, I})$$

satisfying the following property:

$$\tau_I : \mathbb{M}_{\mathbf{3}_\bullet(\tau)(I)} \longrightarrow \mathbb{M}_I$$

is an isomorphism for any increasing $\tau : \{0,\ldots,q\} \to \{0,\ldots,p\}$, and for any $I \in \mathbf{3}_p$ such that the subsets of Σ underlying the $q+1$-tuple $\mathbf{3}_\bullet(\tau)(I)$ and the $p+1$-tuple I are the same.

Remark 5.3. By definition, the components of an object of the category $(\mathbf{MHM}_F\, Z)^{\mathbf{3}}$ represent a finite number of isomorphism classes of Hodge modules on Z.

Observe that the theory of mixed Hodge modules over general simplicial schemes is not well established because for general morphisms, inverse images of mixed Hodge modules are only defined in the derived category. However, in our situation, there is no problem since all the morphisms of the simplicial scheme $Z \times \mathbf{3}_\bullet$ are given by the identity on Z.

Observe also that we can define $(\mathbf{MHM}_F\, Z)^{\mathbf{3}}$ for locally finite Z and infinite coverings $\mathbf{3}$, or even for any simplicial set $\mathbf{3}_\bullet$ not necessarily associated to a Čech covering. An object $(\mathbb{M}_I)_I$ of $(\mathbf{MHM}_F\, Z)^{\mathbf{3}}$ defines a co-simplicial object, denoted $\mathbb{M}_\bullet$, of the category $\mathbf{MHM}_F\, Z$ if $\mathbf{3}$ is finite, and a co-simplicial object of the category $\mathrm{Pro}(\mathbf{MHM}_F\, Z)$ if $\mathbf{3}$ is infinite: put

$$\mathbb{M}_p = \prod_{I \in \mathbf{3}_p} \mathbb{M}_I \ ,$$

with the induced morphisms.

We go back to the hypothesis of Definition 5.2. Thus, Z is of finite type, and $\mathbf{3}$ is finite. The following observation will be used repeatedly:

Proposition 5.4. *Let $f : (\mathbb{M}_I)_I^\bullet \to (\mathbb{N}_I)_I^\bullet$ be a morphism in the category $D^b((\mathbf{MHM}_F\, Z)^{\mathbf{3}})$. Then f is an isomorphism if and only if $f_I : \mathbb{M}_I^\bullet \to \mathbb{N}_I^\bullet$ is an isomorphism in $D^b(\mathbf{MHM}_F\, Z)$, for all $I \in \mathbf{3}_p$, and all $p \geq 0$.*

Next, we need to define functors

$$S_\bullet$$

$$D^b(\mathbf{MHM}_F\, Z) \qquad D^b((\mathbf{MHM}_F\, Z)^{\mathbf{3}})$$

$$\mathrm{Tot}$$

The functor Tot is induced by the exact functor

$$(\mathbf{MHM}_F\, Z)^{\mathbf{3}} \longrightarrow C^+(\mathbf{MHM}_F\, Z) \ ,$$

denoted by the same symbol, that sends a co-simplicial mixed Hodge module $(\mathbb{M}_I)_I$ to the normalized cochain complex of mixed Hodge modules associated

to $\mathbb{M}_\bullet$. Note that by definition of the category $(\mathbf{MHM}_F\, Z)^{\mathbf{3}}$, the resulting functor

$$\text{Tot} : D^b((\mathbf{MHM}_F\, Z)^{\mathbf{3}}) \longrightarrow D^+(\mathbf{MHM}_F\, Z)$$

factorizes through $D^b(\mathbf{MHM}_F\, Z)$.

The construction of the functor $S_\bullet$ depends on the covering $\mathbf{3}$, not only on the index set Σ. For any element $I = (\sigma_0, \ldots, \sigma_p)$ of $\mathbf{3}_p$, set

$$Z_I := \bigcap_{k=0}^p Z_{\sigma_k}\ ,$$

with its reduced scheme structure. We shall write i_I for the closed immersion of Z_I into Z. For any increasing map τ, if $J = \mathbf{3}_\bullet(\tau)(I)$ then Z_I is a closed subset of Z_J. The basic idea for the construction of the functor

$$S_\bullet : D^b(\mathbf{MHM}_F\, Z) \longrightarrow D^b((\mathbf{MHM}_F\, Z)^{\mathbf{3}})$$

is to associate to a complex $\mathbb{M}^\bullet$ of Hodge modules on Z the class of a certain complex of Hodge modules over $Z \times \mathbf{3}_\bullet$ that restricts to

$$(i_I)_* i_I^* \mathbb{M}^\bullet$$

on the component $Z \times \{I\}$. In order to do this rigorously, we recall the definition of $(i_I)_* i_I^*$ in Saito's formalism [Sa, (4.4.1)]: *choose* an open affine covering of the complement $j_I : U_I \hookrightarrow Z$ of Z_I, and use the Čech complex associated to that covering to define the functor $(j_I)_! j_I^*$ on the level of complexes, together with a transformation $(j_I)_! j_I^* \to \text{Id}$. The functor

$$(i_I)_* i_I^* : C^b(\mathbf{MHM}_F\, Z) \longrightarrow C^b(\mathbf{MHM}_F\, Z)$$

maps a complex $\mathbb{M}^\bullet$ to the simple complex associated to

$$(j_I)_! j_I^* \mathbb{M}^\bullet \longrightarrow \mathbb{M}^\bullet\ .$$

This construction descends to the level of derived categories, and the induced functor

$$(i_I)_* i_I^* : D^b(\mathbf{MHM}_F\, Z) \longrightarrow D^b(\mathbf{MHM}_F\, Z)$$

does not depend on the choice of the affine covering of U_I.

In our situation, we can choose the affine coverings for the different closed sub-schemes in such a way that, for every inclusion $Z_I \subset Z_J$, any open affine subset occurring in the covering of U_J is contained in an open affine subset occurring in the covering of U_I. This choice induces a compatible set of morphisms $(i_J)_* i_J^* \mathbb{M}^\bullet \to (i_I)_* i_I^* \mathbb{M}^\bullet$ at the level of complexes. Putting $S_I(\mathbb{M}^\bullet) = (i_I)_* i_I^* \mathbb{M}^\bullet$, we thus obtain a functor

$$S_\bullet := ((i_I)_* i_I^*)_I : C^b(\mathbf{MHM}_F\, Z) \longrightarrow C^b((\mathbf{MHM}_F\, Z)^{\mathbf{3}})\ .$$

This construction descends to the level of derived categories. The induced functor is independent of the choices.

Remark 5.5. For later use, it will be important to observe that the above construction also defines a filtered version of the functor $S_\bullet$:

$$S_\bullet : DF^b(\mathbf{MHM}_F\, Z) \longrightarrow DF^b((\mathbf{MHM}_F\, Z)^3) \, ,$$

where DF^b denote the *filtered bounded derived categories* used in [BBD, Section 3] and [B, Appendix A]. Thus, the term "bounded" refers to boundedness of the complexes as well as finiteness of the filtrations.

Proposition 5.6. *There is a canonical isomorphism of functors*

$$\mathrm{Id} \cong \mathrm{Tot} \circ S_\bullet : D^b(\mathbf{MHM}_F\, Z) \longrightarrow D^b(\mathbf{MHM}_F\, Z) \, .$$

Proof. By construction, the functors $(i_I)_* \, i_I^*$ come with natural transformations $\mathrm{Id} \to (i_I)_* \, i_I^*$, which induce a natural transformation $\mathrm{Id} \to \mathrm{Tot} \circ S_\bullet$. That it is an isomorphism can be checked after application of the forgetful functor to the bounded derived category of perverse sheaves on $Z(\mathbb{C})$. By [B, Main Thm. 1.3], this latter category can be identified with a full sub-category of the derived category of Abelian sheaves on $Z(\mathbb{C})$. Thus, our claim follows from the fact that the Čech complex of any sheaf $\mathfrak{F}$ is a resolution of $\mathfrak{F}$.

q.e.d.

We need to discuss functoriality of our constructions. Let $p : Z \to Y$ be a morphism of reduced schemes, which are separated and of finite type over $\mathbb{C}$, and assume given finite coverings $\{Z_\sigma\}_{\sigma \in \Sigma}$ and $\{Y_\sigma\}_{\sigma \in \Sigma}$ of Z and Y, respectively. Since the index set Σ is the same for the two coverings, we shall write $(\mathbf{MHM}_F\, Z)^3$ and $(\mathbf{MHM}_F\, Y)^3$ for the respective categories defined in 5.2. We have the following co-simplicial version of direct images under p:

Proposition 5.7. *(a) There is a canonical functor*

$$p_*^3 : D^b((\mathbf{MHM}_F\, Z)^3) \longrightarrow D^b((\mathbf{MHM}_F\, Y)^3) \, .$$

(b) Let $q \geq 0$, and $J \in \mathfrak{Z}_q$. There is a natural commutative diagram

$$
\begin{array}{ccc}
D^b((\mathbf{MHM}_F\, Z)^3) & \xrightarrow{(\mathbb{M}_I)_I^\bullet \mapsto \mathbb{M}_J^\bullet} & D^b(\mathbf{MHM}_F\, Z) \\
{\scriptstyle p_*^3} \downarrow & & \downarrow {\scriptstyle p_*} \\
D^b((\mathbf{MHM}_F\, Y)^3) & \xrightarrow{(\mathbb{N}_I)_I^\bullet \mapsto \mathbb{N}_J^\bullet} & D^b(\mathbf{MHM}_F\, Y)
\end{array}
$$

(c) There is a natural commutative diagram

$$
\begin{array}{ccc}
D^b((\mathbf{MHM}_F\, Z)^3) & \xrightarrow{\mathrm{Tot}} & D^b(\mathbf{MHM}_F\, Z) \\
{\scriptstyle p_*^3} \downarrow & & \downarrow {\scriptstyle p_*} \\
D^b((\mathbf{MHM}_F\, Y)^3) & \xrightarrow{\mathrm{Tot}} & D^b(\mathbf{MHM}_F\, Y)
\end{array}
$$

Proof. In order to define p_*^3, we recall part of the definition of p_* in Saito's formalism [Sa, Thm. 4.3]: if $k : V \hookrightarrow Z$ is the immersion of an open affine subset, then $(p \circ k)_*$ is the total left derived functor of the functor $\mathcal{H}^0(p \circ k)_*$ (the definition of $\mathcal{H}^0(p \circ k)_*$ will not be recalled, since it will not be needed). *Choose* a finite open affine covering $\mathfrak{V} = \{V_1, \ldots, V_r\}$ of Z. Call a Hodge module $\mathbb{L}$ on Z p_*-*acyclic with respect to* $\mathfrak{V}$ if the restriction of $\mathbb{L}$ to any intersection of the V_l is $(p \circ k)_*$-acyclic, where k denotes the open immersion of that intersection into Z. We then have (see the proof of [Sa, Thm. 4.3], which in turn is based on [B, Section 3]):

(1) for any Hodge module $\mathbb{M}$ on Z, there is an epimorphism $\mathbb{L} \longrightarrow \mathbb{M}$, whose source is p_*-acyclic with respect to $\mathfrak{V}$.

(2) for any system consisting of Hodge modules $\mathbb{M}_n$ representing a finite number of isomorphism classes, the epimorphism in (1) can be chosen functorially with respect to all morphisms between the $\mathbb{M}_n$. Indeed, this can be seen from [B, proof of Lemma 3.3]; e.g., if Z is quasi-projective, the $\mathbb{L}_n$ can be chosen as $j_! j^* \mathbb{M}_n$, for the open immersion j of some suitable affine open subset U of Z (the same for all n).

Given a bounded complex $\mathbb{M}^\bullet$ of Hodge modules, we use (1) and (2) to construct a complex $\mathbb{L}^\bullet$, all of whose components are p_*-acyclic with respect to $\mathfrak{V}$, and a morphism $\varphi : \mathbb{L}^\bullet \to \mathbb{M}^\bullet$, which becomes an isomorphism in $D^-(\mathbf{MHM}_F Z)$. Observe that $\mathbb{L}^\bullet$ can be chosen to be bounded because of the finite cohomological dimension of the $(p \circ k)_* \, k^*$. (For later use, it will be important to note that furthermore, the morphism of complexes φ can be chosen to be epimorphic in all degrees.) Replace $\mathbb{L}^\bullet$ by the Čech complex $C(\mathbb{L}^\bullet)^\bullet$ associated to $\mathfrak{V}$ (note that since the k are affine, the $k_* \, k^*$ are exact). We get an actual complex $p_* C(\mathbb{L}^\bullet)^\bullet$, whose class in $D^b(\mathbf{MHM}_F Y)$ does not depend on any of the choices. By definition, this class is $p_* \mathbb{M}^\bullet$.

Thanks to (2), and to Remark 5.3, the above can be imitated on the level of complexes of simplicial objects. This is the functor p_*^3, and it satisfies properties (b) and (c). **q.e.d.**

Corollary 5.8. *There is a natural commutative diagram*

$$
\begin{array}{ccc}
D^b(\mathbf{MHM}_F Z) & \xrightarrow{S_\bullet} & D^b((\mathbf{MHM}_F Z)^3) \\
{\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*^3} \\
D^b(\mathbf{MHM}_F Y) & \xleftarrow{\mathrm{Tot}} & D^b((\mathbf{MHM}_F Y)^3)
\end{array}
$$

Proof. This follows from Propositions 5.7 (c) and 5.6. **q.e.d.**

6 Degeneration in relative torus embeddings

The aim of this short section is to study the degeneration of local systems, and of variations of Hodge structure in a relative torus embedding. We are going to use a number of concepts related to torus embeddings as explained in [KKMS, Chap. I] or [P1, 5.1–5].

We shall consider the following situation: B is a scheme over $\mathbb{C}$, and T a complex torus with cocharacter group Y. Fix a smooth rational polyhedral decomposition $\mathfrak{s}$ of $Y_{\mathbb{R}}$. Consider the (constant) torus T_B over B. We get a (partial) compactification $T_{\mathfrak{s}}$ of T_B relative to B, which is naturally endowed with a stratification indexed by the cones in $\mathfrak{s}$. More generally, this is true for any T-torsor X over B. Fix a cone $\theta \in \mathfrak{s}$, and consider the diagram

$$X \overset{\jmath}{\hookrightarrow} X_{\mathfrak{s}} \overset{\imath_\theta}{\hookleftarrow} X_\theta \overset{\jmath_\theta}{\hookleftarrow} X_\theta^\circ$$

$$\imath_\theta^\circ$$

Here, $X_{\mathfrak{s}}$ denotes the (partial) compactification, X_θ° the stratum associated to θ, and X_θ its closure. One refers to $X_{\mathfrak{s}}$ as the *relative torus embedding* over B associated to $\mathfrak{s}$. The stratum X_θ° is itself a torsor under a complex torus T_θ°, and there is a canonical isomorphism

$$Y/(\langle\theta\rangle_{\mathbb{R}} \cap Y) \overset{\sim}{\longrightarrow} Y_\theta \ ,$$

where $\langle\theta\rangle_{\mathbb{R}} \subset Y_{\mathbb{R}}$ denotes the subspace generated by θ, and Y_θ the cocharacter group of T_θ°. Recall that the cocharacter group of a torus is canonically identified with its fundamental group. The above isomorphism between $Y/(\langle\theta\rangle_{\mathbb{R}} \cap Y)$ and Y_θ is induced from a projection from Y to Y_θ, which corresponds to the canonical projection from T to T_θ°.

Now let $\mathfrak{F}^\bullet$ be a complex of F-linear local systems on $X(\mathbb{C})$. Denote by a, resp. a_θ, resp. a_θ° the structure morphisms to B from X, resp. from X_θ, resp. from X_θ°. We have:

Proposition 6.1. *(a) The adjunction morphism*

$$\imath_\theta^* \jmath_* \mathfrak{F}^\bullet \longrightarrow \jmath_{\theta*} \left(\imath_\theta^\circ\right)^* \jmath_* \mathfrak{F}^\bullet$$

is an isomorphism in the derived category of Abelian sheaves on $X_\theta(\mathbb{C})$.

(b) Adjunction induces an isomorphism

$$a_* \mathfrak{F}^\bullet \overset{\sim}{\longrightarrow} (a_\theta)_* \imath_\theta^* \jmath_* \mathfrak{F}^\bullet$$

in the derived category of Abelian sheaves on $B(\mathbb{C})$.

Proof. Since a is locally a projection, we may assume that B is a point, and that $X = T$. Claim (a) can be shown after taking inverse images for all

the strata in the natural stratification of T_θ. These correspond to cones ϕ in $\mathfrak{s}$ containing θ as a face. We have to show that

$$(\imath_\phi^\circ)^* \jmath_* \mathfrak{F}^\bullet \longrightarrow (\imath_\phi^\circ)^* \jmath_{\theta *} (\imath_\theta^\circ)^* \jmath_* \mathfrak{F}^\bullet$$

is an isomorphism. For this, we may assume (by passing to a subset of $\mathfrak{s}$) that ϕ is the unique open cone in $\mathfrak{s}$, or equivalently, that $T_\phi = T_\phi^\circ$ is the unique closed stratum of $T_\mathfrak{s}$. Then the structure morphism of $T_\mathfrak{s}$ factors over the projection to T_ϕ, identifying $T_\mathfrak{s}$ with a relative torus embedding over T_ϕ. As before, we may therefore assume that T_ϕ is a point. Locally around this point, we can choose coordinates $t_1, \ldots, t_n$, and assume that the torus embedding $T_\mathfrak{s}$ equals $\mathbb{A}^n$, with the canonical action of $T = \mathbb{G}_m^n \subset \mathbb{A}^n$, that the stratification is the one induced by the coordinates, that $T_\phi = \{0\}$, and that the intermediate stratum T_θ is defined by the vanishing of the first k of the t_i. We therefore have

$$T_\theta^\circ = \{(t_1, \ldots, t_n) \,,\; t_1 = \ldots = t_k = 0 \,,\; t_{k+1} \cdots t_n \neq 0\} \,.$$

Now recall that the complexes of sheaves $(\imath_\theta^\circ)^* \jmath_*$, $(\imath_\phi^\circ)^* \jmath_{\theta *}$, and $(\imath_\phi^\circ)^* \jmath_*$ can be computed from direct limits over the (analytic) neighbourhoods of T_θ in $T_\mathfrak{s}$, of T_ϕ in T_θ, and of T_ϕ in $T_\mathfrak{s}$, respectively. In each of these direct systems, we can find a cofinal system of neighbourhoods, all of whose members are homotopically equivalent to each other. If we evaluate on complexes of local systems, we see that the direct limits over these cofinal systems are constant.

Now denote by U_i the image of the positively oriented generator of the fundamental group of $\mathbb{G}_m(\mathbb{C})$ under the embedding of $\mathbb{G}_m$ into $\mathbb{G}_m^n$ via the i-th coordinate. Denote by $\mathbf{Loc}_F$ the category of F-linear local systems, and by $\mathbf{Sh}$ the category of Abelian sheaves. Identify local systems on $T(\mathbb{C})$ and $T_\theta^\circ(\mathbb{C})$ with representations of $Y = \langle U_1, \ldots, U_n \rangle_{\mathbb{Z}}$ and of $Y_\theta = Y/(\langle \theta \rangle_{\mathbb{R}} \cap Y) = \langle U_{k+1}, \ldots, U_n \rangle_{\mathbb{Z}}$, respectively. For an abstract group H, denote by $R\Gamma(H, \bullet)$ the derived functor of the H-invariants. From the above discussion, we see:

(1) There is a commutative diagram of functors

$$\begin{array}{ccc}
D^+(\mathbf{Loc}_F\, T(\mathbb{C})) & \xrightarrow{R\Gamma(\langle\theta\rangle_{\mathbb{R}} \cap Y,\, \bullet)} & D^+(\mathbf{Loc}_F\, T_\theta^\circ(\mathbb{C})) \\
\downarrow & & \downarrow \\
D^+(\mathbf{Sh}\, T(\mathbb{C})) & \xrightarrow{\quad(\imath_\theta^\circ)^* \jmath_*\quad} & D^+(\mathbf{Sh}\, T_\theta^\circ(\mathbb{C}))
\end{array}$$

(2) There is a commutative diagram of functors

$$\begin{array}{ccc}
D^+(\mathbf{Loc}_F\, T_\theta^\circ(\mathbb{C})) & \xrightarrow{\quad R\Gamma(Y_\theta,\, \bullet)\quad} & D^+(\mathbf{Loc}_F\, T_\phi(\mathbb{C})) \\
\downarrow & & \downarrow \\
D^+(\mathbf{Sh}\, T_\theta^\circ(\mathbb{C})) & \xrightarrow{\quad(\imath_\phi^\circ)^* \jmath_{\theta *}\quad} & D^+(\mathbf{Sh}\, T_\phi(\mathbb{C}))
\end{array}$$

(3) There is a commutative diagram of functors

$$
\begin{array}{ccc}
D^+(\mathbf{Loc}_F\, T(\mathbb{C})) & \xrightarrow{\ R\Gamma(Y,\,\bullet\,)\ } & D^+(\mathbf{Loc}_F\, T_\phi(\mathbb{C})) \\
\downarrow & & \downarrow \\
D^+(\mathbf{Sh}\, T(\mathbb{C})) & \xrightarrow{\ (\imath_\phi^\circ)^*\, \jmath_*\ } & D^+(\mathbf{Sh}\, T_\phi(\mathbb{C}))
\end{array}
$$

(4) Under the identifications of diagrams (1)–(3), the natural transforma-tion

$$
(\imath_\phi^\circ)^*\, \jmath_* \longrightarrow (\imath_\phi^\circ)^*\, \jmath_{\theta*} \circ (\imath_\theta^\circ)^*\, \jmath_*
$$

of functors from $D^+(\mathbf{Sh}\, T(\mathbb{C}))$ to $D^+(\mathbf{Sh}\, T_\phi(\mathbb{C}))$ restricts to the canon-ical isomorphism

$$
R\Gamma(Y,\,\bullet\,) \xrightarrow{\ \sim\ } R\Gamma(Y_\theta,\,\bullet\,) \circ R\Gamma(\langle\theta\rangle_{\mathbb{R}} \cap Y,\,\bullet\,)
$$

of functors from $D^+(\mathbf{Loc}_F\, T(\mathbb{C}))$ to $D^+(\mathbf{Loc}_F\, T_\phi(\mathbb{C}))$.

This shows claim (a). Claim (b) follows from (a), and the fact that adjunction induces an isomorphism

$$
a_*\, \mathfrak{F}^\bullet \xrightarrow{\ \sim\ } (a_\theta^\circ)_*\, (\imath_\theta^\circ)^*\, \jmath_*\, \mathfrak{F}^\bullet\ .
$$

This in turn results from the fact that both sides are computed by $R\Gamma(Y,\,\bullet\,)$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ **q.e.d.**

Corollary 6.2. *Under the hypotheses of 6.1, assume that B is smooth, and of finite type over $\mathbb{C}$. Let $\mathbb{M}^\bullet$ be a complex of admissible variations of F-Hodge structure on X. Adjunction induces an isomorphism*

$$
a_*\, \mathbb{M}^\bullet \xrightarrow{\ \sim\ } (a_\theta)_*\, \imath_\theta^*\, \jmath_*\, \mathbb{M}^\bullet
$$

in $D^b(\mathbf{MHM}_F\, B)$.

Proof. Isomorphisms in the category $D^b(\mathbf{MHM}_F\, B)$ can be recognized after application of the forgetful functor *rat* to the bounded derived category $D^b(\mathbf{Perv}_F\, B(\mathbb{C}))$ of perverse sheaves on $B(\mathbb{C})$. This is a formal consequence of exact- and faithfulness of *rat* on the level of Abelian categories

$$
rat : \mathbf{MHM}_F\, B \longrightarrow \mathbf{Perv}_F\, B(\mathbb{C})
$$

[Sa, p. 222]. By [B, Main Thm. 1.3], the category $D^b(\mathbf{Perv}_F\, B(\mathbb{C}))$ can be identified with a full sub-category of the derived category of Abelian sheaves on $B(\mathbb{C})$. So the claim follows from part (b) of Proposition 6.1. **q.e.d.**

7 Specialization of local systems, and of variations of Hodge structure

In order to prove the part of Theorem 2.6 concerning the comparison of weight filtrations, it will be necessary to recall the explicit description of the *nearby cycle functor*, as well as fundamental results on *nilpotent orbits*. The main result of this section is Theorem 7.2. It will be used in the form of Corollary 7.3, in the proof of Proposition 9.3. Because of the rather technical nature of the material, we chose to present the main result first, and then recall the theory needed for its proof.

Throughout this section, assume that X is a smooth analytic space, and that $j : U \hookrightarrow X$ is the open immersion of a dense analytic subset, such that the complement Z' of U is a divisor with normal crossings. Let Z be a closed analytic subspace of Z', which is still a divisor in X. Thus, locally on X, the set underlying Z is the union of components of Z'.

Recall Verdier's construction of the *specialization functor Sp_Z* in the analytic context [V2, Section 9]. It preserves perversity [V2, (SP7)], and can thus be seen as an exact functor

$$\mathbf{Perv}_F\, X \longrightarrow \mathbf{Perv}_F\, N_{Z/X}\ ,$$

where $N_{Z/X}$ denotes the (analytic) normal cone of Z in X. Recall from [V2, (SP1)] that the image of Sp_Z is contained in the category of *monodromical* perverse sheaves on $N_{Z/X}$ [V2, p. 356]. Perverse sheaves of the form $Sp_Z\mathbb{W}$ are thus equipped with a canonical *monodromy automorphism T*, and hence also with a nilpotent *monodromy endomorphism*, namely, the logarithm of the unipotent part of T.

When Z is a principal divisor, defined by a holomorphic function g, then we also have the nearby cycle functor ψ_g, which respects perversity up to a shift by $[-1]$ [V3, Section 3, Claim 4)], and is exact. Let us write

$$\psi_g^p := \psi_g[-1] : \mathbf{Perv}_F\, X \longrightarrow \mathbf{Perv}_F\, Z\ .$$

By the very definition of ψ_g^p, perverse sheaves in its image are again canonically equipped with a monodromy automorphism and a monodromy endomorphism. By [V2, (SP6)], ψ_g^p and its monodromy automorphism can be recovered from Sp_Z.

We shall be particularly interested in the composition of Sp_Z, resp. of ψ_g^p with the functor

$$j_* : \mathbf{Perv}_F\, U \longrightarrow \mathbf{Perv}_F\, X\ ,$$

which respects perversity, since j is affine.

For an object $\mathbb{E}$ of an Abelian category $\mathcal{A}$, equipped with a nilpotent endomorphism N, recall the notion of *monodromy weight filtration of N* on $\mathbb{E}$ [D2, (1.6.1)]. If $\mathbb{E}$ is equipped with a finite ascending filtration $W_\bullet$, then one defines the *monodromy weight filtration of N relative to $W_\bullet$* on $\mathbb{E}$ [D2, (1.6.13)]. (Caution! This latter filtration does not always exist.)

Set $D := \{x \in \mathbb{C} \,|\, |x| < 1\}$, and $D^* := D - \{0\}$. Fix a point z of Z'. If locally around z, the divisor Z' is the union of m smooth components, then the fundamental group of $V \cap U$, for small neighbourhoods V of z in X isomorphic to D^n, is free Abelian of rank m, and independent of V. Let us refer to this group as the *local monodromy group around Z' at z*. Call a $\mathbb{Z}$-base $\mathfrak{T} = (T_1, \ldots, T_m)$ of the local monodromy group *adapted to local coordinates* if there is an isomorphism of D^n onto a neighbourhood V of z identifying $(D^*)^m \times D^{n-m}$ with $V \cap U$, and the canonical $\mathbb{Z}$-base of $\pi_1((D^*)^m \times D^{n-m})$ with $(T_1, \ldots, T_m)$.

We say that a local system $\mathbb{V}$ on U has *unipotent local monodromy around Z* if for any point z of $Z \subset Z'$, the action of the local monodromy group around Z' at z is unipotent. Since the local monodromy group is Abelian, its elements act as automorphisms of local systems on the restriction of $\mathbb{V}$ to $V \cap U$, for V as above. Let us write $\mathbb{V}_{\mathfrak{U}^\circ(z)}$ for the direct limit of these restrictions.

Definition 7.1. Let $\mathbb{V}$ be a local system on U with unipotent local monodromy around Z, and $W_\bullet$ a finite filtration of $\mathbb{V}$ by local systems.

(a) We say that the pair $(\mathbb{V}, W_\bullet)$ *satisfies condition (MON) relative to the divisor Z* if for any point z of Z there exists a $\mathbb{Z}$-base $\mathfrak{T}$ of the local monodromy group around Z' at z, adapted to local coordinates, such that the following holds: for any non-empty subsystem $\mathfrak{T}' = (T_{i_1}, \ldots, T_{i_r})$ of $\mathfrak{T}$, denote by $W_\bullet^{\mathfrak{T}'}$ the monodromy weight filtration of $N_{i_1} + \ldots + N_{i_r}$ on $\mathbb{V}_{\mathfrak{U}^\circ(z)}$, where N_j denotes the logarithm of the image of T_j under the action. Then $W_\bullet^{\mathfrak{T}'}$ coincides with the filtration induced by $W_\bullet$ on $\mathbb{V}_{\mathfrak{U}^\circ(z)}$. (In particular, $W_\bullet^{\mathfrak{T}'}$ is independent of $\mathfrak{T}' \neq \emptyset$.)

(b) Let k be an integer. We say that the triple $(\mathbb{V}, k, W_\bullet)$ *satisfies condition (MON) relative to the divisor Z* if the pair $(\mathbb{V}, W[k]_\bullet)$ satisfies condition (MON) relative to the divisor Z.

Here as usual, the shifted filtration $W[k]_\bullet$ is defined by $W[k]_n \mathbb{V} := W_{k+n} \mathbb{V}$. Condition (MON) will be studied in a situation where the local system in question underlies a variation of Hodge structure, pure of weight k. In order to analyze its specialization along Z, it is natural to consider the shift by $-k$ of the monodromy weight filtration, rather than the monodromy weight filtration itself, whence part (b) of the above definition. Let us remark that condition (MON) is very restrictive, but occurs naturally in the context of

toroidal compactifications of Shimura varieties, as we shall see in Section 9. The main result of this section reads as follows:

Theorem 7.2. *Assume that Z is a reduced principal divisor defined by a function g. Let $(\mathbb{V}, k, W_\bullet)$ be a triple satisfying condition (MON) relative to Z. Assume that $\mathbb{V}$ underlies a variation of Hodge structure, pure of weight k. Then the following filtrations on the perverse sheaf $\psi_g^p j_* \mathbb{V}$ on Z coincide:*

(a) *the monodromy weight filtration (of the canonical monodromy endomorphism), shifted by $-k$,*

(b) *the monodromy weight filtration relative to $\psi_g^p j_* W_\bullet$.*

Corollary 7.3. *Assume that Z is a reduced divisor. Let $(\mathbb{V}, k, W_\bullet)$ be a triple satisfying condition (MON) relative to Z. Assume that $\mathbb{V}$ underlies a variation of Hodge structure, pure of weight k. Then the following filtrations on the perverse sheaf $Sp_Z j_* \mathbb{V}$ on $N_{Z/X}$ coincide:*

(a) *the monodromy weight filtration, shifted by $-k$,*

(b) *the monodromy weight filtration relative to $Sp_Z j_* W_\bullet$.*

By what was said before, the theorem follows from its corollary. But we shall prove the results in the above logical order:

Proof of Corollary 7.3, assuming Theorem 7.2. By [V2, (SP0)], the question is local, so we can assume that Z is a principal divisor defined by g. By [V3, Section 3, Claim 1)] and [V3, 2nd proposition of Section 4], the weight filtrations on $Sp_Z j_* \mathbb{V}$ are uniquely determined by the weight filtrations on both $\psi_g^p j_* \mathbb{V}$ and $\phi_g^p j_* \mathbb{V}$, where ϕ_g denotes the *vanishing cycle functor*, and $\phi_g^p := \phi_g[-1]$. Our claim follows from 7.2, and from the fact that there is a canonical isomorphism

$$\mathrm{Var} : \phi_g^p j_* \mathbb{V} \xrightarrow{\sim} \psi_g^p j_* \mathbb{V} \,,$$

which is compatible with the action of the N_i [V3, Section 3, Claim 5)].

q.e.d.

For the proof of Theorem 7.2, we shall use two main ingredients:
(A) First (see Proposition 7.7), the explicit description, due to Galligo–Granger–Maisonobe [GGM], of the full sub-category $(\mathbf{Perv}_{\mathbb{C}} X)_{nc}$ of $\mathbf{Perv}_{\mathbb{C}} X$ of perverse sheaves *of normal crossing type*, when X is a product of unit disks, U the corresponding product of punctured disks, and g a product of coordinates. Actually, we shall restrict ourselves to the unipotent objects in this category. We shall follow the presentation of [Sa, 3.1]. Next (see Proposition 7.8), using the above, the explicit description, due to Saito [Sa, Thm. 3.3], of the functor ψ_g^p, together with its monodromy endomorphism.

(B) The theory of nilpotent orbits, in particular, the comparison of monodromy weight filtrations in nilpotent orbits of several variables, due to Cattani–Kaplan (see Theorem 7.10).

We can conclude, thanks to a result of Kashiwara's (see Proposition 7.11), which can be interpreted as permanence of nilpotent orbits under ψ_g^p.

Remark 7.4. Theorem 7.2 and Corollary 7.3 continue to hold in a larger generality: in the definition of condition (MON), we can allow quasi-unipotent local monodromy; in the hypotheses of 7.2 and 7.3, the divisor Z need not be reduced. We chose to add the conditions which ensure that the monodromy of $\psi_g^p j_* \mathbb{V}$, resp. of $Sp_Z j_* \mathbb{V}$ is unipotent. First, this covers the situation we shall be considering in Section 9. Second, restriction to unipotent objects in the explicit description of $(\mathbf{Perv}_\mathbb{C} X)_{nc}$ simplifies considerably the presentation of the material.

Until the actual proof of Theorem 7.2, we shall study the situation $X = D^n$ and $U = (D^*)^n$, for some $n \geq 1$. Put $Z_i := \{x_i = 0\}$, and

$$Z_I := \bigcap_{i \in I} Z_i \ ,$$

for $I \subset \{1, \ldots, n\}$.

Definition 7.5. (a) Let $(\mathbf{Perv}_\mathbb{C} X)_{nc}$ be the category of perverse sheaves on X, whose characteristic varieties are contained in the union of the conormal bundles of Z_I.

(b) Denote by $(\mathbf{Perv}_\mathbb{C} X)_{ncu}$ the full sub-category of unipotent perverse sheaves, i.e., the objects $\mathbb{W}$ of $(\mathbf{Perv}_\mathbb{C} X)_{nc}$ satisfying the following: the canonical monodromy automorphisms on $\psi_{x_i}^p(\mathbb{W})$ and $\phi_{x_i}^p(\mathbb{W})$ are unipotent, for all i.

Definition 7.6. Define the category $\mathbb{P}(n)_u$ as follows: objects are finite-dimensional $\mathbb{C}$-vector spaces E_I indexed by $I \subset \{1, \ldots, n\}$, together with morphisms

$$
\begin{aligned}
\mathrm{can}_i : E_I &\longrightarrow E_{I \cup \{i\}} \quad \text{for} \quad i \notin I \ , \\
\mathrm{Var}_i : E_I &\longrightarrow E_{I - \{i\}} \quad \text{for} \quad i \in I \ , \\
N_i : E_I &\longrightarrow E_I \quad \text{for} \quad i \in \{1, \ldots, n\} \ ,
\end{aligned}
$$

subject to the following conditions: $N_i = \mathrm{can}_i \circ \mathrm{Var}_i$ and $N_i = \mathrm{Var}_i \circ \mathrm{can}_i$ whenever these compositions make sense; all N_i are nilpotent; furthermore, A_i and B_j commute for $i \neq j$ and $A, B \in \{\mathrm{can}, \mathrm{Var}, N\}$ such that the composition makes sense. Morphisms in $\mathbb{P}(n)_u$ are the morphisms of vector spaces compatible with the can_i, Var_i, and N_i.

We then have the following:

Proposition 7.7 (Galligo–Granger–Maisonobe). *(a) There is a natural equivalence of categories*

$$\Psi^n : (\mathbf{Perv}_{\mathbb{C}} X)_{ncu} \overset{\sim}{\longrightarrow} \mathbb{P}(n)_u \ .$$

It is defined by associating to $\mathbb{W} \in (\mathbf{Perv}_{\mathbb{C}} X)_{ncu}$ *the data* $(E_I)_I$*, where*

$$E_I := \Psi_{x_1,I} \circ \Psi_{x_2,I} \circ \ldots \circ \Psi_{x_n,I}(\mathbb{W}) \ ,$$

with

$$\Psi_{x_i,I} := \begin{cases} \psi^p_{x_i} \, , & \text{if } i \notin I \, , \\ \phi^p_{x_i} \, , & \text{if } i \in I \, . \end{cases}$$

The morphisms can_i*,* Var_i *and* N_i *are the ones naturally associated to* $\psi^p_{x_i}$ *and* $\phi^p_{x_i}$*.*

(b) There is a natural quasi-inverse $(\Psi^n)^{-1}$ *of* Ψ^n*.*

Let us illustrate the effect of the functor Ψ^n for $n = 1$, and for a perverse sheaf on $X = D$ of the form $j_* \mathfrak{F}$, for a unipotent local system $\mathfrak{F}$ on $U = D^*$. We identify $\mathfrak{F}$ with a vector space H, together with a unipotent automorphism T. We then have $E_\emptyset = E_{\{1\}} = H$, the morphism $\mathrm{Var} = \mathrm{Var}_1 : E_{\{1\}} \to E_\emptyset$ is the identity on H, and $\mathrm{can} = \mathrm{can}_1 : E_\emptyset \to E_{\{1\}}$ is the logarithm of T.

Proof of Proposition 7.7. Our claim is in fact a particular case of [GGM, Thm. IV.3]. There, the condition on (1) unipotency of the perverse sheaves (see 7.5 (b)) is dropped, (2) nilpotency of the endomorphisms N_i (see 7.6) is replaced by "the sums $\mathrm{id} + N_i$ are invertible". By splitting the objects into generalized eigenspaces with respect to the commuting operators $\mathrm{id} + N_i$, one gets the description of [Sa, 3.1]. Our situation corresponds to the (multiple) eigenvalues $(1, 1, \ldots, 1)$. In the description of [Sa, 3.1], this means that the components E_I^v are trivial whenever $v \in (\mathbb{C}/\mathbb{Z})^n$ is unequal to zero. **q.e.d.**

Now fix $m \in \{1, \ldots, n\}$, set $g := \prod_{i=1}^m x_i$, and consider the principal reduced divisor $Z = \cup_{i=1}^m Z_i$ defined by g. Using the fact that the direct image of a closed embedding is fully faithful, we may view ψ^p_g as a functor from $\mathbf{Perv}_{\mathbb{C}} X$ to itself. We then have:

Proposition 7.8 (Saito). *For any subset I of $\{1, \ldots, n\}$, denote by $I_\leq$ the intersection $I \cap \{1, \ldots, m\}$, and by $I_>$ the complement of $I_\leq$ in I.*

(a) The functor ψ^p_g respects the sub-category $(\mathbf{Perv}_{\mathbb{C}} X)_{ncu}$ of $\mathbf{Perv}_{\mathbb{C}} X$.

(b) The composition

$$\Psi^n \circ \psi^p_g \circ (\Psi^n)^{-1} : \mathbb{P}(n)_u \longrightarrow \mathbb{P}(n)_u$$

is given as follows: let

$$\mathbb{E} = (E_I, \mathrm{can}_i, \mathrm{Var}_i, N_i)_{I,i}$$

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be an object of $\mathbb{P}(n)_u$. Then

$$\Psi^n \circ \psi_g^p \circ (\Psi^n)^{-1}(\mathbb{E}) = (\widetilde{E}_I, \widetilde{\mathrm{can}}_i, \widetilde{\mathrm{Var}}_i, \widetilde{N}_i)_{I,i} \ ,$$

with

$$\widetilde{E}_I := \mathrm{Coker}\left(\prod_{i \in I_\le} (N_i - N) \mid E_{I_>}[N] \right) \ ,$$

where we define $E_{I_>}[N]$ as the tensor product of $E_{I_>}$ and the polynomial ring $\mathbb{C}[N]$ in one variable N. The variable acts on $\mathbb{C}[N]$ by multiplication. The actions of N_i and of N on $E_{I_>}[N]$ are the ones induced by the tensor product structure. In particular, the endomorphisms $\prod_{i \in I_\le}(N_i - N)$ of $E_{I_>}[N]$ are injective, so we may identify their cokernels with their mapping cones. The morphisms $\widetilde{\mathrm{can}}_i$, $\widetilde{\mathrm{Var}}_i$, and $\widetilde{N}_i$ are given as morphisms of complexes concentrated in two degrees:

$$\widetilde{\mathrm{can}}_i := \begin{cases} (\mathrm{id}, N_i - N) \ , & \text{if } i \notin I \ , \ 1 \le i \le m \ , \\ (\mathrm{can}_i, \mathrm{can}_i) \ , & \text{if } i \notin I \ , \ i > m \ , \end{cases}$$

$$\widetilde{\mathrm{Var}}_i := \begin{cases} (N_i - N, \mathrm{id}) \ , & \text{if } i \in I_\le \ , \\ (\mathrm{Var}_i, \mathrm{Var}_i) \ , & \text{if } i \in I_> \ , \end{cases}$$

$$\widetilde{N}_i := \begin{cases} (N_i - N, N_i - N) \ , & \text{if } i \in I_\le \ , \\ (N_i, N_i) \ , & \text{if } i \in I_> \ . \end{cases}$$

(c) For $\mathbb{E} \in \mathbb{P}(n)_u$, the canonical monodromy endomorphism on

$$\Psi^n \circ \psi_g^p \circ (\Psi^n)^{-1}(\mathbb{E}) = (\widetilde{E}_I, \widetilde{\mathrm{can}}_i, \widetilde{\mathrm{Var}}_i, \widetilde{N}_i)_{I,i} \ ,$$

in the description of (b), is given by the endomorphism $\widetilde{N} := (N, N)$ on all components $\widetilde{E}_I$.

Proof. This is part of the information provided by [Sa, Thm. 3.3]. There, the specialization Sp_Z is described in terms of the categories $\mathbb{P}(n)_u$ and $\mathbb{P}(n+1)_u$. By the last line of [Sa, Thm. 3.3], in order to read off ψ_g^p from the given description, one has to restrict to the components "$0 \notin I$". Note that Saito admits quasi-unipotent objects. Thanks to reducedness of Z, the condition "$E_I^\nu = 0$ whenever v is unequal to zero" is respected by Sp_Z, hence by ψ_g^p. This shows parts (a) and (b) of our claim. As for (c), observe that by [Sa, Thm. 3.3], $\widetilde{N}$ occurs in the explicit description of Sp_Z. More precisely, it is the restriction to the components "$0 \notin I$" of the collection of the $n+1$st nilpotent endomorphisms M_0 of the components of

$$\Psi^{n+1} \circ Sp_Z \circ (\Psi^n)^{-1}(\mathbb{E}) = (F_I, \mathrm{can}_i, \mathrm{Var}_i, M_i)_{I, 0 \le i \le n} \ .$$

Our claim follows thus from [V2, (SP6)]. **q.e.d.**

In order to prove Theorem 7.2, one is thus naturally led to study monodromy weight filtrations on objects of the form

$$\widetilde{E}_I := \operatorname{Coker}\left(\prod_{i \in I_\leq}(N_i - N) \mid E_{I_>}[N]\right),$$

for certain objects $\mathbb{E} = (E_I, \operatorname{can}_i, \operatorname{Var}_i, N_i)_{I,i}$ of $\mathbb{P}(n)_u$. The result we want to use requires an additional structure on $\mathbb{E}$. Recall the notion of nilpotent orbits of a weight $k \in \mathbb{Z}$ and dimension $n \in \mathbb{N}$ (e.g. [CK, (3.1)]; cmp. also [Ka, 4.1]). For such objects $\mathbb{H}$, we shall use the notation

$$\mathbb{H} = ((H, F^\bullet, W_\bullet), N_i \,(1 \leq i \leq n), S) .$$

As for the nature of the components of $\mathbb{H}$, note in particular that H is a finite-dimensional $\mathbb{C}$-vector space, with finite descending, resp. ascending filtrations $F^\bullet$ and $W_\bullet$, S is a sesquilinear form on H, the N_i are mutually commuting nilpotent endomorphisms, and $W_\bullet$ is the monodromy weight filtration of the sum $\sum_{i=1}^n N_i$, shifted by $-k$.

The main motivation for this concept stems from Schmid's Nilpotent Orbit Theorem, which we shall use in the following form:

Theorem 7.9 (Schmid). *Let* $\mathbb{V}$ *be a local system on* U, *which underlies a variation of Hodge structure, pure of weight* k. *Write*

$$\Psi^n(j_*\mathbb{V}) =: \mathbb{E} = (E_I, \operatorname{can}_i, \operatorname{Var}_i, N_i)_{I,i} .$$

Then for any subset I *of* $\{1, \ldots, n\}$, *the data*

$$(E_I, N_i \,(1 \leq i \leq n))$$

underly a nilpotent orbit of weight k.

Proof. $\mathbb{V}$ is given by a vector space H, together with commuting monodromy automorphisms T_i, $1 \leq i \leq n$. We then have $E_\emptyset = H$, and the N_i are the logarithms of the unipotent parts of the T_i. Now apply [Sch, Thm. (4.12)] to show the claim for $I = \emptyset$. But all the other components of $\Psi^n(j_*\mathbb{V})$ are isomorpic to $E_\emptyset$ [V3, Section 3, Claim 5]. **q.e.d.**

On nilpotent orbits, comparison of monodromy weight filtrations is possible thanks to the following result:

Theorem 7.10 (Cattani–Kaplan). *Let*

$$\mathbb{H} = ((H, F^\bullet, W_\bullet), N_i \,(1 \leq i \leq n), S)$$

be a nilpotent orbit of weight k *and dimension* n, *and* I_1 *and* I_2 *two disjoint subsets of* $\{1, \ldots, n\}$. *Denote by* $W_\bullet^{I_1}$ *the monodromy weight filtration of* $\sum_{i \in I_1} N_i$, *shifted by* $-k$. *Then the following filtrations on* H *coincide:*

(a) the monodromy weight filtration of $\sum_{i \in I_1 \cup I_2} N_i$, *shifted by* $-k$,

(b) the monodromy weight filtration of $\sum_{i \in I_2} N_i$ relative to $W_\bullet^{I_1}$.

Proof. This is the content of [CK, Thm. (3.3)]. Note that the original statement of loc. cit. is misprinted; the correct version can be found in [CKS, Prop. (4.72)]. **q.e.d.**

Proposition 7.11 (Kashiwara). *Let*
$$\mathbb{H} = ((H, F^\bullet, W_\bullet), N_i \, (1 \leq i \leq n), S)$$
be a nilpotent orbit of weight k and dimension n, and $\emptyset \neq I \subset \{1, \ldots, n\}$. Set
$$\widetilde{H}_I := \mathrm{Coker}\left(\prod_{i \in I}(N_i - N) \mid H[N]\right) .$$

(a) The vector space $\widetilde{H}_I$ underlies in a natural way a nilpotent orbit of weight $k + 1 - |I|$ and dimension $n + 1$
$$\widetilde{\mathbb{H}} = ((\widetilde{H}_I, F^\bullet, M_\bullet), N, N_i \, (1 \leq i \leq n), \widetilde{S}) .$$
In particular, $M_\bullet$ is the monodromy weight filtration of the sum $N + \sum_{i=1}^n N_i$, shifted by $-(k + 1 - |I|)$.

(b) The filtration $M_\bullet$ coincides with the monodromy weight filtration of N, shifted by $-(k + 1 - |I|)$.

Proof. Part (a) is contained in [Sa, Prop. 3.19]. In order to see that (b) holds, one has to look at Kashiwara's proof of loc. cit. [Sa, (A.3.1), A.4]. **q.e.d.**

Combining the two preceding results, we get:

Corollary 7.12. *Keep the assumptions of Proposition 7.11, and denote by $W_\bullet$ the filtration on the vector space $\widetilde{H}_I$ induced by the filtration $W_\bullet$ on H. (Note that the functor $H \mapsto \widetilde{H}_I$ is exact.) Then the following filtrations on $\widetilde{H}_I$ coincide:*

(a) the monodromy weight filtration of N, shifted by $-k$,

(b) the monodromy weight filtration of N relative to $W_\bullet$.

Finally, we can show the main result of this section:

Proof of Theorem 7.2. Since the question is local, we may assume that we are in the situation discussed in 7.5–7.12. Propositions 7.7 and 7.8 tell us that we need to compare monodromy weight filtrations on the
$$\widetilde{E}_I := \mathrm{Coker}\left(\prod_{i \in I_\leq}(N_i - N) \mid E_{I_>}[N]\right) ,$$

for $\Psi^n(j_*\mathbb{V}) =: \mathbb{E} = (E_I, \mathrm{can}_i, \mathrm{Var}_i, N_i)_{I,i}$. By Theorem 7.9, the $E_{I_>}$ underlying nilpotent orbits of weight k. We omit the N_i with $i > m$, and consider $E_{I_>}$ as nilpotent orbit of dimension m. Now apply Corollary 7.12, with n replaced by m, and I replaced by $I_\leq = \{i_1, \ldots, i_r\}$. Thanks to condition (MON), the filtration $W_\bullet$ in 7.12 coincides with the one induced by the filtration $W_\bullet$ of $\mathbb{V}$. **q.e.d.**

8 Strata in toroidal compactifications

In order to prepare the proof of Theorem 2.6, to be given in Section 9, we need to discuss the geometry of *toroidal compactifications $M^K(\mathfrak{S})$* of M^K. We keep the notations and hypotheses of Section 2. In particular, the subgroup $K \subset G(\mathbb{A}_f)$ is neat, and $(G, \mathfrak{H})$ satisfies $(+)$. Choose a *K-admissible cone decomposition* $\mathfrak{S}$ satisfying the conditions of [P2, (3.9)]. In particular, all cones occurring in $\mathfrak{S}$ are *smooth*, and the decomposition is *complete*.

Let us denote by $M^K(\mathfrak{S}) := M^K(G, \mathfrak{H}, \mathfrak{S})$ the toroidal compactification associated to $\mathfrak{S}$. It is a smooth projective scheme over $\mathbb{C}$, which in a natural way contains M^K as an open sub-scheme. The complement is a union of smooth divisors with normal crossings. The identity on M^K extends uniquely to a surjective morphism

$$p = p_\mathfrak{S} : M^K(\mathfrak{S}) \longrightarrow\!\!\!\rightarrow (M^K)^* .$$

The inverse images under p of the strata described in Section 1 form a stratification of $M^K(\mathfrak{S})$. We follow [P2, (3.10)] for the description of these inverse images: as usual, fix a proper boundary component $(P_1, \mathfrak{X}_1)$ of $(G, \mathfrak{H})$, and an element $g \in G(\mathbb{A}_f)$. To the given data, the following are canonically associated:

(i) an Abelian scheme $A \to M^{\pi(K_1)}$, and an A-torsor $B \to M^{\pi(K_1)}$,

(ii) a torus T, and a T-torsor $X \to B$,

(iii) a rational partial polyhedral decomposition of $Y_*(T)_\mathbb{R}$ ($Y_*(T) :=$ the cocharacter group of T), again denoted by $\mathfrak{S}$, and a non-empty subset $\mathfrak{T} \subset \mathfrak{S}$,

(iv) an action of H_Q on B, X, and T.

These objects satisfy the following properties:

(A) the H_Q-action is equivariant with respect to the group and torsor structures and stabilizes $\mathfrak{S}$ and $\mathfrak{T}$,

(B) the subgroup $P_1(\mathbb{Q})$ of H_Q acts trivially on B, X, and T,

(C) the group $\Delta_1 = H_Q/P_1(\mathbb{Q})$ acts freely on $\mathfrak{T}$,

(D) the pair $(\mathfrak{S}, \mathfrak{T})$ satisfies conditions [P2, (2.3.1–3)] (see below).

Consider the relative torus embedding $X \hookrightarrow X(\mathfrak{S})$. Condition [P2, (2.3.1)] is equivalent to saying that $\mathfrak{T}$ defines a closed sub-scheme Z of $X(\mathfrak{S})$. In order to state the other two conditions, define

$$D := \bigcup_{\sigma \in \mathfrak{T}} \sigma^\circ \,,$$

where for each cone σ we denote by σ° the topological interior of σ inside the linear subspace of $Y_*(T)_\mathbb{R}$ generated by σ. The subset D of $Y_*(T)_\mathbb{R}$ is endowed with the induced topology. Condition [P2, (2.3.2)] says that every point of D admits a neighbourhood U such that $U \cap \sigma \neq \emptyset$ for only a finite number of $\sigma \in \mathfrak{T}$. Condition [P2, (2.3.3)] states that D is contractible.

By (C), the induced action of Δ_1 on Z is free and proper in the sense of [P2, (1.7)]. The geometric quotient $M_{1,\mathfrak{S}}^K$ exists and is canonically isomorphic to the inverse image of M_1^K under p. Furthermore, the analytic space $M_{1,\mathfrak{S}}^K(\mathbb{C})$ is the quotient of $Z(\mathbb{C})$ by Δ_1 in the analytic category. We summarize the situation by the following diagram:

$$\begin{array}{ccccccc}
M^K & \overset{j_\mathfrak{S}}{\hookrightarrow} & M^K(\mathfrak{S}) & \overset{i_\mathfrak{S}}{\hookleftarrow} & M_{1,\mathfrak{S}}^K = \Delta_1\backslash Z & \overset{\tilde{q}}{\longleftarrow} & Z \\
\| & & \downarrow{p} & & \downarrow{p} & & \downarrow{\tilde{p}} \\
M^K & \overset{j}{\hookrightarrow} & (M^K)^* & \overset{i}{\hookleftarrow} & M_1^K = \Delta\backslash M^{\pi(K_1)} & \overset{q}{\longleftarrow} & M^{\pi(K_1)}
\end{array}$$

The left and the middle square are Cartesian, and the maps p are proper.

It will be necessary to consider a refinement of the stratification of $M^K(\mathfrak{S})$. The induced stratification of Z is the natural one given by $\mathfrak{T}$. For any cone $\sigma \in \mathfrak{T}$, denote by

$$i_\sigma^\circ : Z_\sigma^\circ \longrightarrow Z$$

the immersion of the corresponding stratum into Z, and by

$$i_\sigma : Z_\sigma \longrightarrow Z$$

the immersion of its closure. In the same way, we shall write

$$i_{\tilde{\sigma}}^\circ : Z_{\tilde{\sigma}}^\circ \longrightarrow M_{1,\mathfrak{S}}^K \longrightarrow M^K(\mathfrak{S})$$

and

$$i_{\tilde{\sigma}} : Z_{\tilde{\sigma}} \longrightarrow M_{1,\mathfrak{S}}^K \longrightarrow M^K(\mathfrak{S})$$

for the respective immersions into $M_{1,\mathfrak{S}}^K$, or into $M^K(\mathfrak{S})$. These immersions are indexed by the quotient $\tilde{\mathfrak{T}} := \Delta_1\backslash\mathfrak{T}$. Note that $Z_{\tilde{\sigma}}$ is closed in $M_{1,\mathfrak{S}}^K$, but in general not in $M^K(\mathfrak{S})$.

In order to describe the situation on the level of the underlying analytic spaces, let us connect the present notation (which is that of [P2, (3.10)]) to the one of [P1]. Consider the factorization of $\pi : (P_1, \mathfrak{X}_1) \longrightarrow (G_1, \mathfrak{H}_1)$ corresponding to the weight filtration of the unipotent radical W_1:

$$(P_1, \mathfrak{X}_1) \xrightarrow{\pi_t} (P_1', \mathfrak{X}_1') := (P_1, \mathfrak{X}_1)/U_1 \xrightarrow{\pi_a} (G_1, \mathfrak{H}_1)$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\pi}$$

where U_1 denotes the weight -2 part of W_1 [P1, Def. 2.1 (v)]. On the level of Shimura varieties, the picture looks as follows:

$$M^{K_1} = M^{K_1}(P_1, \mathfrak{X}_1) \xrightarrow{\pi_t} M^{\pi_t(K_1)} := M^{\pi_t(K_1)}(P_1', \mathfrak{X}_1') \xrightarrow{\pi_a} M^{\pi(K_1)}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\pi}$$

By [P1, 3.12–3.22 (a)], π_a is in a natural way a torsor under an Abelian scheme, while π_t is a torsor under a torus. In fact, we have (i) $B = M^{\pi_t(K_1)}$, and (ii) $X = M^{K_1}$. Furthermore, the action (iv) of H_Q on B, X and T is induced by the natural action of H_Q on the Shimura data involved in the above factorization of π. Since $P_1(\mathbb{Q})$ acts trivially on the associated Shimura varieties, this explains property (B).

The map $\tilde{p} : Z \to M^{\pi(K_1)}$ thus factors through π_a. [P1, 6.13] contains the definition of a K_1-admissible smooth cone decomposition $\mathfrak{S}_1^0$ canonically associated to $(P_1, \mathfrak{X}_1)$ and g. It is *concentrated in the unipotent fibre* [P1, 6.5 (d)], and thus defines a smooth torus embedding $j_1 : M^{K_1} \hookrightarrow M^{K_1}(\mathfrak{S}_1^0)$ over $M^{\pi_t(K_1)}$. In fact, we have $X(\mathfrak{S}) = M^{K_1}(\mathfrak{S}_1^0)$. Furthermore [P1, 6.13], there is a closed analytic subset $\partial\mathfrak{U} := \partial\mathfrak{U}(P_1, \mathfrak{X}_1, g)$ of $M^{K_1}(\mathfrak{S}_1^0)(\mathbb{C})$ canonically associated to our data. The proof of [P1, Prop. 6.21] shows that $\partial\mathfrak{U} = Z(\mathbb{C})$. In fact, the projection $\tilde{q} : Z \to \Delta_1\backslash Z$ corresponds to the quotient map

$$\partial\mathfrak{U} \longrightarrow \Delta_1\backslash\partial\mathfrak{U}$$

of [P1, 7.3].

Proposition 8.1. *The morphism* $\tilde{q} : Z \longrightarrow M_{1,\mathfrak{S}}^K$ *induces an isomorphism*

$$Z_\sigma \xrightarrow{\sim} Z_{\tilde{\sigma}}$$

for any $\sigma \in \mathfrak{T}$. *In particular, it induces an isomorphism on every irreducible component of* Z.

Proof. By [P1, Cor. 7.17 (a)], the morphism

$$\tilde{q} : Z_\sigma \longrightarrow Z_{\tilde{\sigma}}$$

identifies $Z_{\tilde{\sigma}}$ with the quotient of Z_σ by a certain subgroup $\mathrm{Stab}_{\Delta_1}([\sigma])$ of Δ_1. (The hypotheses of loc. cit. are satisfied because they are implied by the

conditions of [P2, (3.9)], which we assume throughout.) By [W2, Lemma 1.7], condition (+) and neatness of K imply that

$$\mathrm{Stab}_{\Delta_1}([\sigma]) = 1 \ .$$

<div align="right">q.e.d.</div>

Consider the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\ \ i_1\ \ } & X(\mathfrak{S}) = M^{K_1}(\mathfrak{S}_1^0) \\
{\scriptstyle \tilde{q}}\downarrow & & \\
M^K_{1,\mathfrak{S}} & \xhookrightarrow{\ i_\mathfrak{S}\ } & M^K(\mathfrak{S})
\end{array}
$$

By [P1, 6.13], there is an open neighbourhood $\mathfrak{U} := \bar{\mathfrak{U}}(P_1, \mathfrak{X}_1, g)$ of $Z(\mathbb{C})$ in $M^{K_1}(\mathfrak{S}_1^0)(\mathbb{C})$, and a natural extension of $\tilde{q}$ to $\mathfrak{U}$. It will equally be denoted by $\tilde{q}$:

$$
\begin{array}{ccc}
Z(\mathbb{C}) & \xrightarrow{\ \ i_1\ \ } \mathfrak{U} \hookrightarrow & M^{K_1}(\mathfrak{S}_1^0)(\mathbb{C}) \\
{\scriptstyle \tilde{q}}\downarrow & {\scriptstyle \tilde{q}}\downarrow & \\
M^K_{1,\mathfrak{S}}(\mathbb{C}) & \xhookrightarrow{\ i_\mathfrak{S}\ } M^K(\mathfrak{S})(\mathbb{C}) &
\end{array}
$$

Furthermore, the open subset $\mathfrak{U}$ is stable under Δ_1. By [W2, Prop. 1.9], the map $\tilde{q}$ (which in loc. cit. was denoted by f) is open, and we have the equality

$$\tilde{q}^{-1}(M^K(\mathbb{C})) = \mathfrak{U} \cap M^{K_1}(\mathbb{C}) \ .$$

Furthermore [W2, Thm. 1.11 (i)], $\tilde{q}$ is locally biholomorphic near Z. It thus induces an isomorphism between the quotient of the formal analytic completion of $M^{K_1}(\mathfrak{S}_1^0)(\mathbb{C})$ along $Z(\mathbb{C})$ by the free action of Δ_1, and the formal analytic completion of $M^K(\mathfrak{S})(\mathbb{C})$ along $M^K_{1,\mathfrak{S}}(\mathbb{C})$. According to [P2, p. 224], we have:

Proposition 8.2. *This isomorphism is algebraic in the following sense:*

(a) The action of Δ_1 on the formal completion $\mathfrak{F} = \mathfrak{F}_{Z/M^{K_1}(\mathfrak{S}_1^0)}$ of $M^{K_1}(\mathfrak{S}_1^0)$ along Z is free and proper. The geometric quotient $\Delta_1 \backslash \mathfrak{F}$ exists. Furthermore, the analytic space $(\Delta_1 \backslash \mathfrak{F})(\mathbb{C})$ is the quotient of $\mathfrak{F}(\mathbb{C})$ by Δ_1 in the analytic category.

(b) $\tilde{q}$ induces an isomorphism between $\Delta_1 \backslash \mathfrak{F}$ and the formal completion of $M^K(\mathfrak{S})$ along $M^K_{1,\mathfrak{S}}$.

In fact, $\tilde{q}$ descends to the reflex field of our Shimura varieties. In Section 9, the following consequence of Proposition 8.2 will be needed:

Corollary 8.3. *The map $\tilde{q}$ induces an isomorphism*

$$\Delta_1 \backslash N_{Z/M^{K_1}(\mathfrak{S}_1^0)} \xrightarrow{\ \sim\ } N_{M^K_{1,\mathfrak{S}}/M^K(\mathfrak{S})}$$

between the quotient of the normal cone of Z in $M^{K_1}(\mathfrak{S}_1^0)$ by the free and proper action of Δ_1, and the normal cone of $M^K_{1,\mathfrak{S}}$ in $M^K(\mathfrak{S})$.

9 Proof of the main result

Recall the situation considered in Section 8:

$$
\begin{array}{ccccccc}
M^K & \overset{j_\mathfrak{S}}{\hookrightarrow} & M^K(\mathfrak{S}) & \overset{i_\mathfrak{S}}{\longleftarrow} & M^K_{1,\mathfrak{S}} = \Delta_1\backslash Z & \overset{\tilde{q}}{\longleftarrow} & Z \\
\| & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \tilde{p}} \\
M^K & \overset{j}{\hookrightarrow} & (M^K)^* & \overset{i}{\longleftarrow} & M^K_1 = \Delta\backslash\, M^{\pi(K_1)} & \overset{q}{\longleftarrow} & M^{\pi(K_1)}
\end{array}
$$

Proper base change [Sa, (4.4.3)] yields the following:

Proposition 9.1. *There is a canonical isomorphism of functors*

$$
i^* j_* \cong p_* \, i^*_\mathfrak{S} \, j_{\mathfrak{S}*} : D^b(\mathbf{MHM}_F \, M^K) \longrightarrow D^b(\mathbf{MHM}_F \, M^K_1) \ .
$$

We are thus led to study the inverse image

$$
i^*_\mathfrak{S} : D^b(\mathbf{MHM}_F \, M^K(\mathfrak{S})) \longrightarrow D^b(\mathbf{MHM}_F \, M^K_{1,\mathfrak{S}}) \ .
$$

According to Corollary 8.3, the normal cone $N_{M^K_{1,\mathfrak{S}} / M^K(\mathfrak{S})}$ is canonically isomorphic to the quotient of the normal cone $N_{Z/M^{K_1}(\mathfrak{S}^0_1)}$ by the free and proper action of Δ_1. Using Corollary 4.3, we make the following identifications:

$$
D^b(\mathbf{MHM}_F \, M^K_{1,\mathfrak{S}}) = D^b(\Delta_1\text{-}\mathbf{MHM}_F \, Z) \ ,
$$

$$
D^b(\mathbf{MHM}_F \, N_{M^K_{1,\mathfrak{S}} / M^K(\mathfrak{S})}) = D^b(\Delta_1\text{-}\mathbf{MHM}_F \, N_{Z/M^{K_1}(\mathfrak{S}^0_1)}) \ .
$$

Since the action of Δ_1 on $N_{Z/M^{K_1}(\mathfrak{S}^0_1)}$ respects the natural inclusion of Z, we can think of the inverse image

$$
i^*_0 : D^b(\mathbf{MHM}_F \, N_{M^K_{1,\mathfrak{S}} / M^K(\mathfrak{S})}) \longrightarrow D^b(\mathbf{MHM}_F \, M^K_{1,\mathfrak{S}})
$$

as the Δ_1-equivariant inverse image

$$
i^*_0 : D^b(\Delta_1\text{-}\mathbf{MHM}_F \, N_{Z/M^{K_1}(\mathfrak{S}^0_1)}) \longrightarrow D^b(\Delta_1\text{-}\mathbf{MHM}_F \, Z) \ .
$$

Recall the specialization functor

$$
Sp_{M^K_{1,\mathfrak{S}}} : D^b(\mathbf{MHM}_F \, M^K(\mathfrak{S})) \longrightarrow D^b(\mathbf{MHM}_F \, N_{M^K_{1,\mathfrak{S}} / M^K(\mathfrak{S})}) \ .
$$

According to [Sa, 2.30], we have:

Proposition 9.2. *There is a canonical isomorphism of functors*

$$
i^*_\mathfrak{S} \cong i^*_0 \, Sp_{M^K_{1,\mathfrak{S}}} : D^b(\mathbf{MHM}_F \, M^K(\mathfrak{S})) \longrightarrow D^b(\mathbf{MHM}_F \, M^K_{1,\mathfrak{S}}) \ .
$$

We summarize the situation by the following commutative diagram:

$$
\begin{array}{c}
D^b(\Delta_1\text{-}\mathbf{MHM}_F\, N_{Z/\, M^{K_1}(\mathfrak{S}_1^0)}) \\
\| \\
D^b(\mathbf{MHM}_F\, M^K(\mathfrak{S})) \xrightarrow{\ Sp_{M^K_{1,\mathfrak{S}}}\ } D^b(\mathbf{MHM}_F\, N_{M^K_{1,\mathfrak{S}}/\, M^K(\mathfrak{S})}) \\
\downarrow i_0^* \\
D^b(\mathbf{MHM}_F\, M^K_{1,\mathfrak{S}}) \\
\| \\
D^b(\Delta_1\text{-}\mathbf{MHM}_F\, Z)
\end{array}
$$

Recall the open immersion $j_1 : M^{K_1} \hookrightarrow M^{K_1}(\mathfrak{S}_1^0)$ introduced in Section 8. It is a smooth relative torus embedding, hence in particular affine. This allows to define the exact functor

$$j_{1*} : \mathbf{MHM}_F\, M^{K_1} \longrightarrow \mathbf{MHM}_F\, M^{K_1}(\mathfrak{S}_1^0)$$

even though $M^{K_1}(\mathfrak{S}_1^0)$ is only locally of finite type: cover $M^{K_1}(\mathfrak{S}_1^0)$ by open affines, use exactness of the direct image of the restriction of j_1 to each such affine [Sa, 4.2.11], and glue. The same technique allows to define the specialization functor

$$Sp_Z : \mathbf{MHM}_F\, M^{K_1}(\mathfrak{S}_1^0) \longrightarrow \mathbf{MHM}_F\, N_{Z/\, M^{K_1}(\mathfrak{S}_1^0)} \ .$$

Because of the functorial behaviour of j_{1*} and Sp_Z, these functors admit Δ_1-equivariant versions. Since they are exact, they induce functors on the level of bounded derived categories.

Proposition 9.3. *There is a natural commutative diagram*

$$
\begin{array}{ccc}
D^b(\mathbf{Rep}_F\, G) & \xrightarrow{\ \ \mu_K\ \ } & D^b(\mathbf{MHM}_F\, M^K) \\
\ \ \downarrow{\scriptstyle \mathrm{Res}_Q^G} & & \\
D^b(\mathbf{Rep}_F\, Q) & & \\
\downarrow & & \\
D^b(\mathbf{Rep}_F\, P_1, H_Q) & & \Big\downarrow {\scriptstyle j_{\mathfrak{S}*}} \\
\ \ \downarrow{\scriptstyle \mu_{K_1}} & & \\
D^b(\Delta_1\text{-}\mathbf{MHM}_F\, M^{K_1}) & & \\
\ \ \downarrow{\scriptstyle j_{1*}} & & \\
D^b(\Delta_1\text{-}\mathbf{MHM}_F\, M^{K_1}(\mathfrak{S}_1^0)) & & D^b(\mathbf{MHM}_F\, M^K(\mathfrak{S})) \\
\ \ \downarrow{\scriptstyle Sp_Z} & & \Big\downarrow {\scriptstyle Sp_{M^K_{1,\mathfrak{S}}}} \\
D^b(\Delta_1\text{-}\mathbf{MHM}_F\, N_{Z/\, M^{K_1}(\mathfrak{S}_1^0)}) & =\!=\!= & D^b(\mathbf{MHM}_F\, N_{M^K_{1,\mathfrak{S}}/\, M^K(\mathfrak{S})})
\end{array}
$$

Remark 9.4. This result implies a comparison isomorphism on the level of singular cohomology, which is already known. In fact, it can be seen to be equivalent to [HZ1, Prop. (5.6.12)].

Proof of Proposition 9.3.   Since all the functors in the diagram are exact on the level of Abelian categories, it suffices to show the result for objects $\mathbb{V}$ of $\mathbf{Rep}_F\, G$. Recall from Corollary 8.3 that the isomorphism between $\Delta_1 \backslash N_{Z/M^{K_1}(\mathfrak{S}_1^0)}$ and $N_{M^K_{1,\mathfrak{S}}/M^K(\mathfrak{S})}$ is induced by the analytic map

$$\tilde{q} : \mathfrak{U} \longrightarrow M^K(\mathfrak{S})(\mathbb{C}) \;,$$

which is a local analytic isomorphism near Z. The restriction to the pre-image of $M^K(\mathbb{C})$ of $\tilde{q}$ looks as follows (see [P1, 6.10], or the proof of [W2, Prop. 2.1]): we have

$$\tilde{q}^{-1}(M^K(\mathbb{C})) = P_1(\mathbb{Q})\backslash(\mathfrak{X}^+ \times P_1(\mathbb{A}_f)/K_1) \;,$$

for a certain complex manifold $\mathfrak{X}^+$, which is open in both $\mathfrak{H}$ and $\mathfrak{X}_1$. On $\tilde{q}^{-1}(M^K(\mathbb{C}))$, the map $\tilde{q}$ is given by

$$[(x,p_1)] \longmapsto [(x,p_1 g)] \in G(\mathbb{Q})\backslash(\mathfrak{H} \times G(\mathbb{A}_f)/K) = M^K(\mathbb{C}) \;.$$

It follows that the local system $\tilde{q}^{-1} \circ \mu_{K,\mathrm{top}}(\mathbb{V})$ is the restriction to $\tilde{q}^{-1}(M^K(\mathbb{C}))$ of the local system $\mu_{K_1,\mathrm{top}}(\mathrm{Res}_{P_1}^G \mathbb{V})$, and that the natural action of Δ_1 on $\tilde{q}^{-1} \circ \mu_{K,\mathrm{top}}(\mathbb{V})$ corresponds to the action of H_Q on $\mathrm{Res}_Q^G \mathbb{V}$. Since the topological version of specialization can be computed locally [V2, (SP0)], we thus obtain the desired comparison result on the level of perverse sheaves. It remains to show that this isomorphism, call it α, respects the weight and Hodge filtrations.

Denote by $\mathbb{V}_G$ and $\mathbb{V}_Q$ the two Δ_1-equivariant variations on the open subset $\tilde{q}^{-1}(M^K(\mathbb{C}))$ of $M^{K_1}(\mathbb{C})$ obtained by restricting $\mu_K(\mathbb{V})$ and $\mu_{K_1}(\mathrm{Res}_Q^G \mathbb{V})$, respectively. By [P1, Prop. 4.12], the Hodge filtrations on $\mathbb{V}_G$ and $\mathbb{V}_Q$ coincide. By the proof of [Sa, Thm. 3.27], the Hodge filtrations of the mixed Hodge modules $j_{1*}\mathbb{V}_G$ and $j_{1*}\mathbb{V}_Q$ depend only on the Hodge filtrations of $\mathbb{V}_G$ and $\mathbb{V}_Q$ respectively. Therefore, they coincide as well. By definition of the functor Sp_Z (see in particular [Sa, 2.30] and [Sa, 2.3]), the Hodge filtrations of $Sp_Z j_{1*}\mathbb{V}_G$ and $Sp_Z j_{1*}\mathbb{V}_Q$ depend only on the Hodge filtrations of $j_{1*}\mathbb{V}_G$ and $j_{1*}\mathbb{V}_Q$, respectively. They are therefore respected by α.

It remains to compare the weight filtrations of $Sp_Z j_{1*}\mathbb{V}_G$ and $Sp_Z j_{1*}\mathbb{V}_Q$. Recall the *barycentric subdivision* $\mathfrak{S}'$ of $\mathfrak{S}$ (e.g., [P1, 5.24]). By the proof of [P1, Prop. 9.20], the cone decomposition $\mathfrak{S}'$ still satisfies the conditions of [P2, (3.9)]. The refinement induces a projective and surjective morphism

$$M^K(G,\mathfrak{H},\mathfrak{S}') \longrightarrow M^K(\mathfrak{S}) = M^K(G,\mathfrak{H},\mathfrak{S}) \;,$$

and the pre-image Z' of Z is a divisor (with normal crossings). Now recall the definition of Sp_Z via the nearby cycle functor [Sa, 2.30]. Apply projective cohomological base change for the latter [Sa, Thm. 2.14], and the fact that in our situation, the cohomology objects are trivial in degree non-zero. This shows that without loss of generality, we may assume that Z is a divisor with normal crossings.

Because of the semi-simplicity of $\mathbf{Rep}_F\, G$, we may also assume that $\mathbb{V}$ is pure of weight k (say). Via α, we view the local system underlying $\mathbb{V}_G$ as

being equipped with the filtration $W_\bullet$ (coming from $\mathbb{V}_Q$). By definition of the weight filtration on $Sp_Z j_{1*} \mathbb{V}_G$ (see [Sa, 2.3]), it remains to show that the following coincide:

(a) the monodromy weight filtration on $Sp_Z j_{1*} \mathbb{V}_G$, shifted by $-k$,

(b) the monodromy weight filtration on $Sp_Z j_{1*} \mathbb{V}_G$ relative to $Sp_Z j_{1*} W_\bullet$.

By [W2, Prop. 1.3], the triple $(\mathbb{V}_G, k, W_\bullet)$ satisfies condition (MON) relative to the divisor Z. Our claim follows thus from Corollary 7.3. **q.e.d.**

Remark 9.5. We use the opportunity to point out a minor error in [W2]. The proof of loc. cit., Prop. 1.3 relies on loc. cit., Lemma 1.2, which is not correctly stated: the claim "$\mathrm{im}(\iota x) \in U_1(\mathbb{R})(-1)$" should be replaced by "$\mathrm{im}(\iota x) - \mathrm{im}(x) \in U_1(\mathbb{R})(-1)$". As a consequence, the proof of loc. cit., Prop. 1.3 (but not its statement) has to be slightly modified: in line 12 of page 328, replace "maps u_0 to $\pm \frac{1}{2\pi i} T$" by "maps u_0 to $\pm \frac{1}{2\pi i} T \mod U$", where U denotes the weight -2 part of the unipotent radical of the group P. Since U acts trivially on the weight-graded parts of any representation $\mathbb{V}$ of P, the rest of the proof remains unchanged.

By the preceding results, we have to compute the composition of the following three functors: (I) the functor

$$Sp_Z\, j_{1*}\, \mu_{K_1} : D^b(\mathbf{Rep}_F\, P_1, H_Q) \longrightarrow D^b(\Delta_1\text{-}\mathbf{MHM}_F\, N_{Z/M^{K_1}(\mathfrak{S}_1^0)})$$

(see Proposition 9.3); (II) the functor

$$i_0^* : D^b(\Delta_1\text{-}\mathbf{MHM}_F\, N_{Z/M^{K_1}(\mathfrak{S}_1^0)}) \longrightarrow D^b(\Delta_1\text{-}\mathbf{MHM}_F\, Z)\ ,$$

whose target is equal to $D^b(\mathbf{MHM}_F\, M_{1,\mathfrak{S}}^K)$; (III) the functor

$$p_* : D^b(\mathbf{MHM}_F\, M_{1,\mathfrak{S}}^K) \longrightarrow D^b(\mathbf{MHM}_F\, M_1^K)\ .$$

This computation is complicated by the fact that p_* is neither left nor right exact — remember that we are working in a (derived) category of objects which behave like perverse sheaves, hence there are no exactness properties for Grothendieck's functors associated to arbitrary morphisms. This is why we construct a certain factorization of p_*, which will represent it as the composition of a left exact and a right exact functor (Proposition 9.7). In order to do so, consider the diagram

$$
\begin{array}{ccc}
M_{1,\mathfrak{S}}^K = \Delta_1 \backslash Z & \xleftarrow{\ \tilde{q}\ } & Z \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \tilde{p}} \\
M_1^K = \Delta \backslash M^{\pi(K_1)} & \xleftarrow{\ q\ } & M^{\pi(K_1)}
\end{array}
$$

It is *not* Cartesian. However, setting $\bar{Z} := \overline{H}_C \backslash Z$ (remember that thanks to Corollary 1.5, we consider $\overline{H}_C$ as a subgroup of Δ_1 in a natural way), we get

a natural factorization of the morphism $\tilde{q}$, which fits into the diagram

$$
\begin{array}{ccccc}
M_{1,\mathfrak{S}}^K = \Delta_1 \backslash Z & \xleftarrow{\ q\ } & \bar{Z} = \overline{H}_C \backslash Z & \xleftarrow{\ \tilde{q}\ } & Z \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \tilde{p}} \\
M_1^K = \Delta \backslash M^{\pi(K_1)} & \xleftarrow{\ q\ } & M^{\pi(K_1)} & =\!=\!=\!= & M^{\pi(K_1)}
\end{array}
$$

Observe that the left half of this diagram *is* Cartesian, and that the morphisms q are finite Galois coverings, with Galois group Δ. We identify $D^b(\mathbf{MHM}_F\, M_{1,\mathfrak{S}}^K)$ with $D^b(\Delta\text{-}\mathbf{MHM}_F\, \bar{Z})$, and the functor

$$
p_* : D^b(\mathbf{MHM}_F\, M_{1,\mathfrak{S}}^K) \longrightarrow D^b(\mathbf{MHM}_F\, M_1^K)
$$

with its Δ-equivariant version

$$
p_* : D^b(\Delta\text{-}\mathbf{MHM}_F\, \bar{Z}) \longrightarrow D^b(\Delta\text{-}\mathbf{MHM}_F\, M^{\pi(K_1)})\ .
$$

In order to study this last functor, recall the closed covering of Z by the closures Z_σ of the toric strata (Section 8). It induces a (finite!) closed covering of $\bar{Z}$ by the $Z_{\bar{\sigma}}$, for $\bar{\sigma} \in \bar{\mathfrak{T}} := \overline{H}_C \backslash \mathfrak{T}$. By Proposition 8.1, the morphism $\bar{q}$ induces isomorphisms

$$
Z_\sigma \xrightarrow{\ \sim\ } Z_{\bar{\sigma}}\ ,
$$

for any $\sigma \in \mathfrak{T}$. Observe moreover that, for any $\bar{I} \in \bar{\mathfrak{T}}_\bullet$ the intersection $Z_{\bar{I}}$ is either one stratum $Z_{\bar{\sigma}}$ or the empty set.

Definition 9.6. Let M denote one of the varieties $\bar{Z}$ or $M^{\pi(K_1)}$. Define the Abelian category $(\mathbf{MHM}_F\, M)^{\bar{\mathfrak{T}}}$ applying Definition 5.2 to the closed covering $\{Z_{\bar{\sigma}}\}_{\bar{\sigma} \in \bar{\mathfrak{T}}}$ in the case of $\bar{Z}$ and to the trivial closed covering $\{M_{\bar{\sigma}}\}_{\bar{\sigma} \in \bar{\mathfrak{T}}}$ with $M_{\bar{\sigma}} = M^{\pi(K_1)}$ for all $\bar{\sigma} \in \bar{\mathfrak{T}}$, in the case of $M^{\pi(K_1)}$.

Now remember the action of the finite group Δ on our geometric situation. This group acts on the spaces $\bar{Z}$ and $M^{\pi(K_1)}$ and on the set of indexes $\bar{\mathfrak{T}}$, hence on the simplicial schemes $\bar{Z} \times \bar{\mathfrak{T}}_\bullet$ and $M^{\pi(K_1)} \times \bar{\mathfrak{T}}_\bullet$. Therefore, as in 3.18 (c), we can define the categories $\Delta\text{-}(\mathbf{MHM}_F\, \bar{Z})^{\bar{\mathfrak{T}}}$ and $\Delta\text{-}(\mathbf{MHM}_F\, M^{\pi(K_1)})^{\bar{\mathfrak{T}}}$. For instance the former is the category of mixed Hodge modules $\mathbb{M}$ over the simplicial scheme $\bar{Z} \times \bar{\mathfrak{T}}_\bullet$ together with isomorphisms, for $\gamma \in \Delta$,

$$
\rho_\gamma : \gamma^*\mathbb{M} \xrightarrow{\ \sim\ } \mathbb{M},
$$

that satisfy the cocycle condition.

Since Δ respects the stratification of $\bar{Z}$ indexed by $\bar{\mathfrak{T}}$, we can define the equivariant version of the functors $S_\bullet$ and Tot. We leave it to the reader to check that the Δ-equivariant versions of 5.7 and 5.8 hold (in the proof of the analogue of 5.7, choose a finite open affine covering $\mathfrak{V}$ closed under the action of the finite group Δ, and observe that the open subset U occurring in point (2) can be replaced by the intersection of all its translates under Δ). In particular, we have:

Proposition 9.7. *(a) There is a canonical functor*

$$p_*^{\bar{\mathfrak{T}}} : D^b(\Delta\text{-}(\mathbf{MHM}_F\,\bar{Z})^{\bar{\mathfrak{T}}}) \longrightarrow D^b(\Delta\text{-}(\mathbf{MHM}_F\,M^{\pi(K_1)})^{\bar{\mathfrak{T}}})\ .$$

(b) There is a natural commutative diagram

Using $\Delta = \Delta_1/\overline{H}_C$ and $\bar{Z} = \overline{H}_C\setminus Z$, and a slight generalization of Proposition 4.2 we make the identification

$$D^b(\Delta\text{-}\mathbf{MHM}_F\,\bar{Z}) = D^b(\Delta_1\text{-}\mathbf{MHM}_F\,Z)\ .$$

Since the group Δ_1 acts on $M^{\pi(K_1)}$ (by its quotient Δ) and on the set $\mathfrak{T}$, it acts also on $M^{\pi(K_1)}\times\mathfrak{T}_\bullet$. We define the category $\Delta_1\text{-}(\mathbf{MHM}_F\,M^{\pi(K_1)})^{\mathfrak{T}}$, in the same way as we have defined $\Delta\text{-}(\mathbf{MHM}_F\,M^{\pi(K_1)})^{\bar{\mathfrak{T}}}$ but using the infinite version of Definition 5.2. We have the following variant of Proposition 4.2:

Proposition 9.8. *The inverse image*

$$\Delta\text{-}(\mathbf{MHM}_F\,M^{\pi(K_1)})^{\bar{\mathfrak{T}}} \longrightarrow \Delta_1\text{-}(\mathbf{MHM}_F\,M^{\pi(K_1)})^{\mathfrak{T}}$$

is an equivalence of categories, which possesses a canonical pseudo-inverse.

Proof. The group Δ_1 acts freely and properly on the simplicial scheme $M^{\pi(K_1)}\times\mathfrak{T}_\bullet$. Hence so does the subgroup $\overline{H}_C$ of Δ_1. The quotient by $\overline{H}_C$ of $M^{\pi(K_1)}\times\mathfrak{T}_\bullet$ equals $M^{\pi(K_1)}\times\bar{\mathfrak{T}}_\bullet$. The action of $\Delta = \Delta_1/\overline{H}_C$ on this quotient is free and proper, and

$$\Delta\backslash(M^{\pi(K_1)}\times\bar{\mathfrak{T}}_\bullet) = \Delta_1\backslash(M^{\pi(K_1)}\times\mathfrak{T}_\bullet)\ .$$

q.e.d.

We now start to evaluate our functors. Consider the composition

$$\nu := p_*^{\bar{\mathfrak{T}}}\,S_\bullet\,i_0^*\,Sp_Z\,j_{1*}\,\mu_{K_1} : D^b(\mathbf{Rep}_F\,P_1, H_Q) \longrightarrow D^b(\Delta\text{-}(\mathbf{MHM}_F\,M^{\pi(K_1)})^{\bar{\mathfrak{T}}})$$

where we use the identification

$$D^b(\Delta\text{-}\mathbf{MHM}_F\,\bar{Z}) = D^b(\Delta_1\text{-}\mathbf{MHM}_F\,Z)\ .$$

before applying the functor $S_\bullet$. We have a variant of the canonical construction

$$\mu_{\pi(K_1)}^{\mathfrak{T}} : (\mathbf{Rep}_F\,G_1, \overline{H}_Q) \longrightarrow \Delta_1\text{-}(\mathbf{MHM}_F\,M^{\pi(K_1)})^{\mathfrak{T}}\ ,$$

which associates to a representation $\mathbb{V}_1$ the mixed Hodge module, whose component over $M^{\pi(K_1)} \times \{I\}$ is

$$\mathbb{M}_I = \begin{cases} \mu_{\pi(K_1)}(\mathbb{V}_1) \,, & \text{if } Z_I \neq \emptyset \,, \\ 0 \,, & \text{if } Z_I = \emptyset \,. \end{cases}$$

For any increasing map τ and $I \in \mathfrak{T}_{\bullet}$ with $J = \mathfrak{T}_{\bullet}(\tau)(I)$, we put $\tau_I = \mathrm{Id}$ if Z_I is not empty and zero otherwise. For $\gamma \in \Delta_1$ we let the isomorphisms ρ_γ be given by the action of $\overline{H}_Q$.

The functor $\mu^{\mathfrak{T}}_{\pi(K_1)}$ is exact. As before, we let c denote the codimension of M_1^K in $(M^K)^*$, which is the same as the relative dimension of the morphism $\pi : M^{K_1} \to M^{\pi(K_1)}$.

Proposition 9.9. *There is a natural commutative diagram*

$$\begin{array}{ccc} D^b(\mathbf{Rep}_F\, P_1, H_Q) & \xrightarrow{\nu[-c]} & D^b(\Delta\text{-}(\mathbf{MHM}_F\, M^{\pi(K_1)})^{\bar{\mathfrak{T}}}) \\ {\scriptstyle R\Gamma(W_1,\,\bullet)} \Big\downarrow & & \Big\| \\ D^b(\mathbf{Rep}_F\, G_1, \overline{H}_Q) & \xrightarrow{\mu^{\mathfrak{T}}_{\pi(K_1)}} & D^b(\Delta_1\text{-}(\mathbf{MHM}_F\, M^{\pi(K_1)})^{\mathfrak{T}}) \end{array}$$

Proof. Let us first determine the cohomology functors

$$\mathcal{H}^r\nu : (\mathbf{Rep}_F\, P_1, H_Q) \longrightarrow \Delta_1\text{-}(\mathbf{MHM}_F\, M^{\pi(K_1)})^{\mathfrak{T}} \,.$$

Let $\mathbb{V}_1$ be in $(\mathbf{Rep}_F\, P_1, H_Q)$, and $I \in \mathfrak{T}$. By Proposition 5.7 (b), the component $(\mathcal{H}^r\nu(\mathbb{V}_1))_I$ is given by

$$\mathcal{H}^r(\tilde{p} \circ i_I)_* \, i_I^* \, Sp_Z \, j_{1*} \, \mu_{K_1}(\mathbb{V}_1) \,.$$

Now remember the factorization

$$M^{K_1} \xrightarrow{\pi_t} M^{\pi_t(K_1)} \xrightarrow{\pi_a} M^{\pi(K_1)}$$
$$\underbrace{\phantom{M^{K_1} \xrightarrow{\pi_t} M^{\pi_t(K_1)} \xrightarrow{\pi_a} M^{\pi(K_1)}}}_{\pi}$$

of the morphism $\pi : M^{K_1} \to M^{\pi(K_1)}$. As explained in Section 8, the map $\tilde{p} : Z \to M^{\pi(K_1)}$ factors through π_a, and identifies Z with a closed union of strata in a torus embedding of M^{K_1} over $M^{\pi_t(K_1)}$. The above object involves the direct images $(\tilde{p} \circ i_I)_*$ of individual strata of this torus embedding. Note that the corresponding direct image of the generic stratum M^{K_1} equals $\pi_{a*} \circ \pi_{t*} = \pi_*$. Using Corollary 6.2 and the compatibility of Sp_Z with $i_{I*} \circ i_I^*$ [Sa, 2.30], we see that

$$\mathcal{H}^r(\tilde{p} \circ i_I)_* \, i_I^* \, Sp_Z \, j_{1*} \, \mu_{K_1}(\mathbb{V}_1) = \mathcal{H}^r \pi_* \, \mu_{K_1}(\mathbb{V}_1)$$

when Z_I is not empty and zero otherwise. Furthermore, this identification is compatible with the simplicial structure: assume that $\mathfrak{T}_{\bullet}(\tau)(I) = J$ for some increasing map τ and that Z_I is not empty. Then the morphisms

$$(\tau)_I : (\mathcal{H}^r\nu(\mathbb{V}_1))_J \longrightarrow (\mathcal{H}^r\nu(\mathbb{V}_1))_I$$

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correspond to the identity on $\mathcal{H}^r \pi_* \mu_{K_1}(\mathbb{V}_1)$. By [W1, Thm. 2.3], there is a natural isomorphism

$$\mu_{\pi(K_1)}(H^{r+c}(W_1, \mathbb{V}_1)) \overset{\sim}{\longrightarrow} \mathcal{H}^r \pi_* \mu_{K_1}(\mathbb{V}_1) \ .$$

In fact, this is the *canonical* isomorphism given by the universal property of the cohomological derived functor [V1, II.2.1.4], and by the fact that the functors on the right hand side are *effaçable* for $r > -c$. Since this isomorphism is compatible with automorphisms of Shimura data, we see that the natural actions of Δ_1 on both sides are compatible. This proves the claim after passage to the cohomology objects. We see in particular that the functor $\mathcal{H}^{-c}\nu$ is left exact, and that its total right derived functor is equal to

$$\mu_{\pi(K_1)}^{\mathfrak{T}} \circ R\Gamma(W_1, \ \bullet \ ) \ .$$

Let us assume for a moment that ν admits a natural *f-lifting* in the sense of [B, Def. A.1 (c)] to the filtered bounded derived categories (see below). By [B, A.7], this f-lifting induces a natural transformation

$$\eta : \mu_{\pi(K_1)}^{\mathfrak{T}} \circ R\Gamma(W_1, \ \bullet \ ) \longrightarrow \nu[-c]$$

of triangulated functors. Furthermore, the $H^n \eta$ are the natural transformations corresponding to the universal property of the cohomological derived functor. Since we already know that these are isomorphisms, we get the desired conclusion.

It remains to construct the natural extension

$$\nu : DF^b(\mathbf{Rep}_F P_1, H_Q) \longrightarrow DF^b(\Delta\text{-}(\mathbf{MHM}_F M^{\pi(K_1)})^{\bar{\mathfrak{T}}})$$

of ν satisfying the conditions of [B, Def. A.1 (c)]. Since ν is a composition of functors, we need to define such an extension for each of them. For the exact functors μ_{K_1}, j_{1*}, and Sp_Z, there is no problem. For $S_\bullet$, we have Remark 5.5. It remains to consider i_0^* and $p_*^{\bar{\mathfrak{T}}}$. For the construction of

$$i_0^* : DF^b(\mathbf{MHM}_F N_{M_{1,\mathfrak{S}}^K / M^K(\mathfrak{S})}) \longrightarrow DF^b(\mathbf{MHM}_F M_{1,\mathfrak{S}}^K) \ ,$$

observe first that by the construction of $i_{0*} i_0^*$ recalled earlier, this latter functor admits a filtered version:

$$i_{0*} i_0^* : DF^b(\mathbf{MHM}_F N_{M_{1,\mathfrak{S}}^K / M^K(\mathfrak{S})}) \longrightarrow DF^b(\mathbf{MHM}_F N_{M_{1,\mathfrak{S}}^K / M^K(\mathfrak{S})}) \ .$$

In fact, the image of $i_{0*} i_0^*$ is contained in $DF_{M_{1,\mathfrak{S}}^K}^b(\mathbf{MHM}_F N_{M_{1,\mathfrak{S}}^K / M^K(\mathfrak{S})})$, the full triangulated sub-category of $DF^b(\mathbf{MHM}_F N_{M_{1,\mathfrak{S}}^K / M^K(\mathfrak{S})})$ of filtered complexes $(\mathbb{M}, F^\bullet \mathbb{M})$ with support in $M_{1,\mathfrak{S}}^K$, i.e., for which the cohomology objects of all $F^r \mathbb{M}$ are supported in $M_{1,\mathfrak{S}}^K$. It remains to show that the functor

$$i_{0*} : DF^b(\mathbf{MHM}_F M_{1,\mathfrak{S}}^K) \longrightarrow DF_{M_{1,\mathfrak{S}}^K}^b(\mathbf{MHM}_F N_{M_{1,\mathfrak{S}}^K / M^K(\mathfrak{S})})$$

(which exists since the unfiltered i_{0*} is exact) is an equivalence of categories. For this, we need to check (α) full faithfulness and (β) essential surjectivity.

For (α), let $\mathbb{M}$ and $\mathbb{N}$ be two objects of $DF^b(\mathbf{MHM}_F\, M_{1,\mathfrak{S}}^K)$. In order to show that

$$i_{0*} : \operatorname{Hom}_{M_{1,\mathfrak{S}}^K}(\mathbb{M}, \mathbb{N}) \longrightarrow \operatorname{Hom}_{N_{M_{1,\mathfrak{S}}^K\,/\,M^K(\mathfrak{S})}}(i_{0*}\mathbb{M}, i_{0*}\mathbb{N})$$

is an isomorphism, we may, using the exact triangles associated to the (finite!) filtrations of both $\mathbb{M}$ and $\mathbb{N}$, suppose that these are concentrated in single degrees, say m and n. The same is then true for the filtrations of $i_{0*}\mathbb{M}$ and $i_{0*}\mathbb{N}$. By [B, Def. A.1 (a) (iii)], there are no non-trivial morphisms if $m > n$. Furthermore, loc. cit. allows to reduce the case $m \leq n$ to the case $m = n$. But then the morphisms can be calculated in the unfiltered derived categories [B, Def. A.1 (c)], and the claim follows from [Sa, (4.2.10)]. For (β), we use induction on the length of the filtration of a given object $\mathbb{M}$ in $DF^b_{M_{1,\mathfrak{S}}^K}(\mathbf{MHM}_F\, N_{M_{1,\mathfrak{S}}^K\,/\,M^K(\mathfrak{S})})$. If the filtration is concentrated in a single degree, use [B, Def. A.1 (c)] and [Sa, (4.2.10)]. If not, then $\mathbb{M}$ is a cone of a morphism $\mathbb{M}'' \to \mathbb{M}'[1]$ in $DF^b_{M_{1,\mathfrak{S}}^K}(\mathbf{MHM}_F\, N_{M_{1,\mathfrak{S}}^K\,/\,M^K(\mathfrak{S})})$ of two objects in the image of i_{0*}. By (α), this morphism comes from a morphism f in $DF^b(\mathbf{MHM}_F\, M_{1,\mathfrak{S}}^K)$. Thus there is an isomorphism between $\mathbb{M}$ and the image under i_{0*} of a cone of f.

For the construction of

$$p_*^{\bar{\mathfrak{T}}} : DF^b(\Delta\text{-}(\mathbf{MHM}_F\, \bar{Z})^{\bar{\mathfrak{T}}}) \longrightarrow DF^b(\Delta\text{-}(\mathbf{MHM}_F\, M^{\pi(K_1)})^{\bar{\mathfrak{T}}})\,,$$

observe first that the functor $(\mathbb{M}, F^\bullet\mathbb{M}) \mapsto (\mathbb{M}, \mathbb{M}/F^\bullet\mathbb{M})$ identifies the filtered (derived) category of complexes of Hodge modules with the co-filtered (derived) category. This latter point of view will be better adapted to our needs. Next, fix a finite Δ-equivariant open affine covering $\mathfrak{V} = \{V_1, \ldots, V_r\}$ of $\bar{Z}$ as in the proof of Proposition 9.7. Now imitate the proof of Proposition 5.7, using the following observation (see the proof of [Sa, Thm. 4.3]): assume given a diagram of bounded complexes of Δ-equivariant Hodge modules on $\bar{Z}$

$$
\begin{array}{ccc}
 & & \mathbb{N}_0^\bullet \\
 & & \downarrow{\scriptstyle \varphi_0} \\
\mathbb{M}_1^\bullet & \xrightarrow{\;f_M\;} & \mathbb{M}_0^\bullet
\end{array}
$$

where

(1) the morphism $f_M : \mathbb{M}_1^\bullet \to \mathbb{M}_0^\bullet$ is epimorphic in all degrees,

(2) the components of $\mathbb{N}_0^\bullet$ are p_*-acyclic with respect to $\mathfrak{V}$,

(3) the morphism $\varphi_0 : \mathbb{N}_0^\bullet \to \mathbb{M}_0^\bullet$ is epimorphic in all degrees, and becomes an isomorphism in $D^b(\mathbf{MHM}_F\, \bar{Z})$.

Then this diagram can be completed in the following way:

$$
\begin{array}{ccc}
\mathbb{N}_1^\bullet & \xrightarrow{\ f_N\ } & \mathbb{N}_0^\bullet \\
{\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle\varphi_0} \\
\mathbb{M}_1^\bullet & \xrightarrow{\ f_M\ } & \mathbb{M}_0^\bullet
\end{array}
$$

where

(4) the morphism f_N is epimorphic in all degrees,

(5) the components of $\mathbb{N}_1^\bullet$ are p_*-acyclic with respect to $\mathfrak{V}$,

(6) the morphism φ_1 is epimorphic in all degrees, and becomes an isomorphism in $D^b(\mathbf{MHM}_F \bar{Z})$.

$$\textbf{q.e.d.}$$

We can now complete the proof of our main result:

Proof of Theorem 2.6. By Propositions 9.1, 9.2, 9.3, 9.7, and 9.9, all that remains to be proved is that there is a natural commutative diagram

$$
\begin{array}{ccc}
D^b(\mathbf{Rep}_F\, G_1, \overline{H}_Q) & \xrightarrow{\ \mu_{\pi(K_1)}^{\mathfrak{T}}\ } & D^b(\Delta_1\text{-}(\mathbf{MHM}_F\, M^{\pi(K_1)})^{\mathfrak{T}}) \\
{\scriptstyle R\Gamma(\overline{H}_C,\,\bullet)}\downarrow & & \downarrow{\scriptstyle \mathrm{Tot}} \\
D^b(\mathbf{Rep}_F\, G_1, \overline{H}_Q\,/\,\overline{H}_C) & \xrightarrow{\ \mu_{\pi(K_1)}\ } & D^b(\Delta\text{-}(\mathbf{MHM}_F\, M^{\pi(K_1)}))
\end{array}
$$

Recall that we identify Δ_1-$(\mathbf{MHM}_F\, M^{\pi(K_1)})^{\mathfrak{T}}$ and Δ-$(\mathbf{MHM}_F\, M^{\pi(K_1)})^{\bar{\mathfrak{T}}}$, as well as Δ-$(\mathbf{MHM}_F\, M^{\pi(K_1)})$ and $\mathbf{MHM}_F\, M_1^K$, and that the functor Tot is formed with respect to the stratification $\bar{\mathfrak{T}}$ (not with respect to $\mathfrak{T}$).

This is where the conditions [P2, (2.3.1–3)] listed in the beginning of Section 8 enter. For any $\sigma \in \mathfrak{T}$ we denote by $star_{\mathfrak{T}}(\sigma)$ the union of τ° for all $\tau \in \mathfrak{T}$ such that σ is a face of τ. Then the $star_{\mathfrak{T}}(\sigma)$ form an open covering of the set D. Moreover, all these open sets are contractible and the intersection of a finite number of them is also contractible [P2, Lemma (2.4.1)]. We denote by $C_\bullet(\{star_{\mathfrak{T}}(\sigma)\}, \mathbb{Z})$ the Čech chain complex associated to this covering (i.e. the dual of the usual Čech cochain complex). Since the set D is contractible, the natural augmentation

$$C_\bullet(\{star_{\mathfrak{T}}(\sigma)\}, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

is a resolution. The group Δ_1 acts freely and properly on the set $\mathfrak{T}$. Therefore, the Abelian groups $C_p(\{star_{\mathfrak{T}}(\sigma)\}, \mathbb{Z})$ have a natural structure of free $\mathbb{Z}\Delta_1$-modules. Moreover, since the combinatorics of the open covering $\{star_{\mathfrak{T}}(\sigma)\}$ of D agrees with that of the closed covering $\{Z_\sigma\}$ of Z, by the definition of $\mu_{\pi(K_1)}^{\mathfrak{T}}$, the composition of functors $\mathrm{Tot} \circ \mu_{\pi(K_1)}^{\mathfrak{T}}$ agrees with the composition of functors

$$\mu_{\pi(K_1)} \circ (\,\bullet\,)^{\overline{H}_C} \circ \mathrm{Hom}(C_\bullet(\{star_{\mathfrak{T}}(\sigma)\}, \mathbb{Z}),\, \bullet\,),$$

which by Proposition 3.19 agrees with

$$\mu_{\pi(K_1)} \circ R\Gamma(\overline{H}_C, \bullet).$$

<div align="right">**q.e.d.**</div>

Remark 9.10. (a) Our proof uses a *choice* of toroidal compactification. However, as can be seen by passing to simultaneous refinements of two cone decompositions, the isomorphism of Theorem 2.6 does not depend on this choice. We leave the details of the proof to the reader.

(b) We also leave it to the reader to formulate and prove results like [P2, Prop. (4.8.5)] on the behaviour of the isomorphism of 2.6 under change of the subgroup $K \subset G(\mathbb{A}_f)$, and of the element $g \in G(\mathbb{A}_f)$.

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