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Spectral action on $SU_q(2)$

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Abstract

The spectral action on the equivariant real spectral triple over $\mathcal{A}(SU_q(2))$ is computed explicitly. Properties of the differential calculus arising from the Dirac operator are studied and the results are compared to the commutative case of the sphere \mathbb{S}^3 .

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1 Introduction

The quantum group $SU_q(2)$ has already a rather long history of studies [33] being one of the finest examples of quantum deformation. This includes an approach via the noncommutative notion of spectral triple introduced by Connes [10, 15] and various notions of Dirac operators were introduced in [2, 4, 6, 13, 28]. Finally, a real spectral triple, which was exhibited in [21], is invariant by left and right action of $\mathcal{U}_q(su(2))$ and satisfies almost all postulated axioms of triples except the commutant and first-order properties. These, however, remain valid only up to infinitesimal of arbitrary high order. The last presentation generalizes in a straightforward way all geometric construction details of the spinorial spectral triple for the classical three-sphere. In

particular, both the equivariant representation and the symmetries have a $q \rightarrow 1$ proper classical limit.

The goal of this article is to obtain the spectral action defined in [7] by

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) := \text{Tr}(\Phi(\mathcal{D}_A/\Lambda)) \quad (1)$$

where \mathcal{D} is the Dirac operator, A is a selfadjoint one-form, $\mathcal{D}_A = \mathcal{D} + A + JAJ^{-1}$ and J is the reality operator. Here, Φ is any even positive cut-off function which could be replaced by a step function up to some mathematical difficulties investigated in [23]. This means that \mathcal{S} counts the spectral values of $|\mathcal{D}_A|$ less than the mass scale Λ . Actually, as shown in [8]

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{0 < k \in Sd^+} \Phi_k \Lambda^k \int |\mathcal{D}_A|^{-k} + \Phi(0) \zeta_{\mathcal{D}_A}(0) + \mathcal{O}(\Lambda^{-1}), \quad (2)$$

$$\zeta_{\mathcal{D}_A}(0) = \zeta_{\mathcal{D}}(0) + \sum_{k=1}^d \frac{(-1)^k}{k} \int (A\mathcal{D}^{-1})^k. \quad (3)$$

where $D_A = \mathcal{D}_A + P_A$, P_A the projection on $\text{Ker } \mathcal{D}_A$, $\Phi_k = \frac{1}{2} \int_0^\infty \Phi(t) t^{k/2-1} dt$, d is the spectral dimension of the triple and Sd^+ is the strictly positive part of the dimension spectrum Sd of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Here, $Sd^+ = Sd = \{1, 2, 3\}$, so

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{1 \leq k \leq 3} \Phi_k \Lambda^k \int |\mathcal{D}_A|^{-k} + \Phi(0) \zeta_{\mathcal{D}_A}(0). \quad (4)$$

Recall that the tadpole of order Λ^k is the linear term in $A \in \Omega_D^1(\mathcal{A})$ in the Λ^k part of (4). Note that there are no terms in Λ^{-k} , $k > 0$ because the dimension spectrum is bounded below by 1. This spectral action has been computed on few examples: [3, 8, 9, 15, 22, 24–26, 30, 34]. Here, we compute (4) with the main difficulty which is to control the differential calculus generated by the Dirac operator. To proceed, we introduce two presentations of one-forms. The main ingredient is $F = \text{sign}(\mathcal{D})$ which appears to be a one-form up to $OP^{-\infty}$. In section 2, we discuss the spectral action of an arbitrary 3-dimensional spectral triple using cocycles.

In sections 3 and 4 we recall the main results on $SU_q(2)$ of [21] and show that the full spectral action with reality operator given by (4) is completely determined by the terms

$$\int A^q |\mathcal{D}|^{-p}, \quad 1 \leq q \leq p \leq 3.$$

This question of computation of spectral action was addressed in the epilogue of [37]. In section 5, we establish a differential calculus up to some ideal in pseudodifferential operators and apply these results to the precise computation of previous noncommutative integrals. Section 6 is devoted to explicit examples, while in next section are given different comparisons with the commutative case of the 3-sphere corresponding to $SU(2)$.

2 Spectral action in 3-dimension

2.1 Tadpole and cocycles

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple of dimension 3. For $n \in \mathbb{N}^*$ and $a_i \in \mathcal{A}$, define

$$\phi_n(a_0, \dots, a_n) := \int a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} \dots [\mathcal{D}, a_n] \mathcal{D}^{-1}.$$

We also use notational integrals on the universal n -forms $\Omega_u^n(\mathcal{A})$ defined by

$$\int_{\phi_n} a_0 da_1 \cdots da_n := \phi_n(a_0, a_1, \dots, a_n).$$

and the reordering fact that $(da_0)a_1 = d(a_0a_1) - a_0da_1$.

We use the $b - B$ bicomplex defined in [10]: b is the Hochschild coboundary map (and b' is truncated one) defined on n -cochains ϕ by

$$\begin{aligned} b\phi(a_0, \dots, a_{n+1}) &:= b'\phi(a_0, \dots, a_{n+1}) + (-1)^{n+1}\phi(a_{n+1}a_0, a_1, \dots, a_n), \\ b'\phi(a_0, \dots, a_{n+1}) &:= \sum_{j=0}^n (-1)^j \phi(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}). \end{aligned}$$

Recall that B_0 is defined on the normalized cochains ϕ_n by

$$B_0\phi_n(a_0, a_1, \dots, a_{n-1}) := \phi_n(1, a_0, \dots, a_{n-1}), \text{ thus } \int_{\phi_n} d\omega = \int_{B_0\phi_n} \omega \text{ for } \omega \in \Omega_u^{n-1}(\mathcal{A}).$$

Then $B := NB_0$, where $N := 1 + \lambda + \dots + \lambda^n$ is the cyclic skewsymmetrizer on the n -cochains and λ is the cyclic permutation $\lambda\phi(a_0, \dots, a_n) := (-1)^n\phi(a_n, a_0, \dots, a_{n-1})$.

We will also encounter the cyclic 1-cochain $N\phi_1$:

$$N\phi_1(a_0, a_1) := \phi_1(a_0, a_1) - \phi_1(a_1, a_0) \text{ and } \int_{N\phi_1} a_0 da_1 := N\phi_1(a_0, a_1).$$

Remark 2.1. Assume the integrand of f is in OP^{-3} . Since $[\mathcal{D}^{-1}, a] = -\mathcal{D}^{-1}[\mathcal{D}, a]\mathcal{D}^{-1} \in OP^{-2}$, this commutator introduces a integrand in OP^{-4} so has a vanishing integral: under the integral, we can commute \mathcal{D}^{-1} with all $a \in \mathcal{A}$ and all one-forms.

Lemma 2.2. We have

- (i) $b\phi_1 = -\phi_2$.
- (ii) $b\phi_2 = 0$.
- (iii) $b\phi_3 = 0$.
- (iv) $B\phi_1 = 0$.
- (v) $B_0\phi_2 = -(1 - \lambda)\phi_1$.
- (vi) $bB_0\phi_2 = 2\phi_2 + B_0\phi_3$.
- (vii) $B\phi_2 = 0$.
- (viii) $B_0\phi_3 = Nb'\phi_1$.
- (ix) $B\phi_3 = 3B_0\phi_3$.

Proof. (i)

$$\begin{aligned} b\phi_1(a_0, a_1, a_2) &= \int a_0 a_1 [\mathcal{D}, a_2] \mathcal{D}^{-1} - \int a_0 (a_1 [\mathcal{D}, a_2] + [\mathcal{D}, a_1] a_2) \mathcal{D}^{-1} + \int a_2 a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} \\ &= \int a_0 [\mathcal{D}, a_1] (\mathcal{D}^{-1} a_2 - a_2 \mathcal{D}^{-1}) = - \int a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\ &= -\phi_2(a_0, a_1, a_2) \end{aligned}$$

where we have used the trace property of the noncommutative integral.

(ii) $b\phi_2(a_0, a_1, a_2, a_3)$

$$\begin{aligned}
&= \int a_0 a_1 [\mathcal{D}, a_2] \mathcal{D}^{-1} [\mathcal{D}, a_3] \mathcal{D}^{-1} - \int a_0 (a_1 [\mathcal{D}, a_2] + [\mathcal{D}, a_1] a_2) \mathcal{D}^{-1} [\mathcal{D}, a_3] \mathcal{D}^{-1} \\
&\quad + \int a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} (a_2 [\mathcal{D}, a_3] + [\mathcal{D}, a_2] a_3) \mathcal{D}^{-1} - \int a_3 a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\
&= \int a_0 [\mathcal{D}, a_1] (\mathcal{D}^{-1} a_2 - a_2 \mathcal{D}^{-1}) [\mathcal{D}, a_3] \mathcal{D}^{-1} + \int a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] (a_3 \mathcal{D}^{-1} - \mathcal{D}^{-1} a_3) \\
&= - \int a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} [\mathcal{D}, a_3] \mathcal{D}^{-1} + \int a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} [\mathcal{D}, a_3] \mathcal{D}^{-1} \\
&= 0.
\end{aligned}$$

(iii) Using Remark 2.1, we get $\phi_3(a_0, a_1, a_2, a_3) = \int a_0 [\mathcal{D}, a_1] [\mathcal{D}, a_2] [\mathcal{D}, a_3] |\mathcal{D}|^{-3}$, so similar computations as for ϕ_2 gives $b\phi_3 = 0$.

(iv) $B_0\phi_1(a_0) = \int [\mathcal{D}, a_0] \mathcal{D}^{-1} = \int (\mathcal{D} a_0 \mathcal{D}^{-1} - a_0) = 0$.

$$\begin{aligned}
(v) \quad B_0\phi_2(a_0, a_1) &= \int [\mathcal{D}, a_0] \mathcal{D}^{-1} [\mathcal{D}, a_1] \mathcal{D}^{-1} = \int a_0 \mathcal{D}^{-1} [\mathcal{D}, a_1] - \int a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} \\
&= \int a_0 a_1 - \int a_0 \mathcal{D}^{-1} a_1 \mathcal{D} - \int a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} \\
&= - \int a_1 [\mathcal{D}, a_0] \mathcal{D}^{-1} - \int a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} = -\phi_1(a_1, a_0) - \phi_1(a_0, a_1).
\end{aligned}$$

(vi) Since $-b\lambda\phi_1(a_0, a_1, a_2) = \phi_1(a_2, a_0 a_1) - \phi_1(a_1 a_2, a_0) + \phi_1(a_1, a_2 a_0)$, one obtains that

$$-b\lambda\phi_1(a_0, a_1, a_2) = \int a_0 a_1 \mathcal{D}^{-1} a_2 \mathcal{D} + a_0 \mathcal{D}^{-1} a_1 \mathcal{D} a_2 - a_0 \mathcal{D}^{-1} a_1 a_2 \mathcal{D} - a_0 a_1 a_2.$$

So by direct expansion, this is equal to $-\int a_0 \mathcal{D}^{-1} [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2]$ which means that

$$\begin{aligned}
-b\lambda\phi_1(a_0, a_1, a_2) &= \int [\mathcal{D}^{-1}, a_0] [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] - a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\
&= -B_0\phi_3(a_0, a_1, a_2) - \phi_2(a_0, a_1, a_2).
\end{aligned}$$

Now the result follows from (i), (v).

(vii) $B\phi_2 = NB_0\phi_2 = -N(1 - \lambda)\phi_1 = 0$ since $N(1 - \lambda) = 0$.

$$\begin{aligned}
(viii) \quad B_0\phi_3(a_0, a_1, a_2) &= \int [\mathcal{D}, a_0] \mathcal{D}^{-1} [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\
&= \int a_0 \mathcal{D}^{-1} [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] - \int a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\
&= \int a_0 a_1 \mathcal{D}^{-1} [\mathcal{D}, a_2] - \int a_0 \mathcal{D}^{-1} a_1 [\mathcal{D}, a_2] - \int a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\
&= \int a_0 a_1 a_2 - \int a_0 a_1 \mathcal{D}^{-1} a_2 \mathcal{D} - \int a_0 \mathcal{D}^{-1} a_1 \mathcal{D} a_2 + \int a_0 \mathcal{D}^{-1} a_1 a_2 \mathcal{D} \\
&\quad - \int a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\
&= \int a_0 a_1 a_2 - a_2 \mathcal{D} a_1 a_0 \mathcal{D}^{-1} + a_1 a_2 \mathcal{D} a_0 \mathcal{D}^{-1} + a_2 \mathcal{D} a_0 a_1 \mathcal{D}^{-1} \\
&\quad - (a_0 \mathcal{D} a_1 a_2 \mathcal{D}^{-1} - a_0 \mathcal{D} a_1 \mathcal{D}^{-1} - a_0 a_1 \mathcal{D} a_2 \mathcal{D}^{-1} + a_0 a_1 a_2).
\end{aligned}$$

Expanding $(id + \lambda + \lambda^2)b'\phi_1(a_0, a_1, a_2)$, we recover previous expression.
(ix) Consequence of *(viii)*. □

2.2 Scale-invariant term of the spectral action

We know from [8] that the scale-invariant term of the action can be written as

$$\zeta_{D_A}(0) - \zeta_D(0) = -\int AD^{-1} + \frac{1}{2}\int AD^{-1}AD^{-1} - \frac{1}{3}\int AD^{-1}AD^{-1}AD^{-1}. \quad (5)$$

In fact, this action can be expressed in dimension 3 as contributions corresponding to tadpole and the Yang–Mills and Chern–Simons actions in dimension 4:

Proposition 2.3. *For any one-form A ,*

$$\zeta_{D_A}(0) - \zeta_D(0) = -\frac{1}{2}\int_{N\phi_1} A + \frac{1}{2}\int_{\phi_2} (dA + A^2) - \frac{1}{2}\int_{\phi_3} (AdA + \frac{2}{3}A^3). \quad (6)$$

To prove this, we calculate now each terms of the action.

Lemma 2.4. *For any one-form A , we have*

- (i)* $\int_{\phi_2} dA = \int_{B_0\phi_2} A = -\int_{\phi_1} A - \int_{\lambda\phi_1} A$.
- (ii)* $\int AD^{-1} = \int_{\phi_1} A = \frac{1}{2}\int_{N\phi_1} A - \frac{1}{2}\int_{\phi_2} dA$.
- (iii)* $\int AD^{-1}AD^{-1} = -\int_{\phi_3} AdA + \int_{\phi_2} A^2$.
- (iv)* $\int AD^{-1}AD^{-1}AD^{-1} = \int_{\phi_3} A^3$.

Proof. *(i)* and *(ii)* follow directly from Lemma 2.2 *(v)*.

(iii) With the shorthand $A = a_i db_i$ (summation on i)

$$\begin{aligned} \int AD^{-1}AD^{-1} &= \int a_0[\mathcal{D}, b_0]\mathcal{D}^{-1}a_1[\mathcal{D}, b_1]\mathcal{D}^{-1} \\ &= -\int_{\phi_3} AdA + \int a_0[\mathcal{D}, b_0]a_1b_1\mathcal{D}^{-1} - \int a_0[\mathcal{D}, b_0]a_1\mathcal{D}^{-1}b_1. \end{aligned}$$

We calculate further the remaining terms

$$\begin{aligned} \int a_0[\mathcal{D}, b_0]a_1b_1\mathcal{D}^{-1} - \int a_0[\mathcal{D}, b_0]a_1\mathcal{D}^{-1}b_1 &= \int a_0\mathcal{D}b_0a_1b_1\mathcal{D}^{-1} - \int a_0b_0\mathcal{D}a_1b_1\mathcal{D}^{-1} \\ &\quad - \int a_0\mathcal{D}b_0a_1\mathcal{D}^{-1}b_1 + \int a_0b_0\mathcal{D}a_1\mathcal{D}^{-1}b_1, \end{aligned}$$

which are compared with $\int_{\phi_2} A^2 = \int_{\phi_2} a_0(db_0)a_1db_1 = \int_{\phi_2} a_0d(b_0a_1)db_1 - a_0b_0da_1db_1$:

$$\begin{aligned} \int_{\phi_2} A^2 &= \int a_0[\mathcal{D}, b_0a_1]\mathcal{D}^{-1}[\mathcal{D}, b_1]\mathcal{D}^{-1} - \int a_0b_0[\mathcal{D}, a_1]\mathcal{D}^{-1}[\mathcal{D}, b_1]\mathcal{D}^{-1} \\ &= \int a_0\mathcal{D}b_0a_1b_1\mathcal{D}^{-1} - \int a_0\mathcal{D}b_0a_1\mathcal{D}^{-1}b_1 - \int a_0b_0a_1\mathcal{D}b_1\mathcal{D}^{-1} + \int a_0b_0a_1b_1 \\ &\quad - \int a_0b_0\mathcal{D}a_1b_1\mathcal{D}^{-1} + \int a_0b_0\mathcal{D}a_1\mathcal{D}^{-1}b_1 + \int a_0b_0a_1\mathcal{D}b_1\mathcal{D}^{-1} - \int a_0b_0a_1b_1 \\ &= \int a_0\mathcal{D}b_0a_1b_1\mathcal{D}^{-1} - \int b_1a_0\mathcal{D}b_0a_1\mathcal{D}^{-1} - \int a_0b_0\mathcal{D}a_1b_1\mathcal{D}^{-1} + \int b_1a_0b_0\mathcal{D}a_1\mathcal{D}^{-1}. \end{aligned}$$

(iv) Note that

$$\begin{aligned}
\int_{\phi_3} A^3 &= \int_{\phi_3} a_0(db_0)a_1(db_1)a_2db_2 = \int_{\phi_3} a_0d(b_0a_1)d(b_1a_2)db_2 - a_0b_0da_1d(b_1a_2)db_2 \\
&\quad - a_0d(b_0a_1b_1)d(a_2db_2 + a_0b_0d(a_1b_1)da_2db_2) \\
&= \int_{\phi_3} a_0[\mathcal{D}, b_0a_1]\mathcal{D}^{-1}[\mathcal{D}, b_1a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1} - a_0b_0[\mathcal{D}, a_1]\mathcal{D}^{-1}[\mathcal{D}, b_1a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1} \\
&\quad - a_0[\mathcal{D}, b_0a_1b_1]\mathcal{D}^{-1}[\mathcal{D}, a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1} + a_0b_0[\mathcal{D}, a_1b_1]\mathcal{D}^{-1}[\mathcal{D}, a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1}.
\end{aligned}$$

Summing up the first two terms and the last two ones gives

$$\int_{\phi_3} A^3 = \int_{\phi_3} a_0[\mathcal{D}, b_0]a_1\mathcal{D}^{-1}[\mathcal{D}, b_1a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1} - a_0[\mathcal{D}, b_0]a_1b_1\mathcal{D}^{-1}[\mathcal{D}, a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1}.$$

Using Remark 2.1, we can commute under the integral \mathcal{D}^{-1} with all $a \in \mathcal{A}$ and similarly

$$\int AD^{-1}AD^{-1}AD^{-1} = \int a_0[\mathcal{D}, b_0]a_1\mathcal{D}^{-1}[\mathcal{D}, b_1]a_2\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1}$$

which proves (iv). \square

We deduce Proposition 2.3 from (5) using the previous lemma.

3 The $SU_q(2)$ triple

3.1 The spectral triple

We briefly recall the main facts of the real spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, \mathcal{D})$ introduced in [21], see also [4, 5, 13].

The algebra:

Let $\mathcal{A} := \mathcal{A}(SU_q(2))$ be the $*$ -algebra generated polynomially by a and b , subject to the following commutation rules with $0 < q < 1$:

$$ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b, \quad a^*a + q^2b^*b = 1, \quad aa^* + bb^* = 1. \quad (7)$$

Lemma 3.1. *For any representation π of \mathcal{A} ,*

$$\begin{aligned}
Spect(\pi(bb^*)) &= \{0, q^{2k} \mid k \in \mathbb{N}\} \text{ or } \pi(b) = 0, \\
Spect(\pi(aa^*)) &= \{1, 1 - q^{2k} \mid k \in \mathbb{N}\} \text{ or } \pi(b) = 0 \text{ and } \pi(a) \text{ is a unitary.}
\end{aligned}$$

Proof. [31] Since $\{0\} \cup \sigma(\pi(aa^*)) = \{0\} \cup \sigma(\pi(a^*a))$, we get

$$\{1\} \cup B = \{1\} \cup q^2B \quad (8)$$

if $B := \sigma(\pi(bb^*))$. Since $0 \leq \pi(bb^*) \leq 1$, so B is a closed subset of $[0, 1]$. Assume $b \neq 0$. Let $s := \sup(B)$ and suppose $s \neq 1$. Then $s = q^2x$ where $x \in B$. Thus $s = q^2x < x \leq s$ gives $s = 0$ and the contradiction $b = 0$, thus $1 \in B$. Similar argument for $\inf(B)$ implies $0 \in B$. Let $C := \{0, q^{2k} \mid k \in \mathbb{N}\} \subset B$ and assume $B \setminus C \neq \emptyset$. Then $B \setminus C = (q^2B) \setminus C$ by (8) and this is equal to $q^2(B \setminus C)$ since $q^{-2} > 1$. If $s := \sup(B \setminus C)$, then $s = \lim_n(q^2x_n)$ where $x_n \in B \setminus C$ and $s = q^2 \lim_n x_n \leq q^2s$ implying $s = 0$. But $B \setminus C \subset \{0\}$ yields a contradiction, so $B \setminus C = \emptyset$. \square

This lemma is interesting since it shows the appearance of discreteness for $0 \leq q < 1$ while for $q = 1$, $SU_q(2) = SU(2) \simeq \mathbb{S}^3$ and the spectrum of the commuting operator $\pi(aa^*)$ and $\pi(bb^*)$ are equal to $[0, 1]$. Moreover, all foregoing results on noncommutative integrals will involve q^2 and not q .

Any element of \mathcal{A} can be uniquely decomposed as a linear combination of terms of the form $a^\alpha b^\beta b^{*\gamma}$ where $\alpha \in \mathbb{Z}$, $\beta, \gamma \in \mathbb{N}$, with the convention

$$a^{-|\alpha|} := a^{*|\alpha|}.$$

The spinorial Hilbert space:

$\mathcal{H} = \mathcal{H}^\uparrow \oplus \mathcal{H}^\downarrow$ has an orthonormal basis consisting of vectors $|j\mu n^\uparrow\rangle$ with $j = 0, \frac{1}{2}, 1, \dots$, $\mu = -j, \dots, j$ and $n = -j^+, \dots, j^+$, together with $|j\mu n^\downarrow\rangle$ for $j = \frac{1}{2}, 1, \dots$, $\mu = -j, \dots, j$ and $n = -j^-, \dots, j^-$ (here $x^\pm := x \pm \frac{1}{2}$).

It is convenient to use a vector notation, setting:

$$|j\mu n\rangle\rangle := \begin{pmatrix} |j\mu n^\uparrow\rangle \\ |j\mu n^\downarrow\rangle \end{pmatrix} \quad (9)$$

and with the convention that the lower component is zero when $n = \pm(j + \frac{1}{2})$ or $j = 0$.

The representation π and its approximate $\underline{\pi}$:

It is known that representation theory of $SU_q(2)$ is similar to that of $SU(2)$ [39]. The representation π given in [21] is:

$$\begin{aligned} \pi(a) |j\mu n\rangle\rangle &:= \alpha_{j\mu n}^+ |j^+ \mu^+ n^+\rangle\rangle + \alpha_{j\mu n}^- |j^- \mu^+ n^+\rangle\rangle, \\ \pi(b) |j\mu n\rangle\rangle &:= \beta_{j\mu n}^+ |j^+ \mu^+ n^-\rangle\rangle + \beta_{j\mu n}^- |j^- \mu^+ n^-\rangle\rangle, \\ \pi(a^*) |j\mu n\rangle\rangle &:= \tilde{\alpha}_{j\mu n}^+ |j^+ \mu^- n^-\rangle\rangle + \tilde{\alpha}_{j\mu n}^- |j^- \mu^- n^-\rangle\rangle, \\ \pi(b^*) |j\mu n\rangle\rangle &:= \tilde{\beta}_{j\mu n}^+ |j^+ \mu^- n^+\rangle\rangle + \tilde{\beta}_{j\mu n}^- |j^- \mu^- n^+\rangle\rangle \end{aligned} \quad (10)$$

where

$$\begin{aligned} \alpha_{j\mu n}^+ &:= \sqrt{q^{\mu+n-1/2} [j + \mu + 1]} \begin{pmatrix} q^{-j-1/2} \frac{\sqrt{[j+n+3/2]}}{[2j+2]} & 0 \\ q^{1/2} \frac{\sqrt{[j-n+1/2]}}{[2j+1][2j+2]} & q^{-j} \frac{\sqrt{[j+n+1/2]}}{[2j+1]} \end{pmatrix}, \\ \alpha_{j\mu n}^- &:= \sqrt{q^{\mu+n+1/2} [j - \mu]} \begin{pmatrix} q^{j+1} \frac{\sqrt{[j-n+1/2]}}{[2j+1]} & -q^{1/2} \frac{\sqrt{[j+n+1/2]}}{[2j][2j+1]} \\ 0 & q^{j+1/2} \frac{\sqrt{[j-n-1/2]}}{[2j]} \end{pmatrix}, \\ \beta_{j\mu n}^+ &:= \sqrt{q^{\mu+n-1/2} [j + \mu + 1]} \begin{pmatrix} \frac{\sqrt{[j-n+3/2]}}{[2j+2]} & 0 \\ -q^{-j-1} \frac{\sqrt{[j+n+1/2]}}{[2j+1][2j+2]} & q^{-1/2} \frac{\sqrt{[j-n+1/2]}}{[2j+1]} \end{pmatrix}, \\ \beta_{j\mu n}^- &:= \sqrt{q^{\mu+n-1/2} [j - \mu]} \begin{pmatrix} -q^{-1/2} \frac{\sqrt{[j+n+1/2]}}{[2j+1]} & -q^j \frac{\sqrt{[j-n+1/2]}}{[2j][2j+1]} \\ 0 & -\frac{\sqrt{[j+n-1/2]}}{[2j]} \end{pmatrix} \end{aligned}$$

with $\tilde{\alpha}_{j\mu n}^\pm := (\alpha_{j^\mp \mu^\mp n^\mp}^\mp)^*$, $\tilde{\beta}_{j\mu n}^\pm := (\beta_{j^\mp \mu^\mp n^\mp}^\mp)^*$ and with the q -number of $\alpha \in \mathbb{R}$ be defined as

$$[\alpha] := \frac{q^\alpha - q^{-\alpha}}{q - q^{-1}}.$$

For the purpose of this paper it is sufficient to use the approximate spinorial $*$ -representation $\underline{\pi}$ of $SU_q(2)$ presented in [21, 38] instead of the full spinorial one π .

This approximate representation is

$$\underline{\pi}(a) := a_+ + a_-, \quad \underline{\pi}(b) := b_+ + b_-$$

with the following definitions:

$$\begin{aligned} a_+ |j\mu n\rangle\rangle &:= q_{j_+ + \mu^+} \begin{pmatrix} q_{j_+ + n^+ + 1} & 0 \\ 0 & q_{j_+ + n^+} \end{pmatrix} |j^+ \mu^+ n^+\rangle\rangle, \\ a_- |j\mu n\rangle\rangle &:= q^{2j + \mu + n + \frac{1}{2}} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^- \mu^+ n^+\rangle\rangle, \\ b_+ |j\mu n\rangle\rangle &:= q^{j + n - \frac{1}{2}} q_{j_+ + \mu^+} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^+ \mu^+ n^-\rangle\rangle, \\ b_- |j\mu n\rangle\rangle &:= -q^{j + \mu} \begin{pmatrix} q_{j_+ + n^+} & 0 \\ 0 & q_{j_- + n^-} \end{pmatrix} |j^- \mu^+ n^-\rangle\rangle. \end{aligned} \quad (11)$$

All disregarded terms are trace-class and do not influence residue calculations. More precisely, $\pi(x) - \underline{\pi}(x) \in \mathcal{K}_q$ where \mathcal{K}_q is the principal ideal generated by the operators

$$J_q |j\mu n\rangle\rangle := q^j |j\mu n\rangle\rangle. \quad (12)$$

Actually, \mathcal{K}_q is independent of q and is contained in all ideals of operators such that $\mu_n = o(n^{-\alpha})$ (infinitesimal of order α) for any $\alpha > 0$, and $\mathcal{K}_q \subset OP^{-\infty}$.

We define the alternative orthonormal basis $v_{m,l}^{j\uparrow}$ and $v_{m,l}^{j\downarrow}$ and the vector notation

$$v_{m,l}^j := \begin{pmatrix} v_{m,l}^{j\uparrow} \\ v_{m,l}^{j\downarrow} \end{pmatrix} \text{ where } v_{m,l}^{j\uparrow} := |j, m - j, l - j^+, \uparrow\rangle, \quad v_{m,l}^{j\downarrow} := |j, m - j, l - j^-, \downarrow\rangle.$$

Here $j \in \frac{1}{2}\mathbb{N}$, $0 \leq m \leq 2j$, $0 \leq l \leq 2j + 1$ and $v_{m,l}^{j\downarrow}$ is zero whenever $j = 0$ or $l = 2j$ or $2j + 1$. The interest is that now, the operators a_{\pm} and b_{\pm} satisfy the simpler relations

$$\begin{aligned} a_+ v_{m,l}^j &= q_{m+1} q_{l+1} v_{m+1,l+1}^{j^+}, & a_- v_{m,l}^j &= q^{m+l+1} v_{m,l}^{j^-}, \\ b_+ v_{m,l}^j &= q^l q_{m+1} v_{m+1,l}^{j^+}, & b_- v_{m,l}^j &= -q^m q_l v_{m,l-1}^{j^-}. \end{aligned} \quad (13)$$

Thus

$$\begin{aligned} a_+^* v_{m,l}^j &= q_m q_l v_{m-1,l-1}^{j^-}, & a_-^* v_{m,l}^j &= q^{m+l+1} v_{m,l}^{j^+}, \\ b_+^* v_{m,l}^j &= q^l q_m v_{m-1,l}^{j^-}, & b_-^* v_{m,l}^j &= -q^m q_{l+1} v_{m,l+1}^{j^+}. \end{aligned} \quad (14)$$

Moreover, we have

$$\begin{aligned} a_- a_+ &= q^2 a_+ a_-, & b_- b_+ &= q^2 b_+ b_-, & b_+ a_+ &= q a_+ b_+, & b_- a_- &= q a_- b_-, \\ a_-^* a_+ &= q^2 a_+ a_-^*, & a_-^* a_- &= a_- a_-^*, & a_-^* b_+ &= q b_+ a_-^*, & a_-^* b_- &= q b_- a_-^*, \\ a_+^* a_- &= q^2 a_- a_+^*, & b_-^* b_+ &= b_+ b_-^*, & b_-^* a_+ &= q a_+ b_-^*, & a_- b_+ &= q b_+ a_-. \end{aligned} \quad (15)$$

Note for instance that

$$\begin{aligned} a_+ a_+^* v_{m,l}^j &= q_m^2 q_l^2 v_{m,l}^j, & a_+^* a_+ v_{m,l}^j &= q_{m+1}^2 q_{l+1}^2 v_{m,l}^j, \\ b_+ b_+^* v_{m,l}^j &= q^{2l} q_m^2 v_{m,l}^j, & b_+^* b_+ v_{m,l}^j &= q^{2l} q_{m+1}^2 v_{m,l}^j, \end{aligned}$$

so applied to $v_{m,l}^j$, we get the first relation (and similarly for the others)

$$a_+^* a_+ - q^2 a_+ a_+^* + q^2 (b_+^* b_+ - b_+ b_+^*) = 1 - q^2, \quad (16)$$

$$a_+ a_+^* + a_- a_-^* + b_+ b_+^* + b_- b_-^* = 1, \quad (17)$$

$$a_+^* a_+ + a_-^* a_- + q^2 (b_+^* b_+ + b_-^* b_-) = 1, \quad (18)$$

$$a_-^* a_- - q^2 a_- a_-^* + q^2 b_-^* b_- - q^2 b_- b_-^* = 0, \quad (19)$$

$$a_+ a_-^* + b_-^* b_+ = 0, \quad a_-^* a_+ + q^2 b_-^* b_+ = 0, \quad (20)$$

$$a_- a_+^* + b_+^* b_- = 0, \quad a_+^* a_- + q^2 b_+^* b_- = 0, \quad (21)$$

$$b_+ b_+^* - b_+^* b_+ + b_- b_-^* - b_-^* b_- = 0, \quad (22)$$

$$q a_+ b_- - b_- a_+ + q a_- b_+ - b_+ a_- = 0. \quad (23)$$

And two others:

Note that we also use two other infinite dimensional *-representations π_{\pm} of \mathcal{A} on $\ell^2(\mathbb{N})$ defined as follows on the orthonormal basis $\{\varepsilon_n : n \in \mathbb{N}\}$ of $\ell^2(\mathbb{N})$ by

$$\begin{aligned} \pi_{\pm}(a) \varepsilon_n &:= q_{n+1} \varepsilon_{n+1}, & \pi_{\pm}(b) \varepsilon_n &:= \pm q^n \varepsilon_n, \\ q_n &:= \sqrt{1 - q^{2n}}. \end{aligned} \quad (24)$$

These representations are irreducible but not faithful since for instance $\pi_{\pm}(b - b^*) = 0$.

The Dirac operator:

It is chosen the same as in the classical case of a 3-sphere with the round metric:

$$\mathcal{D} |j\mu n\rangle := \begin{pmatrix} 2j + \frac{3}{2} & 0 \\ 0 & -2j - \frac{1}{2} \end{pmatrix} |j\mu n\rangle, \quad (25)$$

which means, with our convention, that $\mathcal{D} v_{ml}^j = \begin{pmatrix} 2j + \frac{3}{2} & 0 \\ 0 & -2j - \frac{1}{2} \end{pmatrix} v_{ml}^j$. Note that this operator is asymptotically diagonal with linear spectrum and

the eigenvalues $2j + \frac{1}{2}$ for $j \in \frac{1}{2}\mathbb{N}$, have multiplicities $(2j + 1)(2j + 2)$,
the eigenvalues $-(2j + \frac{1}{2})$ for $j \in \frac{1}{2}\mathbb{N}^*$, have multiplicities $2j(2j + 1)$.

So this Dirac operator coincide exactly with the classical one on the 3-sphere (see [1, 32]) with a gap around 0.

Let $\mathcal{D} = F|\mathcal{D}|$ be the polar decomposition of \mathcal{D} , thus

$$|\mathcal{D}| |j\mu n\rangle = \begin{pmatrix} d_j + & 0 \\ 0 & d_j \end{pmatrix} |j\mu n\rangle, \quad d_j := 2j + \frac{1}{2}, \quad (26)$$

$$F |j\mu n\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} |j\mu n\rangle, \quad (27)$$

and it follows from (11) and (27) that

$$F \text{ commutes with } a_{\pm}, b_{\pm}. \quad (28)$$

The reality operator:

This antilinear operator J is defined on the basis of \mathcal{H} by

$$J |j, \mu, n, \uparrow\rangle := i^{2(2j+\mu+n)} |j, -\mu, -n, \uparrow\rangle, \quad J |j, \mu, n, \downarrow\rangle := i^{2(2j-\mu-n)} |j, -\mu, -n, \downarrow\rangle \quad (29)$$

thus it satisfies

$$J^{-1} = -J = J^* \text{ and } \mathcal{D}J = J\mathcal{D},$$

$$Jv_{m,l}^{j\uparrow} = i^{2(m+l)-1}v_{2j-m,2j+1-l}^{j\uparrow}, \quad Jv_{m,l}^{j\downarrow} = i^{-2(m+l)+1}v_{2j-m,2j-1-l}^{j\downarrow}.$$

The Hopf map r

For the explicit calculations of residues, we need a *-homomorphism $r : X \rightarrow \pi_+(\mathcal{A}) \otimes \pi_-(\mathcal{A})$ defined by the tensor product in the sense of Hopf algebras of representations π_+ and π_- :

$$\begin{aligned} r(a_+) &:= \pi_+(a) \otimes \pi_-(a), & r(a_-) &:= -q\pi_+(b) \otimes \pi_-(b^*), \\ r(b_+) &:= -\pi_+(a) \otimes \pi_-(b), & r(b_-) &:= -\pi_+(b) \otimes \pi_-(a^*). \end{aligned} \quad (30)$$

In fact, \mathcal{A} is a Hopf *-algebra under the coproduct $\Delta(a) := a \otimes a - qb \otimes b^*$, $\Delta(b) := a \otimes b + b \otimes a^*$. These homomorphisms appeared in [39] with the translation $\alpha \leftrightarrow a^*$, $\gamma \leftrightarrow -b$. In particular, if $U := \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}$ is the canonical generator of the $K_1(\mathcal{A})$ -group $(\Delta a, \Delta b) = (a, b) \dot{\otimes} U$ where the last $\dot{\otimes}$ means the matrix product of tensors of components.

The grading:

According to the shift $j \rightarrow j^\pm$ appearing in formulae (13), (14), we get a \mathbb{Z} -grading on X defined by the degree +1 on a_+, b_+, a_-^*, b_-^* and -1 on a_-, b_-, a_+^*, b_+^* .

Any operator $T \in X$ can be (uniquely) decomposed as $T = \sum_{j \in \mathbb{Z}} T_j$ where T_j is homogeneous of degree j .

For $T \in X$, T° will denote the 0-degree part of T for this grading and by a slight abuse of notations, we write $r(T)^\circ$ instead of $r(T^\circ)$.

The symbol map:

We also use the *-homomorphism $\sigma : \pi_\pm(\mathcal{A}) \rightarrow C^\infty(S^1)$ defined for $z \in S^1$ on the generators by

$$\sigma(\pi_\pm(a))(z) := z, \quad \sigma(\pi_\pm(a^*))(z) := \bar{z}, \quad \sigma(\pi_\pm(b))(z) = \sigma(\pi_\pm(b^*))(z) := 0.$$

The application $(\sigma \otimes \sigma) \circ r$ is defined on X (and so on \mathcal{B}) with values in $C^\infty(S^1) \otimes C^\infty(S^1)$.

We define

$$dT := [\mathcal{D}, T] \text{ and } \delta(T) := [|\mathcal{D}|, T].$$

Lemma 3.2. a_\pm, b_\pm are bounded operators on \mathcal{H} such that for all $p \in \mathbb{N}$,

- (i) $\delta(a_\pm) = \pm a_\pm, \quad \delta(b_\pm) = \pm b_\pm,$
- (ii) $\delta^p(\underline{\pi}(a)) = a_+ + (-1)^p a_-, \quad \delta^p(\underline{\pi}(b)) = b_+ + (-1)^p b_-,$
- (iii) $\delta(a_\pm^p) = \pm p a_\pm^{p-1}, \quad \delta(b_\pm^p) = \pm p b_\pm^{p-1}.$

Proof. (i) By definition, $a_\pm |j\mu n\rangle = \begin{pmatrix} \alpha_\pm & 0 \\ 0 & \beta_\pm \end{pmatrix} |j^\pm \mu^+ n^+\rangle$ where the numbers α_\pm and β_\pm depend on j, μ, n and q , so we get by (26)

$$\begin{aligned} \delta(a_\pm) |j\mu n\rangle &= \begin{pmatrix} (d_{j^\pm})^{\alpha_\pm} & 0 \\ 0 & d_{j^\pm} \beta_\pm \end{pmatrix} |j^\pm \mu^+ n^+\rangle - \begin{pmatrix} (d_{j^\pm})^{\alpha_\pm} & 0 \\ 0 & d_{j^\pm} \beta_\pm \end{pmatrix} |j^\pm \mu^+ n^+\rangle \\ &= \begin{pmatrix} \pm \alpha_\pm & 0 \\ 0 & \pm \beta_\pm \end{pmatrix} |j^\pm \mu^+ n^+\rangle = \pm a_\pm |j\mu n\rangle \end{aligned}$$

and similar proofs for b_\pm .

(ii) and (iii) are straightforward consequences of (i) and definition of $\underline{\pi}$. □

We note

\mathcal{B} the $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the operators in $\delta^k(\pi(\mathcal{A}))$ for all $k \in \mathbb{N}$,
 $\Psi_0^0(\mathcal{A})$ the algebra generated by $\delta^k(\pi(\mathcal{A}))$ and $\delta^k([\mathcal{D}, \pi(\mathcal{A})])$ for all $k \in \mathbb{N}$,
 X the $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ algebraically generated by the set $\{a_\pm, b_\pm\}$.

Remark 3.3. By Lemma 3.2, we see that, modulo $OP^{-\infty}$, X is equal to \mathcal{B} and in particular contains $\pi(\mathcal{A})$.

Using (28), we get that $\mathcal{B} \subset \Psi_0^0(\mathcal{A}) \subset$ algebra generated by \mathcal{B} and $\mathcal{B}F$.

Note that, despite the last inclusion, F is not a priori in $\Psi_0^0(\mathcal{A})$.

3.2 The noncommutative integrals

Recall that for any pseudodifferential operator T , $\mathfrak{f}T := \operatorname{Res}_{s=0} \zeta_{\mathcal{D}}^T(s)$ where $\zeta_{\mathcal{D}}^T(s) := \operatorname{Tr}(T|\mathcal{D}|^{-s})$.

Theorem 3.4. *The dimension spectrum (without reality structure given by J) of the spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$ is simple and equal to $\{1, 2, 3\}$.*

Moreover, the corresponding residues for $T \in \mathcal{B}$ are

$$\begin{aligned} \mathfrak{f} T|\mathcal{D}|^{-3} &= 2(\tau_1 \otimes \tau_1)(r(T)^\circ), \\ \mathfrak{f} T|\mathcal{D}|^{-2} &= 2(\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(r(T)^\circ), \\ \mathfrak{f} T|\mathcal{D}|^{-1} &= (2\tau_0 \otimes \tau_0 - \frac{1}{2}\tau_1 \otimes \tau_1)(r(T)^\circ), \\ \mathfrak{f} F T|\mathcal{D}|^{-3} &= 0, \\ \mathfrak{f} F T|\mathcal{D}|^{-2} &= 0, \\ \mathfrak{f} F T|\mathcal{D}|^{-1} &= (\tau_0 \otimes \tau_1 - \tau_1 \otimes \tau_0)(r(T)^\circ), \end{aligned}$$

where the functionals τ_0, τ_1 are defined for $x \in \pi_\pm(\mathcal{A})$ by

$$\tau_0(x) := \lim_{N \rightarrow \infty} (\operatorname{Tr}_N x - (N+1)\tau_1(x)), \quad \tau_1(x) := \frac{1}{2\pi} \int_0^{2\pi} \sigma(x)(e^{i\theta}) d\theta,$$

with $\operatorname{Tr}_N x = \sum_{n=0}^N \langle \varepsilon_n, x \varepsilon_n \rangle$.

Proof. Consequence of [38, Theorem 4.1 and (4.3)]. □

Remark 3.5. *Since F is not in \mathcal{B} , the equation of Theorem 3.4 are not valid for all $T \in \Psi_0^0(\mathcal{A})$. But when $T \in \Psi_0^0(\mathcal{A})$, $\mathfrak{f} T|\mathcal{D}|^{-k} = 0$ for $k \notin \{1, 2, 3\}$ since the dimension spectrum is $\{1, 2, 3\}$ [38].*

Compared to [38] where we had

$$\tau_0^\uparrow(x) := \lim_{N \rightarrow \infty} \operatorname{Tr}_N x - (N + \frac{3}{2})\tau_1(x), \quad \tau_0^\downarrow(x) := \lim_{N \rightarrow \infty} \operatorname{Tr}_N x - (N + \frac{1}{2})\tau_1(x),$$

we replaced them with τ_0 :

$$\tau_0^\uparrow = \tau_0 - \frac{1}{2}\tau_1, \quad \tau_0^\downarrow = \tau_0 + \frac{1}{2}\tau_1.$$

Note that τ_1 is a trace on $\pi_\pm(\mathcal{A})$ such that $\tau_1(1) = 1$, while τ_0 is not since $\tau_0(1) = 0$ and

$$\tau_0(\pi_\pm(aa^*)) = \lim_{N \rightarrow \infty} \sum_{n=0}^{\infty} (1 - q^{2n}) - (N + 1) = -\frac{1}{1 - q^2}, \quad (31)$$

so, because of the shift, the replacement $a \leftrightarrow a^*$ gives

$$\tau_0(\pi_\pm(a^*a)) = q^2 \tau_0(\pi_\pm(aa^*)). \quad (32)$$

3.3 The tadpole

Lemma 3.6. *For $SU_q(2)$, the condition of the vanishing tadpole (see [15]) is not satisfied.*

Proof. For example, an explicit calculation gives $\int \pi(b)[\mathcal{D}, \pi(b^*)]\mathcal{D}^{-1} = \frac{2}{1 - q^2}$:

Let $x, y \in \underline{\pi}(\mathcal{A})$. Since $[F, x] = 0$, we have

$$\int x[\mathcal{D}, y]\mathcal{D}^{-1} = \int x\delta(y)|\mathcal{D}|^{-1} = \tau'(r(x\delta(y)))^0$$

where $\tau' := 2\tau_0 \otimes \tau_0 - \frac{1}{2}\tau_1 \otimes \tau_1$.

By Lemma 3.2, $\underline{\pi}(b)\delta(\underline{\pi}(b^*)) = (b_+ + b_-)((b_-)^* - (b_+)^*) = -b_+b_+^* + b_-b_-^* + b_+b_-^* - b_-b_+^*$. Since only the first two terms have degree 0, we get, using the formulae from Theorem 3.4

$$\begin{aligned} \tau'(r(-b_+b_+^*)) &= -\tau'(\pi_+(aa^*) \otimes \pi_-(bb^*)) \\ &= -2\tau_0(\pi_+(aa^*))\tau_0(\pi_-(bb^*)) + \frac{1}{2}\tau_1(\pi_+(aa^*))\tau_1(\pi_-(bb^*)) \end{aligned}$$

and $\tau_1(\pi_+(aa^*)) = \frac{1}{2\pi} \int_0^{2\pi} 1 d\theta = 1$, $\tau_1(\pi_-(bb^*)) = 0$. Similarly, using (32)

$$\tau'(r(-b_-b_-^*)) = 2\tau_0(\pi_+(bb^*))\tau_0(\pi_-(a^*a)) = 2q^2\tau_0(\pi_-(aa^*))\tau_0(\pi_+(bb^*)).$$

Since $\tau_0(\pi_\pm(bb^*)) = \text{Tr}(\pi_\pm(bb^*)) = \sum_{n=0}^{\infty} q^{2n} = \frac{1}{1 - q^2}$ and (31),

$$\int \pi(b)[\mathcal{D}, \pi(b^*)]\mathcal{D}^{-1} = 2\frac{1}{1 - q^2} \frac{1}{1 - q^2} + 2q^2 \frac{-1}{1 - q^2} \frac{1}{1 - q^2} = \frac{2}{1 - q^2}. \quad \square$$

In particular the pairing of the tadpole cyclic cocycle ϕ_1 with the generator of K_1 -group is nontrivial:

Remark 3.7. *Other examples: with the shortcut x instead of $\underline{\pi}(x)$,*

$$\begin{aligned} (\tau_1 \otimes \tau_1) r(a\delta(a^*)^\circ) &= -1, & (\tau_1 \otimes \tau_1) r(a^*\delta(a)^\circ) &= 1, \\ (\tau_0 \otimes \tau_0) r(a\delta(a^*)^\circ) &= \frac{1}{q^2 - 1}, & (\tau_0 \otimes \tau_0) r(a^*\delta(a)^\circ) &= \frac{q^2}{q^2 - 1}, \\ \int a\delta(a^*)|\mathcal{D}|^{-1} &= \frac{q^2 + 3}{2(q^2 - 1)}, & \int a^*\delta(a)|\mathcal{D}|^{-1} &= \frac{3q^2 + 1}{2(q^2 - 1)}, \\ \int b\delta(b)|\mathcal{D}|^{-1} &= 0, & \int b^*\delta(b^*)|\mathcal{D}|^{-1} &= 0, \\ \int b\delta(b^*)|\mathcal{D}|^{-1} &= \frac{-2}{q^2 - 1}, & \int b^*\delta(b)|\mathcal{D}|^{-1} &= \frac{-2}{q^2 - 1}. \end{aligned}$$

In particular, $N\Phi_1$ does not vanish on 1-forms since $\int_{N\Phi_1} ada^* = N\Phi_1(a, a^*) = -1$.

Let U be the canonical generator of the $K_1(\mathcal{A})$ -group, $U = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}$ acting on $\mathcal{H} \otimes \mathbb{C}^2$. Then for $A_U := \sum_{k,l=1}^2 \pi(U_{kl}) d\pi(U^*_{kl})$, using above remark, $\int_{\phi_1} A_U = -2$ as obtained in [38, page 391]: in fact, with $P := \frac{1}{2}(1 + F)$,

$$\psi_1(U, U^*) := 2 \sum_{k,l} \int U_{kl} \delta(U^*_{kl}) P |\mathcal{D}|^{-1} - \int U_{kl} \delta^2(U^*_{kl}) P |\mathcal{D}|^{-2} + \frac{2}{3} \int U_{kl} \delta^3(U^*_{kl}) P |\mathcal{D}|^{-3}$$

satisfies $\psi_1(U, U^*) = 2 \sum_{k,l} \int U_{kl} \delta(U^*_{kl}) P |\mathcal{D}|^{-1} = \int_{\phi_1} A_U$.

4 Reality operator and spectral action on $SU_q(2)$

4.1 Spectral action in dimension 3 with $[F, \mathcal{A}] \in OP^{-\infty}$

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ a be real spectral triple of dimension 3. Assume that $[F, \mathcal{A}] \in OP^{-\infty}$, where $F := \mathcal{D} |\mathcal{D}|^{-1}$ (we suppose \mathcal{D} invertible). Let \mathbb{A} be a selfadjoint one form, so \mathbb{A} is of the form $\mathbb{A} = \sum_i a_i db_i$ where $a_i, b_i \in \mathcal{A}$.

Thus, $\mathbb{A} \simeq AF \pmod{OP^{-\infty}}$ where $A := \sum_i a_i \delta(b_i)$ is the δ -one-form associated to \mathbb{A} . Note that A and F commute modulo $OP^{-\infty}$.

We define

$$\begin{aligned} D_{\mathbb{A}} &:= \mathcal{D}_{\mathbb{A}} + P_{\mathbb{A}}, & P_{\mathbb{A}} &\text{ the projection on } \text{Ker } \mathcal{D}_{\mathbb{A}}, \\ \mathcal{D}_{\mathbb{A}} &:= \mathcal{D} + \tilde{\mathbb{A}}, & \tilde{\mathbb{A}} &:= \mathbb{A} + J\mathbb{A}J^{-1}. \end{aligned}$$

Theorem 4.1. *The coefficients of the full spectral action (with reality operator) on any real spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension 3 such that $[F, \mathcal{A}] \in OP^{-\infty}$ are*

$$\begin{aligned} (i) \quad & \int |D_{\mathbb{A}}|^{-3} = \int |\mathcal{D}|^{-3}. \\ (ii) \quad & \int |D_{\mathbb{A}}|^{-2} = \int |\mathcal{D}|^{-2} - 4 \int A |\mathcal{D}|^{-3}. \\ (iii) \quad & \int |D_{\mathbb{A}}|^{-1} = \int |\mathcal{D}|^{-1} - 2 \int A |\mathcal{D}|^{-2} + 2 \int A^2 |\mathcal{D}|^{-3} + 2 \int AJAJ^{-1} |\mathcal{D}|^{-3}. \\ (iv) \quad & \zeta_{D_{\mathbb{A}}}(0) = \zeta_{\mathcal{D}}(0) - 2 \int A |\mathcal{D}|^{-1} + \int A(A + JAJ^{-1}) |\mathcal{D}|^{-2} + \int \delta(A)(A + JAJ^{-1}) |\mathcal{D}|^{-3} \\ & \quad - \frac{2}{3} \int A^3 |\mathcal{D}|^{-3} - 2 \int A^2 JAJ^{-1} |\mathcal{D}|^{-3}. \end{aligned}$$

Proof. (i) We apply [22, Proposition 4.9].

(ii) By [22, Lemma 4.10 (i)], we have $\int |D_{\mathbb{A}}|^{-2} = \int |\mathcal{D}|^{-2} - \int (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2) |\mathcal{D}|^{-4}$. By the trace property of the noncommutative integral and the fact that $\tilde{\mathbb{A}}^2 |\mathcal{D}|^{-4}$ is trace-class, we get $\int |D_{\mathbb{A}}|^{-2} = \int |\mathcal{D}|^{-2} - 2 \int \tilde{\mathbb{A}}\mathcal{D} |\mathcal{D}|^{-4} = \int |\mathcal{D}|^{-2} - 4 \int \mathbb{A}\mathcal{D} |\mathcal{D}|^{-4}$. Since $\mathbb{A}\mathcal{D} \sim A |\mathcal{D}| \pmod{OP^{-\infty}}$, we get the result.

(iii) By [22, Lemma 4.10 (ii)], we have

$$\int |D_{\mathbb{A}}|^{-1} = \int |\mathcal{D}|^{-1} - \frac{1}{2} \int (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2) |\mathcal{D}|^{-3} + \frac{3}{8} \int (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2)^2 |\mathcal{D}|^{-5}.$$

Following arguments of (ii), we get

$$\begin{aligned}\int (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2)|\mathcal{D}|^{-3} &= 4 \int A|\mathcal{D}|^{-2} + 2 \int A^2|\mathcal{D}|^{-3} + 2 \int AJAJ^{-1}|\mathcal{D}|^{-3}, \\ \int (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2)^2|\mathcal{D}|^{-5} &= 8 \int A^2|\mathcal{D}|^{-3} + 8 \int AJAJ^{-1}|\mathcal{D}|^{-3},\end{aligned}$$

and the result follows.

(iv) By [22, Lemma 4.5] gives $\zeta_{D_{\mathbb{A}}}(0) = \sum_{j=1}^3 \frac{(-1)^j}{j} \int (\tilde{\mathbb{A}}\mathcal{D}^{-1})^j$.

Moreover, we have $\int \tilde{\mathbb{A}}\mathcal{D}^{-1} = 2 \int A|\mathcal{D}|^{-1}$ and $\int (\tilde{\mathbb{A}}\mathcal{D}^{-1})^2 = 2 \int (A|\mathcal{D}|^{-1})^2 + 2 \int A|\mathcal{D}|^{-1}AJAJ^{-1}|\mathcal{D}|^{-1}$. Since $\delta(A) \in OP^0$, we can check that $\int (A|\mathcal{D}|^{-1})^2 = \int A^2|\mathcal{D}|^{-2} + \int \delta(A)A|\mathcal{D}|^{-3}$ and, with the same argument, that $\int A|\mathcal{D}|^{-1}AJAJ^{-1}|\mathcal{D}|^{-1} = \int AJAJ^{-1}|\mathcal{D}|^{-2} + \int \delta(A)AJAJ^{-1}|\mathcal{D}|^{-3}$. Thus, we get

$$\int (\tilde{\mathbb{A}}\mathcal{D}^{-1})^2 = 2 \int A(A + AJAJ^{-1})|\mathcal{D}|^{-2} + 2 \int \delta(A)(A + AJAJ^{-1})|\mathcal{D}|^{-3}. \quad (33)$$

The third term to be computed is

$$\int (\tilde{\mathbb{A}}\mathcal{D}^{-1})^3 = 2 \int (A|\mathcal{D}|^{-1})^3 + 4 \int (A|\mathcal{D}|^{-1})^2AJAJ^{-1}|\mathcal{D}|^{-1} + 2 \int A|\mathcal{D}|^{-1}AJAJ^{-1}|\mathcal{D}|^{-1}A|\mathcal{D}|^{-1}.$$

Any operator in OP^{-4} being trace-class here, we get

$$\int (\tilde{\mathbb{A}}\mathcal{D}^{-1})^3 = 2 \int A^3|\mathcal{D}|^{-3} + 4 \int A^2AJAJ^{-1}|\mathcal{D}|^{-3} + 2 \int AJAJ^{-1}A|\mathcal{D}|^{-3}. \quad (34)$$

Since $\int AJAJ^{-1}A|\mathcal{D}|^{-3} = \int A^2AJAJ^{-1}|\mathcal{D}|^{-3}$ by trace property and the fact that $\delta(A) \in OP^0$, the result follows then from (33) and (34). \square

Corollary 4.2. *For the spectral action of \mathbb{A} without the reality operator (i.e. $\mathcal{D}_{\mathbb{A}} = \mathcal{D} + \mathbb{A}$), we get*

$$\begin{aligned}\int |D_{\mathbb{A}}|^{-2} &= \int |\mathcal{D}|^{-2} - 2 \int A|\mathcal{D}|^{-3}, \\ \int |D_{\mathbb{A}}|^{-1} &= \int |\mathcal{D}|^{-1} - \int A|\mathcal{D}|^{-2} + \int A^2|\mathcal{D}|^{-3}, \\ \zeta_{D_{\mathbb{A}}}(0) &= \zeta_{\mathcal{D}}(0) - \int A|\mathcal{D}|^{-1} + \frac{1}{2} \int A^2|\mathcal{D}|^{-2} + \frac{1}{2} \int \delta(A)A|\mathcal{D}|^{-3} - \frac{1}{3} \int A^3|\mathcal{D}|^{-3}.\end{aligned}$$

4.2 Spectral action on $SU_q(2)$: main result

On $SU_q(2)$, since F commutes with a_{\pm} and b_{\pm} , the previous lemma can be used for the spectral action computation.

Here is the main result of this section

Theorem 4.3. *In the full spectral action (4) (with the reality operator) of $SU_q(2)$ for a one-form*

\mathbb{A} and A its associated δ -one-form, the coefficients are:

$$\begin{aligned} \int |D_{\mathbb{A}}|^{-3} &= 2, \\ \int |D_{\mathbb{A}}|^{-2} &= -4 \int A|\mathcal{D}|^{-3}, \\ \int |D_{\mathbb{A}}|^{-1} &= -\frac{1}{2} + 2\left(\int A^2|\mathcal{D}|^{-3} - \int A|\mathcal{D}|^{-2}\right) + \left|\int A|\mathcal{D}|^{-3}\right|^2, \\ \zeta_{D_{\mathbb{A}}}(0) &= -2 \int A|\mathcal{D}|^{-1} + \int A^2|\mathcal{D}|^{-2} - \frac{2}{3} \int A^3|\mathcal{D}|^{-3} \\ &\quad + \overline{\int A|\mathcal{D}|^{-3}} \left(\frac{1}{2} \int A|\mathcal{D}|^{-2} - \int A^2|\mathcal{D}|^{-3}\right) + \frac{1}{2} \int A|\mathcal{D}|^{-3} \overline{\int A|\mathcal{D}|^{-2}}. \end{aligned}$$

In order to prove this theorem, we will use a decomposition of one-forms in the Poincaré–Birkhoff–Witt basis of \mathcal{A} with an extension of previous representations to operators like $TJT'J^{-1}$ where T and T' are in X .

4.3 Balanced components and Poincaré–Birkhoff–Witt basis of \mathcal{A}

Our objective is to compute all integrals in term of A and the computation will lead to functions of A which capture certain symmetries on A .

For convenience, let us introduce now these functions:

Let $\mathbb{A} = \sum_i \pi(x^i) d\pi(y^i)$ on $SU_q(2)$ be one-form and A the associated δ -one-form. The x^i and y^i are in \mathcal{A} and as such they can be uniquely written as finite sums $x^i = \sum_{\alpha} x_{\alpha}^i m^{\alpha}$ and $y^i = \sum_{\beta} y_{\beta}^i m^{\beta}$ where $m^{\alpha} := a^{\alpha_1} b^{\alpha_2} b^{*\alpha_3}$ is the canonical monomial of \mathcal{A} with $\alpha, \beta \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$ based on a fixed Poincaré–Birkhoff–Witt type basis of \mathcal{A} .

Remark 4.4. Any one-form $\mathbb{A} = \sum_i \pi(x^i) d\pi(y^i)$ on $SU_q(2)$ is characterized by a complex valued matrix $A_{\alpha}^{\beta} = \sum_i x_{\alpha}^i y_{\beta}^i$ where $\alpha, \beta \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$. This matrix is such that

$$A = A_{\alpha}^{\beta} M_{\beta}^{\alpha}$$

where $M_{\beta}^{\alpha} := \pi(m^{\alpha}) \delta(\pi(m^{\beta}))$.

In the following, we note

$$\bar{A} := \bar{A}_{\alpha}^{\beta} M_{\beta}^{\alpha}$$

so for any $p \in \mathbb{N}$, $\int \bar{A}|\mathcal{D}|^{-p} = \overline{\int A|\mathcal{D}|^{-p}}$.

This presentation of one-forms is not unique modulo $OP^{-\infty}$ since, as we will see in section 5, $F = \sum_i x_i dy_i$ where $x_i, y_i \in \mathcal{A}$, thus for any generator z , $[F, z] = \sum_i x_i d(y_i z) - x_i y_i dz - z x_i dy_i = 0$ mod $OP^{-\infty}$. We do not know however if this presentation is unique when the $OP^{-\infty}$ part is taken into account.

The δ -one-forms M_{β}^{α} are said to be *canonical*. Any product of n canonical δ -one forms, where $n \in \mathbb{N}^*$, is called a *canonical δ^n -one-form*. Thus, if A is a δ -one-form, $A^n = (A^n)_{\bar{\alpha}}^{\bar{\beta}} M_{\beta}^{\bar{\alpha}}$ where $\bar{\alpha} = (\alpha, \alpha', \dots, \alpha^{(n-1)})$, $\bar{\beta} = (\beta, \beta', \dots, \beta^{(n-1)})$ are in $\mathbb{Z}^n \times \mathbb{N}^n \times \mathbb{N}^n$, $(A^n)_{\bar{\alpha}}^{\bar{\beta}} := A_{\alpha}^{\beta} \cdots A_{\alpha^{(n-1)}}^{\beta^{(n-1)}}$ and $M_{\beta}^{\bar{\alpha}}$ is the canonical δ^n -one form equals to $M_{\beta}^{\alpha} \cdots M_{\beta^{(n-1)}}^{\alpha^{(n-1)}}$.

Definition 4.5. A canonical δ^n -one-form is a -balanced if it is of the form

$$a^{\alpha_1} \delta(a^{\beta_1}) \cdots a^{\alpha_1^{(n-1)}} \delta(a^{\beta_1^{(n-1)}})$$

where $\sum_{i=0}^{n-1} \alpha_1^{(i)} + \beta_1^{(i)} = 0$.

For any δ -one-form A , the a -balanced components of A^n are noted $B_a(A^n)_{\bar{\alpha}}^{\bar{\beta}}$.

Note that

$$B_a(A)_{\bar{\alpha}}^{\bar{\beta}} = A_{-\beta_1 0 0}^{\beta_1 0 0} \delta_{\alpha_1 + \beta_1, 0} \delta_{\alpha_2 + \alpha_3 + \beta_2 + \beta_3, 0}.$$

Definition 4.6. A canonical δ^n -one-form is balanced if it is of the form

$$m^\alpha \delta(m^\beta) \cdots m^{\alpha^{(n-1)}} \delta(m^{\beta^{(n-1)}})$$

where $\sum_{i=0}^{n-1} \alpha_1^{(i)} + \beta_1^{(i)} = 0$ and $\sum_{i=0}^{n-1} \alpha_2^{(i)} + \beta_2^{(i)} = \sum_{i=0}^{n-1} \alpha_3^{(i)} + \beta_3^{(i)}$.

For any δ -one-form A , the balanced components of A^n are noted $B(A^n)_{\bar{\alpha}}^{\bar{\beta}}$.

Note that

$$B(A)_{\bar{\alpha}}^{\bar{\beta}} = A_{-\beta_1 \alpha_2 \alpha_3}^{\beta_1 \beta_2 \beta_3} \delta_{\alpha_1 + \beta_1, 0} \delta_{\alpha_2 + \beta_2, \alpha_3 + \beta_3}.$$

As we will show, a contribution to the k^{th} -coefficient in the spectral action, is only brought by one-forms \mathbb{A} such that A^k is balanced (and even a -balanced in the case $k = 1$).

Note also that if A is balanced, then A^k for $k \geq 1$ is also balanced, whereas the converse is false.

4.4 The reality operator J on $SU_q(2)$

For any $n, p \in \mathbb{N}$,

$$\begin{aligned} q_n &:= \sqrt{1 - q^{2n}}, & q_{-n} &:= 0 \text{ if } n > 0, \\ q_{n,p}^\uparrow &:= q_{n+1} \cdots q_{n+p}, & q_{n,p}^\downarrow &:= q_n \cdots q_{n-(p-1)}, \end{aligned}$$

with the convention $q_{n,0}^\uparrow = q_{n,0}^\downarrow := 1$. Thus, we have the relations

$$\begin{aligned} \pi_\pm(a^p) \varepsilon_n &= q_{n,p}^\uparrow \varepsilon_{n+p}, & \pi_\pm(a^{*p}) \varepsilon_n &= q_{n,p}^\downarrow \varepsilon_{n-p}, \\ \pi_\pm(b^p) \varepsilon_n &= (\pm q^n)^p \varepsilon_n, & \pi_\pm(b^{*p}) \varepsilon_n &= (\pm q^n)^p \varepsilon_n, \end{aligned}$$

where $\varepsilon_k := 0$ if $k < 0$.

The sign of $x \in \mathbb{R}$ is noted η_x . By convention, $a_j := a$, $a_{\pm,j} := a_\pm$ if $j \geq 0$ and $a_j := a^*$, $a_{\pm,j} := a_\pm^*$ if $j < 0$. Note that, with convention

$$q_{n,p}^{\uparrow \alpha_1} := q_{n,p}^\uparrow \text{ if } \alpha_1 > 0, \quad q_{n,p}^{\uparrow \alpha_1} := q_{n,p}^\downarrow \text{ if } \alpha_1 < 0, \text{ and } q_{n,0}^{\uparrow 0} := 1,$$

we have for any $\alpha_1 \in \mathbb{Z}$ and $p \leq \alpha_1$, $\pi_\pm(a_{\alpha_1}^p) \varepsilon_n = q_{n,p}^{\uparrow \alpha_1} \varepsilon_{n+\eta_{\alpha_1} p}$.

Recall that the reality operator J is defined by

$$J v_{m,l}^{j\uparrow} = i^{2(m+l)-1} v_{2j-m, 2j+1-l}^{j\uparrow}, \quad J v_{m,l}^{j\downarrow} = i^{-2(m+l)+1} v_{2j-m, 2j-1-l}^{j\downarrow},$$

thus the real conjugate operators

$$\hat{a}_\pm := J a_\pm J^{-1}, \quad \hat{b}_\pm := J b_\pm J^{-1}$$

satisfy

$$\begin{aligned}\widehat{a}_+ v_{m,l}^j &:= -q_{2j+1-m} \begin{pmatrix} q_{2j+2-l} & 0 \\ 0 & q_{2j-l} \end{pmatrix} v_{m,l}^{j+}, & \widehat{a}_- v_{m,l}^j &:= -q^{2j-m} \begin{pmatrix} q^{2j+2-l} & 0 \\ 0 & q^{2j-l} \end{pmatrix} v_{m-1,l-1}^{j-}, \\ \widehat{b}_+ v_{m,l}^j &:= q_{2j+1-m} \begin{pmatrix} q^{2j+1-l} & 0 \\ 0 & q^{2j-1-l} \end{pmatrix} v_{m,l+1}^{j+}, & \widehat{b}_- v_{m,l}^j &:= -q^{2j-m} \begin{pmatrix} q_{2j+1-l} & 0 \\ 0 & q_{2j-1-l} \end{pmatrix} v_{m-1,l}^{j-}.\end{aligned}$$

So the real conjugate operator behave differently on the up and down part of the Hilbert space. The difference comes from the fact that the index l is not treated uniformly by J on up and down parts.

We note \widehat{X} the algebra generated by $\{\widehat{a}_\pm, \widehat{b}_\pm\}$, \widetilde{X} the algebra generated by $\{a_\pm, b_\pm, \widehat{a}_\pm, \widehat{b}_\pm\}$ and $\mathcal{H}' := \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z})$ and we construct two $*$ -representations $\widehat{\pi}_\pm$ of \mathcal{A} :

The representation $\widehat{\pi}_+$ gives bounded operators on \mathcal{H}' while $\widehat{\pi}_-$ represents \mathcal{A} into $\mathcal{B}(\mathcal{H}' \otimes \mathbb{C}^2)$. The representation $\widehat{\pi}_+$ is defined on the generators by:

$$\widehat{\pi}_+(a) \varepsilon_m \otimes \varepsilon_{2j} := q_{2j+1-m} \varepsilon_m \otimes \varepsilon_{2j+1}, \quad \widehat{\pi}_+(b) \varepsilon_m \otimes \varepsilon_{2j} := -q^{2j-m} \varepsilon_{m+1} \otimes \varepsilon_{2j+1}$$

while $\widehat{\pi}_-$ is defined by:

$$\begin{aligned}\widehat{\pi}_-(a) \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &:= -q_{2j+1\pm 1-l} \varepsilon_l \otimes \varepsilon_{2j+1} \otimes \varepsilon_{\uparrow\downarrow}, \\ \widehat{\pi}_-(b) \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &:= -q^{2j\pm 1-l} \varepsilon_{l+1} \otimes \varepsilon_{2j+1} \otimes \varepsilon_{\uparrow\downarrow},\end{aligned}$$

where $\varepsilon_{\uparrow\downarrow}$ is the canonical basis of \mathbb{C}^2 and the $+$ in \pm corresponds to \uparrow in $\uparrow\downarrow$.

The link between $\widehat{\pi}_\pm$ and π_\pm which explains the notations about these intermediate objects and the fact that $\widehat{\pi}_\pm$ are representations on different Hilbert spaces, is in the parallel between equations (30), (35) and (36).

Let us give immediately a few properties (x_β equals x if the sign β is positive and equals x^* otherwise)

$$\begin{aligned}\widehat{\pi}_+(a_\beta)^p \varepsilon_m \otimes \varepsilon_{2j} &= q_{2j-m,p}^{\uparrow\beta} \varepsilon_m \otimes \varepsilon_{2j+\eta_\beta p}, \\ \widehat{\pi}_-(a_\beta)^p \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &= (-1)^p q_{2j\pm 1-l,p}^{\uparrow\beta} \varepsilon_l \otimes \varepsilon_{2j+\eta_\beta p} \otimes \varepsilon_{\uparrow\downarrow}, \\ \widehat{\pi}_+(b_\beta)^p \varepsilon_m \otimes \varepsilon_{2j} &= (-1)^p q^{(2j-m)p} \varepsilon_{m+\eta_\beta p} \otimes \varepsilon_{2j+\eta_\beta p}, \\ \widehat{\pi}_-(b_\beta)^p \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &= (-1)^p q^{(2j\pm 1-l)p} \varepsilon_{l+\eta_\beta p} \otimes \varepsilon_{2j+\eta_\beta p} \otimes \varepsilon_{\uparrow\downarrow}.\end{aligned}$$

Note that the $\widehat{\pi}_\pm$ representations still contain the shift information, contrary to representations π_\pm . Moreover, $\widehat{\pi}_\pm(b) \neq \widehat{\pi}_\pm(b^*)$ while $\pi_\pm(b) = \pi_\pm(b^*)$.

The operators $\widehat{a}_\pm, \widehat{b}_\pm$ are coded on $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ as the correspondence

$$\begin{aligned}\widehat{a}_+ &\longleftrightarrow \widehat{\pi}_+(a) \otimes \widehat{\pi}_-(a), & \widehat{a}_- &\longleftrightarrow -q \widehat{\pi}_+(b^*) \otimes \widehat{\pi}_-(b^*), \\ \widehat{b}_+ &\longleftrightarrow -\widehat{\pi}_+(a) \otimes \widehat{\pi}_-(b), & \widehat{b}_- &\longleftrightarrow -\widehat{\pi}_+(b^*) \otimes \widehat{\pi}_-(a^*).\end{aligned}\tag{35}$$

We now set the following extension to $\mathcal{B}(\mathcal{H}')$ of π_+ and to $\mathcal{B}(\mathcal{H}' \otimes \mathbb{C}^2)$ of π_- by

$$\begin{aligned}\pi'_+(a) &:= \pi_+(a) \otimes V, & \pi'_+(b) &:= \pi_+(b) \otimes V \quad (V \text{ is the shift of } \ell^2(\mathbb{Z})), \\ \pi'_-(a) &:= \pi_-(a) \otimes V \otimes 1_2, & \pi'_-(b) &:= \pi_-(b) \otimes V \otimes 1_2.\end{aligned}$$

So, we can define a canonical algebra morphism $\widetilde{\rho}$ from \widetilde{X} into the bounded operators on $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$. This morphism is defined on the generators part $\{\widehat{a}_\pm, \widehat{b}_\pm\}$ of \widetilde{X} by preceding correspondence and on the generators part $\{a_\pm, b_\pm\}$ by –see (30):

$$\begin{aligned}a_+ &\longleftrightarrow \pi'_+(a) \otimes \pi'_-(a), & a_- &\longleftrightarrow -q \pi'_+(b^*) \otimes \pi'_-(b^*), \\ b_+ &\longleftrightarrow -\pi'_+(a) \otimes \pi'_-(b), & b_- &\longleftrightarrow -\pi'_+(b^*) \otimes \pi'_-(a^*).\end{aligned}\tag{36}$$

We note S the canonical surjection from $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ onto \mathcal{H} . This surjection is associated to the parameters restrictions on m, j, l, j' . In particular, the index j' associated to the second $\ell^2(\mathbb{N})$ in $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ is set to be equal to j . Any vector in $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ not satisfying these restrictions is sent to 0 in \mathcal{H} .

Denote by I the canonical injection of \mathcal{H} into $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ (the index j is doubled). Thus, $S\tilde{\rho}(\cdot)I$ is the identity on \tilde{X} .

In the computation of residues of $\zeta_{\mathcal{D}}^T$ functions, we can therefore replace the operator T by $S\tilde{\rho}(T)I$.

We now extend τ_0 on $\pi'_{\pm}(\mathcal{A})\hat{\pi}_{\pm}(\mathcal{A})$: For $x, y \in \mathcal{A}$, we set

$$\begin{aligned}\mathrm{Tr}_N(\pi'_+(x)\hat{\pi}_+(y)) &:= \sum_{m=0}^N \langle \varepsilon_m \otimes \varepsilon_N, \pi'_+(x)\hat{\pi}_+(y) \varepsilon_m \otimes \varepsilon_N \rangle, \\ \mathrm{Tr}_N^{\uparrow}(\pi'_-(x)\hat{\pi}_-(y)) &:= \sum_{l=0}^N \langle \varepsilon_l \otimes \varepsilon_{N-1} \otimes \varepsilon_{\uparrow}, \pi'_-(x)\hat{\pi}_-(y) \varepsilon_l \otimes \varepsilon_{N-1} \otimes \varepsilon_{\uparrow} \rangle, \\ \mathrm{Tr}_N^{\downarrow}(\pi'_-(x)\hat{\pi}_-(y)) &:= \sum_{l=0}^N \langle \varepsilon_l \otimes \varepsilon_{N+1} \otimes \varepsilon_{\downarrow}, \pi'_-(x)\hat{\pi}_-(y) \varepsilon_l \otimes \varepsilon_{N+1} \otimes \varepsilon_{\downarrow} \rangle.\end{aligned}$$

Actually, a computation on monomials of \mathcal{A} shows that $\mathrm{Tr}_N^{\downarrow}(\pi'_-(x)\hat{\pi}_-(y)) = \mathrm{Tr}_N^{\uparrow}(\pi'_-(x)\hat{\pi}_-(y))$. For convenience, we shall note $\mathrm{Tr}_N(\pi'_-(x)\hat{\pi}_-(y))$ this functional.

Lemma 4.7. *Let $x, y \in \mathcal{A}$. Then,*

(i) $\tau_0(\pi'_{\pm}(x)\hat{\pi}_{\pm}(y)) := \lim_{N \rightarrow \infty} U_N$ exists where

$$U_N := \mathrm{Tr}_N(\pi'_{\pm}(x)\hat{\pi}_{\pm}(y)) - (N+1)\tau_1(\pi_{\pm}(x))\tau_1(\pi_{\pm}(y)).$$

(ii) $U_N = \tau_0(\pi'_{\pm}(x)\hat{\pi}_{\pm}(y)) + \mathcal{O}(N^{-k})$ for all $k > 0$.

Proof. (i) We can suppose that x and y are monomials, since the result will follow by linearity. We will give a proof for the case of the π_+ representations, the case π_- being similar, with minor changes.

We have $\hat{\pi}_+(y) = (\hat{\pi}_+ a_{\beta_1})^{|\beta_1|} (\hat{\pi}_+ b)^{\beta_2} (\hat{\pi}_+ b^*)^{\beta_3}$. A computation gives

$$\hat{\pi}_+(y) \varepsilon_m \otimes \varepsilon_{2j} = (-1)^{\beta_2+\beta_3} q^{(2j-m)(\beta_2+\beta_3)} q_{2j-m,|\beta_1|}^{\uparrow\beta_1} \varepsilon_{m-\beta_3+\beta_2} \otimes \varepsilon_{2j-\beta_3+\beta_2+\beta_1}$$

and with the notation $t_{2j,m} := \langle \varepsilon_m \otimes \varepsilon_{2j}, \pi'_{\pm}(x)\hat{\pi}_{\pm}(y) \varepsilon_m \otimes \varepsilon_{2j} \rangle$ and $T_{2j} := \sum_{m=0}^{2j} t_{2j,m}$, we get

$$\begin{aligned}t_{2j,m} &= (-1)^{\beta_2+\beta_3} q^{(2j-m)(\beta_2+\beta_3)} q_{2j-m,|\beta_1|}^{\uparrow\beta_1} q_{m-\beta_3+\beta_2,|\alpha_1|}^{\uparrow\alpha_1} q^{(m+\beta_2-\beta_3)(\alpha_2+\alpha_3)} \delta_{\alpha_1+\beta_2-\beta_3,0} \\ &\quad \times \delta_{-\alpha_3+\alpha_2+\beta_1,0} \\ &= (-1)^{\alpha_1} q^{(2j-m)(\beta_2+\beta_3)+(m-\alpha_1)(\alpha_2+\alpha_3)} q_{2j-m,|\beta_1|}^{\uparrow\beta_1} q_{m-\alpha_1,|\alpha_1|}^{\uparrow\alpha_1} \delta_{\alpha_1+\beta_2-\beta_3,0} \delta_{\alpha_2-\alpha_3+\beta_1,0} \\ &=: f_{\alpha,\beta} q^{2j\lambda} t'_{2j,m} =: f_{\alpha,\beta} q^{2j\kappa} t''_{2j,2j-m}\end{aligned}$$

where

$$t'_{2j,m} := q^{m(\kappa-\lambda)} q_{2j-m,|\beta_1|}^{\uparrow\beta_1} q_{m-\alpha_1,|\alpha_1|}^{\uparrow\alpha_1}, \quad (37)$$

$$t''_{2j,m} := q^{m(\lambda-\kappa)} q_{m,|\beta_1|}^{\uparrow\beta_1} q_{2j-m-\alpha_1,|\alpha_1|}^{\uparrow\alpha_1}, \quad (38)$$

with $\lambda := \beta_2 + \beta_3 \geq 0$ and $\kappa := \alpha_2 + \alpha_3 \geq 0$. We will now prove that if $\lambda \neq \kappa$, then (T_{2j}) is a convergent sequence. Suppose $\kappa > \lambda$. Let us note $U'_{2j} := \sum_{m=0}^{2j} t'_{2j,m}$. Since the $t'_{2j,m}$ are positive and $t'_{2j+1,m} \geq t'_{2j,m}$ for all j, m , U'_{2j} is an increasing real sequence. The estimate

$$U'_{2j} \leq \sum_{m=0}^{2j} q^{m(\kappa-\lambda)} \leq \frac{1}{1-q^{\kappa-\lambda}} < \infty$$

proves then that U'_{2j} is a convergent sequence. With $T_{2j} = f_{\alpha,\beta} q^{2j\lambda} U'_{2j}$, we obtain our result.

Suppose now that $\lambda > \kappa$. Let us note $U''_{2j} := \sum_{m=0}^{2j} t''_{2j,m}$. Since the $t''_{2j,m}$ are positive and $t''_{2j+1,m} \geq t''_{2j,m}$ for all j, m , U''_{2j} is an increasing real sequence. The estimate

$$U''_{2j} \leq \sum_{m=0}^{2j} q^{m(\lambda-\kappa)} \leq \frac{1}{1-q^{\lambda-\kappa}} < \infty$$

proves then that U''_{2j} is a convergent sequence. With $T_{2j} = f_{\alpha,\beta} q^{2j\kappa} U''_{2j}$, we have again our result. Moreover, note that if λ and κ are both different from zero, the limit of (T_{2j}) is zero and more precisely,

$$T_{2j} = \mathcal{O}(q^{2j\lambda}) \text{ if } \kappa > \lambda > 0, \quad (39)$$

$$T_{2j} = \mathcal{O}(q^{2j\kappa}) \text{ if } \lambda > \kappa > 0. \quad (40)$$

Suppose now that $\lambda = \kappa \neq 0$. In that case, (T_{2j}) also converges rapidly to zero. Indeed, let us fix $q < \varepsilon < 1$. we have $\varepsilon^{-2j\lambda} T_{2j} = \sum_{m=0}^{2j} c_m d_{2j-m} = c * d(2j)$ where $c_m := f_{\alpha,\beta} (q/\varepsilon)^{\lambda m} q_{m-\alpha_1,|\alpha_1|}^{\uparrow\alpha_1}$ and $d_m := (q/\varepsilon)^{\lambda m} q_{m,|\beta_1|}^{\uparrow\beta_1}$. Since both $\sum_m c_m$ and $\sum_m d_m$ are absolutely convergent series, their Cauchy product $\sum_{2j} \varepsilon^{-2j\lambda} T_{2j}$ is convergent. In particular, $\lim_{j \rightarrow \infty} \varepsilon^{-2j\lambda} T_{2j} = 0$, and

$$T_{2j} = \mathcal{O}(\varepsilon^{2j\lambda}). \quad (41)$$

Finally, T_{2j} has a finite limit in all cases except possibly when $\lambda = \kappa = 0$, which is the case when $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$. In that case, $t_{2j,m} = 1$.

A straightforward computation gives $\tau_1(\pi_{\pm}(x)) \tau_1(\pi_{\pm}(y)) = \delta_{\alpha_1,0} \delta_{\beta_1,0} \delta_{\alpha_2,0} \delta_{\beta_2,0} \delta_{\alpha_3,0} \delta_{\beta_3,0}$.

Thus,

$$U_{2j} = T_{2j} - (2j+1) \delta_{\alpha_1,0} \delta_{\beta_1,0} \delta_{\alpha_2,0} \delta_{\beta_2,0} \delta_{\alpha_3,0} \delta_{\beta_3,0}$$

has always a finite limit when $j \rightarrow \infty$.

(ii) The result is clear if $\lambda = \kappa = 0$ (in that case $U_N = \tau_0 = 0$). Suppose λ or κ is not zero. In that case $U_{2j} = T_{2j}$. By (40), (39) and (41), we see that if $\lambda > \kappa > 0$ or $\kappa > \lambda > 0$ or $\kappa = \lambda$, (T_{2j}) converges to 0 with a rate in $\mathcal{O}(\varepsilon^{2j\alpha})$ where $\alpha > 0$ and $q \leq \varepsilon < 1$. Thus, it only remains to check the cases $(\kappa > 0, \lambda = 0)$ and $(\kappa = 0, \lambda > 0)$. In the first one, we get from (37), $U_{2j} = f_{\alpha,\beta} \sum_{m=0}^{2j} q^{m\kappa} q_{2j-m,|\beta_1|}^{\uparrow\beta_1}$. If $\beta_1 = 0$, we are done.

Suppose $\beta_1 > 0$. We have $q_{2j-m,|\beta_1|}^{\uparrow\beta_1} = \sum_{p=0}^{\infty} l_p q^{r_p} q^{2|p|_1(2j-m)}$ where $p = (p_1, \dots, p_{\beta_1})$ and $l_p = (-1)^{|p|_1} \left(\frac{1}{p}\right)$, $r_p := 2p_1 + \dots + 2\beta_1 p_{\beta_1}$. Thus, cutting the sum in two, we get, noting $L_{2j} := f_{\alpha,\beta} \sum_{m=0}^{2j} q^{m\kappa}$,

$$U_{2j} - L_{2j} = f_{\alpha,\beta} \sum_{|p|_1 > \kappa/2} l_p q^{r_p} \frac{q^{4|p|_1 j - q^{(2j+1)\kappa - 2|p|_1}}}{1 - q^{\kappa - 2|p|_1}} + f_{\alpha,\beta} \sum_{0 \neq |p|_1 \leq \kappa/2} l_p q^{r_p} q^{4|p|_1 j} \sum_{m=0}^{2j} q^{m(\kappa - 2|p|_1)}.$$

Since $\sum_{0 \neq |p|_1 \leq \kappa/2} l_p q^{r_p} q^{4|p|_1 j} \sum_{m=0}^{2j} q^{m(\kappa-2|p|_1)}$ is in $\mathcal{O}_{j \rightarrow \infty}(jq^{4j})$, we have, modulo a rapidly decreasing sequence,

$$U_{2j} - L_{2j} \sim f_{\alpha, \beta} \sum_{|p|_1 > \kappa/2} l_p q^{r_p} \frac{q^{4|p|_1 j - q^{(2j+1)\kappa - 2|p|_1}}}{1 - q^{\kappa - 2|p|_1}} =: f_{\alpha, \beta} q^{2\kappa j} V_{2j}$$

with

$$V_{2j} = \sum_{|p|_1 > \kappa/2} l_p q^{r_p} \frac{1 - q^{(2|p|_1 - \kappa)(2j+1)}}{1 - q^{2|p|_1 - \kappa}} = \sum_{|p|_1 > \kappa/2} \sum_{m=0}^{2j} l_p q^{r_p} q^{(2|p|_1 - \kappa)m}.$$

The family $v_{m,p} := (l_p q^{r_p} q^{(2|p|_1 - \kappa)m})_{(p,m) \in I}$, where $I = \{(p, m) \in \mathbb{N}^{\beta_1} \times \mathbb{N} : |p|_1 > \kappa/2\}$ is (absolutely) summable. Indeed $|v_{m,p}| \leq |l_p| q^{r_p} q^m$ so $|v_{m,p}|$ is summable as the product of two summable families. As a consequence, $\lim_{j \rightarrow \infty} V_{2j}$ exists and is finite, which proves that $(q^{2\kappa j} V_{2j})$, and thus $(U_{2j} - L_{2j})$ converge rapidly to 0.

Suppose now that $\beta_1 < 0$. In that case, $q_{2j-m, |\beta_1|}^{\uparrow \beta_1} = q_{2j-m, |\beta_1|}^{\downarrow} = q_{2j-(m+|\beta_1|), |\beta_1|}^{\uparrow}$ and by (37), we get $U_{2j} = f_{\alpha, \beta} \sum_{m=0}^{2j} q^{m\kappa} q_{2j-(m+|\beta_1|), |\beta_1|}^{\uparrow} = f_{\alpha, \beta} q^{-|\beta_1|\kappa} \sum_{m=|\beta_1|}^{2j+|\beta_1|} q^{m\kappa} q_{2j-m, |\beta_1|}^{\uparrow}$, so the same arguments as in case $\beta_1 > 0$ apply here, the summation on m simply shifted of $|\beta_1|$.

The same proof can be applied for the other case ($\kappa = 0, \lambda > 0$). This time, we only need to use (38) instead of (37) and the preceding arguments follow by replacing κ by λ and β_1 by α_1 . \square

Remark 4.8. *Contrary to the preceding τ_0 , the new functional contains the shift information. In particular, it filters the parts of nonzero degree.*

If $T \in X\widehat{X}$, $\tilde{\rho}(T) \in \pi_+(\mathcal{A})\widehat{\pi}_+(\mathcal{A}) \otimes \pi_-(\mathcal{A})\widehat{\pi}_-(\mathcal{A})$.

For notational convenience, we define τ_1 on $\pi'_{\pm}(\mathcal{A})\widehat{\pi}_{\pm}(\mathcal{A})$ as

$$\tau_1(\pi'_{\pm}(x)\widehat{\pi}_{\pm}(y)) := \tau_1(\pi_{\pm}(x)) \tau_1(\pi_{\pm}(y)).$$

In the following, the symbol \sim_e means equals modulo a entire function.

Theorem 4.9. *Let $T \in X\widehat{X}$. Then*

- (i) $\zeta_D^T(s) \sim_e 2(\tau_1 \otimes \tau_1)(\tilde{\rho}(T)) \zeta(s-2) + 2(\tau_0 \otimes \tau_1 + \tau_1 \otimes \tau_0)(\tilde{\rho}(T)) \zeta(s-1) + 2(\tau_0 \otimes \tau_0 - \frac{1}{2}\tau_1 \otimes \tau_1)(\tilde{\rho}(T)) \zeta(s),$
- (ii) $\int T|\mathcal{D}|^{-3} = 2(\tau_1 \otimes \tau_1)(\tilde{\rho}(T)),$
- (iii) $\int T|\mathcal{D}|^{-2} = 2(\tau_0 \otimes \tau_1 + \tau_1 \otimes \tau_0)(\tilde{\rho}(T)),$
- (iv) $\int T|\mathcal{D}|^{-1} = 2(\tau_0 \otimes \tau_0 - \frac{1}{2}\tau_1 \otimes \tau_1)(\tilde{\rho}(T)).$

Proof. (i) Since $T \in X\widehat{X}$, $\tilde{\rho}(T)$ is a linear combination of terms like $\pi'_+(x)\widehat{\pi}_+(y) \otimes \pi'_-(z)\widehat{\pi}_-(t)$, where $x, y, z, t \in \mathcal{A}$. Such a term is noted in the following $T_+ \otimes T_-$. Linear combination of these term is implicit. With the shortcut $T_{c_1, \dots, c_p} := \langle \varepsilon_{c_1} \otimes \dots \otimes \varepsilon_{c_p}, T \varepsilon_{c_1} \otimes \dots \otimes \varepsilon_{c_p} \rangle$, recalling that

$v_{m,l}^{j,\downarrow}$ is 0 when $j = 0$, or $l \geq 2j$, we get

$$\begin{aligned}
\zeta_{\mathcal{D}}^T(s) &= \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \langle (v_{m,l}^{j,\uparrow}), S\tilde{\rho}(T)I(v_{m,l}^{j,\uparrow}) \rangle d_{j^+}^{-s} + \langle (v_{m,l}^{j,\downarrow}), S\tilde{\rho}(T)I(v_{m,l}^{j,\downarrow}) \rangle d_j^{-s} \\
&= \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \tilde{\rho}(T)_{m,2j,l,2j,\uparrow} d_{j^+}^{-s} + \sum_{2j=1}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j-1} \tilde{\rho}(T)_{m,2j,l,2j,\downarrow} d_j^{-s} \\
&= \sum_{2j=0}^{\infty} (\text{Tr}_{2j}(T_+) \text{Tr}_{2j+1}^{\uparrow}(T_-) + \text{Tr}_{2j+1}(T_+) \text{Tr}_{2j}^{\downarrow}(T_-)) d_{j^+}^{-s}.
\end{aligned}$$

By Lemma 4.7 (ii), for all $k > 0$,

$$\begin{aligned}
\text{Tr}_{2j}(T_{\pm}) &= (2j + \frac{3}{2})\tau_1(T_{\pm}) + \tau_0(T_{\pm}) - \frac{1}{2}\tau_1(T_{\pm}) + \mathcal{O}((2j)^{-k}), \\
\text{Tr}_{2j+1}(T_{\pm}) &= (2j + \frac{3}{2})\tau_1(T_{\pm}) + \tau_0(T_{\pm}) + \frac{1}{2}\tau_1(T_{\pm}) + \mathcal{O}((2j)^{-k}).
\end{aligned}$$

The result follows by noting that the difference of the Hurwitz zeta function $\zeta(s, \frac{3}{2})$ and Riemann zeta function $\zeta(s)$ is an entire function.

(ii, iii, iv) are direct consequences of (i). □

4.5 The smooth algebra $C^{\infty}(SU_q(2))$

In [13, 38], the smooth algebra $C^{\infty}(SU_q(2))$ is defined by pulling back the smooth structure $C^{\infty}(D_{q^{\pm}}^2)$ into the C^* -algebra generated by \mathcal{A} , through the morphism ρ and the application λ (the compression which gives an operator on \mathcal{H} from an operator on $l^2(\mathbb{N}) \otimes l^2(\mathbb{N}) \otimes l^2(\mathbb{Z}) \otimes \mathbb{C}^2$). The important point is that with [13, Lemma 2, p. 69], this algebra is stable by holomorphic calculus. By defining $\rho := \tilde{\rho} \circ c$ and $\lambda(\cdot) := S(\cdot)I$, the same lemma (with same notation) can be applied to our setting, with $c := \pi(x) \mapsto \underline{\pi}(x)$ and

$$\mathcal{C} := C^{\infty}(D_{q^+}^2) \otimes C^{\infty}(S^1) \otimes C^{\infty}(D_{q^+}^2) \otimes C^{\infty}(S^1) \otimes \mathcal{M}_2(\mathbb{C})$$

as algebra stable by holomorphic calculus containing the image of $\tilde{\rho}$. Here, we use Schwartz sequences to define the smooth structures. We finally obtain $C^{\infty}(SU_q(2))$ with real structure as a subalgebra stable by holomorphic calculus of the C^* -algebra generated by $\pi(\mathcal{A}) \cup J\pi(\mathcal{A})J^{-1}$ and containing $\pi(\mathcal{A}) \cup J\pi(\mathcal{A})J^{-1}$.

Corollary 4.10. *The dimension spectrum of the real spectral triple $(C^{\infty}(SU_q(2)), \mathcal{H}, D)$ is simple and given by $\{1, 2, 3\}$. Its KO -dimension is 3.*

Proof. Since F commutes with $\underline{\pi}(\mathcal{A})$, the pseudodifferential operators of order 0 (without the real structure and in the sense of [22]) are exactly (modulo $OP^{-\infty}$) the operators in $\mathcal{B} + \mathcal{B}F$. From Theorem 3.4 we see that the dimension spectrum of $SU_q(2)$ without taking into account the reality operator J is $\{1, 2, 3\}$, in other words, the possible poles of $\zeta_{\mathcal{D}}^b : s \mapsto \text{Tr}(bF^{\varepsilon}|\mathcal{D}|^{-s})$ (with $\varepsilon \in \{0, 1\}$, $b \in \mathcal{B}$) are in $\{1, 2, 3\}$. Theorem 4.9 (i) shows that the possible poles are still $\{1, 2, 3\}$ when we take into account the real structure of $SU_q(2)$, that is to say, when \mathcal{B} is enlarged to $\mathcal{B}J\mathcal{B}J^{-1}$. Indeed, any element of $\mathcal{B}J\mathcal{B}J^{-1}$ is in $X\hat{X}$ and it is clear from the preceding proof that adding F in the previous zeta function do not add any pole to $\{1, 2, 3\}$. All arguments goes true from the polynomial algebra $\mathcal{A}(SU_q(2))$ to the smooth pre- C^* -algebra $C^{\infty}(SU_q(2))$.

KO -dimension refers just to $J^2 = -1$ and $\mathcal{D}J = J\mathcal{D}$ since there is no chirality because spectral dimension is 3. □

4.6 Noncommutative integrals with reality operator and one-forms on $SU_q(2)$

The goal of this section is to obtain the following suppression of J :

Theorem 4.11. *Let A and B be δ -one-forms. Then*

- (i) $\int A J B J^{-1} |\mathcal{D}|^{-3} = \frac{1}{2} \int A |\mathcal{D}|^{-3} \overline{\int B |\mathcal{D}|^{-3}},$
- (ii) $\int A J B J^{-1} |\mathcal{D}|^{-2} = \frac{1}{2} \int A |\mathcal{D}|^{-2} \overline{\int B |\mathcal{D}|^{-3}} + \frac{1}{2} \int A |\mathcal{D}|^{-3} \overline{\int B |\mathcal{D}|^{-2}},$
- (iii) $\int A^2 J B J^{-1} |\mathcal{D}|^{-3} = \frac{1}{2} \int A^2 |\mathcal{D}|^{-3} \overline{\int B |\mathcal{D}|^{-3}},$
- (iv) $\int \delta(A) A |\mathcal{D}|^{-3} = \int \delta(A) J A J^{-1} |\mathcal{D}|^{-3} = 0.$

We gather at the beginning of this section the main notations for technical lemmas which will follow.

For any pair $(k, p) \in \mathbb{N}^3 \times \mathbb{N}^3$ such that $k_i \leq |\alpha_i|$, $p_i \leq |\beta_i|$, where $\alpha, \beta \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$, we define

$$\begin{aligned} v_{k,p} &:= g(p) \binom{|\alpha_1|}{k_1}_{q^{2\eta\alpha_1}} \binom{\alpha_2}{k_2} \binom{\alpha_3}{k_3} \binom{|\beta_1|}{p_1}_{q^{2\eta\beta_1}} \binom{\beta_2}{p_2} \binom{\beta_3}{p_3} (-1)^{k_1+p_1+\alpha_2+\alpha_3+\beta_2+\beta_3} q^{\sigma_{k,p}}, \\ h_{k,p} &:= \alpha_1 + \alpha_2 - \alpha_3 - 2(\eta\alpha_1 k_1 + k_2 - k_3) + g(p), \\ g(p) &:= \beta_1 + \beta_2 - \beta_3 - 2(\eta\beta_1 p_1 + p_2 - p_3), \\ \sigma_{k,p} &:= k_1 + p_1 + \sigma_{k,p}^t + \sigma_{k,p}^u, \\ \sigma_{k,p}^t &:= k_1 \widehat{k}_2 - \widehat{k}_3(k_1 + k_2) + \eta_{\beta_1} \widehat{p}_1 |k|_1 + \widehat{p}_2(|k|_1 + p_1) - \widehat{p}_3(|k|_1 + p_1 + p_2), \\ \sigma_{k,p}^u &:= (k_3 + \eta_{\beta_1} \widehat{p}_1 - p_2 + p_3)(k_1 + \widehat{k}_2 + \widehat{k}_3) - k_2(k_1 + \widehat{k}_2) + (p_1 + \widehat{p}_2)(-p_2 + p_3) + \widehat{p}_3 p_3, \\ t_{k,p} &= a_{\alpha_1}^{\widehat{k}_1} a^{\widehat{k}_2} a^{*\widehat{k}_3} a_{\beta_1}^{\widehat{p}_1} a^{\widehat{p}_2} a^{*\widehat{p}_3} b^{|k|_1+|p|_1}, \\ u_{k,p} &= a_{\alpha_1}^{\widehat{k}_1} a^{*k_2} a^{k_3} a_{\beta_1}^{\widehat{p}_1} a^{*p_2} a^{p_3} b^{|\widehat{k}|_1+|\widehat{p}|_1}. \end{aligned}$$

where we used the notation

$$\widehat{k}_i := |\alpha_i| - k_i, \widehat{p}_i := |\beta_i| - p_i,$$

so $0 \leq \widehat{k}_i \leq |\alpha_i|$, $0 \leq \widehat{p}_i \leq |\beta_i|$. We will also use the shortcut $\widetilde{k} := (k_1, \widehat{k}_2, \widehat{k}_3)$.

For $\beta_1 \in \mathbb{Z}$ and $j \in \mathbb{N}$, we define

$$\begin{aligned} w_1(\beta_1, j) &:= \sum_{n=0}^{\infty} (q^{2jn} (q_{n,|\beta_1|}^{\uparrow\beta_1})^2 - \delta_{j,0}), \\ w_{\beta}^{\alpha} &:= 2\beta_1 q^{\beta_1(2\alpha_3+\beta_3-\beta_2)} w_1(\beta_1, \alpha_3 + \beta_3). \end{aligned}$$

We introduce the following notations:

$$\begin{aligned} q_{k,p,n}^+ &:= q^{n(|k|_1+|p|_1)} q_{n+r_{k,p}^+}^{\uparrow\alpha_1} \widehat{k}_1, \widehat{k}_1 q_{n-\widehat{k}_3+\eta_{\beta_1}\widehat{p}_1+\widehat{p}_2-\widehat{p}_3, \widehat{k}_2}^{\uparrow} q_{n+\eta_{\beta_1}\widehat{p}_1+\widehat{p}_2-\widehat{p}_3, \widehat{k}_3}^{\downarrow} q_{n+\widehat{p}_2-\widehat{p}_3, \widehat{p}_1}^{\uparrow\beta_1} q_{n-\widehat{p}_3, \widehat{p}_2}^{\uparrow} q_{n, \widehat{p}_3}^{\downarrow}, \\ q_{k,p,n}^- &:= q^{n(|\widetilde{k}|_1+|\widetilde{p}|_1)} q_{n+r_{k,p}^-}^{\uparrow\alpha_1} \widehat{k}_1, \widehat{k}_1 q_{n+k_3+\eta_{\beta_1}\widehat{p}_1-p_2+p_3, k_2}^{\downarrow} q_{n+\eta_{\beta_1}\widehat{p}_1-p_2+p_3, k_3}^{\uparrow} q_{n-p_2+p_3, \widehat{p}_1}^{\uparrow\beta_1} q_{n+p_3, p_2}^{\downarrow} q_{n, p_3}^{\uparrow} \\ &\quad \times (-1)^{|\widetilde{k}|_1+|\widetilde{p}|_1}, \\ r_{k,p}^+ &:= \eta_{\alpha_1} \widehat{k}_1 + \widehat{k}_2 - \widehat{k}_3 + \eta_{\beta_1} \widehat{p}_1 + \widehat{p}_2 - \widehat{p}_3, \\ r_{k,p}^- &:= \eta_{\alpha_1} \widehat{k}_1 - k_2 + k_3 + \eta_{\beta_1} \widehat{p}_1 - p_2 + p_3. \end{aligned}$$

Thus, $\pi_+(t_{k,p})\varepsilon_n = q_{k,p,n}^+ \varepsilon_{n+r_{k,p}^+}$ and $\pi_-(u_{k,p})\varepsilon_n = q_{k,p,n}^- \varepsilon_{n+r_{k,p}^-}$.

Lemma 4.12. *We have*

$$r((M_\beta^\alpha)^\circ) = \sum_{k,p} \delta_{h_{k,p},0} v_{k,p} \pi_+(t_{k,p}) \otimes \pi_-(u_{k,p})$$

where the summation is done on k_i, p_i in \mathbb{N} such that $k_i \leq |\alpha_i|, p_i \leq |\beta_i|$ for $i \in \{1, 2, 3\}$.

Proof. Since $\underline{\pi}(m^\alpha) = (a_+ + a_-)^{\alpha_1} (b_+ + b_-)^{\alpha_2} (b_+^* + b_-^*)^{\alpha_3}$, with $v_k := \binom{|\alpha_1|}{k_1} q^{2\eta_{\alpha_1}} \binom{\alpha_2}{k_2} \binom{\alpha_3}{k_3}$,

$$\underline{\pi}(m^\alpha) = \sum_k v_k c_k \text{ where } c_k := a_{+,\alpha_1}^{|\alpha_1|-k_1} a_{-,\alpha_1}^{k_1} b_+^{\alpha_2-k_2} b_-^{k_2} b_+^* \alpha_3-k_3 b_-^* k_3.$$

By Lemma 3.2 (iii) we see that $\delta(\underline{\pi}(m^\beta)) = \sum_p w_p d_p$ where we introduce

$$w_p := \binom{|\beta_1|}{p_1} q^{2\eta_{\beta_1}} \binom{\beta_2}{p_2} \binom{\beta_3}{p_3} \text{ and } d_p := g(p) a_{+,\beta_1}^{|\beta_1|-p_1} a_{-,\beta_1}^{p_1} b_+^{\beta_2-p_2} b_-^{p_2} b_+^* \beta_3-p_3 b_-^* p_3.$$

As a consequence, $(M_\beta^\alpha)^\circ = \sum_{k,p} \delta_{h(k,p),0} g(p) v_k w_p c_{k,p}$ where

$$c_{k,p} = a_{+,\alpha_1}^{\widehat{k}_1} a_{-,\alpha_1}^{k_1} b_+^{\widehat{k}_2} b_-^{k_2} b_+^* \widehat{k}_3 b_-^* k_3 a_{+,\beta_1}^{\widehat{p}_1} a_{-,\beta_1}^{p_1} b_+^{\widehat{p}_2} b_-^{p_2} b_+^* \widehat{p}_3 b_-^* p_3 \quad (42)$$

With (42), we get $r(c_{k,p}) = (-1)^{k_1+p_1+\alpha_2+\alpha_3+\beta_2+\beta_3} q^{k_1+p_1} \pi_+(t'_{k,p}) \otimes \pi_-(u'_{k,p})$ where

$$\begin{aligned} t'_{k,p} &= a_{\alpha_1}^{\widehat{k}_1} b^{k_1} a^{\widehat{k}_2} b^{k_2} a^{*\widehat{k}_3} b^{k_3} a_{\beta_1}^{\widehat{p}_1} b^{p_1} a^{\widehat{p}_2} b^{p_2} a^{*\widehat{p}_3} b^{p_3}, \\ u'_{k,p} &= a_{\alpha_1}^{\widehat{k}_1} b^{k_1} b^{\widehat{k}_2} a^{*k_2} b^{\widehat{k}_3} a^{k_3} a_{\beta_1}^{\widehat{p}_1} b^{p_1} b^{\widehat{p}_2} a^{*p_2} b^{\widehat{p}_3} a^{p_3}. \end{aligned}$$

A recursive use of relation $ba_j = q^{\eta_j} a_j b$ yields the result. □

Lemma 4.13. *We have*

$$(i) (\tau_1 \otimes \tau_1)(r(M_\beta^\alpha)^\circ) = \beta_1 \delta_{\alpha_1, -\beta_1} \delta_{\alpha_2, 0} \delta_{\alpha_3, 0} \delta_{\beta_2, 0} \delta_{\beta_3, 0}.$$

$$(ii) (\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(r(M_\beta^\alpha)^\circ) = \delta_{\alpha_1, -\beta_1} \delta_{\alpha_2+\beta_2, \alpha_3+\beta_3} w_\beta^\alpha.$$

In particular, if A is a δ -one-form, we have

$$\begin{aligned} \oint A|\mathcal{D}|^{-3} &= 2\beta_1 A_{-\beta_1 00}^{\beta_1 00}, \\ \oint A|\mathcal{D}|^{-2} &= 2w_\beta^\alpha B(A)_\alpha^\beta. \end{aligned}$$

where we implicitly summed on all α, β indices.

Proof. (i) Using same notations of Lemma 4.12, we obtain by definition of τ_1 ,

$$\tau_1(\pi_+(t_{k,p})) = \delta_{k,0} \delta_{p,0} \delta_{\alpha_1+\alpha_2-\alpha_3+\beta_1+\beta_2-\beta_3,0}, \quad (43)$$

$$\tau_1(\pi_-(u_{k,p})) = \delta_{\widetilde{k},0} \delta_{\widetilde{p},0} \delta_{\alpha_1-\alpha_2+\alpha_3+\beta_1-\beta_2+\beta_3,0}. \quad (44)$$

We get $\tau_1(\pi_+(t_{k,p})) \tau_1(\pi_-(u_{k,p})) = \delta_{k,0} \delta_{p,0} \delta_{\alpha_2,0} \delta_{\alpha_3,0} \delta_{\beta_2,0} \delta_{\beta_3,0} \delta_{\alpha_1, -\beta_1}$, so Lemma 4.12 gives the result.

(ii) Since $\pi_+(t_{k,p})\varepsilon_n = q_{k,p,n}^+ \varepsilon_{n+r_{k,p}^+}$ and $\pi_-(u_{k,p})\varepsilon_n = q_{k,p,n}^- \varepsilon_{n+r_{k,p}^-}$, we get,

$$\tau_0(\pi_+(t_{k,p})) = \delta_{r_{k,p}^+,0} \sum_{n=0}^{\infty} (q_{k,p,n}^+ - \delta_{k,0} \delta_{p,0} \delta_{\alpha_1+\alpha_2-\alpha_3+\beta_1+\beta_2-\beta_3,0}), \quad (45)$$

$$\tau_0(\pi_-(u_{k,p})) = \delta_{r_{k,p}^-,0} \sum_{n=0}^{\infty} (q_{k,p,n}^- - \delta_{k,0} \delta_{p,0} \delta_{\alpha_1-\alpha_2+\alpha_3+\beta_1-\beta_2+\beta_3,0}). \quad (46)$$

With (43) and (46) we get

$$\begin{aligned} \tau_1(\pi_+(t_{k,p})) \tau_0(\pi_-(u_{k,p})) &= \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} \delta_{\alpha_1,-\beta_1} \sum_{n=0}^{\infty} (\delta_{k,0} \delta_{p,0} q_{k,p,n}^- - \delta_{\alpha_3+\beta_3,0}) \\ &= \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} \delta_{\alpha_1,-\beta_1} w_1(\beta_1, \alpha_3 + \beta_3). \end{aligned}$$

Using (44) and (45),

$$\begin{aligned} \tau_0(\pi_+(t_{k,p})) \tau_1(\pi_-(u_{k,p})) &= \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} \delta_{\alpha_1,-\beta_1} \sum_{n=0}^{\infty} (\delta_{k,0} \delta_{p,0} q_{k,p,n}^+ - \delta_{\alpha_3+\beta_3,0}) \\ &= \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} \delta_{\alpha_1,-\beta_1} w_1(\beta_1, \alpha_3 + \beta_3). \end{aligned}$$

Lemma 4.12 yields the result. \square

With notations of Lemma 4.12, it is direct to check that for given $\bar{\alpha} = (\alpha, \alpha', \dots, \alpha^{(n-1)})$ and $\bar{\beta} = (\beta, \beta', \dots, \beta^{(n-1)})$,

$$r((M_{\bar{\beta}}^{\bar{\alpha}})^{\circ}) = \sum_{K,P} \delta_{h_{K,P},0} v_{K,P} \pi_+(t_{K,P}) \otimes \pi_-(u_{K,P}) \quad (47)$$

where $K = (k, k', \dots, k^{(n-1)})$, $P = (p, p', \dots, p^{(n-1)})$ with $0 \leq k_i^{(j)} \leq |\alpha_i^{(j)}|$, $0 \leq p_i^{(j)} \leq |\beta_i^{(j)}|$,

$$\begin{aligned} t_{K,P} &:= t_{k,p} t_{k',p'} \cdots t_{k^{(n-1)},p^{(n-1)}}, & u_{K,P} &:= u_{k,p} u_{k',p'} \cdots u_{k^{(n-1)},p^{(n-1)}}, \\ v_{K,P} &:= v_{k,p} v_{k',p'} \cdots v_{k^{(n-1)},p^{(n-1)}}, & h_{K,P} &:= h_{k,p} + h_{k',p'} + \cdots + h_{k^{(n-1)},p^{(n-1)}}. \end{aligned}$$

In the following, we will use the shortcuts $A_i := \alpha_i + \alpha'_i + \cdots + \alpha_i^{(n-1)}$, $B_i := \beta_i + \beta'_i + \cdots + \beta_i^{(n-1)}$.

In the case $n = 2$, we also note $r_{K,P}^{\pm} := r_{k,p}^{\pm} + r_{k',p'}^{\pm}$ and $q_{K,P,n}^{\pm} := q_{k',p',n}^{\pm} q_{k,p,n+r_{k',p'}^{\pm}}^{\pm}$.

Thus, we have $\pi_+(t_{K,P})\varepsilon_m = q_{K,P,m}^+ \varepsilon_{m+r_{K,P}^+}$ and $\pi_-(u_{K,P})\varepsilon_m = q_{K,P,n}^- \varepsilon_{m+r_{K,P}^-}$.

We also introduce, still for $n = 2$,

$$\begin{aligned} v_{\beta_1, \alpha'_1, \beta'_1}(l, j) &:= \sum_{n=0}^{\infty} (q^{l+2nj} q_{n+\beta'_1+\alpha'_1+\beta_1, |\beta'_1+\alpha'_1+\beta_1|}^{\uparrow_{-\beta'_1-\alpha'_1-\beta_1}} q_{n+\beta'_1+\alpha'_1, |\beta_1|}^{\uparrow_{\beta_1}} q_{n+\beta'_1, |\alpha'_1|}^{\uparrow_{\alpha'_1}} q_{n, |\beta'_1|}^{\uparrow_{\beta'_1}} - \delta_{j,0}), \\ V_{\bar{\beta}}^{\bar{\alpha}} &:= 2[\beta_1 \beta'_1 + (\beta_2 - \beta_3)(\beta'_2 - \beta'_3)] q^{2\beta_1(\alpha_2+\alpha_3)+2\beta'_1(\alpha'_2+\alpha'_3)} \\ &\quad \times v_{\beta_1, \alpha'_1, \beta'_1}((\alpha_2 + \beta_2 + \alpha_3 + \beta_3)(\alpha'_1 + \beta'_1), A_3 + B_3). \end{aligned}$$

Lemma 4.14. *We have*

$$(i) \quad (\tau_1 \otimes \tau_1) (r(M_{\bar{\beta}}^{\bar{\alpha}} M_{\bar{\beta}'}^{\bar{\alpha}'})^{\circ}) = \beta_1 \beta'_1 \delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}.$$

$$(ii) \quad (\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1) (r(M_{\bar{\beta}}^{\bar{\alpha}} M_{\bar{\beta}'}^{\bar{\alpha}'})^{\circ}) = \delta_{A_2+B_2, A_3+B_3} \delta_{A_1, -B_1} V_{\bar{\beta}}^{\bar{\alpha}}.$$

- (iii) $(\tau_1 \otimes \tau_1) (r(M_\beta^\alpha M_{\beta'}^{\alpha'} M_{\beta''}^{\alpha''})^0) = \beta_1 \beta_1' \beta_1'' \delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}$.
(iv) $(\tau_1 \otimes \tau_1) (r(\delta(M_\beta^\alpha) M_{\beta'}^{\alpha'})^0) = -(\alpha_1' + \beta_1') \beta_1 \beta_1' \delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}$.
(v) In particular, if A is a δ -one-form,

$$\begin{aligned} \int A^2 |\mathcal{D}|^{-3} &= 2\beta_1 \beta_1' B_a(A^2)_{\bar{\alpha}}^{\bar{\beta}}, \\ \int A^2 |\mathcal{D}|^{-2} &= 2V_{\bar{\beta}}^{\bar{\alpha}} B(A^2)_{\bar{\alpha}}^{\bar{\beta}}, \\ \int A^3 |\mathcal{D}|^{-3} &= 2\beta_1 \beta_1' \beta_1'' B_a(A^3)_{\bar{\alpha}}^{\bar{\beta}}, \\ \int \delta(A) A |\mathcal{D}|^{-3} &= \int A \delta(A) |\mathcal{D}|^{-3} = 0. \end{aligned}$$

Proof. We have

$$\tau_1(\pi_+(t_{K,P})) = \delta_{K,0} \delta_{P,0} \delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}, \quad (48)$$

$$\tau_1(\pi_-(u_{K,P})) = \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}. \quad (49)$$

and

$$\tau_0(\pi_+(t_{K,P})) = \delta_{r_{K,P}^+,0} \sum_{n=0}^{\infty} (q_{K,P,n}^+ - \delta_{K,0} \delta_{P,0} \delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}), \quad (50)$$

$$\tau_0(\pi_-(u_{K,P})) = \delta_{r_{K,P}^-,0} \sum_{n=0}^{\infty} (q_{K,P,n}^- - \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}). \quad (51)$$

(i) Equations (48) and (49) give $(\tau_1 \otimes \tau_1) r(\underline{A} \underline{A}')^0 = \delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0} \lambda_{0,0}$. A computation of $v_{0,0}$ with $\delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0} = 1$ gives the result.

(ii) Equations (48) and (51) yield

$$\begin{aligned} \tau_1(\pi_+(t_{K,P})) \tau_0(\pi_-(u_{K,P})) &= \delta_{K,0} \delta_{P,0} \delta_{A_2+B_2, A_3+B_3} \delta_{A_1, -B_1} \\ &\quad \times v_{\beta_1, \alpha_1', \beta_1'}((\alpha_2 + \beta_2 + \alpha_3 + \beta_3)(\alpha_1' + \beta_1'), A_3 + B_3). \end{aligned}$$

Equations (50) and (49) yield

$$\begin{aligned} \tau_0(\pi_+(t_{K,P})) \tau_1(\pi_-(u_{K,P})) &= \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_2+B_2, A_3+B_3} \delta_{A_1, -B_1} \\ &\quad \times v_{\beta_1, \alpha_1', \beta_1'}((\alpha_2 + \beta_2 + \alpha_3 + \beta_3)(\alpha_1' + \beta_1'), A_3 + B_3) \end{aligned}$$

and the result follows.

(iii) With (47) a direct computation gives

$$\tau_1(\pi_+(t_{K,P})) = \delta_{K,0} \delta_{P,0} \delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}, \quad (52)$$

$$\tau_1(\pi_-(u_{K,P})) = \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}. \quad (53)$$

Using (52) and (53), $(\tau_1 \otimes \tau_1) (r(\underline{A} \underline{A}' \underline{A}'')^0) = \delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0} v_{0,0}$. A computation of $v_{0,0}$ with $\delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0} = 1$ gives the result.

(iv) We have $\delta(M_\beta^\alpha) M_{\beta'}^{\alpha'} = \delta(x) \delta(y) x' \delta(y') + x \delta^2(y) x' \delta(y')$ where x, x', y, y' are monomials ($\underline{\pi}$ omitted). Since

$$\underline{\pi}(x) = \sum_k \binom{\alpha}{k} a_{+, \alpha_1}^{\hat{k}_1} a_{-, \alpha_1}^{k_1} b_{+}^{\hat{k}_2} b_{-}^{k_2} b_{+}^{\hat{k}_3} b_{-}^{*k_3} =: \sum_k \binom{\alpha}{k} c_k,$$

we get $\delta(\pi(x)) = \sum_k g(k) \binom{\alpha}{k} c_k$.

Similarly, $\delta(\pi(y)) = \sum_p g(p) \binom{\beta}{p} c_p$ and $\delta^2(\pi(y)) = \sum_p g(p)^2 \binom{\beta}{p} c_p$.

Thus, with $c_{K,P} := c_k c_p c_{k'} c_{p'}$,

$$\delta(x)\delta(y)x'\delta(y') = \sum_{K,P} g(k)g(p)g(p') \binom{\alpha}{K} \binom{\beta}{P} c_{K,P},$$

$$x\delta^2(y)x'\delta(y') = \sum_{K,P} g(p)^2 g(p') \binom{\alpha}{K} \binom{\beta}{P} c_{K,P},$$

$$r(\delta(M_\beta^\alpha)M_{\beta'}^{\alpha'})^0 = \sum_{K,P} \delta_{h_{K,P},0}(g(k) + g(p))g(p)g(p') \binom{\alpha}{K} \binom{\beta}{P} r(c_{K,P}) =: \sum_{K,P} \lambda_{K,P} r(c_{K,P}).$$

Since $r(c_k) = (-q)^{k_1} (-1)^{\alpha_2 + \alpha_3} \pi_+(t_k) \otimes \pi_-(u_k)$ with t_k, u_k defined by

$$t_k := a_{\alpha_1}^{\widehat{k}_1} b^{k_1} a^{\widehat{k}_2} b^{k_2} a^{*\widehat{k}_3} b^{k_3} \quad \text{and} \quad u_k := a_{\alpha_1}^{\widehat{k}_1} b^{k_1} b^{\widehat{k}_2} a^{*k_2} b^{\widehat{k}_3} a^{k_3},$$

we get

$$r(\delta(M_\beta^\alpha)M_{\beta'}^{\alpha'})^0 = \sum_{K,P} \lambda_{K,P} (-q)^{k_1 + k'_1 + p_1 + p'_1} (-1)^{A_2 + A_3 + B_2 + B_3} \pi_+(t_{K,P}) \otimes \pi_-(u_{K,P})$$

where $t_{K,P} = t_k t_p t_{k'} t_{p'}$ and $u_{K,P} = u_k u_p u_{k'} u_{p'}$. Direct computations yield

$$\tau_1(\pi_+(t_{K,P})) = \delta_{K,0} \delta_{P,0} \delta_{A_1 + A_2 - A_3 + B_1 + B_2 - B_3, 0},$$

$$\tau_1(\pi_-(u_{K,P})) = \delta_{\widetilde{K},0} \delta_{\widetilde{P},0} \delta_{A_1 - A_2 + A_3 + B_1 - B_2 + B_3, 0}.$$

The result follows.

(v) For the last equality, note that by (iv)

$$\int \delta(A)A|\mathcal{D}|^{-3} = -2 \sum_{\alpha_1, \alpha'_1, \beta_1, \beta'_1} (\alpha'_1 + \beta'_1) \beta_1 \beta'_1 A_{\alpha_1 000}^{\beta_1 00} A_{\alpha'_1 00}^{\beta'_1 00} \delta_{\alpha_1 + \alpha'_1 + \beta_1 + \beta'_1, 0}.$$

The following change of variables $\alpha_1 \leftrightarrow \alpha'_1$, $\beta_1 \leftrightarrow \beta'_1$, implies by symmetry that this is equal to zero. \square

For a given δ -1-form A , we say that A is homogeneous of degree in a equal to $n \in \mathbb{Z}$ if it is a linear combination of M_β^α such that $\alpha_1 + \beta_1 = n$. From Lemma 4.14 (iv) we get,

Corollary 4.15. *Let A, A' be two δ -1-forms, then*

$$\int (A|\mathcal{D}|^{-1})^2 = \int A^2|\mathcal{D}|^{-2},$$

$$\int A|\mathcal{D}|^{-1}A'|\mathcal{D}|^{-1} = \int AA'|\mathcal{D}|^{-2} - n \int AA'|\mathcal{D}|^{-3}, \quad \text{when } A' \text{ homogenous of degree } n.$$

Lemma 4.16. *We have*

$$(i) (\tau_1 \otimes \tau_1) \widetilde{\rho}(M_\beta^\alpha J M_{\beta'}^{\alpha'} J^{-1}) = \beta_1 \beta'_1 \delta_{\alpha_1, -\beta_1} \delta_{\alpha'_1, -\beta'_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}.$$

$$(ii) (\tau_0 \otimes \tau_1 + \tau_1 \otimes \tau_0) \widetilde{\rho}(M_\beta^\alpha J M_{\beta'}^{\alpha'} J^{-1}) = \delta_{\alpha_1, -\beta_1} \delta_{\alpha'_1, -\beta'_1} (\beta'_1 w_\beta^\alpha \delta_{\alpha_2 + \beta_2 + \alpha_3 + \beta_3, 0} \delta_{\alpha_2 + \beta_2, \alpha_3 + \beta_3} + \beta_1 w_{\beta'}^{\alpha'} \delta_{\alpha_2 + \beta_2 + \alpha_3 + \beta_3, 0} \delta_{\alpha'_2 + \beta'_2, \alpha'_3 + \beta'_3}).$$

- (iii) $(\tau_1 \otimes \tau_1) \tilde{\rho}(M_\beta^\alpha M_{\beta'}^{\alpha'} J M_{\beta''}^{\alpha''} J^{-1}) = \beta_1 \beta_1' \beta_1'' \delta_{\alpha_1 + \alpha_1', -\beta_1 - \beta_1'} \delta_{\alpha_1'', -\beta_1''} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}$.
(iv) $(\tau_1 \otimes \tau_1) \tilde{\rho}(\delta(M_\beta^\alpha) J M_{\beta'}^{\alpha'} J^{-1}) = -(\alpha_1' + \beta_1') \beta_1 \beta_1' \delta_{\alpha_1, -\beta_1} \delta_{\alpha_1', -\beta_1'} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}$.
(v) In particular, if A and A' are δ -one forms,

$$\begin{aligned} \int A J A' J^{-1} |\mathcal{D}|^{-3} &= 2(\beta_1 A_{-\beta_1, 0}^{\beta_1, 00}) (\beta_1 \bar{A}'_{-\beta_1, 0}^{\beta_1, 00}), \\ \int A J A' J^{-1} |\mathcal{D}|^{-2} &= 2(\beta_1 \bar{A}'_{-\beta_1, 0}^{\beta_1, 00}) (w_\beta^\alpha B(A)_\alpha^\beta) + 2(\beta_1 A_{-\beta_1, 0}^{\beta_1, 00}) (w_\beta^\alpha B(\bar{A}')_\alpha^\beta), \\ \int A^2 J A' J^{-1} |\mathcal{D}|^{-3} &= 2(\beta_1 \bar{A}'_{-\beta_1, 0}^{\beta_1, 00}) (\beta_1 \beta_1' B_a(A^2)_{\bar{\alpha}}^{\bar{\beta}}), \\ \int \delta(A) J A J^{-1} &= 0. \end{aligned}$$

Proof. (i) Following notations of Lemma 4.12, we have

$$M_\beta^\alpha J M_{\beta'}^{\alpha'} J^{-1} = \sum_{K, P} v_{K, P} c_{k, p} J c_{k', p'} J^{-1}$$

where $K = (k, k')$, $P = (p, p')$, $\lambda_{K, P} = g(p)g(p')v_k v_{k'} w_p w_{p'}$. Thus,

$$\tilde{\rho}(M_\beta^\alpha J M_{\beta'}^{\alpha'} J^{-1}) = (-1)^{A_2 + A_3 + B_2 + B_3} \sum_{K, P} (-q)^{k_1 + k_1' + p_1 + p_1'} \lambda_{K, P} T_{K, P}^+ \otimes T_{K, P}^-$$

where $T_{K, P}^+ := \pi_+'(t_k t_p) \hat{\pi}_+(t_{k'} t_{p'})$ and $T_{K, P}^- := \pi_-'(u_k u_p) \hat{\pi}_-(u_{k'} u_{p'})$ with

$$\begin{aligned} t_k &:= a_{\alpha_1}^{\hat{k}_1} b_{\alpha_1}^{*k_1} a^{\hat{k}_2} b^{*k_2} a^{*\hat{k}_3} b^{k_3}, \\ u_k &:= a_{\alpha_1}^{\hat{k}_1} b_{\alpha_1}^{*k_1} b^{\hat{k}_2} a^{*k_2} b^{*\hat{k}_3} a^{k_3}. \end{aligned}$$

A direct computation leads to

$$\begin{aligned} \tau_1(T_{K, P}^+) &= \delta_{K, 0} \delta_{P, 0} \delta_{\alpha_1 + \alpha_2 - \alpha_3 + \beta_1 + \beta_2 - \beta_3, 0} \delta_{\alpha_1' + \alpha_2' - \alpha_3' + \beta_1' + \beta_2' - \beta_3', 0}, \\ \tau_1(T_{K, P}^-) &= \delta_{\tilde{K}, 0} \delta_{\tilde{P}, 0} \delta_{\alpha_1 - \alpha_2 + \alpha_3 + \beta_1 - \beta_2 + \beta_3, 0} \delta_{\alpha_1' - \alpha_2' + \alpha_3' + \beta_1' - \beta_2' + \beta_3', 0} \end{aligned}$$

which gives the result.

(ii) Using the commutation relations on \mathcal{A} , we see that there are real functions of (K, P) , noted $\sigma_{K, P}^t$ and $\sigma_{K, P}^u$ such that

$$\begin{aligned} T_{K, P}^+ &= q^{\sigma_{K, P}^t} \pi_+'(t_{k, p}) \hat{\pi}_+(t_{k', p'}), \\ T_{K, P}^- &= q^{\sigma_{K, P}^u} \pi_-'(u_{k, p}) \hat{\pi}_-(u_{k', p'}), \\ t_{k, p} &:= a_{\alpha_1}^{\hat{k}_1} a^{\hat{k}_2} a^{*\hat{k}_3} a_{\beta_1}^{\hat{p}_1} a^{\hat{p}_2} a^{*\hat{p}_3} b_{\alpha_1}^{*k_1} b_{\beta_1}^{*p_1} b^{*k_2 + p_2} b^{k_3 + p_3}, \\ u_{k, p} &:= a_{\alpha_1}^{\hat{k}_1} a^{*k_2} a^{k_3} a_{\beta_1}^{\hat{p}_1} a^{*p_2} a^{p_3} b_{\alpha_1}^{*k_1} b_{\beta_1}^{*p_1} b^{\hat{k}_2 + \hat{p}_2} b^{*\hat{k}_3 + \hat{p}_3}. \end{aligned}$$

We have, under the hypothesis $\tau_1(T_{K, P}^-) = 1$,

$$\begin{aligned} \hat{\pi}_+(t_{k', p'}) \varepsilon_{m, 2j} &= (-1)^{\lambda'} q^{(2j-m)\lambda'} q_{2j-m-s, |\beta_1', |\alpha_1'|}^{\uparrow \alpha_1'} q_{2j-m-s, |\beta_1'|}^{\uparrow \beta_1'} \varepsilon_{m+s, 2j}, \\ s &:= -\alpha_2' + \alpha_3' - \beta_2' + \beta_3' = \alpha_1' + \beta_1', \\ \lambda' &:= \alpha_2' + \alpha_3' + \beta_2' + \beta_3', \\ \lambda &:= \alpha_2 + \alpha_3 + \beta_2 + \beta_3 \\ \tau_1(T_{K, P}^+) &= \delta_{\lambda, 0} \delta_{\lambda', 0}. \end{aligned}$$

and then,

$$\begin{aligned} (T_{K,P}^+)_{m,2j} &= q^{\sigma_{K,P}^t + s\lambda} (-1)^{\lambda'} q^{(2j-m)\lambda' + m\lambda} F_m F'_{2j-m} \delta_{A_1+B_1,0}, \\ F'_{2j-m} &:= q_{2j-m-\alpha'_1, |\alpha'_1|}^{\uparrow\alpha'_1} q_{2j-m-\alpha'_1-\beta'_1, |\beta'_1|}^{\uparrow\beta'_1}, \\ F_m &:= q_{m-\alpha_1, |\alpha_1|}^{\uparrow\alpha_1} q_{m-\beta_1-\alpha_1, |\beta_1|}^{\uparrow\beta_1}. \end{aligned}$$

Following the proof of Lemma 4.7, we see that $\tau_0(T_{K,P}^+)$ is possibly nonzero only in the two cases $\lambda' = 0$ or $\lambda = 0$.

Suppose first $\lambda = \lambda' = 0$. In that case, we have

$$\tau_0(T_{K,P}^+) = \lim_{2j \rightarrow \infty} \sum_{m=0}^{2j} ((q_{m,|\beta_1|}^{\uparrow\beta_1} q_{2j-m,|\beta'_1|}^{\uparrow\beta'_1})^2 - 1) = \sum_{m=0}^{\infty} ((q_{m,|\beta_1|}^{\uparrow\beta_1})^2 - 1) + \sum_{m=0}^{\infty} ((q_{m,|\beta'_1|}^{\uparrow\beta'_1})^2 - 1)$$

where the second equality comes from Lemma 4.17.

In the case $(\lambda = 0, \lambda' > 0)$, we get $\alpha'_1 = -\beta'_1$ and thus,

$$(T_{K,P}^+)_{m,2j} = q^{\sigma_{K,P}^t} q^{m\lambda} (q_{m,|\beta_1|}^{\uparrow\beta_1} q_{2j-m,|\beta'_1|}^{\uparrow\beta'_1})^2 \delta_{\alpha_1+\beta_1,0}.$$

Let us note $U_{2j} = \sum_{m=0}^{2j} q^{m\lambda} (q_{m,|\beta_1|}^{\uparrow\beta_1} q_{2j-m,|\beta'_1|}^{\uparrow\beta'_1})^2$ and $L_{2j} = \sum_{m=0}^{2j} q^{m\lambda} (q_{m,|\beta_1|}^{\uparrow\beta_1})^2$.

Suppose $\beta'_1 > 0$. Since $(q_{2j-m,|\beta'_1|}^{\uparrow\beta'_1})^2 - 1 = \sum_{|p_1| \neq 0, p_i \in \{0,1\}} (-1)^{|p_1|} q^{r_p} q^{2(2j-m)|p_1|}$ where we have $r_p = 2 + \dots + 2\beta'_1$. As in the proof of Lemma 4.7 (ii), we can conclude that $U_{2j} - L_{2j}$ converges to 0. The case $\beta'_1 \leq 0$ is similar.

In the other case $(\lambda > 0, \lambda' = 0)$, the arguments are the same, replacing λ by λ' and α_1, β_1 by α'_1, β'_1 . Finally,

$$\begin{aligned} \tau_0(T_{K,P}^+) \tau_1(T_{K,P}^-) &= \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{\alpha_1, -\beta_1} \delta_{\alpha'_1, -\beta'_1} (\delta_{\lambda',0} \delta_{\alpha_2+\beta_2, \alpha_3+\beta_3} s_{\alpha,\beta} + \delta_{\lambda,0} \delta_{\alpha'_2+\beta'_2, \alpha'_3+\beta'_3} s_{\alpha',\beta'}), \\ s_{\alpha\beta} &:= q^{\beta_1(\alpha_3-\alpha_2)} \sum_{m=0}^{\infty} (q^{m\lambda} (q_{m,|\beta_1|}^{\uparrow\beta_1})^2 - \delta_{\lambda,0}). \end{aligned}$$

A similar computation of $\tau_0(T_{K,P}^-)$ can be done following the same arguments. We find eventually

$$\tau_1(T_{K,P}^+) \tau_0(T_{K,P}^-) = \delta_{K,0} \delta_{P,0} \delta_{\alpha_1, -\beta_1} \delta_{\alpha'_1, -\beta'_1} (\delta_{\lambda',0} \delta_{\alpha_2+\beta_2, \alpha_3+\beta_3} s_{\alpha,\beta} + \delta_{\lambda,0} \delta_{\alpha'_2+\beta'_2, \alpha'_3+\beta'_3} s_{\alpha',\beta'})$$

and the result follows.

(iii) The same arguments of (i) apply here with minor changes.

(iv) follows from a slight modification of the proof of Lemma 4.14 (iv).

(v) is a straightforward consequence of (i, ii, iii, iv). \square

Lemma 4.17. *Let $\beta, \beta' \in \mathbb{Z}$. Then,*

$$\lim_{2j \rightarrow \infty} \sum_{m=0}^{2j} ((q_{m,|\beta|}^{\uparrow\beta} q_{2j-m,|\beta'|}^{\uparrow\beta'})^2 - 1) = \sum_{m=0}^{\infty} ((q_{m,|\beta|}^{\uparrow\beta})^2 - 1) + \sum_{m=0}^{\infty} ((q_{m,|\beta'|}^{\uparrow\beta'})^2 - 1).$$

Proof. We give a proof for β and $\beta' > 0$, the other cases being similar.

Since $(q_{m,|\beta|}^{\uparrow\beta})^2 = \sum_{p_i \in \{0,1\}} (-1)^{|p|_1} q^{r_p} q^{2|p|_1 m}$ where $p = (p_1, \dots, p_\beta)$ and $r_p := 2(p_1 + \dots + \beta p_\beta)$, we get, with the notations $\lambda_{p,p'} := (-1)^{|p+p'|_1} q^{r_p+r_{p'}}$ and $U_{2j} := \sum_{m=0}^{2j} (q_{m,|\beta|}^{\uparrow\beta} q_{2j-m,|\beta'|}^{\uparrow\beta'})^2 - 1$,

$$\begin{aligned} U_{2j} &= \sum_{m=0}^{2j} \sum_{|p+p'|_1 > 0} \lambda_{p,p'} q^{2|p|_1 m + 2|p'|_1 (2j-m)} \\ &= \sum_{|p|_1 \geq |p'|_1, |p|_1 > 0} \lambda_{p,p'} V_{2j,p,p'} + \sum_{|p|_1 < |p'|_1, |p'|_1 > 0} \lambda_{p,p'} V'_{2j,p,p'} \end{aligned}$$

where

$$V_{2j,p,p'} = q^{4j|p'|_1} \sum_{m=0}^{2j} q^{2(|p|_1 - |p'|_1)m}, \quad V'_{2j,p,p'} = q^{4j|p|_1} \sum_{m=0}^{2j} q^{2(|p'|_1 - |p|_1)m}.$$

It is clear that $V_{2j,p,p'}$ has 0 for limit when $j \rightarrow \infty$ when $|p'|_1 > 0$, and $V'_{2j,p,p'}$ has 0 for limit when $j \rightarrow \infty$ when $|p|_1 > 0$. As a consequence,

$$U_{2j} = \sum_{|p|_1 > 0} \lambda_{p,0} V_{2j,p,0} + \sum_{|p'|_1 > 0} \lambda_{0,p'} V'_{2j,0,p'} + o(1).$$

The result follows as

$$\sum_{m=0}^{2j} ((q_{m,|\beta|}^{\uparrow\beta})^2 - 1) = \sum_{|p|_1 > 0} \lambda_{p,0} V_{2j,p,0} \text{ and } \sum_{m=0}^{2j} ((q_{m,|\beta'|}^{\uparrow\beta'})^2 - 1) = \sum_{|p'|_1 > 0} \lambda_{0,p'} V'_{2j,0,p'}. \quad \square$$

Proof of Theorem 4.11. The result follows from Lemmas 4.13, 4.14 (v) and 4.16 (v). \square

4.7 Proof of Theorem 4.3 and corollaries

Lemma 4.18. *We have on $SU_q(2)$,*

- (i) $f|\mathcal{D}|^{-3} = 2$.
- (ii) $f|\mathcal{D}|^{-2} = 0$.
- (iii) $f|\mathcal{D}|^{-1} = -\frac{1}{2}$.
- (iv) $\zeta_{\mathcal{D}}(0) = 0$.

Proof. (iv) We have by definition

$$\zeta_{\mathcal{D}}(s) := \text{Tr}(|\mathcal{D}|^{-s}) = \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \langle v_{m,l}^j, |\mathcal{D}|^{-s} v_{m,l}^j \rangle.$$

Since $|\mathcal{D}|^{-s} v_{m,l}^j = \begin{pmatrix} d_{j+}^{-s} & 0 \\ 0 & d_j^{-s} \end{pmatrix} v_{m,l}^j$ where $d_j := 2j + \frac{1}{2}$, we get

$$\zeta_{\mathcal{D}}(s) = \sum_{2j=0}^{\infty} (2j+1)(2j+2) d_{j+}^{-s} + \sum_{2j=1}^{\infty} (2j+1)(2j) d_j^{-s} = 2 \sum_{2j=0}^{\infty} (2j+1)(2j) d_j^{-s}.$$

With the equalities $(2j+1)(2j) = d_j^2 - \frac{1}{4}$ and $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$ (here $\zeta(s, x) := \sum_{n \in \mathbb{N}} \frac{1}{(n+x)^s}$ is the Hurwitz zeta function and $\zeta(s) := \zeta(s, 1)$ is the Riemann zeta function) we get

$$\zeta_{\mathcal{D}}(s) = 2(2^{s-2} - 1)\zeta(s-2) - \frac{1}{2}(2^s - 1)\zeta(s) \quad (54)$$

which entails that $\zeta_{\mathcal{D}}(0) = 0$.

(i, ii, iii) are direct consequences of equation (54). \square

Proof of Theorem 4.3. It is a consequence of Lemma 4.18 and Theorems 4.1, 4.11. \square

As we have seen, the computation of noncommutative integral on $SU_q(2)$ leads to certain function of A which filter some symmetry on the degree in a, a^*, b, b^* of the canonical decomposition. Precisely, it is the balanced features that appear and the following functions of $A^n, n \in \{1, 2, 3\}$:

$$\int A^n |\mathcal{D}|^{-p} \quad (55)$$

where $1 \leq n \leq p \leq 3$. We will see in the next section a method for the computation of these integrals.

Corollary 4.19. *Let u be a unitary in $C^\infty(SU_q(2))$ and $\gamma_u(\mathbb{A}) := \pi(u)\mathbb{A}\pi(u^*) + \pi(u)d\pi(u^*)$ be a gauge-variant of \mathbb{A} . Then the following term of Theorem 4.3 are gauge invariant*

$$\int A|\mathcal{D}|^{-3}, \quad \int A^2|\mathcal{D}|^{-3} - \int A|\mathcal{D}|^{-2}, \quad -2 \int A|\mathcal{D}|^{-1} + \int A^2|\mathcal{D}|^{-2} - \frac{2}{3} \int A^3|\mathcal{D}|^{-3}.$$

Proof. It is sufficient to remark that all terms $\int |D_{\mathbb{A}}|^{-k}$ and $\zeta_{\mathcal{D}_{\mathbb{A}}}(0)$ in spectral action (4) are gauge invariant. This can also be seen via the computation $D_{\gamma_u(\mathbb{A})} = V_u \mathcal{D} V_u^* + V_u P_0 V_u^*$ where P_0 is the projection on $\text{Ker } \mathcal{D}$ and $V_u = \pi(u)J\pi(u)J^{-1}$ and $\int |D_{\mathbb{A}}|^{n-k} = \text{Res}_{s=n-k} \text{Tr}(|D_{\mathbb{A}}|^{n-k})$ (see [22, Prop. 5.1 (iii) and Prop. 4.8].) \square

Corollary 4.20. *In the case of the spectral action without the reality operator (i.e. $D_{\mathbb{A}} = \mathcal{D} + \mathbb{A}$), we get*

$$\begin{aligned} \int |D_{\mathbb{A}}|^{-3} &= 2, & \int |D_{\mathbb{A}}|^{-2} &= -2 \int A|\mathcal{D}|^{-3}, & \int |D_{\mathbb{A}}|^{-1} &= -\frac{1}{2} - \int A|\mathcal{D}|^{-2} + \int A^2|\mathcal{D}|^{-3}, \\ \zeta_{D_{\mathbb{A}}}(0) &= -\int A|\mathcal{D}|^{-1} + \frac{1}{2} \int A^2|\mathcal{D}|^{-2} - \frac{1}{3} \int A^3|\mathcal{D}|^{-3}. \end{aligned}$$

As a consequence, if \mathbb{A} is a one-form such that $\int A|\mathcal{D}|^{-3} = 0$, then the scale invariant term of the spectral action with or without J is exactly the same modulo a global factor of 2.

5 Differential calculus on $SU_q(2)$ and applications

5.1 The sign of \mathcal{D}

There are multiple differential calculi on $SU_q(2)$, see [33, 39]. Thanks to [36, Theorem 3], the $3\mathcal{D}$ and $4D_{\pm}$ differential calculi do not coincide with the one considers here: the right multiplication of one-forms by an element in the algebra A is a consequence of the chosen Dirac operator which was introduced according to some equivariance properties with respect to the duality between the two Hopf algebras $SU_q(2)$ and $\mathcal{U}_q(su(2))$.

It is known that the Fredholm module associated to $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is one-summable since $[F, \pi(x)]$ is trace-class for all $x \in \mathcal{A}$. In fact, more can be said about F ¹:

Proposition 5.1. *Since*

$$F = \frac{1}{1-q^2} (\underline{\pi}(a^*) d\underline{\pi}(a) + q^2 \underline{\pi}(b) d\underline{\pi}(b^*) + q^2 \underline{\pi}(a) d\underline{\pi}(a^*) + q^2 \underline{\pi}(b^*) d\underline{\pi}(b)), \quad (56)$$

F is a central one-form modulo $OP^{-\infty}$.

¹Note that a similar result for a different spectral triple over $SU_q(2)$ when $q = 0$ was obtained in [13, eq. (48)]

Proof. Forgetting $\underline{\pi}$, this follows from

$$\begin{aligned}
a^* \delta a + q^2 b \delta b^* + q^2 a \delta a^* + q^2 b^* \delta b & \\
&= (a_+^* + a_-^*)(a_+ - a_-) + q^2 (b_+ + b_-)(b_-^* - b_+^*) + q^2 (a_+ + a_-)(a_-^* - a_+^*) \\
&\quad + q^2 (b_+^* + b_-^*)(b_+ - b_-) \\
&= [a_+^* a_+ - q^2 a_+ a_+^* + q^2 b_+^* b_+ - q^2 b_+ b_+^*] + R = (1 - q^2) + R \tag{57}
\end{aligned}$$

by (16) where we check that the remainder R is zero:

$$\begin{aligned}
R &= -[a_+^* a_- + q^2 b_+^* b_-] + [a_-^* a_+ + q^2 b_-^* b_+] - [a_-^* a_- - q^2 a_- a_-^* + q^2 b_-^* b_- - q^2 b_- b_-^*] \\
&\quad + (q^2 a_+ a_-^* + q^2 a_-^* b_+) - (a_+^* a_- + q^2 b_+^* b_-),
\end{aligned}$$

thus, applying (19), (20), (21), $R = +(q^2 a_+ a_-^* + q^2 a_-^* b_+) - (a_+^* a_- + q^2 b_+^* b_-) = 0$ using commutation relations (15).

Now, replacing δ by d in (57) gives (56) since F commute with a_{\pm} , b_{\pm} and F is central by (28). \square

Proposition 5.2. *The one-form in (56) is in fact exactly a function of the Dirac operator D :*

$$\pi(a^*) d\pi(a) + q^2 \pi(b) d\pi(b^*) + q^2 \pi(a) d\pi(a^*) + q^2 \pi(b^*) d\pi(b) = \xi_q(\mathcal{D}) = F \xi_q(|\mathcal{D}|), \tag{58}$$

where $\xi_q(s) := q^{\frac{[2s]-2s}{[s+1/2][s-1/2]}}$.

Moreover, $F = \lim_{q \rightarrow 0} \xi_q(D)$.

Proof. First, let us observe that the one-form ω in (58) is invariant under the action of the $\mathcal{U}_q(su(2)) \times \mathcal{U}_q(su(2))$: $h \triangleright \omega = \epsilon(h) \omega$ for any $h \in \mathcal{U}_q(su(2)) \times \mathcal{U}_q(su(2))$. For instance, using notations of [21]

$$e \triangleright \omega = q^{\frac{1}{2}} a^* db + q^2 \left(-q^{\frac{1}{2}-1} b da^* + q^{-\frac{1}{2}} b da^* - q^{-1-\frac{1}{2}} a^* db \right) = 0 = \epsilon(e) \omega.$$

Therefore, since both the representation π as well as the operator D are equivariant, the image of ω must be diagonal in the spinorial base. A tedious computation with the full spinorial representation π given in (10) yields

$$\begin{aligned}
\langle v_{ml}^{j\uparrow}, \omega v_{ml}^{j\uparrow} \rangle &= \frac{q^{8j+8} - q^{8j+6} - (4j+3)q^{4j+6} + (8j+6)q^{4j+4} - (4j+3)q^{4j+2} - q^2 + 1}{(q^{4j+4}-1)(q^{4j+2}-1)} = \xi_q(2j + \frac{3}{2}), \\
\langle v_{ml}^{j\downarrow}, \omega v_{ml}^{j\downarrow} \rangle &= \frac{-q^{8j+4} + q^{8j+2} + (4j+1)q^{4j+4} - (8j+2)q^{4j+2} + (4j+1)q^{4j} + q^2 - 1}{(q^{4j+2}-1)(q^{4j}-1)} = -\xi_q(2j + \frac{1}{2}).
\end{aligned}$$

These expressions have a clear $q = 0$ limit equal respectively to 1 and -1, so $\omega \rightarrow F$ as $q \rightarrow 0$. \square

In the $q = 1$ limit, these expressions yields identically 0, which is confirmed by the fact that all one-forms are central, it could be expressed as $d(aa^* + bb^*) = d1$.

Note that since the invariant one-form we constructed differs by $OP^{-\infty}$ from F , hence any commutator with it will be itself in $OP^{-\infty}$.

We do not know if a central form ω is automatically invariant by the action of both $U_q(su(2))$, that is: $h \triangleright \omega = \epsilon(h) \omega$.

Proposition 5.3. *The order one calculus up to $OP^{-\infty}$ is not universal.*

Proof. Let us take the one-form ω_F from (56), which gives F . Then, for any $x \in \mathcal{A}(SU_q(2))$ we have $\underline{\pi}(x\omega_F - \omega_F x) = 0$. \square

Corollary 5.4. *Still modulo $OP^{-\infty}$, $1 \in \pi(\Omega_u^2(\mathcal{A}))$.*

Proof. $1 = F^2$ is by definition in $\pi(\Omega_u^2(\mathcal{A}))$. \square

In fact, one checks, using (16), (19), (22) that

$$q^2 da da^* - da^* da = 1 - q^2 \quad (59)$$

showing again that $1 \in \pi(\Omega_u^2(\mathcal{A}))$.

Similarly, using (15) and (17), (22), (23), we get still up to $OP^{-\infty}$

$$\begin{aligned} q da db &= db da, & q da db^* &= db^* da, \\ da^* db &= q db da^*, & da^* db^* &= q db^* da^* \\ db db^* &= db^* db, & da da^* + db db^* &= -1. \end{aligned} \quad (60)$$

The use of the last equality of (60) and (59) gives

Proposition 5.5. *Up to $OP^{-\infty}$, F is not a (universal) closed one-form, as*

$$da^* da + q^2 da da^* + q^2 db^* db + q^2 db db^* = -1 - q^2. \quad (61)$$

5.2 The ideal \mathcal{R}

In order to perform explicit calculations of all terms of the spectral action, we observe that each δ -one-form could be expressed in terms of $x\delta(z)y$, where z is one of the generators a, a^*, b, b^* and x, y are some elements of the algebra $\mathcal{A}(SU_q(2))$.

Then, for the computation of $\int x dz y |D|^{-1}$ we can use the trace property of the noncommutative integral to get:

$$\int x \delta(z) y |D|^{-1} = \int y x \delta(z) |D|^{-1} + \int x \delta(z) |D|^{-1} \delta(y) |D|^{-1}.$$

Therefore, the problem of calculating the tadpole-like integral could be in effect reduced to the calculation of much simpler integrals: $\int x \delta(z)$ for all generators z and the integrals of higher order in $|D|^{-1}$.

However, it appears that the calculations of higher-order terms simplify a lot, when we further restrict the algebra by introducing an ideal, which is *invisible* to the parts of integral at dimension 2 and 3. For instance, consider the space of pseudodifferential operators $T \in \Psi^0(\mathcal{A})$ of order less or equal to zero (see [16]), which satisfy

$$\int T t |D|^{-2} = \int t T |D|^{-2} = \int T t |D|^{-3} = \int t T |D|^{-3} = 0, \quad \forall t \in \Psi_0^0(\mathcal{A}). \quad (62)$$

The elements $a_-, b_- b_+, b_- b_+^*$ and their adjoints are in this space up to $OP^{-\infty}$: this is due to the fact that in Theorem 3.4, $\tau_1 \otimes \tau_1(r(x)) = 0$ when $r(x) \in \pi_{\pm}(\mathcal{A}) \otimes \pi_{\pm}(\mathcal{A}) \bmod OP^{-\infty}$ contains tensor products of $\pi_{\pm}(b)$ or $\pi_{\pm}(b^*)$ since these elements are in the kernel of the grading σ .

Definition 5.6. Let R be the kernel in X of $(\sigma \otimes \sigma) \circ r$ where r is the Hopf-map defined in (30) and σ is the symbol map and let \mathcal{R} be the vector space generated by R and RF .

Note that R is a $*$ -ideal in X and

$$a_-, b_-b_+(=q^2b_+b_-), b_-b_+^* \text{ are in } \mathcal{R}.$$

By construction and Theorem 3.4, any $T \in \mathcal{R}$ satisfies (62) and \mathcal{R} is invariant by F .

Moreover, by (19), $[b_-, b_-^*] \in R$, so by (16) and (22), $a_+^*a_+ - q^2a_+a_+^* - (1 - q^2) \in R$ and by (23), $qa_+b_- - b_-a_+ \in R$.

It is interesting to quote, thanks to Theorem 3.4 that if $x \in R$, then $\int Fx|D|^{-1} = 0$ while a priori, $\int x|D|^{-1} \neq 0$.

Note that $F \in \Psi^0(\mathcal{A})$ also satisfies (62) by Theorem 3.4 while $F \notin \mathcal{R}$ since $F^2 = 1$.

Moreover other elements are in \mathcal{R} like for instance $d(b^*b) = d(bb^*)$:

$$\begin{aligned} \delta(bb^*) &= -\delta(aa^*) = -\delta a a^* - a \delta a^* = -(a_+ - a_-)(a_+^* + a_-^*) - (a_+ + a_-)(a_-^* - a_+^*) \\ &= 2(a_+a_-^* - a_-a_+^*) \end{aligned}$$

is in R since $a_- \in R$ yielding $d(bb^*) \in RF$.

We do not know if \mathcal{R} is equal to the subset of the algebra generated by \mathcal{B} and $\mathcal{B}F$ satisfying (62).

Lemma 5.7. \mathcal{R} is a $*$ -ideal in $\Psi^0(\mathcal{A})$ which is invariant by F , d , δ .

Proof. Since R is an ideal in $X = \mathcal{B} \text{ mod } OP^{-\infty}$ (see Remark 3.3), \mathcal{R} appears to be an ideal in $\Psi^0(\mathcal{A}) \subset$ algebra generated by \mathcal{B} and $\mathcal{B}F$. Since \mathcal{R} is invariant by F , its invariance by d follows from its invariance by δ which is true on the generators of R . \square

Note that, according to Theorem 4.13, $\int da|D|^{-2} = \int da|D|^{-3} = 0$ while $\int a^*da|D|^{-3} = 2$ which emphasize the role of " for all t " in (62).

Lemma 5.8. For any $t \in \Psi_0^0(\mathcal{A})$ and $T \in \mathcal{R}$, we have $\int tT|D|^{-1} = \int Tt|D|^{-1}$.

Proof. For any $t \in \mathcal{B}$, we have $\int Tt|D|^{-1} = \int tT|D|^{-1} + \int T|D|^{-1}\delta(t)|D|^{-1}$ and moreover $\int T|D|^{-1}\delta(t)|D|^{-1} = \int T\delta(t)|D|^{-2} - \int T\delta^2(t)|D|^{-3}$ which comes from

$$\begin{aligned} |D|^{-1}\delta(t)|D|^{-1} &= \delta(t)|D|^{-2} + [|D|^{-1}, \delta(t)]|D|^{-1} = \delta(t)|D|^{-2} - |D|^{-1}\delta^2(t)|D|^{-2} \\ &= \delta(t)|D|^{-2} - \delta^2(t)|D|^{-3} + |D|^{-1}\delta^3(t)|D|^{-3}. \end{aligned}$$

So we get the result because T satisfies (62). \square

Lemma 5.9. If \simeq means equality up to the ideal \mathcal{R} , the following rules with $d(\cdot) = [D, \cdot]$ of the first-order differential calculus hold (forgetting π)

$$\begin{array}{llll} a da \simeq da a, & a^* da \simeq -da^* a, & b da \simeq q da b, & b^* da \simeq q da b^*, \\ a da^* \simeq -da a^*, & a^* da^* \simeq da^* a^*, & b da^* \simeq q^{-1} da^* b, & b^* da^* \simeq q^{-1} da^* b^*, \\ a db \simeq q^{-1} db a, & a^* db \simeq q db a^*, & b db \simeq db b, & b^* db \simeq db b^* \simeq -b db^*, \\ a db^* \simeq q^{-1} db^* a, & a^* db^* \simeq q db^* a^*, & b db^* \simeq db^* b \simeq -b^* db, & b^* db^* \simeq db^* b^*. \end{array}$$

Moreover

$$a^* da - q^2 da a^* \simeq (1 - q^2) F, \quad q^2 a da^* - da^* a \simeq (1 - q^2) F. \quad (63)$$

Proof. The table follows from relations (7) and Lemma 3.2 with (28) (one can also use (15).) For instance, since $a_- \in \mathcal{R}$, using the fact that \mathcal{R} is invariant by F ,

$$\begin{aligned} b da &= (b_+ + b_-)(a_+ - a_-) F \simeq (b_+ + b_-)(a_+ + a_-) F = ba F = q ab F \simeq q (a_+ - a_-) F b \\ &= q da b \end{aligned}$$

or similarly, $a^* da = (a_+^* + a_-^*)(a_+ - a_-) F \simeq (a_+^* - a_-^*)(a_+ + a_-) F = -da^* a$.

The second of equivalence of (63) is just the adjoint of the first one that we prove now:

$$\begin{aligned} a^* da - q^2 da a^* &= (a_+^* + a_-^*)(a_+ - a_-) F - q^2 (a_+ - a_-) F (a_+^* + a_-^*) \\ &\simeq (a_+^* + a_-^*)(a_+ + a_-) F - q^2 (a_+ + a_-) (a_+^* + a_-^*) F = (a^* a - q^2 a a^*) F \\ &= (1 - q^2) F. \end{aligned} \quad \square$$

Remark 5.10. *The above written rules remain valid if dx is replaced by $\delta(x)$ and F by 1.*

Working modulo \mathcal{R} simplifies the writing of a one-form:

Lemma 5.11. (i) *Every one-form A can be, up to elements from \mathcal{R} , presented as*

$$A \simeq x_a da + da^* x_{a^*} + x_b db + db^* x_{b^*},$$

where all x_* are the elements of \mathcal{A} .

(ii) *When A is selfadjoint, A can be written up to \mathcal{R} (not in a unique way, though) as*

$$A \simeq x_a da - da^* (x_a)^* + x_b db - db^* (x_b)^*,$$

where x_a, x_b are arbitrary elements of \mathcal{A} .

Proof. (i) A basis for one-forms consists of the following forms: $a^\alpha b^\beta (b^*)^\gamma d(a^{\alpha'} b^{\beta'} (b^*)^{\gamma'})$, where $\alpha, \alpha' \in \mathbb{Z}$ and $\beta, \gamma, \beta', \gamma' \in \mathbb{N}$.

Using the Leibniz rule and the commutation rules within the algebra (up to the \mathcal{R} according to Lemma 5.9), we reduce the problem to the case of the forms: $(a^\alpha b^\beta (b^*)^\gamma) dx (a^{\alpha'} b^{\beta'} (b^*)^{\gamma'})$, where x can be either of the generators a, a^*, b, b^* . If $x = b$ or $x = b^*$, the straightforward application of the rules of the differential calculus leads to the answer that the one-form could be expressed as: $a^\alpha b^\beta (b^*)^\gamma db$ and $db^* a^\alpha b^\beta (b^*)^\gamma$.

Similar considerations for the case $x = a, a^*$ lead to the remaining terms.

Note that the presentation is not unique, since there still might remain terms, which are in \mathcal{R} , for example: $b^* db + db^* b = d(bb^*) \in \mathcal{R}$.

(ii) is direct. □

Next we can start explicit calculation of the integrals, beginning with the tadpole terms.

Application of the Leibniz rule yields to a presentation of one-forms which is different from the one of previous lemma. Each δ -one-form could be expressed, as a finite sum of the terms $x\delta(z)y$, where z is one of the generators a, a^*, b, b^* and x, y are some elements of the algebra $\mathcal{A}(SU_q(2))$.

Proposition 5.12. *For all $x, y \in \mathcal{A}(SU_q(2))$ and $z \in \{a, a^*, b, b^*\}$ we have*

$$\int x\delta(z)y |D|^{-1} = \int yx\delta(z) |D|^{-1} + \int x\delta(z)\delta(y) |D|^{-2} - \int x\delta(z)\delta^2(y) |D|^{-3}.$$

Proof. This is just the application of the trace property of the noncommutative integral, together with the identity: $|D|^{-1}\delta(z)|D|^{-1} = -[|D|^{-1}, z]$. \square

Remark 5.13. *The computation of tadpole-like integrals is reduced to the integrals $\int x\delta(z)|\mathcal{D}|^{-1}$ for all generators z and the integrals of higher order in $|\mathcal{D}|^{-2}$. However, the calculations of higher-order terms simplify a lot after we use the relations which hold up to the ideal \mathcal{R} : this erases parts of integral depending on $|\mathcal{D}|^{-2}$ and $|\mathcal{D}|^{-3}$. Thus, beside $\int x\delta(z)|\mathcal{D}|^{-1}$, we only need to compute $\int x\delta(z)\delta(z')|\mathcal{D}|^{-2}$ where z and z' are generators, since all the $|\mathcal{D}|^{-3}$ integrals have already been explicitly computed in section 4.6 (these integrals do not depend on q .)*

Besides the tadpole, the only integrals that need to be computed are $\int A|\mathcal{D}|^{-2}$ and $\int A^2|\mathcal{D}|^{-2}$ where A is a δ -1-form. Working modulo \mathcal{R} and using again Leibniz rule, we only need to compute $\int x\delta(z)|\mathcal{D}|^{-2}$ and the previous integrals $\int x\delta(z)\delta(z')|\mathcal{D}|^{-2}$.

5.2.1 Operators L_q and M_q

In the notation $v_{l,m}^j$ of \mathcal{H} , we have already use the j dependence in (12) with $J_q v_{m,l}^j := q^j v_{m,l}^j$. Let L_q and M_q be the similar diagonal operators

$$\begin{aligned} L_q v_{m,l}^j &:= q^{2l} v_{m,l}^j, \\ M_q v_{m,l}^j &:= q^{2m} v_{m,l}^j. \end{aligned}$$

We immediately get

Lemma 5.14. *For $n \in \mathbb{N}^*$, $\int (L_q)^n |\mathcal{D}^{-2}| = \int (M_q)^n |\mathcal{D}^{-2}| = \frac{2}{1-q^{2n}}$.*

Proof. We have

$$\begin{aligned} \text{Tr} (L_q^n |\mathcal{D}|^{-2-s}) &= \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \langle v_{m,l}^j, L_q^n |\mathcal{D}|^{-2-s} v_{m,l}^j \rangle \\ &= \sum_{2j=0}^{\infty} (2j+1) \frac{1-q^{2n(2j+2)}}{1-q^{2n}} d_{j^+}^{-2-s} + \sum_{2j=0}^{\infty} (2j+1) \frac{1-q^{2n(2j+2)}}{1-q^{2n}} d_j^{-2-s} \\ &\sim_0 \frac{1}{1-q^{2n}} \left(\zeta(s+1, \frac{3}{2}) + \zeta(s+1, \frac{1}{2}) \right) \sim_e \frac{2}{1-q^{2n}} \zeta(s+1). \end{aligned}$$

where \sim_0 means modulo a function holomorphic at 0. This gives the result for L_q^n and a similar computation can be done for M_q^n . \square

The interest of these operators stems in

Lemma 5.15. *We have $L_q M_q \in \mathcal{R}$. Moreover,*

$$\begin{aligned} b\delta b^* &\simeq M_q - L_q, & b^*\delta b &\simeq L_q - M_q, & bb^* &\simeq L_q + M_q, \\ a\delta(a^*) &\simeq -aa^* \simeq L_q + M_q - 1, & a^*\delta a &\simeq a^*a \simeq 1 - q^2(L_q + M_q), \\ da da^* &\simeq L_q + M_q - 1, & da^* da &\simeq q^2(L_q + M_q) - 1, \\ b^{n-2}(b^*)^n db db &\simeq (L_q)^n + (M_q)^n, \\ b^{n-1}(b^*)^{n-1} db db^* &\simeq -(L_q)^n - (M_q)^n, \\ b^n(b^*)^{n-2} db^* db^* &\simeq (L_q)^n + (M_q)^n. \end{aligned}$$

Proof. Since $L_q M_q = q^2 a_- a_-^* \in \mathcal{R}$, we compute up to the ideal \mathcal{R}

$$b \delta b^* = (b_+ + b_-)(b_-^* - b_+^*) \simeq -b_+ b_+^* + b_- b_-^* = M_q - L_q + L_q M_q (1 - q^2) \simeq M_q - L_q$$

and similarly for the other relations. \square

5.2.2 Automorphisms of the algebra and symmetries of integrals

Proposition 5.16. *For any $n \in \mathbb{N}^*$,*

$$\begin{aligned} \int (bb^*)^n |\mathcal{D}|^{-1} &= \frac{-2(1+q^{2n})}{(1-q^{2n})^2}, \\ \int (bb^*)^n b^* \delta b |\mathcal{D}|^{-1} &= \int (bb^*)^n b \delta b^* |\mathcal{D}|^{-1} = \frac{2}{1-q^{2n+2}}, \\ \int (bb^*)^n a da^* \mathcal{D}^{-1} &= \frac{-2q^{4n+2} - 2q^{4n} - 2q^{2n+2} + 6q^{2n}}{(1-q^{2n})^2(1-q^{2n+2})}, \\ \int (bb^*)^n a^* da \mathcal{D}^{-1} &= \frac{6q^{2n+2} - 2q^{2n} - 2q^2 - 2}{(1-q^{2n})^2(1-q^{2n+2})}. \end{aligned}$$

Note that the knowledge of these integral is enough for the computation of any term of the form $\int x \delta(z) |\mathcal{D}|^{-1}$, where z is a generator, since any other δ -one-form will be unbalanced.

To show this proposition, we will use few symmetries, properties of the ideal \mathcal{R} and replacement of δ -one-forms in terms of L_q, M_q as above.

Let U be the following unitary operator on the Hilbert space:

$$U v_{m,l}^{j\uparrow} = (-1)^{m+l} v_{l,m}^{j\downarrow}, \quad U v_{m,l}^{j\downarrow} = (-1)^{m+l} v_{l,m}^{j\uparrow}.$$

Then, by explicit computations we have

$$U^* a U = a, \quad U^* a^* U = a^*, \quad U^* b U = b^*, \quad U^* b^* U = b, \quad \text{and} \quad U^* \mathcal{D} U = -\mathcal{D}.$$

Lemma 5.17. *Each noncommutative integral (55) of an element of the algebra or differential forms is (up to sign) invariant under the algebra automorphism ρ defined by*

$$\rho(a) := a, \quad \rho(a^*) := a^*, \quad \rho(b) := b^*, \quad \rho(b^*) := b. \quad (64)$$

Proof. For any homogeneous polynomial p and any $k \in \mathbb{N}$,

$$\begin{aligned} \int p(a, a^*, b, b^*, \mathcal{D}) \mathcal{D}^{-k} &= \int U^* p(a, a^*, b, b^*, \mathcal{D}) \mathcal{D}^{-n} U \\ &= (-1)^k \int p(U^* a U, U^* a^* U, U^* b U, U^* b^* U, U^* \mathcal{D} U) \mathcal{D}^{-k} \\ &= (-1)^{k+d} \int p(\rho(a), \rho(a^*), \rho(b), \rho(b^*), \mathcal{D}) \mathcal{D}^{-k}, \end{aligned}$$

where d is the degree of p with respect to D . \square

Corollary 5.18. *For any $n \in \mathbb{N}$, $\int (bb^*)^n b^* db \mathcal{D}^{-1} = \int (bb^*)^n b db^* \mathcal{D}^{-1}$.*

Lemma 5.19. For any $x, y \in \Psi^0(\mathcal{A})$,

- (i) $\int xy|\mathcal{D}|^{-1} = \int yx|\mathcal{D}|^{-1} + \int x\delta(y)|\mathcal{D}|^{-2} - \int x\delta^2(y)|\mathcal{D}|^{-2}.$
- (ii) $\int zx\mathcal{D}^{-1}y\mathcal{D}^{-1} = \int zxy\mathcal{D}^{-2}$, where $z \in \mathcal{A}$ contains b or b^* .

Proof. (i) is direct consequence of the trace property of \int and the fact that OP^{-4} operators are trace-class.

ii) We calculate:

$$\begin{aligned} \int zx\mathcal{D}^{-1}y\mathcal{D}^{-1} &= \int zx(y\mathcal{D}^{-1} - \mathcal{D}^{-1}[\mathcal{D}, y]\mathcal{D}^{-1})\mathcal{D}^{-1} = \int zxy\mathcal{D}^{-2} - \int zx\mathcal{D}^{-1}[\mathcal{D}, y]\mathcal{D}^{-2} \\ &= \int zxy\mathcal{D}^{-2}. \end{aligned}$$

The last step is based on the observation that any integral with \mathcal{D}^{-3} vanishes if the expression integrated contains b or b^* . \square

Lemma 5.20. For any $n \in \mathbb{N}$,

- (i) $\int (bb^*)^n b^* db \mathcal{D}^{-1} = \frac{2}{1-q^{2n+2}}.$
- (ii) $\int (bb^*)^n d(bb^*) \mathcal{D}^{-1} = 0.$
- (iii) $\int (bb^*)^n |\mathcal{D}|^{-1} = \frac{-2(1+q^{2n})}{(1-q^{2n})^2}.$

Proof. (i) With $n > 1$, we begin with $\int d((bb^*)^n) \mathcal{D}^{-1} = 0$, which follows directly from the trace property of the noncommutative integral. Expanding the expression using Leibniz rule and the commutation

$$x\mathcal{D}^{-1} = \mathcal{D}^{-1}x + \mathcal{D}^{-1}[D, x]\mathcal{D}^{-1}, \quad (65)$$

we obtain

$$\begin{aligned} 0 &= \sum_{k=0}^{n-1} \int b^k db b^{n-k-1} (b^*)^n \mathcal{D}^{-1} + \sum_{k=0}^{n-1} \int b^n (b^*)^k db^* (b^*)^{n-k-1} \mathcal{D}^{-1} \\ &= n \left(\int b^{n-1} (b^*)^n db \mathcal{D}^{-1} + \int b^n (b^*)^{n-1} db^* \mathcal{D}^{-1} \right) \\ &\quad + \sum_{k=0}^{n-1} \int \left(b^k db \mathcal{D}^{-1} d(b^{n-k-1} (b^*)^n) \mathcal{D}^{-1} + b^n (b^*)^k db^* \mathcal{D}^{-1} d((b^*)^{n-k-1}) \mathcal{D}^{-1} \right). \end{aligned}$$

Using Lemma 5.19,

$$\begin{aligned} 0 &= n \int (bb^*)^{n-1} (b^* db + b db^*) \mathcal{D}^{-1} \\ &\quad + \int \left(\frac{1}{2}n(n-1)b^{n-2}(b^*)^n db db + n^2 b^{n-1} (b^*)^{n-1} db db^* + \frac{1}{2}n(n-1)b^n (b^*)^{n-2} db^* db^* \right) \mathcal{D}^{-2}. \end{aligned}$$

The integrals with \mathcal{D}^{-2} could be easily calculated when we take restrict ourselves to calculations modulo ideal \mathcal{R} :

$$n \int (bb^*)^{n-1} (b^* db + b db^*) \mathcal{D}^{-1} = -2 (n(n-1) - 2n^2 + n(n-1)) \frac{1}{1-q^{2n}} = 4n \frac{1}{1-q^{2n}}.$$

Hence $\int (bb^*)^{n-1} (b^* db + b db^*) \mathcal{D}^{-1} = \frac{4}{1-q^{2n}}$, which together with Corollary 5.18 proves *i*).

(ii) In a similar way, $\int (bb^*)^{n-1} d(bb^*) \mathcal{D}^{-1} = 0 = \int (bb^*)^{n-1} d(aa^*) \mathcal{D}^{-1}$ implies:

$$\begin{aligned} 0 &= \sum_{k=0}^{n-1} (bb^*)^{n-k-1} d(bb^*) (bb^*)^k \mathcal{D}^{-1} \\ &= n \int (bb^*)^{n-1} d(bb^*) \mathcal{D}^{-1} + \frac{1}{2} n(n-1) \int (bb^*)^{n-2} d(bb^*) d(bb^*) \mathcal{D}^{-2} \\ &= n \int (bb^*)^{n-1} d(bb^*) \mathcal{D}^{-1}, \end{aligned}$$

where in the last step we used that $d(bb^*) \in \mathcal{R}$. The identity (ii) now follows from the equality $aa^* = 1 - bb^*$.

(iii) Using Lemma 5.19, we get

$$A_n := \int (bb^*)^n |D|^{-1} = \int (bb^*)^n (aa^* + bb^*) |D|^{-1}$$

and we push now a^* through $|D|^{-1}$ and from cyclicity of the trace through $(bb^*)^n$,

$$= A_{n+1} + \int (bb^*)^n q^{2n} a^* a |D|^{-1} + \int (bb^*)^n q^{2n} a \delta(a^*) |D|^{-2}$$

the last term being calculated explicitly, since up to ideal \mathcal{R} , $a\delta(a^*) \simeq L_q + M_q - 1$,

$$= A_{n+1}(1 - q^{2n+2}) + q^{2n} A_n + 4 \left(\frac{1}{1-q^{2n+2}} - \frac{1}{1-q^{2n}} \right),$$

which leads to

$$A_n(1 - q^{2n}) + \frac{4}{1-q^{2n}} = A_{n+1}(1 - q^{2n+2}) + \frac{4}{1-q^{2n+2}}.$$

Assuming $A_n = \frac{f_n}{(1-q^{2n})^2}$ we have $\frac{f_n+4}{1-q^{2n}} = \frac{f_{n+1}+4}{1-q^{2n+2}}$, and taking into account that $A_0 = -2 \frac{1+q^2}{(1-q^2)^2}$, we obtain $A_n = -2 \frac{1+q^{2n}}{(1-q^{2n})^2}$. \square

Finally, to get Proposition 5.16, it remains to prove

Lemma 5.21. *For $n \geq 1$,*

$$\begin{aligned} \int (bb^*)^n a da^* \mathcal{D}^{-1} &= \frac{-2q^{4n+2} - 2q^{4n} - 2q^{2n+2} + 6q^{2n}}{(1-q^{2n})^2(1-q^{2n+2})}, \\ \int (bb^*)^n a^* da \mathcal{D}^{-1} &= \frac{6q^{2n+2} - 2q^{2n} - 2q^2 - 2}{(1-q^{2n})^2(1-q^{2n+2})}. \end{aligned}$$

Proof. First, using Leibniz rule, (65) and Lemma 5.19 we have (for $n \geq 1$)

$$\int (bb^*)^n a da^* \mathcal{D}^{-1} = -q^{2n} \int (bb^*)^n a^* da - \int (bb^*)^n da da^* \mathcal{D}^{-2}.$$

Further, we use the identity (56):

$$\int (bb^*)^n (a^* da + q^2 a da^* + q^2 b db^* + q^2 b^* db) \mathcal{D}^{-1} = (1 - q^2) \int (bb^*)^n |D|^{-1}.$$

taking into account that $F\mathcal{D} = |D|$.

These equations give together a system of linear equations

$$\begin{aligned} \int (bb^*)^n a da^* \mathcal{D}^{-1} + q^{2n} \int (bb^*)^n a^* da \mathcal{D}^{-1} &= -4 \left(\frac{1}{1-q^{2n+2}} - \frac{1}{1-q^{2n}} \right), \\ q^2 \int (bb^*)^n a da^* \mathcal{D}^{-1} + \int (bb^*)^n a^* da \mathcal{D}^{-1} &= -2(1 - q^2) \frac{1+q^{2n}}{(1-q^{2n})^2} - \frac{4q^2}{1-q^{2n+2}} \end{aligned}$$

which is solved by the expressions stated in the lemma. \square

5.2.3 The noncommutative integrals at $|D|^{-2}$

We need to separate this task into two problems. First, we shall to calculate all integrals $\int x \delta(z) |D|^{-2}$, with $x \in \mathcal{A}(SU_q(2))$ and z being one of the generators. The second problem is to calculate $\int x \delta(y) \delta(z) |D|^{-2}$, with both y and z being the generators $\{a, a^*, b, b^*\}$.

Lemma 5.22. *The only a priori non-vanishing integrals of the type $\int x \delta(z) |D|^{-2}$ are for $n \in \mathbb{N}$:*

$$\begin{aligned} \int (bb^*)^n b^* \delta(b) |D|^{-2} &= \int (bb^*)^n b \delta(b^*) |D|^{-2} = 0, \\ \int (bb^*)^n a \delta(a^*) |D|^{-2} &= \frac{4q^{2n}(1-q^2)}{(q^{2n+2}-1)(1-q^{2n})}, \quad n > 0 \\ \int (bb^*)^n a^* \delta(a) |D|^{-2} &= \frac{4(1-q^2)}{(1-q^{2n+2})(1-q^{2n})}. \end{aligned}$$

Proof. Since $a\delta(a^*) \simeq L_q + M_q - 1$ and $(bb^*)^n \simeq L_q^n + M_q^n$, we get

$$(bb^*)^n a\delta(a^*) \simeq L_q^{n+1} + M_q^{n+1} - L_q^n - M_q^n$$

and the second result is obtained from Lemma 5.14. The other integrals are computed in a similar way. \square

Lemma 5.23. *The only a priori non-vanishing integrals of the type $\int x dy dz |D|^{-2}$ are for $n \in \mathbb{N}$:*

$$\begin{aligned} \int (bb^*)^n (b^*)^2 db db |D|^{-2} &= \frac{4}{1-q^{2n+4}}, \\ \int (bb^*)^n db db^* |D|^{-2} &= \frac{4}{1-q^{2n+2}}, \\ \int (bb^*)^n (a^* b^*) (da db) |D|^{-2} &= 0, \\ \int (bb^*)^n (ab^*) (da^* db) |D|^{-2} &= 0, \\ \int (bb^*)^n (a^* b) (da db^*) |D|^{-2} &= 0, \\ \int (bb^*)^n (ab) (da^* db^*) |D|^{-2} &= 0, \\ \int (bb^*)^n (da da^*) |D|^{-2} &= \frac{4(q^{2n+2}-q^{2n})}{(1-q^{2n+2})(1-q^{2n})}, \quad n > 0 \\ \int (bb^*)^n (da^* da) |D|^{-2} &= \frac{4(q^2-1)}{(1-q^{2n+2})(1-q^{2n})}. \end{aligned}$$

Proof. This follows from Lemma 5.14 with the equivalences up to \mathcal{R} gathered in Lemma 5.15. \square

6 Examples of spectral action

It is clear from Theorem 4.3 that any one-form of the form ada , bdb , adb , a^*db , etc... do not contribute to the spectral action. Indeed, only the balanced parts of one-forms give a possibly nonzero term in the coefficients. Let us now give the values of the terms $\int A^n |D|^{-p}$ and the full $\zeta_{\mathcal{D}_\mathbb{A}}(0)$ for few examples

Table 1: Values of noncommutative integrals

\mathbb{A}	$\int A D ^{-3}$	$\int A^2 D ^{-3}$	$\int A^3 D ^{-3}$	$\int A D ^{-2}$	$\int A^2 D ^{-2}$	$\int A D ^{-1}$	$\zeta_{\mathcal{D}_\mathbb{A}}(0)$
a^*da	2	2	2	$\frac{4q^2}{q^2-1}$	$\frac{4q^2(q^2+2)}{q^4-1}$	$\frac{3q^2+1}{2(q^2-1)}$	$\frac{11q^4+36q^2+13}{3(q^4-1)}$
b^*db	0	0	0	0	$\frac{-4}{q^4-1}$	$\frac{-2}{q^2-1}$	$\frac{4q^2}{q^4-1}$
ada^*	-2	2	-2	$\frac{-4}{q^2-1}$	$\frac{4(2q^2+1)}{q^4-1}$	$\frac{q^2+3}{2(q^2-1)}$	$\frac{13q^4+36q^2+11}{3(q^4-1)}$
bdb^*	0	0	0	0	$\frac{-4}{q^4-1}$	$\frac{-2}{q^2-1}$	$\frac{4q^2}{q^4-1}$

1) Clearly the spectral action depends on q : for instance,

$$\mathcal{S}(\mathcal{D}_{a^*da}, \Phi, \Lambda) = 2\Phi_3 \Lambda^3 - 8\Phi_2 \Lambda^2 + \frac{q^2+15}{2(1-q^2)} \Phi_1 \Lambda^1 + \frac{11q^4+36q^2+13}{3(q^4-1)} \Phi(0).$$

2) Moreover, for $B := a \delta a^*$ and $A := B + B^*$, we get since $B \simeq B^* \pmod{\mathcal{R}}$,

$$\int A^p |D|^{-k} = 2^p \int B^p |D|^{-k}, \quad 1 \leq p \leq k \leq 3. \quad (66)$$

Thus the spectral action of the selfadjoint one-form $\mathbb{A} := ada^* + (ada^*)^*$ is

$$\mathcal{S}(\mathcal{D}_{\mathbb{A}}, \Phi, \Lambda) = 2 \Phi_3 \Lambda^3 + 16 \Phi_2 \Lambda^2 + \frac{q^2-33}{2(1-q^2)} \Phi_1 \Lambda^1 + \frac{122q^4+168q^2-2}{3(q^4-1)} \Phi(0).$$

3) When $B_n := (bb^*)^n b \delta b^*$, then by Lemma (5.15), $B_n \simeq B_n^*$, so for $A_n := B_n + B_n^*$, the equation (66) is still valid and $\int B_n^p |\mathcal{D}|^{-k}$ are all zero but $\int B_n |\mathcal{D}|^{-1} = \frac{2}{1-q^{2n+2}}$ and $\int B_n^2 |\mathcal{D}|^{-2} = \frac{4}{1-q^{4n+4}}$, so

$$\mathcal{S}(\mathcal{D}_{A_n}, \Phi, \Lambda) = 2 \Phi_3 \Lambda^3 - \frac{1}{2} \Phi_1 \Lambda^1 + \frac{8}{1+q^{2n+2}} \Phi(0). \quad (67)$$

Remark that this spectral action still exists as $q \rightarrow 1$!

Note however that the symmetrization process (66) is not true in general, for instance if $B := a \delta b$ and $A := B + B^*$, then $\int A^2 |\mathcal{D}|^{-1} = \frac{8(q^4-q^2-1)}{(1-q^4)^2}$ while $\int B^2 |\mathcal{D}|^{-1} = 0$ or $\int [B, B^*] |\mathcal{D}|^{-1} = \frac{4}{1-q^4}$.

4) The spectral action can be also independent of q : for instance, if $\mathbb{A} = \frac{1}{1-q^2} \xi(\mathcal{D})$ is the q -dependent selfadjoint one-form given in (58), then,

$$\mathcal{S}(\mathcal{D}_{\mathbb{A}}, \Phi, \Lambda) = 2 \Phi_3 \Lambda^3 - 8 \Phi_2 \Lambda^2 + \frac{15}{2} \Phi_1 \Lambda^1 - \frac{13}{3}.$$

7 The commutative sphere \mathbb{S}^3

Since $SU(2) \simeq \mathbb{S}^3$, we get a concrete spinorial representation of the algebra $\mathcal{A} := C^\infty(\mathbb{S}^3)$ on the same Hilbert space \mathcal{H} and same Dirac operator \mathcal{D} with (10) where $q = 1$ which means that q -numbers are trivial: $[\alpha] = \alpha$. So

$$\begin{aligned} \pi(a) |j\mu n\rangle &:= \alpha_{j\mu n}^+ |j^+ \mu^+ n^+\rangle + \alpha_{j\mu n}^- |j^- \mu^+ n^+\rangle, \\ \pi(b) |j\mu n\rangle &:= \beta_{j\mu n}^+ |j^+ \mu^+ n^-\rangle + \beta_{j\mu n}^- |j^- \mu^+ n^-\rangle, \\ \pi(a^*) |j\mu n\rangle &:= \tilde{\alpha}_{j\mu n}^+ |j^+ \mu^- n^-\rangle + \tilde{\alpha}_{j\mu n}^- |j^- \mu^- n^-\rangle, \\ \pi(b^*) |j\mu n\rangle &:= \tilde{\beta}_{j\mu n}^+ |j^+ \mu^- n^+\rangle + \tilde{\beta}_{j\mu n}^- |j^- \mu^- n^+\rangle \end{aligned} \quad (68)$$

where

$$\begin{aligned} \alpha_{j\mu n}^+ &:= \sqrt{j+\mu+1} \begin{pmatrix} \frac{\sqrt{j+n+3/2}}{2j+2} & 0 \\ \frac{\sqrt{j-n+1/2}}{(2j+1)(2j+2)} & \frac{\sqrt{j+n+1/2}}{2j+1} \end{pmatrix}, \\ \alpha_{j\mu n}^- &:= \sqrt{j-\mu} \begin{pmatrix} \frac{\sqrt{j-n+1/2}}{2j+1} & -\frac{\sqrt{j+n+1/2}}{2j(2j+1)} \\ 0 & \frac{\sqrt{j-n-1/2}}{2j} \end{pmatrix}, \\ \beta_{j\mu n}^+ &:= \sqrt{j+\mu+1} \begin{pmatrix} \frac{\sqrt{j-n+3/2}}{2j+2} & 0 \\ -\frac{\sqrt{j+n+1/2}}{(2j+1)(2j+2)} & \frac{\sqrt{j-n+1/2}}{2j+1} \end{pmatrix}, \\ \beta_{j\mu n}^- &:= \sqrt{j-\mu} \begin{pmatrix} -\frac{\sqrt{j+n+1/2}}{2j+1} & -\frac{\sqrt{j-n+1/2}}{2j(2j+1)} \\ 0 & -\frac{\sqrt{j+n-1/2}}{2j} \end{pmatrix} \end{aligned}$$

with $\tilde{\alpha}_{j\mu n}^\pm := (\alpha_{j^\pm \mu^\pm n^\pm}^\mp)^*$, $\tilde{\beta}_{j\mu n}^\pm := (\beta_{j^\pm \mu^\pm n^\pm}^\mp)^*$.

Note that the representation on the vectors $v_{m,l}^j$ is not as convenient as in (11).

One can check that the generators $\pi(a)$, $\pi(b)$ and their adjoint commute and that $[x, [\mathcal{D}', y]] = 0$ for any $x, y \in \mathcal{A}$.

7.1 Translation of Dirac operator

In general the Dirac operator is defined in a more symmetric way than that we did. So, although not absolutely necessary here, we define for the interested reader the unbounded self-adjoint translated operator \mathcal{D}' on \mathcal{H} by the constant λ as

$$\mathcal{D}' := \mathcal{D} + \lambda.$$

For instance, this gives for $\lambda = -\frac{1}{2}$ in the case of \mathbb{S}^3 , see [32], $\mathcal{D}' v_{m,l}^j = (2j+1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v_{m,l}^j$ so $v_{m,l}^j$ is an eigenvector of $|\mathcal{D}'|$.

As the following lemma shows, the computation of noncommutative integrals involving \mathcal{D} can be reduced to the computation of certain noncommutative integrals involving \mathcal{D}' :

Lemma 7.1. *If $\oint' T := \text{Res}_{s=0} \text{Tr} (T|\mathcal{D}'|^{-s})$, then for any 1-form A on a spectral triple of dimension n ,*

$$\begin{aligned} \oint A |\mathcal{D}|^{-(n-2)} &= \oint' A |\mathcal{D}'|^{-(n-2)} + \lambda(n-2) \oint' A \mathcal{D}' |\mathcal{D}'|^{-n} + \lambda^2 \frac{(n-1)(n-2)}{2} \oint' A |\mathcal{D}'|^{-n}, \\ \oint A \mathcal{D}^{-(n-2)} &= \oint' A \mathcal{D}'^{-(n-2)} + \lambda(n-2) \oint' A \mathcal{D}'^{-(n-1)} + \lambda^2 \frac{(n-1)(n-2)}{2} \oint' A \mathcal{D}'^{-n}. \end{aligned}$$

Proof. Recall from [22, Proposition 4.8] that for any pseudodifferential operator P ,

$$\oint P |\mathcal{D}|^{-r} = \text{Res}_{s=0} \text{Tr} (P |\mathcal{D}|^{-r} |\mathcal{D}'|^{-s}).$$

Moreover, by [22, Lemma 4.3], for any $s \in \mathbb{C}$ and $N \in \mathbb{N}^*$

$$|\mathcal{D}|^{-s} = |\mathcal{D}'|^{-s} + \sum_{p=1}^N K_{p,s} Y^p |\mathcal{D}'|^{-s} \pmod{OP^{-N-1-\Re(s)}} \quad (69)$$

where $Y = \sum_{k=1}^N \frac{(-1)^{k+1}}{k} (-2\lambda \mathcal{D}' + \lambda^2)^k \mathcal{D}'^{-2k} \pmod{OP^{-N-1}}$ and $K_{p,s}$ are complex numbers that can be explicitly computed. Precisely, we find $K_{p,s} = (-\frac{s}{2})^p V(p)$ where $V(p)$ is the volume of the p -simplex. Since the spectral dimension is n , we work modulo $OP^{-(n+1)}$, and for $s = n-2$, we get from (69): $|\mathcal{D}|^{-(n-2)} = |\mathcal{D}'|^{-(n-2)} + \lambda(n-2) \mathcal{D}' |\mathcal{D}'|^{-n} + \lambda^2 \frac{(n-1)(n-2)}{2} |\mathcal{D}'|^{-n} \pmod{OP^{-(n+1)}}$. As a consequence, we have for $P \in OP^0$ (the OP^0 spaces are the same for \mathcal{D} or \mathcal{D}'),

$$\oint P |\mathcal{D}|^{-(n-2)} = \oint' P |\mathcal{D}'|^{-(n-2)} + \lambda(n-2) \oint' P \mathcal{D}' |\mathcal{D}'|^{-n} + \lambda^2 \frac{(n-1)(n-2)}{2} \oint' P |\mathcal{D}'|^{-n}.$$

Since A and AF are in OP^0 , we get both formulae. \square

7.2 Tadpole and spectral action on \mathbb{S}^3

We consider now the commutative spectral triple $(C^\infty(\mathbb{S}^3), \mathcal{H}, \mathcal{D})$. It is 1-summable since $\langle\langle j\mu n s | [F, \pi(x)] | j\mu n s \rangle\rangle = 0$ when $x = a, a^*, b, b^*$ for any $j, \mu, n, s = \uparrow, \downarrow$.

All integrals of above lemma are zero for \mathbb{S}^3 :

Proposition 7.2. *There is no tadpole of any order on the commutative real spectral triple $(C^\infty(\mathbb{S}^3), \mathcal{H}, \mathcal{D})$. In fact, for any one-form or δ -one-form A , $\oint AD^{-p} = 0$ for $p \in \{1, 2, 3\}$.*

Proof. We first want to prove $\int AD^{-p} = 0$ for $p \in \{1, 2, 3\}$ and any one-form A . Since the representation is real, that is any matrix elements of the generators are real, so must be the trace of AD^{-p} . Hence $\int AD^{-p} = \int A^*D^{-p}$.

The reality operator J introduced in (29) satisfies, when $q = 1$, the commutative relation $JxJ^{-1} = x^*$ for $x \in \mathcal{A}$. Thus $JAJ^{-1} = -A^*$, so $\int AD^{-p} = \int J(A^*D^{-p})J^{-1} = -\int A^*D^{-p}$ and $\int AD^{-p} = 0$. Similar proof for the δ -form AF . \square

For any selfadjoint one-form A , $\mathcal{D}_A := \mathcal{D} + \tilde{A} = \mathcal{D}$. Thus, the spectral action for the real spectral triple $(C^\infty(\mathbb{S}^3), \mathcal{H}, \mathcal{D})$ for \mathcal{D}_A is trivialized by

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 2\Phi_3\Lambda^3 - \frac{1}{2}\Phi_1\Lambda^1 + \mathcal{O}(\Lambda^{-1}). \quad (70)$$

But it is more natural to compare with the spectral action of $\mathcal{D} + A$. This is obtained respectively from Lemma 4.18 and general heat kernel approach [27]:

$$\mathcal{S}(\mathcal{D} + A, \Phi, \Lambda) = 2\Phi_3\Lambda^3 + \int |\mathcal{D} + A|^{-1} \Phi_1 \Lambda^1 + \mathcal{O}(\Lambda^{-1})$$

since all terms of (2) in Λ^{n-k} are zero for k odd and $\zeta_{\mathcal{D}+A}(0) = 0$ when n is odd: as a verification, $\int |\mathcal{D} + A|^{-2}$ is zero according to [22, Lemma 4.10], Lemmas 4.18 and Proposition 7.2. Similarly, $\zeta_{\mathcal{D}+A}(0) = 0$ because in (3), all terms with k odd are zero (same proof as in Proposition 7.2) but for k even, it is not that easy to show that $\int AD^{-1}AD^{-1} = 0$.

Moreover, the curvature term does not depend on A :

Lemma 7.3. *For any one-form A on a commutative spectral triple of dimension n based on a compact Riemannian spin^c manifold without boundary, we have*

$$\int |\mathcal{D} + A|^{-(n-2)} = \int |\mathcal{D}|^{-(n-2)}. \quad (71)$$

Proof. Follows from [23, first formula page 511] with $\rho := A = A^*$, $N(\rho) = \rho$ (the constraint $J\rho J^{-1} = \pm\rho$ is not used.)

One can also use [15, Proposition 1.149]. \square

From [22, Lemma 4.10] $\int |\mathcal{D} + A|^{-(n-2)} = \int |\mathcal{D}|^{-(n-2)} + \frac{n(n-2)}{4} \int (AF)^2 |\mathcal{D}|^{-3} + \frac{(n-2)^2}{4} \int A^2 |\mathcal{D}|^{-3}$ using $X := AD + DA + A^2$ and $[|\mathcal{D}|, A] \in OP^0$, but again, it is not that easy to show that the last two terms cancelled: for instance here, for $B = b[\mathcal{D}, b^*]$, we obtain by direct computation (using the easiest translated Dirac operator \mathcal{D}')

$$\begin{aligned} \text{Tr}(B^2|\mathcal{D}'|^{-3-s}) &= \text{Tr}((B^*)^2|\mathcal{D}'|^{-3-s}) = \frac{1}{2} \text{Tr}(BB^*|\mathcal{D}'|^{-3-s}) = \frac{1}{2} \text{Tr}(B^*B|\mathcal{D}'|^{-3-s}) \\ &= \frac{4}{3} \sum_{2j \in \mathbb{N}} \frac{j+1}{(2j+1)^{2+s}}, \end{aligned}$$

so $\int B^2|\mathcal{D}'|^{-3} = \frac{2}{3}$. Similarly, one checks that $\int (BF)^2|\mathcal{D}'|^{-3} = \frac{1}{2} \int BFB^*F|\mathcal{D}'|^{-3} = -\frac{2}{9}$. Thus if $A := B + B^*$, $\int A^2|\mathcal{D}'|^{-3} = \int B^2|\mathcal{D}'|^{-3} = \frac{2}{3}$ and $\int (AF)^2|\mathcal{D}'|^{-3} = -\frac{4}{3}$ which yields to (71).

Thus for any one-form A on the 3-sphere,

$$\mathcal{S}(\mathcal{D} + A, \Phi, \Lambda) = 2\Phi_3\Lambda^3 - \frac{1}{2}\Phi_1\Lambda^1 + \mathcal{O}(\Lambda^{-1}, A)$$

which as (70) is not identical to (67) which contains a nonzero constant term Λ^0 for $q = 1$.

8 Conclusion

We computed in this paper the full spectral action on the $SU_q(2)$ spectral triple of [21] with the reality operator J (notice the change of definition for pseudodifferential operators.) The dimension spectrum being a finite set, there is only a finite number of terms in the spectral action expansion. The tadpole hypothesis is not satisfied on $SU_q(2)$. We saw that that the action depends on q and the limit $q \rightarrow 1$ does not exist automatically. When it exists, such limit does not lead to the associated action on the commutative sphere \mathbb{S}^3 . The sign F of the Dirac operator has special properties: first, it commutes modulo $OP^{-\infty}$ with elements of the algebra, and second, it can be seen as a one-form, giving terms independent of q in the spectral action. Here, we were interested in the computation of the spectral action of a quantum group. Naturally, it would be interesting to investigate other related cases like the Podleś spheres [17, 19] or the Euclidean quantum spheres [20, 35], especially the 4-sphere [18].

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