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▶ **To cite this version:**

Bruno Iochum, A. Sitarz, Cyril Levy. Spectral action on $SU_q(2)$. Communications in Mathematical Physics, 2009, 289, pp.107-155. hal-00261850

HAL Id: hal-00261850

https://hal.science/hal-00261850v1

Submitted on 11 Jan 2024

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CENTRE DE PHYSIQUE THÉORIQUE [1]
CNRS–Luminy, Case 907
13288 Marseille Cedex 9
FRANCE

arXiv:0803.1058v2 [math-ph] 17 Mar 2008

Spectral action on $SU_q(2)$

B. Iochum[1,2], C. Levy[1,2] and A. Sitarz[3,4]

Abstract

The spectral action on the equivariant real spectral triple over $\mathcal{A}\big(SU_q(2)\big)$ is computed explicitly. Properties of the differential calculus arising from the Dirac operator are studied and the results are compared to the commutative case of the sphere $\mathbb{S}^3$.

PACS numbers: 11.10.Nx, 02.30.Sa, 11.15.Kc
MSC–2000 classes: 46H35, 46L52, 58B34
CPT-P06-2007

[1] UMR 6207
– Unité Mixte de Recherche du CNRS et des Universités Aix-Marseille I, Aix-Marseille II et de l'Université du Sud Toulon-Var
– Laboratoire affilié à la FRUMAM – FR 2291
[2] Also at Université de Provence, iochum@cpt.univ-mrs.fr, levy@cpt.univ-mrs.fr
[3] Institute of Physics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland, sitarz@if.uj.edu.pl
[4] Partially supported by Polish Government grants 189/6.PRUE/2007/7; 115/E-343/SPB/6.PR UE/DIE and N 201 1770 33

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1 Introduction

The quantum group $SU_q(2)$ has already a rather long history of studies [33] being one of the finest examples of quantum deformation. This includes an approach via the noncommutative notion of spectral triple introduced by Connes [10, 15] and various notions of Dirac operators were introduced in [2, 4, 6, 13, 28]. Finally, a real spectral triple, which was exhibited in [21], is invariant by left and right action of $\mathcal{U}_q(su(2))$ and satisfies almost all postulated axioms of triples except the commutant and first-order properties. These, however, remain valid only up to infinitesimal of arbitrary high order. The last presentation generalizes in a straightforward way all geometric construction details of the spinorial spectral triple for the classical three-sphere. In

particular, both the equivariant representation and the symmetries have a $q \to 1$ proper classical limit.
The goal of this article is to obtain the spectral action defined in [7] by

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) := \mathrm{Tr}\left(\Phi(\mathcal{D}_A/\Lambda)\right) \tag{1}$$

where $\mathcal{D}$ is the Dirac operator, A is a selfadjoint one-form, $\mathcal{D}_A = \mathcal{D} + A + JAJ^{-1}$ and J is the reality operator. Here, Φ is any even positive cut-off function which could be replaced by a step function up to some mathematical difficulties investigated in [23]. This means that $\mathcal{S}$ counts the spectral values of $|\mathcal{D}_A|$ less than the mass scale Λ. Actually, as shown in [8]

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{0<k\in Sd^+} \Phi_k\, \Lambda^k \fint |D_A|^{-k} + \Phi(0)\, \zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-1}), \tag{2}$$

$$\zeta_{D_A}(0) = \zeta_D(0) + \sum_{k=1}^{d} \tfrac{(-1)^k}{k} \fint (A\mathcal{D}^{-1})^k. \tag{3}$$

where $D_A = \mathcal{D}_A + P_A$, P_A the projection on $\mathrm{Ker}\,\mathcal{D}_A$, $\Phi_k = \frac{1}{2}\int_0^\infty \Phi(t)\, t^{k/2-1}\, dt$, d is the spectral dimension of the triple and Sd^+ is the strictly positive part of the dimension spectrum Sd of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Here, $Sd^+ = Sd = \{\,1, 2, 3\,\}$, so

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) \;=\; \sum_{1\leq k\leq 3} \Phi_k\, \Lambda^k \fint |D_A|^{-k} + \Phi(0)\, \zeta_{D_A}(0). \tag{4}$$

Recall that the tadpole of order Λ^k is the linear term in $A \in \Omega^1_D(\mathcal{A})$ in the Λ^k part of (4).
Note that there are no terms in Λ^{-k}, $k > 0$ because the dimension spectrum is bounded below by 1. This spectral action has been computed on few examples: [3, 8, 9, 15, 22, 24–26, 30, 34].
Here, we compute (4) with the main difficulty which is to control the differential calculus generated by the Dirac operator. To proceed, we introduce two presentations of one-forms. The main ingredient is $F = \mathrm{sign}\,(\mathcal{D})$ which appears to be a one-form up to $OP^{-\infty}$.
In section 2, we discuss the spectral action of an arbitrary 3-dimensional spectral triple using cocycles.
In sections 3 and 4 we recall the main results on $SU_q(2)$ of [21] and show that the full spectral action with reality operator given by (4) is completely determined by the terms

$$\fint A^q |\mathcal{D}|^{-p}, \quad 1 \leq q \leq p \leq 3\,.$$

This question of computation of spectral action was addressed in the epilogue of [37].
In section 5, we establish a differential calculus up to some ideal in pseudodifferential operators and apply these results to the precise computation of previous noncommutative integrals.
Section 6 is devoted to explicit examples, while in next section are given different comparisons with the commutative case of the 3-sphere corresponding to $SU(2)$.

2 Spectral action in 3-dimension

2.1 Tadpole and cocycles

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple of dimension 3. For $n \in \mathbb{N}^*$ and $a_i \in \mathcal{A}$, define

$$\phi_n(a_0, \cdots, a_n) := \fint a_0[\mathcal{D}, a_1]\mathcal{D}^{-1} \cdots [\mathcal{D}, a_n]\mathcal{D}^{-1}.$$

We also use notational integrals on the universal n-forms $\Omega_u^n(\mathcal{A})$ defined by

$$\int_{\phi_n} a_0 da_1 \cdots da_n := \phi_n(a_0, a_1, \cdots, a_n).$$

and the reordering fact that $(da_0)a_1 = d(a_0 a_1) - a_0 da_1$.
We use the $b - B$ bicomplex defined in [10]: b is the Hochschild coboundary map (and b' is truncated one) defined on n-cochains ϕ by

$$b\phi(a_0, \ldots, a_{n+1}) := b'\phi(a_0, \ldots, a_{n+1}) + (-1)^{n+1}\phi(a_{n+1}a_0, a_1, \ldots, a_n),$$
$$b'\phi(a_0, \ldots, a_{n+1}) := \sum_{j=0}^{n}(-1)^j \phi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}).$$

Recall that B_0 is defined on the normalized cochains ϕ_n by

$$B_0\phi_n(a_0, a_1, \ldots, a_{n-1}) := \phi_n(1, a_0, \ldots, a_{n-1}), \text{ thus } \int_{\phi_n} d\omega = \int_{B_0\phi_n} \omega \text{ for } \omega \in \Omega_u^{n-1}(\mathcal{A}).$$

Then $B := NB_0$, where $N := 1 + \lambda + \ldots \lambda^n$ is the cyclic skewsymmetrizer on the n-cochains and λ is the cyclic permutation $\lambda\phi(a_0, \ldots, a_n) := (-1)^n \phi(a_n, a_0, \ldots, a_{n-1})$.
We will also encounter the cyclic 1-cochain $N\phi_1$:

$$N\phi_1(a_0, a_1) := \phi_1(a_0, a_1) - \phi_1(a_1, a_0) \text{ and } \int_{N\phi_1} a_0 da_1 := N\phi_1(a_0, a_1).$$

Remark 2.1. *Assume the integrand of $⨍$ is in OP^{-3}. Since $[\mathcal{D}^{-1}, a] = -\mathcal{D}^{-1}[\mathcal{D}, a]\mathcal{D}^{-1} \in OP^{-2}$, this commutator introduces a integrand in OP^{-4} so has a vanishing integral: under the integral, we can commute $\mathcal{D}^{-1}$ with all $a \in \mathcal{A}$ and all one-forms.*

Lemma 2.2. *We have*

- (i) $b\phi_1 = -\phi_2$.
- (ii) $b\phi_2 = 0$.
- (iii) $b\phi_3 = 0$.
- (iv) $B\phi_1 = 0$.
- (v) $B_0\phi_2 = -(1 - \lambda)\phi_1$.
- (vi) $bB_0\phi_2 = 2\phi_2 + B_0\phi_3$.
- (vii) $B\phi_2 = 0$.
- $(viii)$ $B_0\phi_3 = Nb'\phi_1$.
- (ix) $B\phi_3 = 3B_0\phi_3$.

Proof. (i)

$$\begin{aligned} b\phi_1(a_0, a_1, a_2) &= ⨍ a_0 a_1 [\mathcal{D}, a_2]\mathcal{D}^{-1} - ⨍ a_0 \left(a_1[\mathcal{D}, a_2] + [\mathcal{D}, a_1]a_2\right)\mathcal{D}^{-1} + ⨍ a_2 a_0 [\mathcal{D}, a_1]\mathcal{D}^{-1} \\ &= ⨍ a_0 [\mathcal{D}, a_1]\left(\mathcal{D}^{-1}a_2 - a_2\mathcal{D}^{-1}\right) = -⨍ a_0 [\mathcal{D}, a_1]\mathcal{D}^{-1}[\mathcal{D}, a_2]\mathcal{D}^{-1} \\ &= -\phi_2(a_0, a_1, a_2) \end{aligned}$$

where we have used the trace property of the noncommutative integral.
(ii) $b\phi_2(a_0,a_1,a_2,a_3)$

$$
\begin{aligned}
&= \fint a_0a_1[\mathcal{D},a_2]\mathcal{D}^{-1}[\mathcal{D},a_3]\mathcal{D}^{-1} - \fint a_0\left(a_1[\mathcal{D},a_2]+[\mathcal{D},a_1]a_2\right)\mathcal{D}^{-1}[\mathcal{D},a_3]\mathcal{D}^{-1}\\
&\quad + \fint a_0[\mathcal{D},a_1]\mathcal{D}^{-1}(a_2[\mathcal{D},a_3]+[\mathcal{D},a_2]a_3)\mathcal{D}^{-1} - \fint a_3a_0[\mathcal{D},a_1]\mathcal{D}^{-1}[\mathcal{D},a_2]\mathcal{D}^{-1}\\
&= \fint a_0[\mathcal{D},a_1]\left(\mathcal{D}^{-1}a_2 - a_2\mathcal{D}^{-1}\right)[\mathcal{D},a_3]\mathcal{D}^{-1} + \fint a_0[\mathcal{D},a_1]\mathcal{D}^{-1}[\mathcal{D},a_2]\left(a_3\mathcal{D}^{-1}-\mathcal{D}^{-1}a_3\right)\\
&= -\fint a_0[\mathcal{D},a_1]\mathcal{D}^{-1}[\mathcal{D},a_2]\mathcal{D}^{-1}[D,a_3]\mathcal{D}^{-1} + \fint a_0[\mathcal{D},a_1]\mathcal{D}^{-1}[\mathcal{D},a_2]\mathcal{D}^{-1}[D,a_3]\mathcal{D}^{-1}\\
&= 0.
\end{aligned}
$$

(iii) Using Remark 2.1, we get $\phi_3(a_0,a_1,a_2,a_3) = \fint a_0[\mathcal{D},a_1][\mathcal{D},a_2][\mathcal{D},a_3]|\mathcal{D}|^{-3}$, so similar computations as for ϕ_2 gives $b\phi_3 = 0$.
(iv) $\quad B_0\phi_1(a_0) = \fint[\mathcal{D},a_0]\mathcal{D}^{-1} = \fint\left(\mathcal{D}a_0\mathcal{D}^{-1}-a_0\right) = 0.$

$$
\begin{aligned}
(v)\quad B_0\phi_2(a_0,a_1) &= \fint[\mathcal{D},a_0]\mathcal{D}^{-1}[\mathcal{D},a_1]\mathcal{D}^{-1} = \fint a_0\mathcal{D}^{-1}[\mathcal{D},a_1] - \fint a_0[\mathcal{D},a_1]\mathcal{D}^{-1}\\
&= \fint a_0a_1 - \fint a_0\mathcal{D}^{-1}a_1\mathcal{D} - \fint a_0[\mathcal{D},a_1]\mathcal{D}^{-1}\\
&= -\fint a_1[\mathcal{D},a_0]\mathcal{D}^{-1} - \fint a_0[\mathcal{D},a_1]\mathcal{D}^{-1} = -\phi_1(a_1,a_0)-\phi_1(a_0,a_1).
\end{aligned}
$$

(vi) Since $-b\lambda\phi_1(a_0,a_1,a_2) = \phi_1(a_2,a_0a_1) - \phi_1(a_1a_2,a_0) + \phi_1(a_1,a_2a_0)$, one obtains that

$$
-b\lambda\phi_1(a_0,a_1,a_2) = \fint a_0a_1\mathcal{D}^{-1}a_2\mathcal{D} + a_0\mathcal{D}^{-1}a_1\mathcal{D}a_2 - a_0\mathcal{D}^{-1}a_1a_2\mathcal{D} - a_0a_1a_2.
$$

So by direct expansion, this is equal to $-\fint a_0\mathcal{D}^{-1}[\mathcal{D},a_1]\mathcal{D}^{-1}[\mathcal{D},a_2]$ which means that

$$
\begin{aligned}
-b\lambda\phi_1(a_0,a_1,a_2) &= \fint[\mathcal{D}^{-1},a_0][\mathcal{D},a_1]\mathcal{D}^{-1}[\mathcal{D},a_2] - a_0[\mathcal{D},a_1]\mathcal{D}^{-1}[\mathcal{D},a_2]\mathcal{D}^{-1}\\
&= -B_0\phi_3(a_0,a_1,a_2) - \phi_2(a_0,a_1,a_2).
\end{aligned}
$$

Now the result follows from $(i),(v)$.
(vii) $B\phi_2 = NB_0\phi_2 = -N(1-\lambda)\phi_1 = 0$ since $N(1-\lambda) = 0$.

$$
\begin{aligned}
(viii)\quad B_0\phi_3(a_0,a_1,a_2) &= \fint[\mathcal{D},a_0]\mathcal{D}^{-1}[\mathcal{D},a_1]\mathcal{D}^{-1}[\mathcal{D},a_2]\mathcal{D}^{-1}\\
&= \fint a_0\mathcal{D}^{-1}[\mathcal{D},a_1]\mathcal{D}^{-1}[\mathcal{D},a_2] - \fint a_0[\mathcal{D},a_1]\mathcal{D}^{-1}[\mathcal{D},a_2]\mathcal{D}^{-1}\\
&= \fint a_0a_1\mathcal{D}^{-1}[\mathcal{D},a_2] - \fint a_0\mathcal{D}^{-1}a_1[\mathcal{D},a_2] - \fint a_0[\mathcal{D},a_1]\mathcal{D}^{-1}[\mathcal{D},a_2]\mathcal{D}^{-1}\\
&= \fint a_0a_1a_2 - \fint a_0a_1\mathcal{D}^{-1}a_2\mathcal{D} - \fint a_0\mathcal{D}^{-1}a_1\mathcal{D}a_2 + \fint a_0\mathcal{D}^{-1}a_1a_2\mathcal{D}\\
&\qquad - \fint a_0[\mathcal{D},a_1]\mathcal{D}^{-1}[\mathcal{D},a_2]\mathcal{D}^{-1}\\
&= \fint a_0a_1a_2 - a_2\mathcal{D}a_1a_0\mathcal{D}^{-1} + a_1a_2\mathcal{D}a_0\mathcal{D}^{-1} + a_2\mathcal{D}a_0a_1\mathcal{D}^{-1}\\
&\qquad - \left(a_0\mathcal{D}a_1a_2\mathcal{D}^{-1} - a_0\mathcal{D}a_1\mathcal{D}^{-1} - a_0a_1\mathcal{D}a_2\mathcal{D}^{-1} + a_0a_1a_2\right).
\end{aligned}
$$

Expanding $(id+\lambda+\lambda^2)b'\phi_1(a_0,a_1,a_2)$, we recover previous expression.
(ix) Consequence of $(viii)$. □

2.2 Scale-invariant term of the spectral action

We know from [8] that the scale-invariant term of the action can be written as

$$\zeta_{D_A}(0)-\zeta_D(0)=-⨍ A\mathcal{D}^{-1}+\tfrac{1}{2}⨍ A\mathcal{D}^{-1}A\mathcal{D}^{-1}-\tfrac{1}{3}⨍ A\mathcal{D}^{-1}A\mathcal{D}^{-1}A\mathcal{D}^{-1}. \tag{5}$$

In fact, this action can be expressed in dimension 3 as contributions corresponding to tadpole and the Yang–Mills and Chern–Simons actions in dimension 4:

Proposition 2.3. *For any one-form A,*

$$\zeta_{D_A}(0)-\zeta_D(0)=-\tfrac{1}{2}\int_{N\phi_1}A+\tfrac{1}{2}\int_{\phi_2}(dA+A^2)-\tfrac{1}{2}\int_{\phi_3}(AdA+\tfrac{2}{3}A^3). \tag{6}$$

To prove this, we calculate now each terms of the action.

Lemma 2.4. *For any one-form A, we have*
(i) $\int_{\phi_2}dA=\int_{B_0\phi_2}A=-\int_{\phi_1}A-\int_{\lambda\phi_1}A$.
(ii) $⨍ A\mathcal{D}^{-1}=\int_{\phi_1}A=\frac{1}{2}\int_{N\phi_1}A-\frac{1}{2}\int_{\phi_2}dA$.
(iii) $⨍ A\mathcal{D}^{-1}A\mathcal{D}^{-1}=-\int_{\phi_3}AdA+\int_{\phi_2}A^2$.
(iv) $⨍ A\mathcal{D}^{-1}A\mathcal{D}^{-1}A\mathcal{D}^{-1}=\int_{\phi_3}A^3$.

Proof. (i) and (ii) follow directly from Lemma 2.2 (v).
(iii) With the shorthand $A=a_idb_i$ (summation on i)

$$\begin{aligned}⨍ A\mathcal{D}^{-1}A\mathcal{D}^{-1}&=⨍ a_0[\mathcal{D},b_0]\mathcal{D}^{-1}a_1[\mathcal{D},b_1]\mathcal{D}^{-1}\\&=-\int_{\phi_3}AdA+⨍ a_0[\mathcal{D},b_0]a_1b_1\mathcal{D}^{-1}-⨍ a_0[\mathcal{D},b_0]a_1\mathcal{D}^{-1}b_1.\end{aligned}$$

We calculate further the remaining terms

$$\begin{aligned}⨍ a_0[\mathcal{D},b_0]a_1b_1\mathcal{D}^{-1}-⨍ a_0[\mathcal{D},b_0]a_1\mathcal{D}^{-1}b_1&=⨍ a_0\mathcal{D}b_0a_1b_1\mathcal{D}^{-1}-⨍ a_0b_0\mathcal{D}a_1b_1\mathcal{D}^{-1}\\&\quad-⨍ a_0\mathcal{D}b_0a_1\mathcal{D}^{-1}b_1+⨍ a_0b_0\mathcal{D}a_1\mathcal{D}^{-1}b_1,\end{aligned}$$

which are compared with $\int_{\phi_2}A^2=\int_{\phi_2}a_0(db_0)a_1db_1=\int_{\phi_2}a_0d(b_0a_1)db_1-a_0b_0da_1db_1$:

$$\begin{aligned}\int_{\phi_2}A^2&=⨍ a_0[\mathcal{D},b_0a_1]\mathcal{D}^{-1}[\mathcal{D},b_1]\mathcal{D}^{-1}-⨍ a_0b_0[\mathcal{D},a_1]\mathcal{D}^{-1}[\mathcal{D},b_1]\mathcal{D}^{-1}\\&=⨍ a_0\mathcal{D}b_0a_1b_1\mathcal{D}^{-1}-⨍ a_0\mathcal{D}b_0a_1\mathcal{D}^{-1}b_1-⨍ a_0b_0a_1\mathcal{D}b_1\mathcal{D}^{-1}+⨍ a_0b_0a_1b_1\\&\quad-⨍ a_0b_0\mathcal{D}a_1b_1\mathcal{D}^{-1}+⨍ a_0b_0\mathcal{D}a_1\mathcal{D}^{-1}b_1+⨍ a_0b_0a_1\mathcal{D}b_1\mathcal{D}^{-1}-⨍ a_0b_0a_1b_1\\&=⨍ a_0\mathcal{D}b_0a_1b_1\mathcal{D}^{-1}-⨍ b_1a_0\mathcal{D}b_0a_1\mathcal{D}^{-1}-⨍ a_0b_0\mathcal{D}a_1b_1\mathcal{D}^{-1}+⨍ b_1a_0b_0\mathcal{D}a_1\mathcal{D}^{-1}.\end{aligned}$$

(iv) Note that

$$\begin{aligned}\int_{\phi_3} A^3 &= \int_{\phi_3} a_0(db_0)a_1(db_1)a_2db_2 = \int_{\phi_3} a_0d(b_0a_1)d(b_1a_2)db_2 - a_0b_0da_1d(b_1a_2)db_2 \\ &\quad - a_0d(b_0a_1b_1)d(a_2db_2 + a_0b_0d(a_1b_1)da_2db_2 \\ &= \fint a_0[\mathcal{D}, b_0a_1]\mathcal{D}^{-1}[\mathcal{D}, b_1a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1} - a_0b_0[\mathcal{D}, a_1]\mathcal{D}^{-1}[\mathcal{D}, b_1a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1} \\ &\quad - a_0[\mathcal{D}, b_0a_1b_1]\mathcal{D}^{-1}[\mathcal{D}, a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1} + a_0b_0[\mathcal{D}, a_1b_1]\mathcal{D}^{-1}[\mathcal{D}, a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1}.\end{aligned}$$

Summing up the first two terms and the last two ones gives

$$\int_{\phi_3} A^3 = \fint a_0[\mathcal{D}, b_0]a_1\mathcal{D}^{-1}[\mathcal{D}, b_1a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1} - a_0[\mathcal{D}, b_0]a_1b_1\mathcal{D}^{-1}[\mathcal{D}, a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1}.$$

Using Remark 2.1, we can commute under the integral $\mathcal{D}^{-1}$ with all $a \in \mathcal{A}$ and similarly

$$\fint A\mathcal{D}^{-1}A\mathcal{D}^{-1}A\mathcal{D}^{-1} = \fint a_0[\mathcal{D}, b_0]a_1\mathcal{D}^{-1}[\mathcal{D}, b_1]a_2\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1}$$

which proves (iv). □

We deduce Proposition 2.3 from (5) using the previous lemma.

3 The $SU_q(2)$ triple

3.1 The spectral triple

We briefly recall the main facts of the real spectral triple $\big(\mathcal{A}(SU_q(2)), \mathcal{H}, \mathcal{D}\big)$ introduced in [21], see also [4, 5, 13].

The algebra:
Let $\mathcal{A} := \mathcal{A}(SU_q(2))$ be the *-algebra generated polynomially by a and b, subject to the following commutation rules with $0 < q < 1$:

$$ba = q\,ab, \qquad b^*a = q\,ab^*, \qquad bb^* = b^*b, \quad a^*a + q^2\,b^*b = 1, \qquad aa^* + bb^* = 1\,. \tag{7}$$

Lemma 3.1. *For any representation π of $\mathcal{A}$,*

$$\begin{aligned}&Spect\big(\pi(bb^*)\big) = \{\,0, q^{2k} \,|\, |k \in \mathbb{N}\,\} \text{ or } \pi(b) = 0,\\ &Spect\big(\pi(aa^*)\big) = \{\,1, 1 - q^{2k} \,|\, k \in \mathbb{N}\,\} \text{ or } \pi(b) = 0 \text{ and } \pi(a) \text{ is a unitary.}\end{aligned}$$

Proof. [31] Since $\{\,0\,\} \cup \sigma\big(\pi(aa^*)\big) = \{\,0\,\} \cup \sigma\big(\pi(a^*a)\big)$, we get

$$\{\,1\,\} \cup B = \{\,1\,\} \cup q^2B \tag{8}$$

if $B := \sigma\big(\pi(bb^*)\big)$. Since $0 \le \pi(bb^*) \le 1$, so B is a closed subset of $[0, 1]$. Assume $b \neq 0$.
Let $s := \sup(B)$ and suppose $s \neq 1$. Then $s = q^2x$ where $x \in B$. Thus $s = q^2x < x \le s$ gives $s = 0$ and the contradiction $b = 0$, thus $1 \in B$. Similar argument for $\inf(B)$ implies $0 \in B$.
Let $C := \{\,0, q^{2k} \,|\, k \in \mathbb{N}\,\} \subset B$ and assume $B\backslash C \neq \emptyset$. Then $B\backslash C = (q^2B)\backslash C$ by (8) and this is equal to $q^2(B\backslash C)$ since $q^{-2} > 1$. If $s := \sup(B\backslash C)$, then $s = \lim_n(q^2x_n)$ where $x_n \in B\backslash C$ and $s = q^2 \lim_n x_n \le q^2s$ implying $s = 0$. But $B\backslash C \subset \{\,0\,\}$ yields a contradiction, so $B\backslash C = \emptyset$. □

This lemma is interesting since it shows the appearance of discreteness for $0 \leq q < 1$ while for $q = 1$, $SU_q(2) = SU(2) \simeq \mathbb{S}^3$ and the spectrum of the commuting operator $\pi(aa^*)$ and $\pi(bb^*)$ are equal to $[0, 1]$. Moreover, all foregoing results on noncommutative integrals will involve q^2 and not q.

Any element of $\mathcal{A}$ can be uniquely decomposed as a linear combination of terms of the form $a^\alpha b^\beta b^{*\gamma}$ where $\alpha \in \mathbb{Z}$, $\beta, \gamma \in \mathbb{N}$, with the convention

$$a^{-|\alpha|} := a^{*|\alpha|}.$$

The spinorial Hilbert space:
$\mathcal{H} = \mathcal{H}^\uparrow \oplus \mathcal{H}^\downarrow$ has an orthonormal basis consisting of vectors $|j\mu n\uparrow\rangle$ with $j = 0, \frac{1}{2}, 1, \ldots$, $\mu = -j, \ldots, j$ and $n = -j^+, \ldots, j^+$, together with $|j\mu n\downarrow\rangle$ for $j = \frac{1}{2}, 1, \ldots$, $\mu = -j, \ldots, j$ and $n = -j^-, \ldots, j^-$ (here $x^\pm := x \pm \frac{1}{2}$).
It is convenient to use a vector notation, setting:

$$|j\mu n\rangle\!\rangle := \left(\begin{smallmatrix} |j\mu n\uparrow\rangle \\ |j\mu n\downarrow\rangle \end{smallmatrix}\right) \tag{9}$$

and with the convention that the lower component is zero when $n = \pm(j + \frac{1}{2})$ or $j = 0$.

The representation π and its approximate $\underline{\pi}$:
It is known that representation theory of $SU_q(2)$ is similar to that of $SU(2)$ [39]. The representation π given in [21] is:

$$\begin{aligned}
\pi(a)\,|j\mu n\rangle\!\rangle &:= \alpha^+_{j\mu n}\,|j^+\mu^+n^+\rangle\!\rangle + \alpha^-_{j\mu n}\,|j^-\mu^+n^+\rangle\!\rangle, \\
\pi(b)\,|j\mu n\rangle\!\rangle &:= \beta^+_{j\mu n}\,|j^+\mu^+n^-\rangle\!\rangle + \beta^-_{j\mu n}\,|j^-\mu^+n^-\rangle\!\rangle, \\
\pi(a^*)\,|j\mu n\rangle\!\rangle &:= \tilde{\alpha}^+_{j\mu n}\,|j^+\mu^-n^-\rangle\!\rangle + \tilde{\alpha}^-_{j\mu n}\,|j^-\mu^-n^-\rangle\!\rangle, \\
\pi(b^*)\,|j\mu n\rangle\!\rangle &:= \tilde{\beta}^+_{j\mu n}\,|j^+\mu^-n^+\rangle\!\rangle + \tilde{\beta}^-_{j\mu n}\,|j^-\mu^-n^+\rangle\!\rangle
\end{aligned} \tag{10}$$

where

$$\alpha^+_{j\mu n} := \sqrt{q^{\mu+n-1/2}[j+\mu+1]}\begin{pmatrix} q^{-j-1/2}\frac{\sqrt{[j+n+3/2]}}{[2j+2]} & 0 \\ q^{1/2}\frac{\sqrt{[j-n+1/2]}}{[2j+1][2j+2]} & q^{-j}\frac{\sqrt{[j+n+1/2]}}{[2j+1]} \end{pmatrix},$$

$$\alpha^-_{j\mu n} := \sqrt{q^{\mu+n+1/2}[j-\mu]}\begin{pmatrix} q^{j+1}\frac{\sqrt{[j-n+1/2]}}{[2j+1]} & -q^{1/2}\frac{\sqrt{[j+n+1/2]}}{[2j][2j+1]} \\ 0 & q^{j+1/2}\frac{\sqrt{[j-n-1/2]}}{[2j]} \end{pmatrix},$$

$$\beta^+_{j\mu n} := \sqrt{q^{\mu+n-1/2}[j+\mu+1]}\begin{pmatrix} \frac{\sqrt{[j-n+3/2]}}{[2j+2]} & 0 \\ -q^{-j-1}\frac{\sqrt{[j+n+1/2]}}{[2j+1][2j+2]} & q^{-1/2}\frac{\sqrt{[j-n+1/2]}}{[2j+1]} \end{pmatrix},$$

$$\beta^-_{j\mu n} := \sqrt{q^{\mu+n-1/2}[j-\mu]}\begin{pmatrix} -q^{-1/2}\frac{\sqrt{[j+n+1/2]}}{[2j+1]} & -q^{j}\frac{\sqrt{[j-n+1/2]}}{[2j][2j+1]} \\ 0 & -\frac{\sqrt{[j+n-1/2]}}{[2j]} \end{pmatrix}$$

with $\tilde{\alpha}^\pm_{j\mu n} := (\alpha^\mp_{j^\pm\mu^-n^-})^*$, $\tilde{\beta}^\pm_{j\mu n} := (\beta^\mp_{j^\pm\mu^-n^+})^*$ and with the q-number of $\alpha \in \mathbb{R}$ be defined as

$$[\alpha] := \tfrac{q^\alpha - q^{-\alpha}}{q - q^{-1}}.$$

For the purpose of this paper it is sufficient to use the approximate spinorial *-representation $\underline{\pi}$ of $SU_q(2)$ presented in [21, 38] instead of the full spinorial one π.
This approximate representation is

$$\underline{\pi}(a) := a_+ + a_-, \quad \underline{\pi}(b) := b_+ + b_-$$

with the following definitions:

$$\begin{aligned}
a_+ \,|j\mu n\rangle\!\rangle &:= q_{j^++\mu^+} \begin{pmatrix} q_{j^++n^++1} & 0 \\ 0 & q_{j^++n} \end{pmatrix} |j^+\mu^+n^+\rangle\!\rangle, \\
a_- \,|j\mu n\rangle\!\rangle &:= q^{2j+\mu+n+\frac{1}{2}} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^-\mu^+n^+\rangle\!\rangle, \\
b_+ \,|j\mu n\rangle\!\rangle &:= q^{j+n-\frac{1}{2}} q_{j^++\mu^+} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^+\mu^+n^-\rangle\!\rangle, \\
b_- \,|j\mu n\rangle\!\rangle &:= -q^{j+\mu} \begin{pmatrix} q_{j^++n} & 0 \\ 0 & q_{j^-+n} \end{pmatrix} |j^-\mu^+n^-\rangle\!\rangle.
\end{aligned} \tag{11}$$

All disregarded terms are trace-class and do not influence residue calculations. More precisely, $\pi(x) - \underline{\pi}(x) \in \mathcal{K}_q$ where $\mathcal{K}_q$ is the principal ideal generated by the operators

$$J_q \,|j\mu n\rangle\!\rangle := q^j \,|j\mu n\rangle\!\rangle. \tag{12}$$

Actually, $\mathcal{K}_q$ is independent of q and is contained in all ideals of operators such that $\mu_n = o(n^{-\alpha})$ (infinitesimal of order α) for any $\alpha > 0$, and $\mathcal{K}_q \subset OP^{-\infty}$.

We define the alternative orthonormal basis $v^{j\uparrow}_{m,l}$ and $v^{j\downarrow}_{m,l}$ and the vector notation

$$v^j_{m,l} := \begin{pmatrix} v^{j\uparrow}_{m,l} \\ v^{j\downarrow}_{m,l} \end{pmatrix} \text{ where } v^{j\uparrow}_{m,l} := |j, m-j, l-j^+, \uparrow\rangle, \quad v^{j\downarrow}_{m,l} := |j, m-j, l-j^-, \downarrow\rangle.$$

Here $j \in \frac{1}{2}\mathbb{N}$, $0 \le m \le 2j$, $0 \le l \le 2j+1$ and $v^{\downarrow,j}_{m,l}$ is zero whenever $j = 0$ or $l = 2j$ or $2j+1$. The interest is that now, the operators $a_\pm$ and $b_\pm$ satisfy the simpler relations

$$\begin{aligned}
a_+ \, v^j_{m,l} &= q_{m+1}\, q_{l+1}\, v^{j^+}_{m+1,l+1}\,, & a_- \, v^j_{m,l} &= q^{m+l+1}\, v^{j^-}_{m,l}\,, \\
b_+ \, v^j_{m,l} &= q^l\, q_{m+1}\, v^{j^+}_{m+1,l}\,, & b_- \, v^j_{m,l} &= -q^m\, q_l\, v^{j^-}_{m,l-1}\,.
\end{aligned} \tag{13}$$

Thus

$$\begin{aligned}
a_+^* \, v^j_{m,l} &= q_m\, q_l\, v^{j^-}_{m-1,l-1}\,, & a_-^* \, v^j_{m,l} &= q^{m+l+1}\, v^{j^+}_{m,l}\,, \\
b_+^* \, v^j_{m,l} &= q^l\, q_m\, v^{j^-}_{m-1,l}\,, & b_-^* \, v^j_{m,l} &= -q^m\, q_{l+1}\, v^{j^+}_{m,l+1}\,.
\end{aligned} \tag{14}$$

Moreover, we have

$$\begin{aligned}
a_-a_+ &= q^2\, a_+\, a_-\,, & b_-b_+ &= q^2\, b_+b_-\,, & b_+a_+ &= q\, a_+b_+\,, & b_-a_- &= q\, a_-b_-\,, \\
a_-^*a_+ &= q^2\, a_+a_-^*\,, & a_-^*a_- &= a_-a_-^*\,, & a_-^*b_+ &= q\, b_+a_-^*\,, & a_-^*b_- &= q\, b_-a_-^*\,, \\
a_+^*a_- &= q^2\, a_-a_+^*\,, & b_-^*b_+ &= b_+b_-^*\,, & b_-^*a_+ &= q\, a_+b_-^*\,, & a_-b_+ &= q\, b_+a_-\,.
\end{aligned} \tag{15}$$

Note for instance that

$$\begin{aligned}
a_+a_+^* \, v^j_{m,l} &= q_m^2 q_l^2\, v^j_{m,l}\,, & a_+^*a_+ \, v^j_{m,l} &= q_{m+1}^2 q_{l+1}^2\, v^j_{m,l}\,, \\
b_+b_+^* \, v^j_{m,l} &= q^{2l} q_m^2\, v^j_{m,l}\,, & b_+^*b_+ \, v^j_{m,l} &= q^{2l} q_{m+1}^2\, v^j_{m,l}\,,
\end{aligned}$$

so applied to $v^j_{m,l}$, we get the first relation (and similarly for the others)

$$a_+^* a_+ - q^2\, a_+ a_+^* + q^2\,(b_+^* b_+ - b_+ b_+^*) = 1 - q^2, \tag{16}$$
$$a_+ a_+^* + a_- a_-^* + b_+ b_+^* + b_- b_-^* = 1, \tag{17}$$
$$a_+^* a_+ + a_-^* a_- + q^2\,(b_+^* b_+ + b_-^* b_-) = 1, \tag{18}$$
$$a_-^* a_- - q^2\, a_- a_-^* + q^2\, b_-^* b_- - q^2\, b_- b_-^* = 0, \tag{19}$$
$$a_+ a_-^* + b_-^* b_+ = 0, \qquad a_-^* a_+ + q^2\, b_-^* b_+ = 0, \tag{20}$$
$$a_- a_+^* + b_+^* b_- = 0, \qquad a_+^* a_- + q^2\, b_+^* b_- = 0, \tag{21}$$
$$b_+ b_+^* - b_+^* b_+ + b_- b_-^* - b_-^* b_- = 0, \tag{22}$$
$$q\, a_+ b_- - b_- a_+ + q\, a_- b_+ - b_+ a_- = 0. \tag{23}$$

And two others:
Note that we also use two other infinite dimensional *-representations $\pi_\pm$ of $\mathcal{A}$ on $\ell^2(\mathbb{N})$ defined as follows on the orthonormal basis $\{\, \varepsilon_n : n \in \mathbb{N} \,\}$ of $\ell^2(\mathbb{N})$ by

$$\pi_\pm(a)\, \varepsilon_n := q_{n+1}\, \varepsilon_{n+1}, \qquad \pi_\pm(b)\, \varepsilon_n := \pm q^n\, \varepsilon_n, \tag{24}$$
$$q_n := \sqrt{1 - q^{2n}}.$$

These representations are irreducible but not faithful since for instance $\pi_\pm(b - b^*) = 0$.

The Dirac operator:
It is chosen the same as in the classical case of a 3-sphere with the round metric:

$$\mathcal{D}\, |j\mu n\rangle\!\rangle := \left(\begin{smallmatrix} 2j+\frac{3}{2} & 0 \\ 0 & -2j-\frac{1}{2} \end{smallmatrix}\right) |j\mu n\rangle\!\rangle, \tag{25}$$

which means, with our convention, that $\mathcal{D}\, v^j_{ml} = \left(\begin{smallmatrix} 2j+\frac{3}{2} & 0 \\ 0 & -2j-\frac{1}{2} \end{smallmatrix}\right) v^j_{ml}$. Note that this operator is asymptotically diagonal with linear spectrum and

the eigenvalues $2j + \frac{1}{2}$ for $j \in \frac{1}{2}\mathbb{N}$, have multiplicities $(2j+1)(2j+2)$,
the eigenvalues $-(2j + \frac{1}{2})$ for $j \in \frac{1}{2}\mathbb{N}^*$, have multiplicities $2j(2j+1)$.

So this Dirac operator coincide exactly with the classical one on the 3-sphere (see [1, 32]) with a gap around 0.
Let $\mathcal{D} = F|\mathcal{D}|$ be the polar decomposition of $\mathcal{D}$, thus

$$|\mathcal{D}|\, |j\mu n\rangle\!\rangle = \left(\begin{smallmatrix} d_j & 0 \\ 0 & d_j \end{smallmatrix}\right) |j\mu n\rangle\!\rangle, \quad d_j := 2j + \tfrac{1}{2}\,, \tag{26}$$
$$F\, |j\mu n\rangle\!\rangle = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) |j\mu n\rangle\!\rangle, \tag{27}$$

and it follows from (11) and (27) that

$$F \text{ commutes with } a_\pm,\, b_\pm. \tag{28}$$

The reality operator:
This antilinear operator J is defined on the basis of $\mathcal{H}$ by

$$J\, |j, \mu, n, \uparrow\rangle := i^{2(2j+\mu+n)}\, |j, -\mu, -n, \uparrow\rangle, \qquad J\, |j, \mu, n, \downarrow\rangle := i^{2(2j-\mu-n)}\, |j, -\mu, -n, \downarrow\rangle \tag{29}$$

thus it satisfies

$$J^{-1} = -J = J^* \text{ and } \mathcal{D}J = J\mathcal{D},$$
$$J\, v^{j\uparrow}_{m,l} = i^{2(m+l)-1} v^{j\uparrow}_{2j-m,2j+1-l}\,, \qquad J\, v^{j\downarrow}_{m,l} = i^{-2(m+l)+1} v^{j\downarrow}_{2j-m,2j-1-l}\,.$$

The Hopf map r
For the explicit calculations of residues, we need a *-homomorphism $r : X \to \pi_+(\mathcal{A}) \otimes \pi_-(\mathcal{A})$ defined by the tensor product in the sense of Hopf algebras of representations π_+ and π_-:

$$r(a_+) := \pi_+(a) \otimes \pi_-(a), \qquad r(a_-) := -q\, \pi_+(b) \otimes \pi_-(b^*),$$
$$r(b_+) := -\pi_+(a) \otimes \pi_-(b), \qquad r(b_-) := -\pi_+(b) \otimes \pi_-(a^*). \tag{30}$$

In fact, $\mathcal{A}$ is a Hopf *-algebra under the coproduct $\Delta(a) := a \otimes a - q\, b \otimes b^*$, $\Delta(b) := a \otimes b + b \otimes a^*$. These homomorphisms appeared in [39] with the translation $\alpha \leftrightarrow a^*$, $\gamma \leftrightarrow -b$. In particular, if $U := \left(\begin{smallmatrix} a & b \\ -qb^* & a^* \end{smallmatrix}\right)$ is the canonical generator of the $K_1(\mathcal{A})$-group $(\Delta a,\, \Delta b) = (a,\, b) \dot{\otimes} U$ where the last $\dot{\otimes}$ means the matrix product of tensors of components.

The grading:
According to the shift $j \to j^{\pm}$ appearing in formulae (13), (14), we get a $\mathbb{Z}$-grading on X defined by the degree $+1$ on $a_+, b_+, a_-{}^*, b_-{}^*$ and -1 on $a_-, b_-, a_+{}^*, b_+{}^*$.
Any operator $T \in X$ can be (uniquely) decomposed as $T = \sum_{j \in J \subset \mathbb{Z}} T_j$ where T_j is homogeneous of degree j.
For $T \in X$, T° will denote the 0-degree part of T for this grading and by a slight abuse of notations, we write $r(T)^\circ$ instead of $r(T^\circ)$.

The symbol map:
We also use the *-homomorphism σ: $\pi_\pm(\mathcal{A}) \to C^\infty(S^1)$ defined for $z \in S^1$ on the generators by

$$\sigma\big(\pi_\pm(a)\big)(z) := z, \quad \sigma\big(\pi_\pm(a^*)\big)(z) := \bar{z}, \quad \sigma\big(\pi_\pm(b)\big)(z) = \sigma\big(\pi_\pm(b^*)\big)(z) := 0.$$

The application $(\sigma \otimes \sigma) \circ r$ is defined on X (and so on $\mathcal{B}$) with values in $C^\infty(S^1) \otimes C^\infty(S^1)$.

We define

$$dT := [\mathcal{D}, T] \text{ and } \delta(T) := [|\mathcal{D}|, T].$$

Lemma 3.2. *$a_\pm$, $b_\pm$ are bounded operators on $\mathcal{H}$ such that for all $p \in \mathbb{N}$,*
(i) $\delta(a_\pm) = \pm a_\pm\,, \quad \delta(b_\pm) = \pm b_\pm\,,$
(ii) $\delta^p(\underline{\pi}(a)) = a_+ + (-1)^p a_-\,, \quad \delta^p(\underline{\pi}(b)) = b_+ + (-1)^p b_-\,,$
(iii) $\delta(a_\pm^p) = \pm p\, a_\pm^p\,, \quad \delta(b_\pm^p) = \pm p\, b_\pm^p\,.$

Proof. (i) By definition, $a_\pm\, |j\mu n\rangle\!\rangle = \left(\begin{smallmatrix} \alpha_\pm & 0 \\ 0 & \beta_\pm \end{smallmatrix}\right) |j^\pm \mu^+ n^+\rangle\!\rangle$ where the numbers $\alpha_\pm$ and $\beta_\pm$ depend on j, μ, n and q, so we get by (26)

$$\delta(a_\pm)|j\mu n\rangle\!\rangle = \left(\begin{smallmatrix} (d_{j^+}{}^{\pm})\alpha_\pm & 0 \\ 0 & d_{j^\pm}\, \beta_\pm \end{smallmatrix}\right) |j^\pm \mu^+ n^+\rangle\!\rangle - \left(\begin{smallmatrix} (d_{j^+})\alpha_\pm & 0 \\ 0 & d_j\, \beta_\pm \end{smallmatrix}\right) |j^\pm \mu^+ n^+\rangle\!\rangle$$
$$= \left(\begin{smallmatrix} \pm\alpha_\pm & 0 \\ 0 & \pm\beta_\pm \end{smallmatrix}\right) |j^\pm \mu^+ n^+\rangle\!\rangle = \pm a_\pm\, |j\mu n\rangle\!\rangle$$

and similar proofs for $b_\pm$.
(ii) and (iii) are straightforward consequences of (i) and definition of $\underline{\pi}$. ∎

We note
$\mathcal{B}$ the *-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the operators in $\delta^k(\pi(\mathcal{A}))$ for all $k \in \mathbb{N}$,
$\Psi_0^0(\mathcal{A})$ the algebra generated by $\delta^k\big(\pi(\mathcal{A})\big)$ and $\delta^k([\mathcal{D}, \pi(\mathcal{A})])$ for all $k \in \mathbb{N}$,
X the *-subalgebra of $\mathcal{B}(\mathcal{H})$ algebraically generated by the set $\{\, a_\pm, b_\pm \,\}$.

Remark 3.3. *By Lemma 3.2, we see that, modulo $OP^{-\infty}$, X is equal to $\mathcal{B}$ and in particular contains $\pi(\mathcal{A})$.*
Using (28)*, we get that $\mathcal{B} \subset \Psi_0^0(\mathcal{A}) \subset$ algebra generated by $\mathcal{B}$ and $\mathcal{B}F$.*

Note that, despite the last inclusion, F is not a priori in $\Psi_0^0(\mathcal{A})$.

3.2 The noncommutative integrals

Recall that for any pseudodifferential operator T, $⨍ T := \operatorname*{Res}_{s=0} \zeta_{\mathcal{D}}^T(s)$ where $\zeta_{\mathcal{D}}^T(s) := \operatorname{Tr}(T|\mathcal{D}|^{-s})$.

Theorem 3.4. *The dimension spectrum (without reality structure given by J) of the spectral triple $\big(\mathcal{A}(SU_q(2)), \mathcal{H}, D\big)$ is simple and equal to $\{1, 2, 3\}$.*
Moreover, the corresponding residues for $T \in \mathcal{B}$ are

$$
\begin{aligned}
⨍ T|\mathcal{D}|^{-3} &= 2(\tau_1 \otimes \tau_1)\big(r(T)^\circ\big),\\
⨍ T|\mathcal{D}|^{-2} &= 2\big(\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1\big)\big(r(T)^\circ\big),\\
⨍ T|\mathcal{D}|^{-1} &= \big(2\,\tau_0 \otimes \tau_0 - \tfrac{1}{2}\,\tau_1 \otimes \tau_1\big)\big(r(T)^\circ\big),\\
⨍ F\,T|\mathcal{D}|^{-3} &= 0,\\
⨍ F\,T|\mathcal{D}|^{-2} &= 0,\\
⨍ F\,T|\mathcal{D}|^{-1} &= \big(\tau_0 \otimes \tau_1 - \tau_1 \otimes \tau_0\big)\big(r(T)^\circ\big),
\end{aligned}
$$

where the functionals τ_0, τ_1 are defined for $x \in \pi_\pm(\mathcal{A})$ by

$$
\tau_0(x) := \lim_{N\to\infty} \big(\operatorname{Tr}_N x - (N+1)\,\tau_1(x)\big), \qquad \tau_1(x) := \tfrac{1}{2\pi}\int_0^{2\pi} \sigma(x)(e^{i\theta})\,d\theta,
$$

with $\operatorname{Tr}_N x = \sum_{n=0}^N \langle \varepsilon_n, x\,\varepsilon_n\rangle$.

Proof. Consequence of [38, Theorem 4.1 and (4.3)]. □

Remark 3.5. *Since F is not in $\mathcal{B}$, the equation of Theorem 3.4 are not valid for all $T \in \Psi_0^0(\mathcal{A})$. But when $T \in \Psi_0^0(\mathcal{A})$, $⨍ T|\mathcal{D}|^{-k} = 0$ for $k \notin \{\,1, 2, 3\,\}$ since the dimension spectrum is $\{\,1, 2, 3\,\}$ [38].*

Compared to [38] where we had

$$
\tau_0^{\uparrow}(x) := \lim_{N\to\infty} \operatorname{Tr}_N x - (N + \tfrac{3}{2})\,\tau_1(x), \qquad \tau_0^{\downarrow}(x) := \lim_{N\to\infty} \operatorname{Tr}_N x - (N + \tfrac{1}{2})\,\tau_1(x),
$$

we replaced them with τ_0:

$$\tau_0^{\uparrow} = \tau_0 - \tfrac{1}{2}\tau_1, \qquad \tau_0^{\downarrow} = \tau_0 + \tfrac{1}{2}\tau_1.$$

Note that τ_1 is a trace on $\pi_\pm(\mathcal{A})$ such that $\tau_1(1) = 1$, while τ_0 is not since $\tau_0(1) = 0$ and

$$\tau_0\big(\pi_\pm(aa^*)\big) = \lim_{N\to\infty} \sum_{n=0}^{\infty} (1 - q^{2n}) - (N+1) = -\tfrac{1}{1-q^2}, \tag{31}$$

so, because of the shift, the replacement $a \leftrightarrow a^*$ gives

$$\tau_0\big(\pi_\pm(a^*a)\big) = q^2\, \tau_0\big(\pi_\pm(aa^*)\big). \tag{32}$$

3.3 The tadpole

Lemma 3.6. *For $SU_q(2)$, the condition of the vanishing tadpole (see [15]) is not satisfied.*

Proof. For example, an explicit calculation gives $⨍ \pi(b)[\mathcal{D}, \pi(b^*)]\mathcal{D}^{-1} = \frac{2}{1-q^2}$:
Let $x, y \in \underline{\pi}(\mathcal{A})$. Since $[F, x] = 0$, we have

$$⨍ x[\mathcal{D}, y]D^{-1} = ⨍ x\delta(y)|\mathcal{D}|^{-1} = \tau'\big(r(x\delta(y))^0\big)$$

where $\tau' := 2\,\tau_0 \otimes \tau_0 - \frac{1}{2}\,\tau_1 \otimes \tau_1$.
By Lemma 3.2, $\underline{\pi}(b)\delta\big(\underline{\pi}(b^*)\big) = (b_+ + b_-)\big((b_-)^* - (b_+)^*\big) = -b_+{b_+}^* + b_-{b_-}^* + b_+{b_-}^* - b_-{b_+}^*$.
Since only the first two terms have degree 0, we get, using the formulae from Theorem 3.4

$$\begin{aligned}\tau'\big(r(-b_+{b_+}^*)\big) &= -\tau'\big(\pi_+(aa^*) \otimes \pi_-(bb^*)\big)\\ &= -2\tau_0\big(\pi_+(aa^*)\big)\tau_0\big(\pi_-(bb^*)\big) + \tfrac{1}{2}\tau_1\big(\pi_+(aa^*)\big)\tau_1\big(\pi_-(bb^*)\big)\end{aligned}$$

and $\tau_1\big(\pi_+(aa^*)\big) = \frac{1}{2\pi}\int_0^{2\pi} 1\, d\theta = 1$, $\tau_1\big(\pi_-(bb^*)\big) = 0$. Similarly, using (32)

$$\tau'\big(r(-b_-{b_-}^*)\big) = 2\tau_0\big(\pi_+(bb^*)\big)\tau_0\big(\pi_-(a^*a)\big) = 2q^2\tau_0\big(\pi_-(aa^*)\big)\tau_0\big(\pi_+(bb^*\big).$$

Since $\tau_0\big(\pi_\pm(bb^*)\big) = \mathrm{Tr}\,\big(\pi_\pm(bb^*)\big) = \sum_{n=0}^{\infty} q^{2n} = \frac{1}{1-q^2}$ and (31),

$$⨍ \pi(b)[\mathcal{D}, \pi(b^*)]\mathcal{D}^{-1} = 2\tfrac{1}{1-q^2}\tfrac{1}{1-q^2} + 2q^2\tfrac{-1}{1-q^2}\tfrac{1}{1-q^2} = \tfrac{2}{1-q^2}. \qquad \square$$

In particular the pairing of the tadpole cyclic cocycle ϕ_1 with the generator of K_1-group is nontrivial:

Remark 3.7. *Other examples: with the shortcut x instead of $\underline{\pi}(x)$,*

$$\begin{aligned}
&(\tau_1 \otimes \tau_1)\, r\big(a\delta(a^*)^\circ\big) = -1, && (\tau_1 \otimes \tau_1) r\big(a^*\delta(a)^\circ\big) = 1,\\
&(\tau_0 \otimes \tau_0)\, r\big(a\delta(a^*)^\circ\big) = \tfrac{1}{q^2-1}\,, && (\tau_0 \otimes \tau_0)\, r\big(a^*\delta(a)^\circ\big) = \tfrac{q^2}{q^2-1}\,,\\
&⨍ a\delta(a^*)|\mathcal{D}|^{-1} = \tfrac{q^2+3}{2(q^2-1)}\,, && ⨍ a^*\delta(a)|\mathcal{D}|^{-1} = \tfrac{3q^2+1}{2(q^2-1)}\,,\\
&⨍ b\delta(b)|\mathcal{D}|^{-1} = 0, && ⨍ b^*\delta(b^*)|\mathcal{D}|^{-1} = 0,\\
&⨍ b\delta(b^*)|\mathcal{D}|^{-1} = \tfrac{-2}{q^2-1}\,, && ⨍ b^*\delta(b)|\mathcal{D}|^{-1} = \tfrac{-2}{q^2-1}\,.
\end{aligned}$$

In particular, $N\Phi_1$ does not vanish on 1-forms since $\int_{N\Phi_1} a da^* = N\Phi_1(a, a^*) = -1$.

Let U be the canonical generator of the $K_1(\mathcal{A})$-group, $U = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}$ acting on $\mathcal{H} \otimes \mathbb{C}^2$. Then for $A_U := \sum_{k,l=1}^{2} \pi(U_{kl})\, d\pi(U^*{}_{kl})$, using above remark, $\int_{\phi_1} A_U = -2$ as obtained in [38, page 391]: in fact, with $P := \frac{1}{2}(1+F)$,

$$\psi_1(U,U^*) := 2\sum_{k,l} ⨍ U_{kl}\delta(U^*_{kl})P|\mathcal{D}|^{-1} - ⨍ U_{kl}\delta^2(U^*_{kl})P|\mathcal{D}|^{-2} + \tfrac{2}{3} ⨍ U_{kl}\delta^3(U^*_{kl})P|\mathcal{D}|^{-3}$$

satisfies $\psi_1(U,U^*) = 2\sum_{k,l} ⨍ U_{kl}\delta(U^*_{kl})P|\mathcal{D}|^{-1} = \int_{\phi_1} A_U$.

4 Reality operator and spectral action on $SU_q(2)$

4.1 Spectral action in dimension 3 with $[F,\mathcal{A}] \in OP^{-\infty}$

Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ a be real spectral triple of dimension 3. Assume that $[F,\mathcal{A}] \in OP^{-\infty}$, where $F := \mathcal{D}|\mathcal{D}|^{-1}$ (we suppose $\mathcal{D}$ invertible). Let $\mathbb{A}$ be a selfadjoint one form, so $\mathbb{A}$ is of the form $\mathbb{A} = \sum_i a_i db_i$ where $a_i, b_i \in \mathcal{A}$.
Thus, $\mathbb{A} \simeq AF \mod OP^{-\infty}$ where $A := \sum_i a_i\delta(b_i)$ is the δ-one-form associated to $\mathbb{A}$. Note that A and F commute modulo $OP^{-\infty}$.
We define

$$\begin{aligned} &D_{\mathbb{A}} := \mathcal{D}_{\mathbb{A}} + P_{\mathbb{A}}, \quad P_{\mathbb{A}} \text{ the projection on } \operatorname{Ker}\mathcal{D}_{\mathbb{A}}, \\ &\mathcal{D}_{\mathbb{A}} := \mathcal{D} + \widetilde{\mathbb{A}}, \quad \widetilde{\mathbb{A}} := \mathbb{A} + J\mathbb{A}J^{-1}. \end{aligned}$$

Theorem 4.1. *The coefficients of the full spectral action (with reality operator) on any real spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ of dimension 3 such that $[F,\mathcal{A}] \in OP^{-\infty}$ are*

$$(i) \qquad ⨍ |D_{\mathbb{A}}|^{-3} = ⨍ |\mathcal{D}|^{-3}.$$

$$(ii) \qquad ⨍ |D_{\mathbb{A}}|^{-2} = ⨍ |\mathcal{D}|^{-2} - 4⨍ A|\mathcal{D}|^{-3}.$$

$$(iii) \qquad ⨍ |D_{\mathbb{A}}|^{-1} = ⨍ |\mathcal{D}|^{-1} - 2⨍ A|\mathcal{D}|^{-2} + 2⨍ A^2|\mathcal{D}|^{-3} + 2⨍ AJAJ^{-1}|\mathcal{D}|^{-3}.$$

$$\begin{aligned}(iv) \qquad \zeta_{D_{\mathbb{A}}}(0) = \zeta_D(0) &- 2⨍ A|\mathcal{D}|^{-1} + ⨍ A(A+JAJ^{-1})|\mathcal{D}|^{-2} + ⨍ \delta(A)(A+JAJ^{-1})|\mathcal{D}|^{-3} \\ &- \tfrac{2}{3}⨍ A^3|\mathcal{D}|^{-3} - 2⨍ A^2JAJ^{-1}|\mathcal{D}|^{-3}.\end{aligned}$$

Proof. (i) We apply [22, Proposition 4.9].
(ii) By [22, Lemma 4.10 (i)], we have $⨍ |D_{\mathbb{A}}|^{-2} = ⨍ |\mathcal{D}|^{-2} - ⨍(\widetilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\widetilde{\mathbb{A}} + \widetilde{\mathbb{A}}^2)|\mathcal{D}|^{-4}$. By the trace property of the noncommutative integral and the fact that $\widetilde{\mathbb{A}}^2|\mathcal{D}|^{-4}$ is trace-class, we get $⨍ |D_{\mathbb{A}}|^{-2} = ⨍ |\mathcal{D}|^{-2} - 2⨍ \widetilde{\mathbb{A}}\mathcal{D}|\mathcal{D}|^{-4} = ⨍ |\mathcal{D}|^{-2} - 4⨍ \mathbb{A}\mathcal{D}|\mathcal{D}|^{-4}$. Since $\mathbb{A}\mathcal{D} \sim A|\mathcal{D}| \mod OP^{-\infty}$, we get the result.
(iii) By [22, Lemma 4.10 (ii)], we have

$$⨍ |D_{\mathbb{A}}|^{-1} = ⨍ |\mathcal{D}|^{-1} - \tfrac{1}{2}⨍(\widetilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\widetilde{\mathbb{A}} + \widetilde{\mathbb{A}}^2)|\mathcal{D}|^{-3} + \tfrac{3}{8}⨍(\widetilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\widetilde{\mathbb{A}} + \widetilde{\mathbb{A}}^2)^2|\mathcal{D}|^{-5}.$$

Following arguments of (ii), we get

$$\fint(\widetilde{\mathbb{A}}\mathcal{D}+\mathcal{D}\widetilde{\mathbb{A}}+\widetilde{\mathbb{A}}^2)|\mathcal{D}|^{-3}=4\fint A|\mathcal{D}|^{-2}+2\fint A^2|\mathcal{D}|^{-3}+2\fint AJAJ^{-1}|\mathcal{D}|^{-3},$$
$$\fint(\widetilde{\mathbb{A}}\mathcal{D}+\mathcal{D}\widetilde{\mathbb{A}}+\widetilde{\mathbb{A}}^2)^2|\mathcal{D}|^{-5}=8\fint A^2|\mathcal{D}|^{-3}+8\fint AJAJ^{-1}|\mathcal{D}|^{-3},$$

and the result follows.
(iv) By [22, Lemma 4.5] gives $\zeta_{D_{\mathbb{A}}}(0)=\sum_{j=1}^{3}\frac{(-1)^j}{j}\fint(\widetilde{\mathbb{A}}\mathcal{D}^{-1})^j$.
Moreover, we have $\fint\widetilde{\mathbb{A}}\mathcal{D}^{-1}=2\fint A|\mathcal{D}|^{-1}$ and $\fint(\widetilde{\mathbb{A}}\mathcal{D}^{-1})^2=2\fint(A|\mathcal{D}|^{-1})^2+2\fint A|\mathcal{D}|^{-1}JAJ^{-1}|\mathcal{D}|^{-1}$. Since $\delta(A)\in OP^0$, we can check that $\fint(A|\mathcal{D}|^{-1})^2=\fint A^2|\mathcal{D}|^{-2}+\fint\delta(A)A|\mathcal{D}|^{-3}$ and, with the same argument, that $\fint A|\mathcal{D}|^{-1}JAJ^{-1}|\mathcal{D}|^{-1}=\fint AJAJ^{-1}|\mathcal{D}|^{-2}+\fint\delta(A)JAJ^{-1}|\mathcal{D}|^{-3}$. Thus, we get

$$\fint(\widetilde{\mathbb{A}}\mathcal{D}^{-1})^2=2\fint A(A+JAJ^{-1})|\mathcal{D}|^{-2}+2\fint\delta(A)(A+JAJ^{-1})|\mathcal{D}|^{-3}. \tag{33}$$

The third term to be computed is

$$\fint(\widetilde{\mathbb{A}}\mathcal{D}^{-1})^3=2\fint(A|\mathcal{D}|^{-1})^3+4\fint(A|\mathcal{D}|^{-1})^2JAJ^{-1}|\mathcal{D}|^{-1}+2\fint A|\mathcal{D}|^{-1}JAJ^{-1}|\mathcal{D}|^{-1}A|\mathcal{D}|^{-1}.$$

Any operator in OP^{-4} being trace-class here, we get

$$\fint(\widetilde{\mathbb{A}}\mathcal{D}^{-1})^3=2\fint A^3|\mathcal{D}|^{-3}+4\fint A^2JAJ^{-1}|\mathcal{D}|^{-3}+2\fint AJAJ^{-1}A|\mathcal{D}|^{-3}. \tag{34}$$

Since $\fint AJAJ^{-1}A|\mathcal{D}|^{-3}=\fint A^2JAJ^{-1}|\mathcal{D}|^{-3}$ by trace property and the fact that $\delta(A)\in OP^0$, the result follows then from (33) and (34). □

Corollary 4.2. *For the spectral action of $\mathbb{A}$ without the reality operator (i.e. $\mathcal{D}_{\mathbb{A}}=\mathcal{D}+\mathbb{A}$), we get*

$$\fint|D_{\mathbb{A}}|^{-2}=\fint|\mathcal{D}|^{-2}-2\fint A|\mathcal{D}|^{-3},$$
$$\fint|D_{\mathbb{A}}|^{-1}=\fint|\mathcal{D}|^{-1}-\fint A|\mathcal{D}|^{-2}+\fint A^2|\mathcal{D}|^{-3},$$
$$\zeta_{D_{\mathbb{A}}}(0)=\zeta_{\mathcal{D}}(0)-\fint A|\mathcal{D}|^{-1}+\tfrac{1}{2}\fint A^2|\mathcal{D}|^{-2}+\tfrac{1}{2}\fint\delta(A)A|\mathcal{D}|^{-3}-\tfrac{1}{3}\fint A^3|\mathcal{D}|^{-3}.$$

4.2 Spectral action on $SU_q(2)$: main result

On $SU_q(2)$, since F commutes with $a_\pm$ and $b_\pm$, the previous lemma can be used for the spectral action computation.
Here is the main result of this section

Theorem 4.3. *In the full spectral action* (4) *(with the reality operator) of $SU_q(2)$ for a one-form*

$\mathbb{A}$ and A its associated δ-one-form, the coefficients are:

$$\begin{aligned}
&⨍ |D_{\mathbb{A}}|^{-3} = 2\,,\\
&⨍ |D_{\mathbb{A}}|^{-2} = -4 ⨍ A|\mathcal{D}|^{-3},\\
&⨍ |D_{\mathbb{A}}|^{-1} = -\tfrac{1}{2} + 2\big(⨍ A^2|\mathcal{D}|^{-3} - ⨍ A|\mathcal{D}|^{-2}\big) + \big|⨍ A|\mathcal{D}|^{-3}\big|^2,\\
&\zeta_{D_{\mathbb{A}}}(0) = -2⨍ A|\mathcal{D}|^{-1} + ⨍ A^2|\mathcal{D}|^{-2} - \tfrac{2}{3}⨍ A^3|\mathcal{D}|^{-3}\\
&\qquad\qquad + \overline{⨍ A|\mathcal{D}|^{-3}}\big(\tfrac{1}{2}⨍ A|\mathcal{D}|^{-2} - ⨍ A^2|\mathcal{D}|^{-3}\big) + \tfrac{1}{2}⨍ A|\mathcal{D}|^{-3}\overline{⨍ A|\mathcal{D}|^{-2}}\,.
\end{aligned}$$

In order to prove this theorem, we will use a decomposition of one-forms in the Poincaré-Birkhoff–Witt basis of $\mathcal{A}$ with an extension of previous representations to operators like $TJT'J^{-1}$ where T and T' are in X.

4.3 Balanced components and Poincaré–Birkhoff–Witt basis of $\mathcal{A}$

Our objective is to compute all integrals in term of A and the computation will lead to functions of A which capture certain symmetries on A.
For convenience, let us introduce now these functions:
Let $\mathbb{A} = \sum_i \pi(x^i)d\pi(y^i)$ on $SU_q(2)$ be one-form and A the associated δ-one-form. The x^i and y^i are in $\mathcal{A}$ and as such they can be uniquely written as finite sums $x^i = \sum_\alpha x^i_\alpha m^\alpha$ and $y^i = \sum_\beta y^i_\beta m^\beta$ where $m^\alpha := a^{\alpha_1}b^{\alpha_2}b^{*\alpha_3}$ is the canonical monomial of $\mathcal{A}$ with $\alpha, \beta \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N}$ based on a fixed Poincaré–Birkhoff–Witt type basis of $\mathcal{A}$.

Remark 4.4. *Any one-form $\mathbb{A} = \sum_i \pi(x^i)d\pi(y^i)$ on $SU_q(2)$ is characterized by a complex valued matrix $A^\beta_\alpha = \sum_i x^i_\alpha\, y^i_\beta$ where $\alpha, \beta \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N}$. This matrix is such that*

$$A = A^\beta_\alpha\, M^\alpha_\beta$$

where $M^\alpha_\beta := \pi(m^\alpha)\delta\big(\pi(m^\beta)\big)$.
In the following, we note

$$\bar{A} := \bar{A}^\beta_\alpha\, M^\alpha_\beta$$

so for any $p \in \mathbb{N}$, $⨍ \bar{A}|\mathcal{D}|^{-p} = \overline{⨍ A|\mathcal{D}|^{-p}}$.

This presentation of one-forms is not unique modulo $OP^{-\infty}$ since, as we will see in section 5, $F = \sum_i x_i dy_i$ where $x_i, y_i \in \mathcal{A}$, thus for any generator z, $[F, z] = \sum_i x_i d(y_i z) - x_i y_i dz - z x_i dy_i = 0$ mod $OP^{-\infty}$. We do not know however if this presentation is unique when the $OP^{-\infty}$ part is taken into account.

The δ-one-forms M^α_β are said to be *canonical*. Any product of n canonical δ-one forms, where $n \in \mathbb{N}^*$, is called a *canonical δ^n-one-form*. Thus, if A is a δ-one-form, $A^n = (A^n)^{\bar\beta}_{\bar\alpha}\, M^{\bar\alpha}_{\bar\beta}$ where $\bar\alpha = (\alpha, \alpha', \cdots, \alpha^{(n-1)})$, $\bar\beta = (\beta, \beta', \cdots, \beta^{(n-1)})$ are in $\mathbb{Z}^n \times \mathbb{N}^n \times \mathbb{N}^n$, $(A^n)^{\bar\beta}_{\bar\alpha} := A^\beta_\alpha \cdots A^{\beta^{(n-1)}}_{\alpha^{(n-1)}}$ and $M^{\bar\alpha}_{\bar\beta}$ is the canonical δ^n-one form equals to $M^\alpha_\beta \cdots M^{\alpha^{(n-1)}}_{\beta^{(n-1)}}$.

Definition 4.5. *A canonical δ^n-one-form is a-balanced if it is of the form*

$$a^{\alpha_1}\delta(a^{\beta_1})\cdots a^{\alpha_1^{(n-1)}}\delta(a^{\beta_1^{(n-1)}})$$

where $\sum_{i=0}^{n-1}\alpha_1^{(i)}+\beta_1^{(i)}=0$.
For any δ-one-form A, the a-balanced components of A^n are noted $B_a(A^n)_{\bar\alpha}^{\bar\beta}$.

Note that

$$B_a(A)_{\bar\alpha}^{\bar\beta}=A_{-\beta_1 00}^{\beta_1 00}\,\delta_{\alpha_1+\beta_1,0}\,\delta_{\alpha_2+\alpha_3+\beta_2+\beta_3,0}.$$

Definition 4.6. *A canonical δ^n-one-form is balanced if it is of the form*

$$m^{\alpha}\delta(m^{\beta})\cdots m^{\alpha^{(n-1)}}\delta(m^{\beta^{(n-1)}})$$

where $\sum_{i=0}^{n-1}\alpha_1^{(i)}+\beta_1^{(i)}=0$ *and* $\sum_{i=0}^{n-1}\alpha_2^{(i)}+\beta_2^{(i)}=\sum_{i=0}^{n-1}\alpha_3^{(i)}+\beta_3^{(i)}$.
For any δ-one-form A, the balanced components of A^n are noted $B(A^n)_{\bar\alpha}^{\bar\beta}$.

Note that

$$B(A)_{\bar\alpha}^{\bar\beta}=A_{-\beta_1\alpha_2\alpha_3}^{\beta_1\beta_2\beta_3}\,\delta_{\alpha_1+\beta_1,0}\,\delta_{\alpha_2+\beta_2,\alpha_3+\beta_3}.$$

As we will show, a contribution to the kth-coefficient in the spectral action, is only brought by one-forms $\mathbb{A}$ such that A^k is balanced (and even a-balanced in the case $k=1$).
Note also that if A is balanced, then A^k for $k\geq 1$ is also balanced, whereas the converse is false.

4.4 The reality operator J on $SU_q(2)$

For any $n,p\in\mathbb{N}$,

$$q_n:=\sqrt{1-q^{2n}},\qquad q_{-n}:=0 \text{ if } n>0,$$
$$q_{n,p}^{\uparrow}:=q_{n+1}\cdots q_{n+p}\,,\qquad q_{n,p}^{\downarrow}:=q_n\cdots q_{n-(p-1)}\,,$$

with the convention $q_{n,0}^{\uparrow}=q_{n,0}^{\downarrow}:=1$. Thus, we have the relations

$$\pi_\pm(a^p)\,\varepsilon_n=q_{n,p}^{\uparrow}\,\varepsilon_{n+p}\,,\qquad \pi_\pm(a^{*p})\,\varepsilon_n=q_{n,p}^{\downarrow}\,\varepsilon_{n-p}\,,$$
$$\pi_\pm(b^p)\,\varepsilon_n=(\pm q^n)^p\,\varepsilon_n\,,\qquad \pi_\pm(b^{*p})\,\varepsilon_n=(\pm q^n)^p\,\varepsilon_n\,,$$

where $\varepsilon_k:=0$ if $k<0$.
The sign of $x\in\mathbb{R}$ is noted η_x. By convention, $a_j:=a$, $a_{\pm,j}:=a_\pm$ if $j\geq 0$ and $a_j:=a^*$, $a_{\pm,j}:=a_\pm^*$ if $j<0$. Note that, with convention

$$q_{n,p}^{\uparrow_{\alpha_1}}:=q_{n,p}^{\uparrow}\text{ if }\alpha_1>0,\quad q_{n,p}^{\uparrow_{\alpha_1}}:=q_{n,p}^{\downarrow}\text{ if }\alpha_1<0,\text{ and }q_{n,p}^{\uparrow_0}:=1,$$

we have for any $\alpha_1\in\mathbb{Z}$ and $p\leq\alpha_1$, $\pi_\pm(a_{\alpha_1}^p)\,\varepsilon_n=q_{n,p}^{\uparrow_{\alpha_1}}\,\varepsilon_{n+\eta_{\alpha_1}p}$.
Recall that the reality operator J is defined by

$$J\,v_{m,l}^{j\uparrow}=i^{2(m+l)-1}v_{2j-m,2j+1-l}^{j\uparrow}\,,\qquad J\,v_{m,l}^{j\downarrow}=i^{-2(m+l)+1}v_{2j-m,2j-1-l}^{j\downarrow}\,,$$

thus the real conjugate operators

$$\widehat{a}_\pm:=Ja_\pm J^{-1},\quad \widehat{b}_\pm:=Jb_\pm J^{-1}$$

satisfy

$$\widehat{a}_+\, v^j_{m,l} := -q_{2j+1-m} \left(\begin{smallmatrix} q_{2j+2-l} & 0 \\ 0 & q_{2j-l} \end{smallmatrix}\right) v^{j+}_{m,l}\,, \qquad \widehat{a}_-\, v^j_{m,l} := -q^{2j-m} \left(\begin{smallmatrix} q^{2j+2-l} & 0 \\ 0 & q^{2j-l} \end{smallmatrix}\right) v^{j-}_{m-1,l-1},$$
$$\widehat{b}_+\, v^j_{m,l} := q_{2j+1-m} \left(\begin{smallmatrix} q^{2j+1-l} & 0 \\ 0 & q^{2j-1-l} \end{smallmatrix}\right) v^{j+}_{m,l+1}\,, \qquad \widehat{b}_-\, v^j_{m,l} := -q^{2j-m} \left(\begin{smallmatrix} q_{2j+1-l} & 0 \\ 0 & q_{2j-1-l} \end{smallmatrix}\right) v^{j-}_{m-1,l}\,.$$

So the real conjugate operator behave differently on the up and down part of the Hilbert space. The difference comes from the fact that the index l is not treated uniformly by J on up and down parts.
We note $\widehat{X}$ the algebra generated by $\{\,\widehat{a}_\pm, \widehat{b}_\pm\,\}$, $\widetilde{X}$ the algebra generated by $\{\,a_\pm, b_\pm, \widehat{a}_\pm, \widehat{b}_\pm\,\}$ and $\mathcal{H}' := \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z})$ and we construct two $*$-representations $\widehat{\pi}_\pm$ of $\mathcal{A}$:
The representation $\widehat{\pi}_+$ gives bounded operators on $\mathcal{H}'$ while $\widehat{\pi}_-$ represents $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}' \otimes \mathbb{C}^2)$.
The representation $\widehat{\pi}_+$ is defined on the generators by:

$$\widehat{\pi}_+(a)\, \varepsilon_m \otimes \varepsilon_{2j} := q_{2j+1-m}\, \varepsilon_m \otimes \varepsilon_{2j+1}, \qquad \widehat{\pi}_+(b)\, \varepsilon_m \otimes \varepsilon_{2j} := -q^{2j-m}\, \varepsilon_{m+1} \otimes \varepsilon_{2j+1}$$

while $\widehat{\pi}_-$ is defined by:

$$\widehat{\pi}_-(a)\, \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} := -q_{2j+1\pm1-l}\, \varepsilon_l \otimes \varepsilon_{2j+1} \otimes \varepsilon_{\uparrow\downarrow}\,,$$
$$\widehat{\pi}_-(b)\, \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} := -q^{2j\pm1-l}\, \varepsilon_{l+1} \otimes \varepsilon_{2j+1} \otimes \varepsilon_{\uparrow\downarrow}\,,$$

where $\varepsilon_{\uparrow\downarrow}$ is the canonical basis of $\mathbb{C}^2$ and the $+$ in $\pm$ corresponds to $\uparrow$ in $\uparrow\downarrow$.
The link between $\widehat{\pi}_\pm$ and $\pi_\pm$ which explains the notations about these intermediate objects and the fact that $\widehat{\pi}_\pm$ are representations on different Hilbert spaces, is in the parallel between equations (30), (35) and (36).
Let us give immediately a few properties (x_β equals x if the sign β is positive and equals x^* otherwise)

$$\begin{aligned}
\widehat{\pi}_+(a_\beta)^p\, \varepsilon_m \otimes \varepsilon_{2j} &= q^{\uparrow_\beta}_{2j-m,p}\, \varepsilon_m \otimes \varepsilon_{2j+\eta_\beta p}\,,\\
\widehat{\pi}_-(a_\beta)^p\, \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &= (-1)^p\, q^{\uparrow_\beta}_{2j\pm1-l,p}\, \varepsilon_l \otimes \varepsilon_{2j+\eta_\beta p} \otimes \varepsilon_{\uparrow\downarrow}\,,\\
\widehat{\pi}_+(b_\beta)^p\, \varepsilon_m \otimes \varepsilon_{2j} &= (-1)^p\, q^{(2j-m)p}\, \varepsilon_{m+\eta_\beta p} \otimes \varepsilon_{2j+\eta_\beta p}\,,\\
\widehat{\pi}_-(b_\beta)^p\, \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &= (-1)^p\, q^{(2j\pm1-l)p}\, \varepsilon_{l+\eta_\beta p} \otimes \varepsilon_{2j+\eta_\beta p} \otimes \varepsilon_{\uparrow\downarrow}\,.
\end{aligned}$$

Note that the $\widehat{\pi}_\pm$ representations still contain the shift information, contrary to representations $\pi_\pm$. Moreover, $\widehat{\pi}_\pm(b) \neq \widehat{\pi}_\pm(b^*)$ while $\pi_\pm(b) = \pi_\pm(b^*)$.
The operators $\widehat{a}_\pm, \widehat{b}_\pm$ are coded on $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ as the correspondence

$$\begin{aligned}
&\widehat{a}_+ \longleftrightarrow \widehat{\pi}_+(a) \otimes \widehat{\pi}_-(a), && \widehat{a}_- \longleftrightarrow -q\, \widehat{\pi}_+(b^*) \otimes \widehat{\pi}_-(b^*),\\
&\widehat{b}_+ \longleftrightarrow -\widehat{\pi}_+(a) \otimes \widehat{\pi}_-(b), && \widehat{b}_- \longleftrightarrow -\widehat{\pi}_+(b^*) \otimes \widehat{\pi}_-(a^*).
\end{aligned} \tag{35}$$

We now set the following extension to $\mathcal{B}(\mathcal{H}')$ of π_+ and to $\mathcal{B}(\mathcal{H}' \otimes \mathbb{C}^2)$ of π_- by

$$\begin{aligned}
&\pi'_+(a) := \pi_+(a) \otimes V, \quad \pi'_+(b) := \pi_+(b) \otimes V \quad \text{(V is the shift of } \ell^2(\mathbb{Z})),\\
&\pi'_-(a) := \pi_-(a) \otimes V \otimes 1_2, \quad \pi'_-(b) := \pi_-(b) \otimes V \otimes 1_2.
\end{aligned}$$

So, we can define a canonical algebra morphism $\widetilde{\rho}$ from $\widetilde{X}$ into the bounded operators on $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$. This morphism is defined on the generators part $\{\,\widehat{a}_\pm, \widehat{b}_\pm\,\}$ of $\widetilde{X}$ by preceding correspondence and on the generators part $\{\,a_\pm, b_\pm\,\}$ by $-$see (30):

$$\begin{aligned}
&a_+ \longleftrightarrow \pi'_+(a) \otimes \pi'_-(a), && a_- \longleftrightarrow -q\, \pi'_+(b^*) \otimes \pi'_-(b^*),\\
&b_+ \longleftrightarrow -\pi'_+(a) \otimes \pi'_-(b), && b_- \longleftrightarrow -\pi'_+(b^*) \otimes \pi'_-(a^*).
\end{aligned} \tag{36}$$

We note S the canonical surjection from $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ onto $\mathcal{H}$. This surjection is associated to the parameters restrictions on m, j, l, j'. In particular, the index j' associated to the second $\ell^2(\mathbb{N})$ in $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ is set to be equal to j. Any vector in $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ not satisfying these restrictions is sent to 0 in $\mathcal{H}$.
Denote by I the canonical injection of $\mathcal{H}$ into $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ (the index j is doubled). Thus, $S\widetilde{\rho}(\cdot)I$ is the identity on $\widetilde{X}$.
In the computation of residues of $\zeta_{\mathcal{D}}^T$ functions, we can therefore replace the operator T by $S\widetilde{\rho}(T)I$.
We now extend τ_0 on $\pi'_{\pm}(\mathcal{A})\widehat{\pi}_{\pm}(\mathcal{A})$: For $x, y \in \mathcal{A}$, we set

$$\begin{aligned}
\mathrm{Tr}_N\left(\pi'_+(x)\widehat{\pi}_+(y)\right) &:= \sum_{m=0}^{N} \langle \varepsilon_m \otimes \varepsilon_N \,,\, \pi'_+(x)\widehat{\pi}_+(y)\, \varepsilon_m \otimes \varepsilon_N \rangle \,, \\
\mathrm{Tr}_N^{\uparrow}\left(\pi'_-(x)\widehat{\pi}_-(y)\right) &:= \sum_{l=0}^{N} \langle \varepsilon_l \otimes \varepsilon_{N-1} \otimes \varepsilon_{\uparrow} \,,\, \pi'_-(x)\widehat{\pi}_-(y)\, \varepsilon_l \otimes \varepsilon_{N-1} \otimes \varepsilon_{\uparrow} \rangle \,, \\
\mathrm{Tr}_N^{\downarrow}\left(\pi'_-(x)\widehat{\pi}_-(y)\right) &:= \sum_{l=0}^{N} \langle \varepsilon_l \otimes \varepsilon_{N+1} \otimes \varepsilon_{\downarrow} \,,\, \pi'_-(x)\widehat{\pi}_-(y)\, \varepsilon_l \otimes \varepsilon_{N+1} \otimes \varepsilon_{\downarrow} \rangle \,.
\end{aligned}$$

Actually, a computation on monomials of $\mathcal{A}$ shows that $\mathrm{Tr}_N^{\downarrow}\left(\pi'_-(x)\widehat{\pi}_-(y)\right) = \mathrm{Tr}_N^{\uparrow}\left(\pi'_-(x)\widehat{\pi}_-(y)\right)$. For convenience, we shall note $\mathrm{Tr}_N\left(\pi'_-(x)\widehat{\pi}_-(y)\right)$ this functional.

Lemma 4.7. *Let $x, y \in \mathcal{A}$. Then,*
(i) $\tau_0\big(\pi'_{\pm}(x)\widehat{\pi}_{\pm}(y)\big) := \lim_{N\to\infty} U_N$ exists where

$$U_N := \mathrm{Tr}_N\left(\pi'_{\pm}(x)\widehat{\pi}_{\pm}(y)\right) - (N+1)\tau_1\big(\pi_{\pm}(x)\big)\,\tau_1\big(\pi_{\pm}(y)\big).$$

(ii) $U_N = \tau_0\big(\pi'_{\pm}(x)\widehat{\pi}_{\pm}(y)\big) + \mathcal{O}(N^{-k})$ for all $k > 0$.

Proof. (i) We can suppose that x and y are monomials, since the result will follow by linearity. We will give a proof for the case of the π_+ representations, the case π_- being similar, with minor changes.
We have $\widehat{\pi}_+(y) = (\widehat{\pi}_+ a_{\beta_1})^{|\beta_1|}\,(\widehat{\pi}_+ b)^{\beta_2}\,(\widehat{\pi}_+ b^*)^{\beta_3}$. A computation gives

$$\widehat{\pi}_+(y)\,\varepsilon_m \otimes \varepsilon_{2j} = (-1)^{\beta_2+\beta_3}\, q^{(2j-m)(\beta_2+\beta_3)}\, q_{2j-m,|\beta_1|}^{\uparrow_{\beta_1}}\, \varepsilon_{m-\beta_3+\beta_2} \otimes \varepsilon_{2j-\beta_3+\beta_2+\beta_1}$$

and with the notation $t_{2j,m} := \langle \varepsilon_m \otimes \varepsilon_{2j} \,,\, \pi'_{\pm}(x)\widehat{\pi}_{\pm}(y)\, \varepsilon_m \otimes \varepsilon_{2j} \rangle$ and $T_{2j} := \sum_{m=0}^{2j} t_{2j,m}$, we get

$$\begin{aligned}
t_{2j,m} &= (-1)^{\beta_2+\beta_3}\, q^{(2j-m)(\beta_2+\beta_3)}\, q_{2j-m,|\beta_1|}^{\uparrow_{\beta_1}}\, q_{m-\beta_3+\beta_2,|\alpha_1|}^{\uparrow_{\alpha_1}}\, q^{(m+\beta_2-\beta_3)(\alpha_2+\alpha_3)}\, \delta_{\alpha_1+\beta_2-\beta_3,0} \\
&\qquad \times \delta_{-\alpha_3+\alpha_2+\beta_1,0} \\
&= (-1)^{\alpha_1}\, q^{(2j-m)(\beta_2+\beta_3)+(m-\alpha_1)(\alpha_2+\alpha_3)}\, q_{2j-m,|\beta_1|}^{\uparrow_{\beta_1}}\, q_{m-\alpha_1,|\alpha_1|}^{\uparrow_{\alpha_1}}\, \delta_{\alpha_1+\beta_2-\beta_3,0}\, \delta_{\alpha_2-\alpha_3+\beta_1,0} \\
&=: f_{\alpha,\beta}\; q^{2j\lambda}\, t'_{2j,m} =: f_{\alpha,\beta}\; q^{2j\kappa}\, t''_{2j,2j-m}
\end{aligned}$$

where

$$t'_{2j,m} := q^{m(\kappa-\lambda)}\, q_{2j-m,|\beta_1|}^{\uparrow_{\beta_1}}\, q_{m-\alpha_1,|\alpha_1|}^{\uparrow_{\alpha_1}} \,, \tag{37}$$

$$t''_{2j,m} := q^{m(\lambda-\kappa)}\, q_{m,|\beta_1|}^{\uparrow_{\beta_1}}\, q_{2j-m-\alpha_1,|\alpha_1|}^{\uparrow_{\alpha_1}} \,, \tag{38}$$

with $\lambda := \beta_2 + \beta_3 \geq 0$ and $\kappa := \alpha_2 + \alpha_3 \geq 0$. We will now prove that if $\lambda \neq \kappa$, then (T_{2j}) is a convergent sequence. Suppose $\kappa > \lambda$. Let us note $U'_{2j} := \sum_{m=0}^{2j} t'_{2j,m}$. Since the $t'_{2j,m}$ are positive and $t'_{2j+1,m} \geq t'_{2j,m}$ for all j, m, U'_{2j} is an increasing real sequence. The estimate

$$U'_{2j} \leq \sum_{m=0}^{2j} q^{m(\kappa-\lambda)} \leq \tfrac{1}{1-q^{\kappa-\lambda}} < \infty$$

proves then that U'_{2j} is a convergent sequence. With $T_{2j} = f_{\alpha,\beta}\, q^{2j\lambda}\, U'_{2j}$, we obtain our result. Suppose now that $\lambda > \kappa$. Let us note $U''_{2j} := \sum_{m=0}^{2j} t''_{2j,m}$. Since the $t''_{2j,m}$ are positive and $t''_{2j+1,m} \geq t''_{2j,m}$ for all j, m, U''_{2j} is an increasing real sequence. The estimate

$$U''_{2j} \leq \sum_{m=0}^{2j} q^{m(\lambda-\kappa)} \leq \tfrac{1}{1-q^{\lambda-\kappa}} < \infty$$

proves then that U''_{2j} is a convergent sequence. With $T_{2j} = f_{\alpha,\beta}\, q^{2j\kappa}\, U''_{2j}$, we have again our result. Moreover, note that if λ and κ are both different from zero, the limit of (T_{2j}) is zero and more precisely,

$$T_{2j} = \mathcal{O}(q^{2j\lambda}) \text{ if } \kappa > \lambda > 0, \tag{39}$$

$$T_{2j} = \mathcal{O}(q^{2j\kappa}) \text{ if } \lambda > \kappa > 0. \tag{40}$$

Suppose now that $\lambda = \kappa \neq 0$. In that case, (T_{2j}) also converges rapidly to zero. Indeed, let us fix $q < \varepsilon < 1$. we have $\varepsilon^{-2j\lambda}\, T_{2j} = \sum_{m=0}^{2j} c_m d_{2j-m} = c * d(2j)$ where $c_m := f_{\alpha,\beta}\, (q/\varepsilon)^{\lambda m}\, q^{\uparrow_{\alpha_1}}_{m-\alpha_1,|\alpha_1|}$ and $d_m := (q/\varepsilon)^{\lambda m} q^{\uparrow_{\beta_1}}_{m,|\beta_1|}$. Since both $\sum_m c_m$ and $\sum_m d_m$ are absolutely convergent series, their Cauchy product $\sum_{2j} \varepsilon^{-2j\lambda}\, T_{2j}$ is convergent. In particular, $\lim_{j\to\infty} \varepsilon^{-2j\lambda}\, T_{2j} = 0$, and

$$T_{2j} = \mathcal{O}(\varepsilon^{2j\lambda}). \tag{41}$$

Finally, T_{2j} has a finite limit in all cases except possibly when $\lambda = \kappa = 0$, which is the case when $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$. In that case, $t_{2j,m} = 1$.
A straightforward computation gives $\tau_1\big(\pi_\pm(x)\big)\, \tau_1\big(\pi_\pm(y)\big) = \delta_{\alpha_1,0}\, \delta_{\beta_1,0}\, \delta_{\alpha_2,0}\, \delta_{\beta_2,0}\, \delta_{\alpha_3,0}\, \delta_{\beta_3,0}$. Thus,

$$U_{2j} = T_{2j} - (2j+1)\delta_{\alpha_1,0}\, \delta_{\beta_1,0}\, \delta_{\alpha_2,0}\, \delta_{\beta_2,0}\, \delta_{\alpha_3,0}\, \delta_{\beta_3,0}$$

has always a finite limit when $j \to \infty$.
(ii) The result is clear if $\lambda = \kappa = 0$ (in that case $U_N = \tau_0 = 0$). Suppose λ or κ is not zero. In that case $U_{2j} = T_{2j}$. By (40), (39) and (41), we see that if $\lambda > \kappa > 0$ or $\kappa > \lambda > 0$ or $\kappa = \lambda$, (T_{2j}) converges to 0 with a rate in $\mathcal{O}(\varepsilon^{2j\alpha})$ where $\alpha > 0$ and $q \leq \varepsilon < 1$. Thus, it only remains to check the cases $(\kappa > 0, \lambda = 0)$ and $(\kappa = 0, \lambda > 0)$. In the first one, we get from (37), $U_{2j} = f_{\alpha,\beta} \sum_{m=0}^{2j} q^{m\kappa} q^{\uparrow_{\beta_1}}_{2j-m,|\beta_1|}$. If $\beta_1 = 0$, we are done.
Suppose $\beta_1 > 0$. We have $q^{\uparrow_{\beta_1}}_{2j-m,|\beta_1|} = \sum_{p=0}^{\infty} l_p\, q^{r_p}\, q^{2|p|_1(2j-m)}$ where $p = (p_1, \cdots, p_{\beta_1})$ and $l_p = (-1)^{|p|_1} \binom{\frac{1}{2}}{p}$, $r_p := 2p_1 + \cdots + 2\beta_1 p_{\beta_1}$. Thus, cutting the sum in two, we get, noting $L_{2j} := f_{\alpha,\beta} \sum_{m=0}^{2j} q^{m\kappa}$,

$$U_{2j} - L_{2j} = f_{\alpha,\beta} \sum_{|p|_1 > \kappa/2} l_p\, q^{r_p}\, \tfrac{q^{4|p|_1 j} - q^{(2j+1)\kappa - 2|p|_1}}{1-q^{\kappa-2|p|_1}} + f_{\alpha,\beta} \sum_{0 \neq |p|_1 \leq \kappa/2} l_p\, q^{r_p}\, q^{4|p|_1 j} \sum_{m=0}^{2j} q^{m(\kappa-2|p|_1)}.$$

Since $\sum_{0\neq|p|_1\leq\kappa/2} l_p\, q^{r_p}\, q^{4|p|_1 j} \sum_{m=0}^{2j} q^{m(\kappa-2|p|_1)}$ is in $\mathcal{O}_{j\to\infty}(jq^{4j})$, we have, modulo a rapidly decreasing sequence,

$$U_{2j} - L_{2j} \sim f_{\alpha,\beta} \sum_{|p|_1>\kappa/2} l_p\, q^{r_p}\, \tfrac{q^{4|p|_1 j} - q^{(2j+1)\kappa-2|p|_1}}{1-q^{\kappa-2|p|_1}} =: f_{\alpha,\beta} q^{2\kappa j} V_{2j}$$

with

$$V_{2j} = \sum_{|p|_1>\kappa/2} l_p\, q^{r_p}\, \tfrac{1-q^{(2|p|_1-\kappa)(2j+1)}}{1-q^{2|p|_1-\kappa}} = \sum_{|p|_1>\kappa/2} \sum_{m=0}^{2j} l_p\, q^{r_p}\, q^{(2|p|_1-\kappa)m}.$$

The family $v_{m,p} := (l_p\, q^{r_p}\, q^{(2|p|_1-\kappa)m})_{(p,m)\in I}$, where $I = \{\, (p,m) \in \mathbb{N}^{\beta_1} \times \mathbb{N} \;:\; |p|_1 > \kappa/2 \,\}$ is (absolutely) summable. Indeed $|v_{m,p}| \leq |l_p| q^{r_p}\, q^m$ so $|v_{m,p}|$ is summable as the product of two summable families. As a consequence, $\lim_{j\to\infty} V_{2j}$ exists and is finite, which proves that $(q^{2\kappa j} V_{2j})$, and thus $(U_{2j} - L_{2j})$ converge rapidly to 0.
Suppose now that $\beta_1 < 0$. In that case, $q^{\uparrow_{\beta_1}}_{2j-m,|\beta_1|} = q^{\downarrow}_{2j-m,|\beta_1|} = q^{\uparrow}_{2j-(m+|\beta_1|),|\beta_1|}$ and by (37), we get $U_{2j} = f_{\alpha,\beta} \sum_{m=0}^{2j} q^{m\kappa} q^{\uparrow}_{2j-(m+|\beta_1|),|\beta_1|} = f_{\alpha,\beta}\, q^{-|\beta_1|\kappa} \sum_{m=|\beta_1|}^{2j+|\beta_1|} q^{m\kappa} q^{\uparrow}_{2j-m,|\beta_1|}$, so the same arguments as in case $\beta_1 > 0$ apply here, the summation on m simply shifted of $|\beta_1|$.
The same proof can be applied for the other case ($\kappa = 0, \lambda > 0$). This time, we only need to use (38) instead of (37) and the preceding arguments follow by replacing κ by λ and β_1 by α_1. □

Remark 4.8. *Contrary to the preceding τ_0, the new functional contains the shift information. In particular, it filters the parts of nonzero degree.*

If $T \in X\widehat{X}$, $\widetilde{\rho}(T) \in \pi_+(\mathcal{A})\widehat{\pi}_+(\mathcal{A}) \otimes \pi_-(\mathcal{A})\widehat{\pi}_-(\mathcal{A})$.
For notational convenience, we define τ_1 on $\pi'_\pm(\mathcal{A})\widehat{\pi}_\pm(\mathcal{A})$ as

$$\tau_1\big(\pi'_\pm(x)\widehat{\pi}_\pm(y)\big) := \tau_1\big(\pi_\pm(x)\big)\, \tau_1\big(\pi_\pm(y)\big).$$

In the following, the symbol $\sim_e$ means equals modulo a entire function.

Theorem 4.9. *Let $T \in X\widehat{X}$. Then*

$$\begin{aligned}
(i)\qquad \zeta^T_{\mathcal{D}}(s) \sim_e\; & 2(\tau_1\otimes\tau_1)\big(\widetilde{\rho}(T)\big)\,\zeta(s-2) + 2(\tau_0\otimes\tau_1 + \tau_1\otimes\tau_0)\big(\widetilde{\rho}(T)\big)\,\zeta(s-1)\\
& + 2(\tau_0\otimes\tau_0 - \tfrac{1}{2}\tau_1\otimes\tau_1)\big(\widetilde{\rho}(T)\big)\,\zeta(s),\\
(ii)\qquad \fint T|\mathcal{D}|^{-3} &= 2(\tau_1\otimes\tau_1)\big(\widetilde{\rho}(T)\big),\\
(iii)\qquad \fint T|\mathcal{D}|^{-2} &= 2(\tau_0\otimes\tau_1 + \tau_1\otimes\tau_0)\big(\widetilde{\rho}(T)\big),\\
(iv)\qquad \fint T|\mathcal{D}|^{-1} &= 2(\tau_0\otimes\tau_0 - \tfrac{1}{2}\tau_1\otimes\tau_1)\big(\widetilde{\rho}(T)\big).
\end{aligned}$$

Proof. (i) Since $T \in X\widehat{X}$, $\widetilde{\rho}(T)$ is a linear combination of terms like $\pi'_+(x)\widehat{\pi}_+(y) \otimes \pi'_-(z)\widehat{\pi}_-(t)$, where $x, y, z, t \in \mathcal{A}$. Such a term is noted in the following $T_+ \otimes T_-$. Linear combination of these term is implicit. With the shortcut $T_{c_1,\cdots,c_p} := \langle \varepsilon_{c_1} \otimes \cdots \otimes \varepsilon_{c_p}, T\varepsilon_{c_1} \otimes \cdots \otimes \varepsilon_{c_p}\rangle$, recalling that

$v^{j,\downarrow}_{m,l}$ is 0 when $j=0$, or $l \geq 2j$, we get

$$\begin{aligned}
\zeta^T_{\mathcal{D}}(s) &= \sum_{2j=0}^{\infty}\sum_{m=0}^{2j}\sum_{l=0}^{2j+1} \langle \left(\begin{smallmatrix} v^{j,\uparrow}_{m,l} \\ 0 \end{smallmatrix}\right), S\widetilde{\rho}(T) I \left(\begin{smallmatrix} v^{j,\uparrow}_{m,l} \\ 0 \end{smallmatrix}\right)\rangle \, d_{j^+}^{-s} + \langle \left(\begin{smallmatrix} 0 \\ v^{j,\downarrow}_{m,l} \end{smallmatrix}\right), S\widetilde{\rho}(T) I \left(\begin{smallmatrix} 0 \\ v^{j,\downarrow}_{m,l} \end{smallmatrix}\right)\rangle \, d_{j}^{-s} \\
&= \sum_{2j=0}^{\infty}\sum_{m=0}^{2j}\sum_{l=0}^{2j+1} \widetilde{\rho}(T)_{m,2j,l,2j,\uparrow} \, d_{j^+}^{-s} + \sum_{2j=1}^{\infty}\sum_{m=0}^{2j}\sum_{l=0}^{2j-1} \widetilde{\rho}(T)_{m,2j,l,2j,\downarrow} \, d_{j}^{-s} \\
&= \sum_{2j=0}^{\infty} \big(\mathrm{Tr}_{2j}(T_+) \, \mathrm{Tr}^{\uparrow}_{2j+1}(T_-) + \mathrm{Tr}_{2j+1}(T_+) \, \mathrm{Tr}^{\downarrow}_{2j}(T_-) \big) \, d_{j^+}^{-s} \, .
\end{aligned}$$

By Lemma 4.7 (ii), for all $k>0$,

$$\begin{aligned}
\mathrm{Tr}_{2j}(T_\pm) &= (2j+\tfrac{3}{2})\tau_1(T_\pm) + \tau_0(T_\pm) - \tfrac{1}{2}\tau_1(T_\pm) + \mathcal{O}\big((2j)^{-k}\big), \\
\mathrm{Tr}_{2j+1}(T_\pm) &= (2j+\tfrac{3}{2})\tau_1(T_\pm) + \tau_0(T_\pm) + \tfrac{1}{2}\tau_1(T_\pm) + \mathcal{O}\big((2j)^{-k}\big) \, .
\end{aligned}$$

The result follows by noting that the difference of the Hurwitz zeta function $\zeta(s,\frac{3}{2})$ and Riemann zeta function $\zeta(s)$ is an entire function.
(ii, iii, iv) are direct consequences of (i). □

4.5 The smooth algebra $C^\infty(SU_q(2))$

In [13, 38], the smooth algebra $C^\infty(SU_q(2)$ is defined by pulling back the smooth structure $C^\infty(D^2_{q^\pm})$ into the C^*-algebra generated by $\mathcal{A}$, through the morphism ρ and the application λ (the compression which gives an operator on $\mathcal{H}$ from an operator on $l^2(\mathbb{N})\otimes l^2(\mathbb{N})\otimes l^2(\mathbb{Z})\otimes\mathbb{C}^2$). The important point is that with [13, Lemma 2, p. 69], this algebra is stable by holomorphic calculus. By defining $\rho := \widetilde{\rho}\circ c$ and $\lambda(\cdot) := S(\cdot)I$, the same lemma (with same notation) can be applied to our setting, with $c := \pi(x) \mapsto \underline{\pi}(x)$ and

$$\mathcal{C} := C^\infty(D^2_{q^+}) \otimes C^\infty(S^1) \otimes C^\infty(D^2_{q^+}) \otimes C^\infty(S^1) \otimes \mathcal{M}_2(\mathbb{C})$$

as algebra stable by holomorphic calculus containing the image of $\widetilde{\rho}$. Here, we use Schwartz sequences to define the smooth structures. We finally obtain $C^\infty(SU_q(2))$ with real structure as a subalgebra stable by holomorphic calculus of the C^*-algebra generated by $\pi(\mathcal{A})\cup J\pi(\mathcal{A})J^{-1}$ and containing $\pi(\mathcal{A})\cup J\pi(\mathcal{A})J^{-1}$.

Corollary 4.10. *The dimension spectrum of the real spectral triple* $\big(C^\infty(SU_q(2)), \mathcal{H}, D\big)$ *is simple and given by* $\{1,2,3\}$. *Its KO-dimension is* 3.

Proof. Since F commutes with $\underline{\pi}(\mathcal{A})$, the pseudodifferential operators of order 0 (without the real structure and in the sense of [22]) are exactly (modulo $OP^{-\infty}$) the operators in $\mathcal{B}+\mathcal{B}F$. From Theorem 3.4 we see that the dimension spectrum of $SU_q(2)$ without taking into account the reality operator J is $\{1,2,3\}$, in other words, the possible poles of $\zeta^b_{\mathcal{D}} : s \mapsto \mathrm{Tr}(bF^\varepsilon|\mathcal{D}|^{-s})$ (with $\varepsilon\in\{0,1\}$, $b\in\mathcal{B}$) are in $\{1,2,3\}$. Theorem 4.9 (i) shows that the possible poles are still $\{1,2,3\}$ when we take into account the real structure of $SU_q(2)$, that is to say, when $\mathcal{B}$ is enlarged to $\mathcal{B}J\mathcal{B}J^{-1}$. Indeed, any element of $\mathcal{B}J\mathcal{B}J^{-1}$ is in $X\widehat{X}$ and it is clear from the preceding proof that adding F in the previous zeta function do not add any pole to $\{1,2,3\}$.
All arguments goes true from the polynomial algebra $\mathcal{A}(SU_q(2))$ to the smooth pre-C*-algebra $C^\infty(SU_q(2))$.
KO-dimension refers just to $J^2=-1$ and $\mathcal{D}\,J = J\,\mathcal{D}$ since there is no chirality because spectral dimension is 3. □

4.6 Noncommutative integrals with reality operator and one-forms on $SU_q(2)$

The goal of this section is to obtain the following suppression of J:

Theorem 4.11. *Let A and B be δ-one-forms. Then*

$$(i)\quad \fint AJBJ^{-1}|\mathcal{D}|^{-3} = \tfrac{1}{2}\fint A|\mathcal{D}|^{-3}\overline{\fint B|\mathcal{D}|^{-3}},$$

$$(ii)\quad \fint AJBJ^{-1}|\mathcal{D}|^{-2} = \tfrac{1}{2}\fint A|\mathcal{D}|^{-2}\overline{\fint B|\mathcal{D}|^{-3}} + \tfrac{1}{2}\fint A|\mathcal{D}|^{-3}\overline{\fint B|\mathcal{D}|^{-2}},$$

$$(iii)\quad \fint A^2JBJ^{-1}|\mathcal{D}|^{-3} = \tfrac{1}{2}\fint A^2|\mathcal{D}|^{-3}\overline{\fint B|\mathcal{D}|^{-3}},$$

$$(iv)\quad \fint \delta(A)A|\mathcal{D}|^{-3} = \fint \delta(A)JAJ^{-1}|\mathcal{D}|^{-3} = 0.$$

We gather at the beginning of this section the main notations for technical lemmas which will follow.

For any pair $(k,p)\in\mathbb{N}^3\times\mathbb{N}^3$ such that $k_i\le|\alpha_i|$, $p_i\le|\beta_i|$, where $\alpha,\beta\in\mathbb{Z}\times\mathbb{N}\times\mathbb{N}$, we define

$$\begin{aligned}
v_{k,p} &:= g(p)\ \tbinom{|\alpha_1|}{k_1}_{q^{2\eta\alpha_1}}\tbinom{\alpha_2}{k_2}\tbinom{\alpha_3}{k_3}\tbinom{|\beta_1|}{p_1}_{q^{2\eta\beta_1}}\tbinom{\beta_2}{p_2}\tbinom{\beta_3}{p_3}\,(-1)^{k_1+p_1+\alpha_2+\alpha_3+\beta_2+\beta_3}\,q^{\sigma_{k,p}},\\
h_{k,p} &:= \alpha_1+\alpha_2-\alpha_3-2(\eta_{\alpha_1}k_1+k_2-k_3)+g(p)\,,\\
g(p) &:= \beta_1+\beta_2-\beta_3-2(\eta_{\beta_1}p_1+p_2-p_3)\,,\\
\sigma_{k,p} &:= k_1+p_1+\sigma^t_{k,p}+\sigma^u_{k,p}\,,\\
\sigma^t_{k,p} &:= k_1\widehat{k}_2-\widehat{k}_3(k_1+k_2)+\eta_{\beta_1}\widehat{p}_1|k|_1+\widehat{p}_2(|k|_1+p_1)-\widehat{p}_3(|k|_1+p_1+p_2)\,,\\
\sigma^u_{k,p} &:= (k_3+\eta_{\beta_1}\widehat{p}_1-p_2+p_3)(k_1+\widehat{k}_2+\widehat{k}_3)-k_2(k_1+\widehat{k}_2)+(p_1+\widehat{p}_2)(-p_2+p_3)+\widehat{p}_3p_3\,,\\
t_{k,p} &= a^{\widehat{k}_1}_{\alpha_1}\,a^{\widehat{k}_2}\,a^{*\widehat{k}_3}\,a^{\widehat{p}_1}_{\beta_1}\,a^{\widehat{p}_2}\,a^{*\widehat{p}_3}\,b^{|k|_1+|p|_1}\,,\\
u_{k,p} &= a^{\widehat{k}_1}_{\alpha_1}\,a^{*k_2}\,a^{k_3}\,a^{\widehat{p}_1}_{\beta_1}\,a^{*p_2}\,a^{p_3}\,b^{|\widetilde{k}|_1+|\widetilde{p}|_1}\,.
\end{aligned}$$

where we used the notation

$$\widehat{k}_i := |\alpha_i|-k_i,\ \widehat{p}_i := |\beta_i|-p_i,$$

so $0\le\widehat{k}_i\le|\alpha_i|$, $0\le\widehat{p}_i\le|\beta_i|$. We will also use the shortcut $\widetilde{k} := (k_1,\widehat{k}_2,\widehat{k}_3)$.

For $\beta_1\in\mathbb{Z}$ and $j\in\mathbb{N}$, we define

$$\begin{aligned}
w_1(\beta_1,j) &:= \sum_{n=0}^{\infty}\big(q^{2jn}(q^{\uparrow_{\beta_1}}_{n,|\beta_1|})^2-\delta_{j,0}\big),\\
w^\alpha_\beta &:= 2\beta_1\,q^{\beta_1(2\alpha_3+\beta_3-\beta_2)}\,w_1(\beta_1,\alpha_3+\beta_3).
\end{aligned}$$

We introduce the following notations:

$$\begin{aligned}
q^+_{k,p,n} &:= q^{n(|k|_1+|p|_1)}q^{\uparrow_{\alpha_1}}_{n+r^+_{k,p}-\eta_{\alpha_1}\widehat{k}_1,\widehat{k}_1}\,q^{\uparrow}_{n-\widehat{k}_3+\eta_{\beta_1}\widehat{p}_1+\widehat{p}_2-\widehat{p}_3,\widehat{k}_2}\,q^{\downarrow}_{n+\eta_{\beta_1}\widehat{p}_1+\widehat{p}_2-\widehat{p}_3,\widehat{k}_3}\,q^{\uparrow_{\beta_1}}_{n+\widehat{p}_2-\widehat{p}_3,\widehat{p}_1}\,q^{\uparrow}_{n-\widehat{p}_3,\widehat{p}_2}\,q^{\downarrow}_{n,\widehat{p}_3}\,,\\
q^-_{k,p,n} &:= q^{n(|\widetilde{k}|_1+|\widetilde{p}|_1)}q^{\uparrow_{\alpha_1}}_{n+r^-_{k,p}-\eta_{\alpha_1}\widehat{k}_1,\widehat{k}_1}\,q^{\downarrow}_{n+k_3+\eta_{\beta_1}\widehat{p}_1-p_2+p_3,k_2}\,q^{\uparrow}_{n+\eta_{\beta_1}\widehat{p}_1-p_2+p_3,k_3}\,q^{\uparrow_{\beta_1}}_{n-p_2+p_3,\widehat{p}_1}\,q^{\downarrow}_{n+p_3,p_2}\,q^{\uparrow}_{n,p_3}\\
&\qquad\times(-1)^{|\widetilde{k}|_1+|\widetilde{p}|_1},\\
r^+_{k,p} &:= \eta_{\alpha_1}\widehat{k}_1+\widehat{k}_2-\widehat{k}_3+\eta_{\beta_1}\widehat{p}_1+\widehat{p}_2-\widehat{p}_3\,,\\
r^-_{k,p} &:= \eta_{\alpha_1}\widehat{k}_1-k_2+k_3+\eta_{\beta_1}\widehat{p}_1-p_2+p_3\,.
\end{aligned}$$

Thus, $\pi_+(t_{k,p})\varepsilon_n = q^+_{k,p,n}\varepsilon_{n+r^+_{k,p}}$ and $\pi_-(u_{k,p})\varepsilon_n = q^-_{k,p,n}\varepsilon_{n+r^-_{k,p}}$.

Lemma 4.12. *We have*

$$r\left((M^\alpha_\beta)^\circ\right) = \sum_{k,p} \delta_{h_{k,p},0}\, v_{k,p}\, \pi_+(t_{k,p}) \otimes \pi_-(u_{k,p})$$

where the summation is done on k_i, p_i in $\mathbb{N}$ such that $k_i \le |\alpha_i|, p_i \le |\beta_i|$ for $i \in \{1,2,3\}$.

Proof. Since $\underline{\pi}(m^\alpha) = (a_+ + a_-)^{\alpha_1}(b_+ + b_-)^{\alpha_2}(b^*_+ + b^*_-)^{\alpha_3}$, with $v_k := \binom{|\alpha_1|}{k_1}_{q^{2\eta_{\alpha_1}}}\binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3}$,

$$\underline{\pi}(m^\alpha) = \sum\nolimits_k v_k\, c_k \text{ where } c_k := a^{|\alpha_1|-k_1}_{+,\alpha_1}\, a^{k_1}_{-,\alpha_1}\, b^{\alpha_2-k_2}_+\, b^{k_2}_-\, b^{*\,\alpha_3-k_3}_+\, b^{*\,k_3}_-.$$

By Lemma 3.2 (iii) we see that $\delta(\underline{\pi}(m^\beta)) = \sum_p w_p\, d_p$ where we introduce $w_p := \binom{|\beta_1|}{p_1}_{q^{2\eta_{\beta_1}}}\binom{\beta_2}{p_2}\binom{\beta_3}{p_3}$ and $d_p := g(p)\, a^{|\beta_1|-p_1}_{+,\beta_1}\, a^{p_1}_{-,\beta_1}\, b^{\beta_2-p_2}_+\, b^{p_2}_-\, b^{*\,\beta_3-p_3}_+\, b^{*\,p_3}_-$.
As a consequence, $(M^\alpha_\beta)^\circ = \sum_{k,p} \delta_{h(k,p),0}\; g(p)\; v_k\, w_p\, c_{k,p}$ where

$$c_{k,p} = a^{\widehat{k}_1}_{+,\alpha_1}\, a^{k_1}_{-,\alpha_1}\, b^{\widehat{k}_2}_+\, b^{k_2}_-\, b^{*\,\widehat{k}_3}_+\, b^{*\,k_3}_-\, a^{\widehat{p}_1}_{+,\beta_1}\, a^{p_1}_{-,\beta_1}\, b^{\widehat{p}_2}_+\, b^{p_2}_-\, b^{*\,\widehat{p}_3}_+\, b^{*\,p_3}_- \tag{42}$$

With (42), we get $r(c_{k,p}) = (-1)^{k_1+p_1+\alpha_2+\alpha_3+\beta_2+\beta_3}\, q^{k_1+p_1}\, \pi_+(t'_{k,p}) \otimes \pi_-(u'_{k,p})$ where

$$t'_{k,p} = a^{\widehat{k}_1}_{\alpha_1}\, b^{k_1}\, a^{\widehat{k}_2}\, b^{k_2}\, a^{*\widehat{k}_3}\, b^{k_3} a^{\widehat{p}_1}_{\beta_1}\, b^{p_1}\, a^{\widehat{p}_2}\, b^{p_2}\, a^{*\widehat{p}_3}\, b^{p_3}\,,$$
$$u'_{k,p} = a^{\widehat{k}_1}_{\alpha_1}\, b^{k_1}\, b^{\widehat{k}_2}\, a^{*k_2}\, b^{\widehat{k}_3} a^{k_3}\, a^{\widehat{p}_1}_{\beta_1}\, b^{p_1}\, b^{\widehat{p}_2}\, a^{*p_2}\, b^{\widehat{p}_3}\, a^{p_3}\,.$$

A recursive use of relation $ba_j = q^{\eta_j} a_j b$ yields the result. □

Lemma 4.13. *We have*
(i) $(\tau_1 \otimes \tau_1)\big(r(M^\alpha_\beta)^\circ\big) = \beta_1\; \delta_{\alpha_1,-\beta_1}\, \delta_{\alpha_2,0}\, \delta_{\alpha_3,0}\, \delta_{\beta_2,0}\, \delta_{\beta_3,0}\,.$
(ii) $(\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)\big(r(M^\alpha_\beta)^\circ\big) = \delta_{\alpha_1,-\beta_1}\, \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3}\, w^\alpha_\beta.$
In particular, if A is a δ-one-form, we have

$$\fint A|\mathcal{D}|^{-3} = 2\beta_1\, A^{\beta_1 00}_{-\beta_1 00}\,,$$
$$\fint A|\mathcal{D}|^{-2} = 2w^\alpha_\beta\, B(A)^\beta_\alpha\,.$$

where we implicitly summed on all α, β indices.

Proof. (i) Using same notations of Lemma 4.12, we obtain by definition of τ_1,

$$\tau_1\big(\pi_+(t_{k,p})\big) = \delta_{k,0}\, \delta_{p,0}\; \delta_{\alpha_1+\alpha_2-\alpha_3+\beta_1+\beta_2-\beta_3,0}\,, \tag{43}$$
$$\tau_1\big(\pi_-(u_{k,p})\big) = \delta_{\widetilde{k},0}\, \delta_{\widetilde{p},0}\; \delta_{\alpha_1-\alpha_2+\alpha_3+\beta_1-\beta_2+\beta_3,0}\,. \tag{44}$$

We get $\tau_1\big(\pi_+(t_{k,p})\big)\, \tau_1\big(\pi_-(u_{k,p})\big) = \delta_{k,0}\, \delta_{p,0}\, \delta_{\alpha_2,0}\, \delta_{\alpha_3,0}\, \delta_{\beta_2,0}\, \delta_{\beta_3,0}\, \delta_{\alpha_1,-\beta_1}$, so Lemma 4.12 gives the result.

(ii) Since $\pi_+(t_{k,p})\varepsilon_n = q^+_{k,p,n}\varepsilon_{n+r^+_{k,p}}$ and $\pi_-(u_{k,p})\varepsilon_n = q^-_{k,p,n}\varepsilon_{n+r^-_{k,p}}$, we get,

$$\tau_0\big(\pi_+(t_{k,p})\big) = \delta_{r^+_{k,p},0}\sum_{n=0}^{\infty}\big(q^+_{k,p,n} - \delta_{k,0}\,\delta_{p,0}\,\delta_{\alpha_1+\alpha_2-\alpha_3+\beta_1+\beta_2-\beta_3,0}\big)\,, \tag{45}$$

$$\tau_0\big(\pi_-(u_{k,p})\big) = \delta_{r^-_{k,p},0}\sum_{n=0}^{\infty}\big(q^-_{k,p,n} - \delta_{\widetilde{k},0}\,\delta_{\widetilde{p},0}\,\delta_{\alpha_1-\alpha_2+\alpha_3+\beta_1-\beta_2+\beta_3,0}\big)\,. \tag{46}$$

With (43) and (46) we get

$$\begin{aligned}\tau_1\big(\pi_+(t_{k,p})\big)\,\tau_0\big(\pi_-(u_{k,p})\big) &= \delta_{k,0}\,\delta_{p,0}\,\delta_{\alpha_2+\beta_2,\alpha_3+\beta_3}\,\delta_{\alpha_1,-\beta_1}\sum_{n=0}^{\infty}\big(\delta_{k,0}\,\delta_{p,0}\,q^-_{k,p,n} - \delta_{\alpha_3+\beta_3,0}\big)\\ &= \delta_{k,0}\,\delta_{p,0}\,\delta_{\alpha_2+\beta_2,\alpha_3+\beta_3}\,\delta_{\alpha_1,-\beta_1}w_1(\beta_1,\alpha_3+\beta_3).\end{aligned}$$

Using (44) and (45),

$$\begin{aligned}\tau_0\big(\pi_+(t_{k,p})\big)\,\tau_1\big(\pi_-(u_{k,p})\big) &= \delta_{\widetilde{k},0}\,\delta_{\widetilde{p},0}\,\delta_{\alpha_2+\beta_2,\alpha_3+\beta_3}\,\delta_{\alpha_1,-\beta_1}\sum_{n=0}^{\infty}\big(\delta_{\widetilde{k},0}\,\delta_{\widetilde{p},0}\,q^+_{k,p,n} - \delta_{\alpha_3+\beta_3,0}\big)\\ &= \delta_{\widetilde{k},0}\,\delta_{\widetilde{p},0}\,\delta_{\alpha_2+\beta_2,\alpha_3+\beta_3}\,\delta_{\alpha_1,-\beta_1}w_1(\beta_1,\alpha_3+\beta_3).\end{aligned}$$

Lemma 4.12 yields the result. □

With notations of Lemma 4.12, it is direct to check that for given $\bar\alpha = (\alpha,\alpha',\cdots,\alpha^{(n-1)})$ and $\bar\beta = (\beta,\beta',\cdots,\beta^{(n-1)})$,

$$r\big((M^{\bar\alpha}_{\bar\beta})^\circ\big) = \sum_{K,P}\delta_{h_{K,P},0}\,v_{K,P}\,\pi_+(t_{K,P})\otimes\pi_-(u_{K,P}) \tag{47}$$

where $K = (k,k',\cdots k^{(n-1)})$, $P = (p,p',\cdots,p^{(n-1)})$ with $0\le k^{(j)}_i \le |\alpha^{(j)}_i|$, $0\le p^{(j)}_i\le|\beta^{(j)}_i|$,

$$\begin{aligned} t_{K,P} &:= t_{k,p}\,t_{k',p'}\cdots t_{k^{(n-1)},p^{(n-1)}}\,, & u_{K,P} &:= u_{k,p}u_{k',p'}\cdots u_{k^{(n-1)},p^{(n-1)}}\,,\\ v_{K,P} &:= v_{k,p}\,v_{k',p'}\cdots v_{k^{(n-1)},p^{(n-1)}}\,, & h_{K,P} &:= h_{k,p}+h_{k',p'}+\cdots h_{k^{(n-1)},p^{(n-1)}}\,.\end{aligned}$$

In the following, we will use the shortcuts $A_i := \alpha_i+\alpha'_i+\cdots+\alpha^{(n-1)}_i$, $B_i := \beta_i+\beta'_i+\cdots+\beta^{(n-1)}_i$.
In the case $n=2$, we also note $r^\pm_{K,P} := r^\pm_{k,p}+r^\pm_{k',p'}$ and $q^\pm_{K,P,n} := q^\pm_{k',p',n}\,q^\pm_{k,p,n+r^\pm_{k',p'}}$.
Thus, we have $\pi_+(t_{K,P})\,\varepsilon_m = q^+_{K,P,m}\,\varepsilon_{m+r^+_{K,P}}$ and $\pi_-(u_{K,P})\,\varepsilon_m = q^-_{K,P,n}\,\varepsilon_{m+r^-_{K,P}}$.
We also introduce, still for $n=2$,

$$\begin{aligned} v_{\beta_1,\alpha'_1,\beta'_1}(l,j) &:= \sum_{n=0}^{\infty}\big(q^{l+2nj}\,q^{\uparrow_{-\beta'_1-\alpha'_1-\beta_1}}_{n+\beta'_1+\alpha'_1+\beta_1,|\beta'_1+\alpha'_1+\beta_1|}q^{\uparrow_{\beta_1}}_{n+\beta'_1+\alpha'_1,|\beta_1|}q^{\uparrow_{\alpha'_1}}_{n+\beta'_1,|\alpha'_1|}q^{\uparrow_{\beta'_1}}_{n,|\beta'_1|} - \delta_{j,0}\big),\\ V^{\bar\alpha}_{\bar\beta} &:= 2[\beta_1\beta'_1 + (\beta_2-\beta_3)(\beta'_2-\beta'_3)]\,q^{2\beta_1(\alpha_2+\alpha_3)+2\beta'_1(\alpha'_2+\alpha'_3)}\\ &\qquad\times v_{\beta_1,\alpha'_1,\beta'_1}((\alpha_2+\beta_2+\alpha_3+\beta_3)(\alpha'_1+\beta'_1),A_3+B_3).\end{aligned}$$

Lemma 4.14. *We have*
(i) $(\tau_1\otimes\tau_1)\big(r(M^\alpha_\beta M^{\alpha'}_{\beta'})^\circ\big) = \beta_1\beta'_1\;\delta_{A_1,-B_1}\,\delta_{A_2,0}\,\delta_{A_3,0}\,\delta_{B_2,0}\delta_{B_3,0}\,.$
(ii) $(\tau_1\otimes\tau_0+\tau_0\otimes\tau_1)\big(r(M^\alpha_\beta M^{\alpha'}_{\beta'})^\circ\big) = \delta_{A_2+B_2,A_3+B_3}\delta_{A_1,-B_1}V^{\bar\alpha}_{\bar\beta}\,.$

(iii) $(\tau_1 \otimes \tau_1)\left(r(M^{\alpha}_{\beta} M^{\alpha'}_{\beta'} M^{\alpha''}_{\beta''})^0\right) = \beta_1\beta_1'\beta_1''\ \delta_{A_1,-B_1}\,\delta_{A_2,0}\,\delta_{A_3,0}\,\delta_{B_2,0}\delta_{B_3,0}$.
(iv) $(\tau_1 \otimes \tau_1)\left(r(\delta(M^{\alpha}_{\beta})M^{\alpha'}_{\beta'})^0\right) = -(\alpha_1' + \beta_1')\beta_1\beta_1'\ \delta_{A_1,-B_1}\,\delta_{A_2,0}\,\delta_{A_3,0}\,\delta_{B_2,0}\delta_{B_3,0}$.
(v) In particular, if A is a δ-one-form,

$$\fint A^2|\mathcal{D}|^{-3} = 2\beta_1\beta_1'\ B_a(A^2)^{\bar\beta}_{\bar\alpha}\,,$$
$$\fint A^2|\mathcal{D}|^{-2} = 2V^{\bar\alpha}_{\bar\beta}\ B(A^2)^{\bar\beta}_{\bar\alpha}\,,$$
$$\fint A^3|\mathcal{D}|^{-3} = 2\beta_1\beta_1'\beta_1''\ B_a(A^3)^{\bar\beta}_{\bar\alpha}\,,$$
$$\fint \delta(A)A|\mathcal{D}|^{-3} = \fint A\delta(A)|\mathcal{D}|^{-3} = 0.$$

Proof. We have

$$\tau_1(\pi_+(t_{K,P})) = \delta_{K,0}\,\delta_{P,0}\,\delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}\,, \tag{48}$$
$$\tau_1(\pi_-(u_{K,P})) = \delta_{\widetilde{K},0}\,\delta_{\widetilde{P},0}\,\delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}\,. \tag{49}$$

and

$$\tau_0(\pi_+(t_{K,P})) = \delta_{r^+_{K,P},0}\sum_{n=0}^{\infty}\left(q^+_{K,P,n} - \delta_{K,0}\,\delta_{P,0}\,\delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}\right), \tag{50}$$
$$\tau_0(\pi_-(u_{K,P})) = \delta_{r^-_{K,P},0}\sum_{n=0}^{\infty}\left(q^-_{K,P,n} - \delta_{\widetilde{K},0}\,\delta_{\widetilde{P},0}\,\delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}\right). \tag{51}$$

(i) Equations (48) and (49) give $(\tau_1 \otimes \tau_1)\, r(\underline{A}\,\underline{A}')^0 = \delta_{A_1,-B_1}\delta_{A_2,0}\delta_{A_3,0}\delta_{B_2,0}\delta_{B_3,0}\,\lambda_{0,0}$. A computation of $v_{0,0}$ with $\delta_{A_1,-B_1}\,\delta_{A_2,0}\,\delta_{A_3,0}\,\delta_{B_2,0}\,\delta_{B_3,0} = 1$ gives the result.
(ii) Equations (48) and (51) yield

$$\begin{aligned}\tau_1(\pi_+(t_{K,P}))\,\tau_0(\pi_-(u_{K,P})) = {} & \delta_{K,0}\,\delta_{P,0}\,\delta_{A_2+B_2,A_3+B_3}\,\delta_{A_1,-B_1}\\ & \times v_{\beta_1,\alpha_1',\beta_1'}((\alpha_2+\beta_2+\alpha_3+\beta_3)(\alpha_1'+\beta_1'), A_3+B_3).\end{aligned}$$

Equations (50) and (49) yield

$$\begin{aligned}\tau_0(\pi_+(t_{K,P}))\,\tau_1(\pi_-(u_{K,P})) = {} & \delta_{\widetilde{K},0}\,\delta_{\widetilde{P},0}\,\delta_{A_2+B_2,A_3+B_3}\,\delta_{A_1,-B_1}\\ & \times v_{\beta_1,\alpha_1',\beta_1'}((\alpha_2+\beta_2+\alpha_3+\beta_3)(\alpha_1'+\beta_1'), A_3+B_3)\end{aligned}$$

and the result follows.
(iii) With (47) a direct computation gives

$$\tau_1(\pi_+(t_{K,P})) = \delta_{K,0}\,\delta_{P,0}\,\delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}\,, \tag{52}$$
$$\tau_1(\pi_-(u_{K,P})) = \delta_{\widetilde{K},0}\,\delta_{\widetilde{P},0}\,\delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}\,. \tag{53}$$

Using (52) and (53), $(\tau_1 \otimes \tau_1)\left(r(\underline{A}\,\underline{A}'\,\underline{A}'')^\circ\right) = \delta_{A_1,-B_1}\,\delta_{A_2,0}\,\delta_{A_3,0}\,\delta_{B_2,0}\,\delta_{B_3,0}\,v_{0,0}$. A computation of $v_{0,0}$ with $\delta_{A_1,-B_1}\,\delta_{A_2,0}\,\delta_{A_3,0}\,\delta_{B_2,0}\,\delta_{B_3,0} = 1$ gives the result.
(iv) We have $\delta(M^{\alpha}_{\beta})M^{\alpha'}_{\beta'} = \delta(x)\delta(y)x'\delta(y') + x\delta^2(y)x'\delta(y')$ where x, x', y, y' are monomials ($\underline{\pi}$ omitted). Since

$$\underline{\pi}(x) = \sum_k \tbinom{\alpha}{k} a^{\widehat{k}_1}_{+,\alpha_1} a^{k_1}_{-,\alpha_1} b^{\widehat{k}_2}_{+} b^{k_2}_{-} b^{*\ \widehat{k}_3}_{+} b^{*\ k_3}_{-} =: \sum_k \tbinom{\alpha}{k} c_k,$$

we get $\delta(\underline{\pi}(x)) = \sum_k g(k)\binom{\alpha}{k} c_k$.
Similarly, $\delta(\underline{\pi}(y)) = \sum_p g(p)\binom{\beta}{p} c_p$ and $\delta^2(\underline{\pi}(y)) = \sum_p g(p)^2\binom{\beta}{p} c_p$.
Thus, with $c_{K,P} := c_k\, c_p\, c_{k'}\, c_{p'}$,

$$\delta(x)\delta(y)x'\delta(y') = \sum_{K,P} g(k)\, g(p)\, g(p')\binom{\alpha}{K}\binom{\beta}{P} c_{K,P}\,,$$
$$x\delta^2(y)x'\delta(y') = \sum_{K,P} g(p)^2 g(p')\binom{\alpha}{K}\binom{\beta}{P} c_{K,P}\,,$$
$$r(\delta(M^\alpha_\beta)M^{\alpha'}_{\beta'})^0 = \sum_{K,P} \delta_{h_{K,P},0}\big(g(k)+g(p)\big)\, g(p)\, g(p')\binom{\alpha}{K}\binom{\beta}{P} r(c_{K,P}) =: \sum_{K,P} \lambda_{K,P}\, r(c_{K,P})\,.$$

Since $r(c_k) = (-q)^{k_1}(-1)^{\alpha_2+\alpha_3}\pi_+(t_k)\otimes\pi_-(u_k)$ with t_k, u_k defined by

$$t_k := a^{\widehat{k}_1}_{\alpha_1}\, b^{k_1}\, a^{\widehat{k}_2}\, b^{k_2}\, a^{*\widehat{k}_3}\, b^{k_3} \text{ and } u_k := a^{\widehat{k}_1}_{\alpha_1}\, b^{k_1}\, b^{\widehat{k}_2}\, a^{*k_2}\, b^{\widehat{k}_3}\, a^{k_3},$$

we get

$$r(\delta(M^\alpha_\beta)M^{\alpha'}_{\beta'})^0 = \sum_{K,P} \lambda_{K,P}\,(-q)^{k_1+k'_1+p_1+p'_1}(-1)^{A_2+A_3+B_2+B_3}\pi_+(t_{K,P})\otimes\pi_-(u_{K,P})$$

where $t_{K,P} = t_k t_p t_{k'} t_{p'}$ and $u_{K,P} = u_k u_p u_{k'} u_{p'}$. Direct computations yield

$$\tau_1\big(\pi_+(t_{K,P})\big) = \delta_{K,0}\,\delta_{P,0}\,\delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}\,,$$
$$\tau_1\big(\pi_-(u_{K,P})\big) = \delta_{\widetilde{K},0}\,\delta_{\widetilde{P},0}\,\delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}\,.$$

The result follows.
(v) For the last equality, note that by (iv)

$$\fint \delta(A)A|\mathcal{D}|^{-3} = -2\sum_{\alpha_1,\alpha'_1,\beta_1,\beta'_1} (\alpha'_1+\beta'_1)\beta_1\beta'_1\, A^{\beta_1 00}_{\alpha_1 000}\, A^{\beta'_1 00}_{\alpha'_1 00}\,\delta_{\alpha_1+\alpha'_1+\beta_1+\beta'_1,0}.$$

The following change of variables $\alpha_1 \leftrightarrow \alpha'_1$, $\beta_1 \leftrightarrow \beta'_1$, implies by symmetry that this is equal to zero. □

For a given δ-1-form A, we say that A is homogeneous of degree in a equal to $n \in \mathbb{Z}$ if it is a linear combination of M^α_β such that $\alpha_1+\beta_1 = n$. From Lemma 4.14 (iv) we get,

Corollary 4.15. *Let A, A' be two δ-1-forms, then*

$$\fint (A|\mathcal{D}|^{-1})^2 = \fint A^2|\mathcal{D}|^{-2}\,,$$
$$\fint A|\mathcal{D}|^{-1}A'|\mathcal{D}|^{-1} = \fint AA'|\mathcal{D}|^{-2} - n\fint AA'|\mathcal{D}|^{-3}\,,\ \textit{when } A' \textit{ homogenous of degree } n.$$

Lemma 4.16. *We have*
(i) $(\tau_1\otimes\tau_1)\,\widetilde{\rho}\big(M^\alpha_\beta J M^{\alpha'}_{\beta'} J^{-1}\big) = \beta_1\beta'_1\ \delta_{\alpha_1,-\beta_1}\,\delta_{\alpha'_1,-\beta'_1}\,\delta_{A_2,0}\,\delta_{A_3,0}\,\delta_{B_2,0}\delta_{B_3,0}\,.$
(ii) $(\tau_0\otimes\tau_1+\tau_1\otimes\tau_0)\,\widetilde{\rho}\big(M^\alpha_\beta J M^{\alpha'}_{\beta'} J^{-1}\big) = \delta_{\alpha_1,-\beta_1}\,\delta_{\alpha'_1,-\beta'_1}\big(\beta'_1 w^\alpha_\beta \delta_{\alpha'_2+\beta'_2+\alpha'_3+\beta'_3,0}\,\delta_{\alpha_2+\beta_2,\alpha_3+\beta_3}$
$+\beta_1 w^{\alpha'}_{\beta'}\delta_{\alpha_2+\beta_2+\alpha_3+\beta_3,0}\,\delta_{\alpha'_2+\beta'_2,\alpha'_3+\beta'_3}\big)\,.$

(iii) $(\tau_1\otimes\tau_1)\,\widetilde{\rho}\big(M^{\alpha}_{\beta}M^{\alpha'}_{\beta'}JM^{\alpha''}_{\beta''}J^{-1}\big)=\beta_1\beta'_1\beta''_1\;\delta_{\alpha_1+\alpha'_1,-\beta_1-\beta'_1}\,\delta_{\alpha''_1,-\beta''_1}\,\delta_{A_2,0}\,\delta_{A_3,0}\,\delta_{B_2,0}\delta_{B_3,0}\,.$
(iv) $(\tau_1\otimes\tau_1)\;\widetilde{\rho}\big(\delta(M^{\alpha}_{\beta})JM^{\alpha'}_{\beta'}J^{-1}\big)=-(\alpha'_1+\beta'_1)\beta_1\beta'_1\;\delta_{\alpha_1,-\beta_1}\,\delta_{\alpha'_1,-\beta'_1}\,\delta_{A_2,0}\,\delta_{A_3,0}\,\delta_{B_2,0}\delta_{B_3,0}\;.$
(v) In particular, if A and A' are δ-one forms,

$$\fint AJA'J^{-1}|\mathcal{D}|^{-3}=2(\beta_1A^{\beta_100}_{-\beta_100})(\beta_1\bar{A'}^{\beta_100}_{-\beta_100}),$$
$$\fint AJA'J^{-1}|\mathcal{D}|^{-2}=2(\beta_1\bar{A'}^{\beta_100}_{-\beta_100})(w^{\alpha}_{\beta}B(A)^{\beta}_{\alpha})+2(\beta_1A^{\beta_100}_{-\beta_100})(w^{\alpha}_{\beta}B(\bar{A'})^{\beta}_{\alpha}),$$
$$\fint A^2JA'J^{-1}|\mathcal{D}|^{-3}=2(\beta_1\bar{A'}^{\beta_100}_{-\beta_100})(\beta_1\beta'_1B_a(A^2)^{\bar{\beta}}_{\bar{\alpha}}),$$
$$\fint \delta(A)JAJ^{-1}=0.$$

Proof. (i) Following notations of Lemma 4.12, we have

$$M^{\alpha}_{\beta}JM^{\alpha'}_{\beta'}J^{-1}=\sum_{K,P}v_{K,P}\,c_{k,p}Jc_{k',p'}J^{-1}$$

where $K=(k,k')$, $P=(p,p')$, $\lambda_{K,P}=g(p)g(p')v_kv_{k'}w_pw_{p'}$. Thus,

$$\widetilde{\rho}(M^{\alpha}_{\beta}JM^{\alpha'}_{\beta'}J^{-1})=(-1)^{A_2+A_3+B_2+B_3}\sum_{K,P}(-q)^{k_1+k'_1+p_1+p'_1}\lambda_{K,P}\,T^{+}_{K,P}\otimes T^{-}_{K,P}$$

where $T^{+}_{K,P}:=\pi'_{+}(t_kt_p)\widehat{\pi}_{+}(t_{k'}t_{p'})$ and $T^{-}_{K,P}:=\pi'_{-}(u_ku_p)\widehat{\pi}_{-}(u_{k'}u_{p'})$ with

$$t_k:=a^{\widehat{k}_1}_{\alpha_1}\,b^{*k_1}_{\alpha_1}\,a^{\widehat{k}_2}\,b^{*k_2}\,a^{*\widehat{k}_3}b^{k_3},$$
$$u_k:=a^{\widehat{k}_1}_{\alpha_1}\,b^{*k_1}_{\alpha_1}\,b^{\widehat{k}_2}\,a^{*k_2}\,b^{*\widehat{k}_3}a^{k_3}.$$

A direct computation leads to

$$\tau_1(T^{+}_{K,P})=\delta_{K,0}\,\delta_{P,0}\;\delta_{\alpha_1+\alpha_2-\alpha_3+\beta_1+\beta_2-\beta_3,0}\,\delta_{\alpha'_1+\alpha'_2-\alpha'_3+\beta'_1+\beta'_2-\beta'_3,0}\,,$$
$$\tau_1(T^{-}_{K,P})=\delta_{\widetilde{K},0}\,\delta_{\widetilde{P},0}\;\delta_{\alpha_1-\alpha_2+\alpha_3+\beta_1-\beta_2+\beta_3,0}\,\delta_{\alpha'_1-\alpha'_2+\alpha'_3+\beta'_1-\beta'_2+\beta'_3,0}$$

which gives the result.
(ii) Using the commutation relations on $\mathcal{A}$, we see that there are real functions of (K,P), noted $\sigma^{t}_{K,P}$ and $\sigma^{u}_{K,P}$ such that

$$T^{+}_{K,P}=q^{\sigma^{t}_{K,P}}\,\pi'_{+}(t_{k,p})\widehat{\pi}_{+}(t_{k',p'}),$$
$$T^{-}_{K,P}=q^{\sigma^{u}_{K,P}}\,\pi'_{-}(u_{k,p})\widehat{\pi}_{-}(u_{k',p'}),$$
$$t_{k,p}:=a^{\widehat{k}_1}_{\alpha_1}\,a^{\widehat{k}_2}\,a^{*\widehat{k}_3}\,a^{\widehat{p}_1}_{\beta_1}\,a^{\widehat{p}_2}\,a^{*\widehat{p}_3}\,b^{*k_1}_{\alpha_1}\,b^{*p_1}_{\beta_1}\,b^{*k_2+p_2}b^{k_3+p_3},$$
$$u_{k,p}:=a^{\widehat{k}_1}_{\alpha_1}\,a^{*k_2}\,a^{k_3}\,a^{\widehat{p}_1}_{\beta_1}\,a^{*p_2}\,a^{p_3}\,b^{*k_1}_{\alpha_1}\,b^{*p_1}_{\beta_1}\,b^{\widehat{k}_2+\widehat{p}_2}b^{*\widehat{k}_3+\widehat{p}_3}.$$

We have, under the hypothesis $\tau_1(T^{-}_{K,P})=1$,

$$\widehat{\pi}_{+}(t_{k',p'})\varepsilon_{m,2j}=(-1)^{\lambda'}q^{(2j-m)\lambda'}\,q^{\uparrow_{\alpha'_1}}_{2j-m-s+\beta'_1,|\alpha'_1|}\,q^{\uparrow_{\beta'_1}}_{2j-m-s,|\beta'_1|}\,\varepsilon_{m+s,2j}\,,$$
$$s:=-\alpha'_2+\alpha'_3-\beta'_2+\beta'_3=\alpha'_1+\beta'_1\,,$$
$$\lambda':=\alpha'_2+\alpha'_3+\beta'_2+\beta'_3\,,$$
$$\lambda:=\alpha_2+\alpha_3+\beta_2+\beta_3$$
$$\tau_1(T^{+}_{K,P})=\delta_{\lambda,0}\,\delta_{\lambda',0}\,.$$

and then,

$$
\begin{aligned}
&(T^+_{K,P})_{m,2j} = q^{\sigma^t_{K,P}+s\lambda}(-1)^{\lambda'} q^{(2j-m)\lambda'+m\lambda}\, F_m\, F'_{2j-m}\, \delta_{A_1+B_1,0}\,,\\
&F'_{2j-m} := q^{\uparrow_{\alpha'_1}}_{2j-m-\alpha'_1,|\alpha'_1|}\, q^{\uparrow_{\beta'_1}}_{2j-m-\alpha'_1-\beta'_1,|\beta'_1|}\,,\\
&F_m := q^{\uparrow_{\alpha_1}}_{m-\alpha_1,|\alpha_1|} q^{\uparrow_{\beta_1}}_{m-\beta_1-\alpha_1,|\beta_1|}\,.
\end{aligned}
$$

Following the proof of Lemma 4.7, we see that $\tau_0(T^+_{K,P})$ is possibly nonzero only in the two cases $\lambda' = 0$ or $\lambda = 0$.
Suppose first $\lambda = \lambda' = 0$. In that case, we have

$$
\tau_0(T^+_{K,P}) = \lim_{2j\to\infty} \sum_{m=0}^{2j} \big((q^{\uparrow_{\beta_1}}_{m,|\beta_1|} q^{\uparrow_{\beta'_1}}_{2j-m,|\beta'_1|})^2 - 1\big) = \sum_{m=0}^{\infty} \big((q^{\uparrow_{\beta_1}}_{m,|\beta_1|})^2 - 1\big) + \sum_{m=0}^{\infty} \big((q^{\uparrow_{\beta'_1}}_{m,|\beta'_1|})^2 - 1\big)
$$

where the second equality comes from Lemma 4.17.
In the case $(\lambda = 0, \lambda' > 0)$, we get $\alpha'_1 = -\beta'_1$ and thus,

$$
(T^+_{K,P})_{m,2j} = q^{\sigma^t_{K,P}} q^{m\lambda}\, (q^{\uparrow_{\beta_1}}_{m,|\beta_1|} q^{\uparrow_{\beta'_1}}_{2j-m,|\beta'_1|})^2 \delta_{\alpha_1+\beta_1,0}.
$$

Let us note $U_{2j} = \sum_{m=0}^{2j} q^{m\lambda}\, (q^{\uparrow_{\beta_1}}_{m,|\beta_1|} q^{\uparrow_{\beta'_1}}_{2j-m,|\beta'_1|})^2$ and $L_{2j} = \sum_{m=0}^{2j} q^{m\lambda}\, (q^{\uparrow_{\beta_1}}_{m,|\beta_1|})^2$.
Suppose $\beta'_1 > 0$. Since $(q^{\uparrow_{\beta'_1}}_{2j-m,|\beta'_1|})^2 - 1 = \sum_{|p|_1\neq 0, p_i\in\{0,1\}} (-1)^{|p|_1} q^{r_p}\, q^{2(2j-m)|p|_1}$ where we have $r_p = 2 + \cdots + 2\beta'_1$. As in the proof of Lemma 4.7 (ii), we can conclude that $U_{2j} - L_{2j}$ converges to 0. The case $\beta'_1 \leq 0$ is similar.
In the other case $(\lambda > 0, \lambda' = 0)$, the arguments are the same, replacing λ by λ' and α_1, β_1 by α'_1, β'_1. Finally,

$$
\begin{aligned}
&\tau_0(T^+_{K,P})\tau_1(T^-_{K,P}) = \delta_{\widetilde{K},0}\, \delta_{\widetilde{P},0}\, \delta_{\alpha_1,-\beta_1}\, \delta_{\alpha'_1,-\beta'_1} (\delta_{\lambda',0}\, \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3}\, s_{\alpha,\beta} + \delta_{\lambda,0}\, \delta_{\alpha'_2+\beta'_2,\alpha'_3+\beta'_3}\, s_{\alpha',\beta'}),\\
&s_{\alpha\beta} := q^{\beta_1(\alpha_3-\alpha_2)} \sum_{m=0}^{\infty} \big(q^{m\lambda}\, (q^{\uparrow_{\beta_1}}_{m,|\beta_1|})^2 - \delta_{\lambda,0}\big)\,.
\end{aligned}
$$

A similar computation of $\tau_0(T^-_{K,P})$ can be done following the same arguments. We find eventually

$$
\tau_1(T^+_{K,P})\tau_0(T^-_{K,P}) = \delta_{K,0}\, \delta_{P,0}\, \delta_{\alpha_1,-\beta_1}\, \delta_{\alpha'_1,-\beta'_1} (\delta_{\lambda',0}\, \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3}\, s_{\alpha,\beta} + \delta_{\lambda,0}\, \delta_{\alpha'_2+\beta'_2,\alpha'_3+\beta'_3}\, s_{\alpha',\beta'})
$$

and the result follows.
(iii) The same arguments of (i) apply here with minor changes.
(iv) follows from a slight modification of the proof of Lemma 4.14 (iv).
(v) is a straightforward consequence of (i, ii, iii, iv). □

Lemma 4.17. *Let $\beta, \beta' \in \mathbb{Z}$. Then,*

$$
\lim_{2j\to\infty} \sum_{m=0}^{2j} \big((q^{\uparrow_{\beta}}_{m,|\beta|} q^{\uparrow_{\beta'}}_{2j-m,|\beta'|})^2 - 1\big) = \sum_{m=0}^{\infty} \big((q^{\uparrow_{\beta}}_{m,|\beta|})^2 - 1\big) + \sum_{m=0}^{\infty} \big((q^{\uparrow_{\beta'}}_{m,|\beta'|})^2 - 1\big).
$$

Proof. We give a proof for β and $\beta' > 0$, the other cases being similar.
Since $(q^{\uparrow_\beta}_{m,|\beta|})^2 = \sum_{p_i \in \{0,1\}} (-1)^{|p|_1} q^{r_p} q^{2|p|_1 m}$ where $p = (p_1, \cdots, p_\beta)$ and $r_p := 2(p_1 + \cdots + \beta p_\beta)$, we get, with the notations $\lambda_{p,p'} := (-1)^{|p+p'|_1} q^{r_p + r_{p'}}$ and $U_{2j} := \sum_{m=0}^{2j} (q^{\uparrow_\beta}_{m,|\beta|} q^{\uparrow_{\beta'}}_{2j-m,|\beta'|})^2 - 1$,

$$\begin{aligned} U_{2j} &= \sum_{m=0}^{2j} \sum_{|p+p'|_1 > 0} \lambda_{p,p'} q^{2|p|_1 m + 2|p'|_1 (2j-m)} \\ &= \sum_{|p|_1 \geq |p'|_1, |p|_1 > 0} \lambda_{p,p'} V_{2j,p,p'} + \sum_{|p|_1 < |p'|_1, |p'|_1 > 0} \lambda_{p,p'} V'_{2j,p,p'} \end{aligned}$$

where

$$V_{2j,p,p'} = q^{4j|p'|_1} \sum_{m=0}^{2j} q^{2(|p|_1 - |p'|_1)m}, \qquad V'_{2j,p,p'} = q^{4j|p|_1} \sum_{m=0}^{2j} q^{2(|p'|_1 - |p|_1)m}.$$

It is clear that $V_{2j,p,p'}$ has 0 for limit when $j \to \infty$ when $|p'|_1 > 0$, and $V'_{2j,p,p'}$ has 0 for limit when $j \to \infty$ when $|p|_1 > 0$. As a consequence,

$$U_{2j} = \sum_{|p|_1 > 0} \lambda_{p,0} V_{2j,p,0} + \sum_{|p'|_1 > 0} \lambda_{0,p'} V'_{2j,0,p'} + o(1).$$

The result follows as
$\sum_{m=0}^{2j} \left((q^{\uparrow_\beta}_{m,|\beta|})^2 - 1 \right) = \sum_{|p|_1 > 0} \lambda_{p,0} V_{2j,p,0}$ and $\sum_{m=0}^{2j} \left((q^{\uparrow_{\beta'}}_{m,|\beta'|})^2 - 1 \right) = \sum_{|p'|_1 > 0} \lambda_{0,p'} V'_{2j,0,p'}$. □

Proof of Theorem 4.11. The result follows from Lemmas 4.13, 4.14 (v) and 4.16 (v). □

4.7 Proof of Theorem 4.3 and corollaries

Lemma 4.18. *We have on $SU_q(2)$,*
(i) $⨍ |\mathcal{D}|^{-3} = 2$.
(ii) $⨍ |\mathcal{D}|^{-2} = 0$.
(iii) $⨍ |\mathcal{D}|^{-1} = -\frac{1}{2}$.
(iv) $\zeta_{\mathcal{D}}(0) = 0$.

Proof. (iv) We have by definition

$$\zeta_{\mathcal{D}}(s) := \mathrm{Tr}(|\mathcal{D}|^{-s}) = \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \langle v^j_{m,l} \,,\, |\mathcal{D}|^{-s} v^j_{m,l} \rangle.$$

Since $|\mathcal{D}|^{-s} v^j_{m,l} = \left(\begin{smallmatrix} d_{j^+}^{-s} & 0 \\ 0 & d_j^{-s} \end{smallmatrix}\right) v^j_{m,l}$ where $d_j := 2j + \frac{1}{2}$, we get

$$\zeta_{\mathcal{D}}(s) = \sum_{2j=0}^{\infty} (2j+1)(2j+2)\, d_{j^+}^{-s} + \sum_{2j=1}^{\infty} (2j+1)(2j)\, d_j^{-s} = 2 \sum_{2j=0}^{\infty} (2j+1)(2j)\, d_j^{-s}.$$

With the equalities $(2j+1)(2j) = d_j^2 - \frac{1}{4}$ and $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$ (here $\zeta(s,x) := \sum_{n \in \mathbb{N}} \frac{1}{(n+x)^s}$ is the Hurwitz zeta function and $\zeta(s) := \zeta(s,1)$ is the Riemann zeta function) we get

$$\zeta_{\mathcal{D}}(s) = 2(2^{s-2} - 1)\zeta(s-2) - \tfrac{1}{2}(2^s - 1)\zeta(s) \tag{54}$$

which entails that $\zeta_{\mathcal{D}}(0) = 0$.
(i, ii, iii) are direct consequences of equation (54). □

Proof of Theorem 4.3. It is a consequence of Lemma 4.18 and Theorems 4.1, 4.11. □

As we have seen, the computation of noncommutative integral on $SU_q(2)$ leads to certain function of A which filter some symmetry on the degree in a, a^*, b, b^* of the canonical decomposition. Precisely, it is the balanced features that appear and the following functions of A^n, $n \in \{1, 2, 3\}$:

$$\fint A^n |\mathcal{D}|^{-p} \tag{55}$$

where $1 \leq n \leq p \leq 3$. We will see in the next section a method for the computation of these integrals.

Corollary 4.19. *Let u be a unitary in $C^\infty\big(SU_q(2)\big)$ and $\gamma_u(\mathbb{A}) := \pi(u)\mathbb{A}\pi(u^*) + \pi(u)d\pi(u^*)$ be a gauge-variant of $\mathbb{A}$. Then the following term of Theorem 4.3 are gauge invariant*

$$\fint A|\mathcal{D}|^{-3}\,, \qquad \fint A^2|\mathcal{D}|^{-3} - \fint A|\mathcal{D}|^{-2}\,, \qquad -2\fint A|\mathcal{D}|^{-1} + \fint A^2|\mathcal{D}|^{-2} - \tfrac{2}{3}\fint A^3|\mathcal{D}|^{-3}.$$

Proof. It is sufficient to remark that all terms $\fint |D_{\mathbb{A}}|^{-k}$ and $\zeta_{\mathcal{D}_{\mathbb{A}}}(0)$ in spectral action (4) are gauge invariant. This can also be seen via the computation $D_{\gamma_u(\mathbb{A})} = V_u \mathcal{D} V_u^* + V_u P_0 V_u^*$ where P_0 is the projection on $\operatorname{Ker}\mathcal{D}$ and $V_u = \pi(u)J\pi(u)J^{-1}$ and $\fint |D_{\mathbb{A}}|^{n-k} = \operatorname{Res}_{s=n-k} \operatorname{Tr}\big(|D_{\mathbb{A}}|^{n-k}\big)$ (see [22, Prop. 5.1 (iii) and Prop. 4.8].) □

Corollary 4.20. *In the case of the spectral action without the reality operator (i.e. $D_{\mathbb{A}} = \mathcal{D} + \mathbb{A}$), we get*

$$\fint |D_{\mathbb{A}}|^{-3} = 2, \qquad \fint |D_{\mathbb{A}}|^{-2} = -2\fint A|\mathcal{D}|^{-3}\,, \qquad \fint |D_{\mathbb{A}}|^{-1} = -\tfrac{1}{2} - \fint A|\mathcal{D}|^{-2} + \fint A^2|\mathcal{D}|^{-3},$$
$$\zeta_{D_{\mathbb{A}}}(0) = -\fint A|\mathcal{D}|^{-1} + \tfrac{1}{2}\fint A^2|\mathcal{D}|^{-2} - \tfrac{1}{3}\fint A^3|\mathcal{D}|^{-3}\,.$$

As a consequence, if $\mathbb{A}$ is a one-form such that $\fint A|\mathcal{D}|^{-3} = 0$, then the scale invariant term of the spectral action with or without J is exactly the same modulo a global factor of 2.

5 Differential calculus on $SU_q(2)$ and applications

5.1 The sign of $\mathcal{D}$

There are multiple differential calculi on $SU_q(2)$, see [33,39]. Thanks to [36, Theorem 3], the $3D$ and $4D_\pm$ differential calculi do not coincide with the one considers here: the right multiplication of one-forms by an element in the algebra A is a consequence of the chosen Dirac operator which was introduced according to some equivariance properties with respect to the duality between the two Hopf algebras $SU_q(2)$ and $\mathcal{U}_q(su(2))$.
It is known that the Fredholm module associated to $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is one-summable since $[F, \pi(x)]$ is trace-class for all $x \in \mathcal{A}$. In fact, more can be said about F [1]:

Proposition 5.1. *Since*

$$F = \tfrac{1}{1-q^2}\,\big(\underline{\pi}(a^*)\,d\underline{\pi}(a) + q^2\,\underline{\pi}(b)\,d\underline{\pi}(b^*) + q^2\,\underline{\pi}(a)\,d\underline{\pi}(a^*) + q^2\,\underline{\pi}(b^*)\,d\underline{\pi}(b)\big)\,, \tag{56}$$

F is a central one-form modulo $OP^{-\infty}$.

[1] Note that a similar result for a different spectral triple over $SU_q(2)$ when $q = 0$ was obtained in [13, eq. (48)]

Proof. Forgetting $\underline{\pi}$, this follows from

$$\begin{aligned} a^*\,\delta a + q^2\, b\,\delta b^* + q^2\, a\,\delta a^* + q^2\, b^*\,\delta b \\ &= (a_+^* + a_-^*)(a_+ - a_-) + q^2\,(b_+ + b_-)(b_-^* - b_+^*) + q^2\,(a_+ + a_-)(a_-^* - a_+^*) \\ &\qquad + q^2\,(b_+^* + b_-^*)(b_+ - b_-) \\ &= [a_+^* a_+ - q^2\, a_+ a_+^* + q^2\, b_+^* b_+ - q^2\, b_+ b_+^*] + R = (1-q^2) + R \end{aligned} \tag{57}$$

by (16) where we check that the remainder R is zero:

$$\begin{aligned} R = &- [a_+^* a_- + q^2\, b_+^* b_-] + [a_-^* a_+ + q^2\, b_-^* b_+] - [a_-^* a_- - q^2\, a_- a_-^* + q^2\, b_-^* b_- - q^2\, b_- b_-^*] \\ &+ (q^2\, a_+ a_-^* + q^2\, q_-^* b_+) - (a_+^* a_- + q^2\, b_+^* b_-), \end{aligned}$$

thus, applying (19), (20), (21), $R = +(q^2\, a_+ a_-^* + q^2\, q_-^* b_+) - (a_+^* a_- + q^2\, b_+^* b_-) = 0$ using commutation relations (15).
Now, replacing δ by d in (57) gives (56) since F commute with $a_\pm$, $b_\pm$ and F is central by (28). □

Proposition 5.2. *The one-form in* (56) *is in fact exactly a function of the Dirac operator D:*

$$\pi(a^*)\, d\pi(a) + q^2\, \pi(b)\, d\pi(b^*) + q^2\, \pi(a)\, d\pi(a^*) + q^2\, \pi(b^*)\, d\pi(b) = \xi_q(\mathcal{D}) = F\, \xi_q(|\mathcal{D}|), \tag{58}$$

where $\xi_q(s) := q\frac{[2s]-2s}{[s+1/2][s-1/2]}$.
Moreover, $F = \lim_{q\to 0} \xi_q(D)$.

Proof. First, let us observe that the one-form ω in (58) is invariant under the action of the $\mathcal{U}_q(su(2)) \times \mathcal{U}_q(su(2))$: $h \triangleright \omega = \epsilon(h)\,\omega$ for any $h \in \mathcal{U}_q(su(2)) \times \mathcal{U}_q(su(2))$. For instance, using notations of [21]

$$e \triangleright \omega = q^{\frac{1}{2}} a^* db + q^2 \left(-q^{\frac{1}{2}-1} b da^* + q^{-\frac{1}{2}} b da^* - q^{-1-\frac{1}{2}} a^* db \right) = 0 = \epsilon(e)\,\omega.$$

Therefore, since both the representation π as well as the operator D are equivariant, the image of ω must be diagonal in the spinorial base. A tedious computation with the full spinorial representation π given in (10) yields

$$\begin{aligned} \langle v_{ml}^{j\uparrow},\, \omega\, v_{ml}^{j\uparrow} \rangle &= \tfrac{q^{8j+8} - q^{8j+6} - (4j+3)\, q^{4j+6} + (8j+6)\, q^{4j+4} - (4j+3)\, q^{4j+2} - q^2 + 1}{(q^{4j+4}-1)(q^{4j+2}-1)} = \xi_q(2j + \tfrac{3}{2}), \\ \langle v_{ml}^{j\downarrow},\, \omega\, v_{ml}^{j\downarrow} \rangle &= \tfrac{-q^{8j+4} + q^{8j+2} + (4j+1)\, q^{4j+4} - (8j+2)\, q^{4j+2} + (4j+1)\, q^{4j} + q^2 - 1}{(q^{4j+2}-1)(q^{4j}-1)} = -\xi_q(2j + \tfrac{1}{2}). \end{aligned}$$

These expressions have a clear $q = 0$ limit equal respectively to 1 and -1, so $\omega \to F$ as $q \to 0$. □

In the $q = 1$ limit, these expressions yields identically 0, which is confirmed by the fact that all one-forms are central, it could be expressed as $d(aa^* + bb^*) = d\,1$.
Note that since the invariant one-form we constructed differs by $OP^{-\infty}$ from F, hence any commutator with it will be itself in $OP^{-\infty}$.
We do not know if a central form ω is automatically invariant by the action of both $U_q(su(2))$, that is: $h \triangleright \omega = \epsilon(h)\omega$.

Proposition 5.3. *The order one calculus up to $OP^{-\infty}$ is not universal.*

Proof. Let us take the one-form ω_F from (56), which gives F. Then, for any $x \in \mathcal{A}(SU_q(2))$ we have $\underline{\pi}(x\omega_F - \omega_F x) = 0$. □

Corollary 5.4. *Still modulo $OP^{-\infty}$, $1 \in \pi\big(\Omega^2_u(\mathcal{A})\big)$.*

Proof. $1 = F^2$ is by definition in $\pi\big(\Omega^2_u(\mathcal{A})\big)$. □

In fact, one checks, using (16), (19), (22) that

$$q^2\, da\, da^* - da^*\, da = 1 - q^2 \tag{59}$$

showing again that $1 \in \pi\big(\Omega^2_u(\mathcal{A})\big)$.
Similarly, using (15) and (17), (22), (23), we get still up to $OP^{-\infty}$

$$\begin{aligned} &q\, da\, db = db\, da, && q\, da\, db^* = db^*\, da, \\ &da^*\, db = q\, db\, da^*, && da^*\, db^* = q\, db^*\, da^* \\ &db\, db^* = db^*\, db, && da\, da^* + db\, db^* = -1. \end{aligned} \tag{60}$$

The use of the last equality of (60) and (59) gives

Proposition 5.5. *Up to $OP^{-\infty}$, F is not a (universal) closed one-form, as*

$$da^*\, da + q^2\, da\, da^* + q^2\, db^*\, db + q^2\, db\, db^* = -1 - q^2. \tag{61}$$

5.2 The ideal $\mathcal{R}$

In order to perform explicit calculations of all terms of the spectral action, we observe that each δ-one-form could be expressed in terms of $x\delta(z)y$, where z is one of the generators a, a^*, b, b^* and x, y are some elements of the algebra $\mathcal{A}(SU_q(2))$.
Then, for the computation of $⨍ x dz y |D|^{-1}$ we can use the trace property of the noncommutative integral to get:

$$⨍ x\delta(z)y\, |D|^{-1} = ⨍ yx\delta(z)\, |D|^{-1} + ⨍ x\delta(z)\, |D|^{-1}\delta(y)\, |D|^{-1}.$$

Therefore, the problem of calculating the tadpole-like integral could be in effect reduced to the calculation of much simpler integrals: $⨍ x\delta(z)$ for all generators z and the integrals of higher order in $|D|^{-1}$.
However, it appears that the calculations of higher-order terms simplify a lot, when we further restrict the algebra by introducing an ideal, which is *invisible* to the parts of integral at dimension 2 and 3. For instance, consider the space of pseudodifferential operators $T \in \Psi^0(\mathcal{A})$ of order less or equal to zero (see [16]), which satisfy

$$⨍ T\, t\, |D|^{-2} = ⨍ t\, T\, |D|^{-2} = ⨍ T\, t\, |D|^{-3} = ⨍ t\, T\, |D|^{-3} = 0,\ \forall t \in \Psi^0_0(\mathcal{A}). \tag{62}$$

The elements a_-, b_-b_+, $b_-b_+^*$ and their adjoints are in this space up to $OP^{-\infty}$: this is due to the fact that in Theorem 3.4, $\tau_1 \otimes \tau_1\big(r(x)\big) = 0$ when $r(x) \in \pi_\pm(\mathcal{A}) \otimes \pi_\pm(\mathcal{A})$ mod $OP^{-\infty}$ contains tensor products of $\pi_\pm(b)$ or $\pi_\pm(b^*)$ since these elements are in the kernel of the grading σ.

Definition 5.6. *Let R be the kernel in X of $(\sigma\otimes\sigma)\circ r$ where r is the Hopf-map defined in* (30) *and σ is the symbol map and let $\mathcal{R}$ be the vector space generated by R and $R\,F$.*

Note that R is a *-ideal in X and

$$a_-,\ b_-b_+(=q^2\,b_+b_-),\ b_-b_+^* \text{ are in } \mathcal{R}.$$

By construction and Theorem 3.4, any $T\in\mathcal{R}$ satisfies (62) and $\mathcal{R}$ is invariant by F.
Moreover, by (19), $[b_-,b_-^*]\in R$, so by (16) and (22), $a_+^*a_+-q^2\,a_+a_+^*-(1-q^2)\in R$ and by (23), $q\,a_+b_--b_-a_+\in R$.
It is interesting to quote, thanks to Theorem 3.4 that if $x\in R$, then $⨍\,F\,x\,|\mathcal{D}|^{-1}=0$ while a priori, $⨍\,x\,|\mathcal{D}|^{-1}\neq 0$.
Note that $F\in\Psi^0(\mathcal{A})$ also satisfies (62) by Theorem 3.4 while $F\notin\mathcal{R}$ since $F^2=1$.
Moreover other elements are in $\mathcal{R}$ like for instance $d(b^*b)=d(bb^*)$:

$$\begin{aligned}\delta(bb^*)&=-\delta(aa^*)=-\delta a\,a^*-a\,\delta a^*=-(a_+-a_-)(a_+^*+a_-^*)-(a_++a_-)(a_-^*-a_+^*)\\&=2(a_+a_-^*-a_-a_+^*)\end{aligned}$$

is in R since $a_-\in R$ yielding $d(bb^*)\in R\,F$.
We do not know if $\mathcal{R}$ is equal to the subset of the algebra generated by $\mathcal{B}$ and $\mathcal{B}\,F$ satisfying (62).

Lemma 5.7. *$\mathcal{R}$ is a $*$-ideal in $\Psi^0(\mathcal{A})$ which is invariant by F, d, δ.*

Proof. Since R is an ideal in $X=\mathcal{B}$ mod $OP^{-\infty}$ (see Remark 3.3), $\mathcal{R}$ appears to be an ideal in $\Psi^0(\mathcal{A})\subset$ algebra generated by $\mathcal{B}$ and $\mathcal{B}\,F$. Since $\mathcal{R}$ is invariant by F, its invariance by d follows from its invariance by δ which is true on the generators of R. □

Note that, according to Theorem 4.13, $⨍\,da\,|D|^{-2}=⨍\,da\,|D|^{-3}=0$ while $⨍\,a^*da\,|D|^{-3}=2$ which emphasize the role of "for all t" in (62).

Lemma 5.8. *For any $t\in\Psi_0^0(\mathcal{A})$ and $T\in\mathcal{R}$, we have $⨍\,t\,T\,|D|^{-1}=⨍\,T\,t\,|D|^{-1}$.*

Proof. For any $t\in\mathcal{B}$, we have $⨍\,T\,t\,|D|^{-1}=⨍\,t\,T\,|D|^{-1}+⨍\,T\,|D|^{-1}\,\delta(t)\,|D|^{-1}$ and moreover $⨍\,T\,|D|^{-1}\,\delta(t)\,|D|^{-1}=⨍\,T\,\delta(t)\,|D|^{-2}-⨍\,T\,\delta^2(t)\,|D|^{-3}$ which comes from

$$\begin{aligned}|D|^{-1}\delta(t)|D|^{-1}&=\delta(t)|D|^{-2}+[|D|^{-1},\delta(t)]|D|^{-1}=\delta(t)|D|^{-2}-|D|^{-1}\delta^2(t)|D|^{-2}\\&=\delta(t)|D|^{-2}-\delta^2(t)|D|^{-3}+|D|^{-1}\delta^3(t)|D|^{-3}.\end{aligned}$$

So we get the result because T satisfies (62). □

Lemma 5.9. *If $\simeq$ means equality up to the ideal $\mathcal{R}$, the following rules with $d(.)=[\mathcal{D},.]$ of the first-order differential calculus hold (forgetting $\underline{\pi}$)*

$$\begin{array}{llll}a\,da\simeq da\,a, & a^*\,da\simeq -da^*\,a, & b\,da\simeq q\,da\,b, & b^*\,da\simeq q\,da\,b^*,\\ a\,da^*\simeq -da\,a^*, & a^*\,da^*\simeq da^*\,a^*, & b\,da^*\simeq q^{-1}\,da^*\,b, & b^*\,da^*\simeq q^{-1}\,da^*\,b^*,\\ a\,db\simeq q^{-1}\,db\,a, & a^*\,db\simeq q\,db\,a^*, & b\,db\simeq db\,b, & b^*\,db\simeq db\,b^*\simeq -b\,db^*,\\ a\,db^*\simeq q^{-1}\,db^*\,a, & a^*\,db^*\simeq q\,db^*\,a^*, & b\,db^*\simeq db^*\,b\simeq -b^*\,db, & b^*\,db^*\simeq db^*\,b^*.\end{array}$$

Moreover

$$a^*\,da-q^2da\,a^*\simeq(1-q^2)\,F,\qquad q^2\,a\,da^*-da^*\,a\simeq(1-q^2)\,F. \tag{63}$$

Proof. The table follows from relations (7) and Lemma 3.2 with (28) (one can also use (15).) For instance, since $a_- \in \mathcal{R}$, using the fact that $\mathcal{R}$ is invariant by F,

$$\begin{aligned} b\,da &= (b_+ + b_-)(a_+ - a_-)\,F \simeq (b_+ + b_-)(a_+ + a_-)\,F = ba\,F = q\,ab\,F \simeq q\,(a_+ - a_-)\,F\,b \\ &= q\,da\,b \end{aligned}$$

or similarly, $a^*\,da = (a_+^* + a_-^*)(a_+ - a_-)F \simeq (a_+^* - a_-^*)(a_+ + a_-)F = -da^*\,a$.
The second of equivalence of (63) is just the adjoint of the first one that we prove now:

$$\begin{aligned} a^*\,da - q^2 da\,a^* &= (a_+^* + a_-^*)(a_+ - a_-)F - q^2\,(a_+ - a_-)F(a_+^* + a_-^*) \\ &\simeq (a_+^* + a_-^*)(a_+ + a_-)F - q^2\,(a_+ + a_-)(a_+^* + a_-^*)F = (a^*a - q^2\,aa^*)\,F \\ &= (1 - q^2)\,F. \end{aligned}$$

□

Remark 5.10. *The above written rules remain valid if dx is replaced by $\delta(x)$ and F by 1.*

Working modulo $\mathcal{R}$ simplifies the writing of a one-form:

Lemma 5.11. *(i) Every one-form A can be, up to elements from $\mathcal{R}$, presented as*

$$A \simeq x_a\,da + da^*\,x_{a^*} + x_b\,db + db^*\,x_{b^*},$$

where all x_* are the elements of $\mathcal{A}$.
(ii) When A is selfadjoint, A can be written up to $\mathcal{R}$ (not in a unique way, though) as

$$A \simeq x_a\,da - da^*\,(x_a)^* + x_b\,db - db^*\,(x_b)^*\,,$$

where x_a, x_b are arbitrary elements of $\mathcal{A}$.

Proof. (i) A basis for one-forms consists of the following forms: $a^\alpha b^\beta (b^*)^\gamma\,d\big(a^{\alpha'} b^{\beta'} (b^*)^{\gamma'}\big)$, where $\alpha, \alpha' \in \mathbb{Z}$ and $\beta, \gamma, \beta', \gamma' \in \mathbb{N}$.
Using the Leibniz rule and the commutation rules within the algebra (up to the $\mathcal{R}$ according to Lemma 5.9), we reduce the problem to the case of the forms: $\big(a^\alpha b^\beta (b^*)^\gamma\big)\,dx\,\big(a^{\alpha'} b^{\beta'} (b^*)^{\gamma'}\big)$, where x can be either of the generators a, a^*, b, b^*. If $x = b$ or $x = b^*$, the straightforward application of the rules of the differential calculus leads to the answer that the one-form could be expressed as: $a^\alpha b^\beta (b^*)^\gamma\,db$ and $db^*\,a^\alpha b^\beta (b^*)^\gamma$.
Similar considerations for the case $x = a, a^*$ lead to the remaining terms.
Note that the presentation is not unique, since there still might remain terms, which are in $\mathcal{R}$, for example: $b^*db + db^*b = d(bb^*) \in \mathcal{R}$.
(ii) is direct. □

Next we can start explicit calculation of the integrals, beginning with the tadpole terms.
Application of the Leibniz rule yields to a presentation of one-forms which is different from the one of previous lemma. Each δ-one-form could be expressed, as a finite sum of the terms $x\delta(z)y$, where z is one of the generators a, a^*, b, b^* and x, y are some elements of the algebra $\mathcal{A}\big(SU_q(2)\big)$.

Proposition 5.12. *For all $x, y \in \mathcal{A}\big(SU_q(2)\big)$ and $z \in \{a, a^*, b, b^*\}$ we have*

$$⨍ x\delta(z)y\,|D|^{-1} = ⨍ yx\delta(z)\,|D|^{-1} + ⨍ x\delta(z)\delta(y)\,|D|^{-2} - ⨍ x\delta(z)\delta^2(y)\,|\mathcal{D}|^{-3}.$$

Proof. This is just the application of the trace property of the noncommutative integral, together with the identity: $|D|^{-1}\delta(z)|D|^{-1} = -\left[|D|^{-1}, z\right]$. □

Remark 5.13. *The computation of tadpole-like integrals is reduced to the integrals* $⨍ x\delta(z)|\mathcal{D}|^{-1}$ *for all generators* z *and the integrals of higher order in* $|D|^{-2}$*. However, the calculations of higher-order terms simplify a lot after we use the relations which hold up to the ideal* $\mathcal{R}$*: this erases parts of integral depending on* $|\mathcal{D}|^{-2}$ *and* $|\mathcal{D}|^{-3}$*. Thus, beside* $⨍ x\delta(z)\,|\mathcal{D}|^{-1}$*, we only need to compute* $⨍ x\delta(z)\delta(z')\,|\mathcal{D}|^{-2}$ *where* z *and* z' *are generators, since all the* $|\mathcal{D}|^{-3}$ *integrals have already been explicitly computed in section 4.6 (these integrals do not depend on* q*.)*
Besides the tadpole, the only integrals that need to be computed are $⨍ A\,|\mathcal{D}|^{-2}$ *and* $⨍ A^2\,|\mathcal{D}|^{-2}$ *where* A *is a* δ*-1-form. Working modulo* $\mathcal{R}$ *and using again Leibniz rule, we only need to compute* $⨍ x\delta(z)\,|\mathcal{D}|^{-2}$ *and the previous integrals* $⨍ x\delta(z)\delta(z')\,|\mathcal{D}|^{-2}$*.*

5.2.1 Operators L_q and M_q

In the notation $v^j_{l,m}$ of $\mathcal{H}$, we have already use the j dependence in (12) with $J_q\, v^j_{m,l} := q^j\, v^j_{m,l}$. Let L_q and M_q be the similar diagonal operators

$$L_q\, v^j_{m,l} := q^{2l}\, v^j_{m,l}\,,$$
$$M_q\, v^j_{m,l} := q^{2m}\, v^j_{m,l}\,.$$

We immediately get

Lemma 5.14. *For* $n \in \mathbb{N}^*$, $⨍ (L_q)^n\,|\mathcal{D}^{-2}| = ⨍ (M_q)^n\,|\mathcal{D}^{-2}| = \frac{2}{1-q^{2n}}$.

Proof. We have

$$\begin{aligned}
\mathrm{Tr}\left(L_q^n|\mathcal{D}|^{-2-s}\right) &= \sum_{2j=0}^{\infty}\sum_{m=0}^{2j}\sum_{l=0}^{2j+1}\langle v^j_{m,l}, L_q^n|\mathcal{D}|^{-2-s}v^j_{m,l}\rangle \\
&= \sum_{2j=0}^{\infty}(2j+1)\tfrac{1-q^{2n(2j+2)}}{1-q^{2n}}d_{j^+}^{-2-s} + \sum_{2j=0}^{\infty}(2j+1)\tfrac{1-q^{2n(2j+2)}}{1-q^{2n}}d_{j^-}^{-2-s} \\
&\sim_0 \tfrac{1}{1-q^{2n}}\left(\zeta(s+1,\tfrac{3}{2}) + \zeta(s+1,\tfrac{1}{2}\right) \sim_e \tfrac{2}{1-q^{2n}}\zeta(s+1)\,.
\end{aligned}$$

where $\sim_0$ means modulo a function holomorphic at 0. This gives the result for L_q^n and a similar computation can be done for M_q^n. □

The interest of these operators stems in

Lemma 5.15. *We have* $L_qM_q \in \mathcal{R}$*. Moreover,*

$$\begin{aligned}
&b\,\delta b^* \simeq M_q - L_q, \quad b^*\delta b \simeq L_q - M_q, \quad bb^* \simeq L_q + M_q,\\
&a\,\delta(a^*) \simeq -aa^* \simeq L_q + M_q - 1, \quad a^*\,\delta a \simeq a^*a \simeq 1 - q^2(L_q + M_q),\\
&da\,da^* \simeq L_q + M_q - 1, \quad da^*\,da \simeq q^2(L_q + M_q) - 1,\\
&b^{n-2}(b^*)^n\,db\,db \simeq (L_q)^n + (M_q)^n,\\
&b^{n-1}\,(b^*)^{n-1}\,db\,db^* \simeq -(L_q)^n - (M_q)^n,\\
&b^n(b^*)^{n-2}\,db^*\,db^* \simeq (L_q)^n + (M_q)^n.
\end{aligned}$$

Proof. Since $L_q M_q = q^2\, a_- a_-^* \in \mathcal{R}$, we compute up to the ideal $\mathcal{R}$

$$b\,\delta b^* = (b_+ + b_-)(b_-^* - b_+^*) \simeq -b_+ b_+^* + b_- b_-^* = M_q - L_q + L_q M_q (1-q^2) \simeq M_q - L_q$$

and similarly for the other relations. □

5.2.2 Automorphisms of the algebra and symmetries of integrals

Proposition 5.16. *For any $n \in \mathbb{N}^*$,*

$$\begin{aligned}
&⨍ (bb^*)^n\, |\mathcal{D}|^{-1} = \tfrac{-2(1+q^{2n})}{(1-q^{2n})^2}\,,\\
&⨍ (bb^*)^n b^*\, \delta b\, |\mathcal{D}|^{-1} = ⨍ (bb^*)^n b\, \delta b^*\, |\mathcal{D}|^{-1} = \tfrac{2}{1-q^{2n+2}}\,,\\
&⨍ (bb^*)^n a\, da^*\, \mathcal{D}^{-1} = \tfrac{-2q^{4n+2}-2q^{4n}-2q^{2n+2}+6q^{2n}}{(1-q^{2n})^2(1-q^{2n+2})}\,,\\
&⨍ (bb^*)^n a^*\, da\, \mathcal{D}^{-1} = \tfrac{6q^{2n+2}-2q^{2n}-2q^2-2}{(1-q^{2n})^2(1-q^{2n+2})}\,.
\end{aligned}$$

Note that the knowledge of these integral is enough for the computation of any term of the form $⨍ x\delta(z)|\mathcal{D}|^{-1}$, where z is a generator, since any other δ-one-form will be unbalanced.
To show this proposition, we will use few symmetries, properties of the ideal $\mathcal{R}$ and replacement of δ-one-forms in terms of L_q, M_q as above.
Let U be the following unitary operator on the Hilbert space:

$$U\, v_{m,l}^{j\uparrow} = (-1)^{m+l}\, v_{l,m}^{j^+\downarrow}, \quad U\, v_{m,l}^{j\downarrow} = (-1)^{m+l}\, v_{l,m}^{j^-\uparrow}.$$

Then, by explicit computations we have

$$U^* a U = a, \qquad U^* a^* U = a^*, \qquad U^* b U = b^*, \qquad U^* b^* U = b, \qquad \text{and} \qquad U^* \mathcal{D} U = -\mathcal{D}.$$

Lemma 5.17. *Each noncommutative integral* (55) *of an element of the algebra or differential forms is (up to sign) invariant under the algebra automorphism ρ defined by*

$$\rho(a) := a, \;\; \rho(a^*) := a^*, \;\; \rho(b) := b^*, \;\; \rho(b^*) := b. \tag{64}$$

Proof. For any homogeneous polynomial p and any $k \in \mathbb{N}$,

$$\begin{aligned}
⨍ p(a, a^*, b, b^*, \mathcal{D})\, \mathcal{D}^{-k} &= ⨍ U^* p(a, a^*, b, b^*, \mathcal{D})\, \mathcal{D}^{-n} U\\
&= (-1)^k ⨍ p(U^*aU, U^*a^*U, U^*bU, U^*b^*U, U^*\mathcal{D}U)\, \mathcal{D}^{-k}\\
&= (-1)^{k+d} ⨍ p(\rho(a), \rho(a^*), \rho(b), \rho(b^*), \mathcal{D})\, \mathcal{D}^{-k},
\end{aligned}$$

where d is the degree of p with respect to D. □

Corollary 5.18. *For any $n \in \mathbb{N}$, $⨍ (bb^*)^n\, b^*\, db\, \mathcal{D}^{-1} = ⨍ (bb^*)^n\, b\, db^*\, \mathcal{D}^{-1}$.*

Lemma 5.19. *For any $x, y \in \Psi^0(\mathcal{A})$,*

$$(i) \quad \fint xy|\mathcal{D}|^{-1} = \fint yx|\mathcal{D}|^{-1} + \fint x\delta(y)|\mathcal{D}|^{-2} - \fint x\delta^2(y)\,|\mathcal{D}|^{-2}.$$

$$(ii) \quad \fint z\,x\,\mathcal{D}^{-1}y\,\mathcal{D}^{-1} = \fint z\,xy\,\mathcal{D}^{-2}, \text{ where } z \in \mathcal{A} \text{ contains } b \text{ or } b^*.$$

Proof. (i) is direct consequence of the trace property of $\fint$ and the fact that OP^{-4} operators are trace-class.
$ii)$ We calculate:

$$\begin{aligned}\fint z\,x\,\mathcal{D}^{-1}y\,\mathcal{D}^{-1} &= \fint z\,x\left(y\,\mathcal{D}^{-1} - \mathcal{D}^{-1}[\mathcal{D}, y]\,\mathcal{D}^{-1}\right)\mathcal{D}^{-1} = \fint z\,xy\,\mathcal{D}^{-2} - \fint z\,x\mathcal{D}^{-1}[D, y]\,\mathcal{D}^{-2}\\ &= \fint z\,xy\,\mathcal{D}^{-2}.\end{aligned}$$

The last step is based on the observation that any integral with $\mathcal{D}^{-3}$ vanishes if the expression integrated contains b or b^*. □

Lemma 5.20. *For any $n \in \mathbb{N}$,*

$$(i) \quad \fint (bb^*)^n\,b^*\,db\,\mathcal{D}^{-1} = \tfrac{2}{1-q^{2n+2}}\,.$$

$$(ii) \quad \fint (bb^*)^n d(bb^*)\,\mathcal{D}^{-1} = 0.$$

$$(iii) \quad \fint (bb^*)^n|D|^{-1} = \tfrac{-2(1+q^{2n})}{(1-q^{2n})^2}\,.$$

Proof. (i) With $n > 1$, we begin with $\fint d\left((b\,b^*)^n\right)\mathcal{D}^{-1} = 0$, which follows directly from the trace property of the noncommutative integral. Expanding the expression using Leibniz rule and the commutation

$$x\mathcal{D}^{-1} = \mathcal{D}^{-1}x + \mathcal{D}^{-1}[D, x]\mathcal{D}^{-1}, \tag{65}$$

we obtain

$$\begin{aligned}0 =& \sum_{k=0}^{n-1}\fint b^k\,db\,b^{n-k-1}(b^*)^n\,\mathcal{D}^{-1} + \sum_{k=0}^{n-1}\fint b^n(b^*)^k\,db^*\,(b^*)^{n-k-1}\,\mathcal{D}^{-1}\\ =& n\left(\fint b^{n-1}(b^*)^n\,db\,\mathcal{D}^{-1} + \fint b^n(b^*)^{n-1}\,db^*\,\mathcal{D}^{-1}\right)\\ &+ \sum_{k=0}^{n-1}\fint\left(b^k\,db\,\mathcal{D}^{-1}d(b^{n-k-1}(b^*)^n)\mathcal{D}^{-1} + b^n(b^*)^k\,db^*\,\mathcal{D}^{-1}d((b^*)^{n-k-1})\mathcal{D}^{-1}\right).\end{aligned}$$

Using Lemma 5.19,

$$\begin{aligned}0 =& n\fint (bb^*)^{n-1}(b^*\,db + b\,db^*)\,\mathcal{D}^{-1}\\ &+ \fint\left(\tfrac{1}{2}n(n-1)b^{n-2}(b^*)^n\,db\,db + n^2b^{n-1}\,(b^*)^{n-1}\,db\,db^* + \tfrac{1}{2}n(n-1)b^n(b^*)^{n-2}\,db^*\,db^*\right)\mathcal{D}^{-2}.\end{aligned}$$

The integrals with $\mathcal{D}^{-2}$ could be easily calculated when we take restrict ourselves to calculations modulo ideal $\mathcal{R}$:

$$n \fint (bb^*)^{n-1}(b^*\,db + b\,db^*)\,\mathcal{D}^{-1} = -2\left(n(n-1) - 2n^2 + n(n-1)\right)\tfrac{1}{1-q^{2n}} = 4n\tfrac{1}{1-q^{2n}}\,.$$

Hence $\fint (bb^*)^{n-1}(b^*\,db + b\,db^*)\,\mathcal{D}^{-1} = \frac{4}{1-q^{2n}}$, which together with Corollary 5.18 proves i).
(ii) In a similar way, $\fint (bb^*)^{n-1} d(bb^*)\,\mathcal{D}^{-1} = 0 = \fint (bb^*)^{n-1} d(aa^*)\,\mathcal{D}^{-1}$ implies:

$$\begin{aligned}
0 =& \sum_{k=0}^{n-1} (bb^*)^{n-k-1} d(bb^*)(bb^*)^k\,\mathcal{D}^{-1}\\
=& n \fint (bb^*)^{n-1} d(bb^*)\,\mathcal{D}^{-1} + \tfrac{1}{2}n(n-1)\fint (bb^*)^{n-2} d(bb^*)\,d(bb^*)\,\mathcal{D}^{-2}\\
=& n \fint (bb^*)^{n-1} d(bb^*)\,\mathcal{D}^{-1},
\end{aligned}$$

where in the last step we used that $d(bb^*) \in \mathcal{R}$. The identity (ii) now follows from the equality $aa^* = 1 - bb^*$.
(iii) Using Lemma 5.19, we get

$$A_n := \fint (bb^*)^n |D|^{-1} = \fint (bb^*)^n (aa^* + bb^*)|D|^{-1}$$

and we push now a^* through $|D|^{-1}$ and from cyclicity of the trace through $(bb^*)^n$,

$$= A_{n+1} + \fint (bb^*)^n q^{2n} a^* a\,|D|^{-1} + \fint (bb^*)^n q^{2n} a\delta(a^*)\,|D|^{-2}$$

the last term being calculated explicitly, since up to ideal $\mathcal{R}$, $a\delta(a^*) \simeq L_q + M_q - 1$,

$$= A_{n+1}(1 - q^{2n+2}) + q^{2n} A_n + 4\left(\tfrac{1}{1-q^{2n+2}} - \tfrac{1}{1-q^{2n}}\right),$$

which leads to

$$A_n(1 - q^{2n}) + \tfrac{4}{1-q^{2n}} = A_{n+1}(1 - q^{2n+2}) + \tfrac{4}{1-q^{2n+2}}\,.$$

Assuming $A_n = \frac{f_n}{(1-q^{2n})^2}$ we have $\frac{f_n+4}{1-q^{2n}} = \frac{f_{n+1}+4}{1-q^{2n+2}}$, and taking into account that $A_0 = -2\frac{1+q^2}{(1-q^2)^2}$, we obtain $A_n = -2\frac{1+q^{2n}}{(1-q^{2n})^2}$. □

Finally, to get Proposition 5.16, it remains to prove

Lemma 5.21. *For $n \geq 1$,*

$$\begin{aligned}
\fint (bb^*)^n a\,da^*\,\mathcal{D}^{-1} &= \tfrac{-2q^{4n+2}-2q^{4n}-2q^{2n+2}+6q^{2n}}{(1-q^{2n})^2(1-q^{2n+2})}\,,\\
\fint (bb^*)^n a^*\,da\,\mathcal{D}^{-1} &= \tfrac{6q^{2n+2}-2q^{2n}-2q^2-2}{(1-q^{2n})^2(1-q^{2n+2})}\,.
\end{aligned}$$

Proof. First, using Leibniz rule, (65) and Lemma 5.19 we have (for $n \geq 1$)

$$\fint (bb^*)^n a\,da^*\,\mathcal{D}^{-1} = -q^{2n}\fint (bb^*)^n a^*\,da - \fint (bb^*)^n da\,da^*\,\mathcal{D}^{-2}.$$

Further, we use the identity (56):

$$\fint (bb^*)^n \left(a^*\, da + q^2 a\, da^* + q^2 b\, db^* + q^2 b^*\, db\right) \mathcal{D}^{-1} = (1-q^2) \fint (bb^*)^n\, |D|^{-1}.$$

taking into account that $F\,\mathcal{D} = |\mathcal{D}|$.
These equations give together a system of linear equations

$$\begin{aligned} \fint (bb^*)^n a\, da^*\, \mathcal{D}^{-1} + q^{2n} \fint (bb^*)^n a^*\, da\, \mathcal{D}^{-1} &= -4\left(\tfrac{1}{1-q^{2n+2}} - \tfrac{1}{1-q^{2n}}\right), \\ q^2 \fint (bb^*)^n a\, da^*\, \mathcal{D}^{-1} + \fint (bb^*)^n a^*\, da\, \mathcal{D}^{-1} &= -2(1-q^2)\tfrac{1+q^{2n}}{(1-q^{2n})^2} - \tfrac{4q^2}{1-q^{2n+2}} \end{aligned}$$

which is solved by the expressions stated in the lemma. □

5.2.3 The noncommutative integrals at $|D|^{-2}$

We need to separate this task into two problems. First, we shall to calculate all integrals $\fint x\, \delta(z) |D|^{-2}$, with $x \in \mathcal{A}(SU_q(2))$ and z being one of the generators. The second problem is to calculate $\fint x\, \delta(y)\, \delta(z)\, |D|^{-2}$, with both y and z being the generators $\{a, a^*, b, b^*\}$.

Lemma 5.22. *The only a priori non-vanishing integrals of the type* $\fint x\, \delta(z)\, |D|^{-2}$ *are for* $n \in \mathbb{N}$*:*

$$\begin{aligned} \fint (bb^*)^n b^* \delta(b)\, |\mathcal{D}|^{-2} &= \fint (bb^*)^n b \delta(b^*)\, |\mathcal{D}|^{-2} = 0, \\ \fint (bb^*)^n a \delta(a^*)\, |\mathcal{D}|^{-2} &= \tfrac{4q^{2n}(1-q^2)}{(q^{2n+2}-1)(1-q^{2n})}, \quad n > 0 \\ \fint (bb^*)^n a^* \delta(a)\, |\mathcal{D}|^{-2} &= \tfrac{4(1-q^2)}{(1-q^{2n+2})(1-q^{2n})}. \end{aligned}$$

Proof. Since $a\delta(a^*) \simeq L_q + M_q - 1$ and $(bb^*)^n \simeq L_q^n + M_q^n$, we get

$$(bb^*)^n a \delta(a^*) \simeq L_q^{n+1} + M_q^{n+1} - L_q^n - M_q^n$$

and the second result is obtained from Lemma 5.14. The other integrals are computed in a similar way. □

Lemma 5.23. *The only a priori non-vanishing integrals of the type* $⨍ x\, dy\, dz\, |D|^{-2}$ *are for* $n \in \mathbb{N}$*:*

$$⨍ (bb^*)^n\, (b^*)^2 db\, db\, |\mathcal{D}|^{-2} = \tfrac{4}{1-q^{2n+4}},$$
$$⨍ (bb^*)^n\, db\, db^*\, |\mathcal{D}|^{-2} = \tfrac{4}{1-q^{2n+2}},$$
$$⨍ (bb^*)^n\, (a^*b^*)(da\, db)\, |\mathcal{D}|^{-2} = 0,$$
$$⨍ (bb^*)^n\, (ab^*)(da^*\, db)\, |\mathcal{D}|^{-2} = 0,$$
$$⨍ (bb^*)^n\, (a^*b)(da\, db^*)\, |\mathcal{D}|^{-2} = 0,$$
$$⨍ (bb^*)^n\, (ab)(da^*\, db^*)\, |\mathcal{D}|^{-2} = 0,$$
$$⨍ (bb^*)^n\, (da\, da^*)\, |\mathcal{D}|^{-2} = \tfrac{4(q^{2n+2}-q^{2n})}{(1-q^{2n+2})(1-q^{2n})}\,, \quad n > 0$$
$$⨍ (bb^*)^n\, (da^*\, da)\, |\mathcal{D}|^{-2} = \tfrac{4(q^2-1)}{(1-q^{2n+2})(1-q^{2n})}\,.$$

Proof. This follows from Lemma 5.14 with the equivalences up to $\mathcal{R}$ gathered in Lemma 5.15. □

6 Examples of spectral action

It is clear from Theorem 4.3 that any one-form of the form ada, bdb, adb, a^*db, etc... do not contribute to the spectral action. Indeed, only the balanced parts of one-forms give a possibly nonzero term in the coefficients. Let us now give the values of the terms $⨍ A^n\, |\mathcal{D}|^{-p}$ and the full $\zeta_{\mathcal{D}_{\mathbb{A}}}(0)$ for few examples

Table 1: Values of noncommutative integrals

$\mathbb{A}$	$⨍ A\, \lvert\mathcal{D}\rvert^{-3}$	$⨍ A^2\, \lvert\mathcal{D}\rvert^{-3}$	$⨍ A^3\, \lvert\mathcal{D}\rvert^{-3}$	$⨍ A\, \lvert\mathcal{D}\rvert^{-2}$	$⨍ A^2\, \lvert\mathcal{D}\rvert^{-2}$	$⨍ A\, \lvert\mathcal{D}\rvert^{-1}$	$\zeta_{\mathcal{D}_{\mathbb{A}}}(0)$
a^*da	2	2	2	$\frac{4q^2}{q^2-1}$	$\frac{4q^2(q^2+2)}{q^4-1}$	$\frac{3q^2+1}{2(q^2-1)}$	$\frac{11q^4+36q^2+13}{3(q^4-1)}$
b^*db	0	0	0	0	$\frac{-4}{q^4-1}$	$\frac{-2}{q^2-1}$	$\frac{4q^2}{q^4-1}$
ada^*	-2	2	-2	$\frac{-4}{q^2-1}$	$\frac{4(2q^2+1)}{q^4-1}$	$\frac{q^2+3}{2(q^2-1)}$	$\frac{13q^4+36q^2+11}{3(q^4-1)}$
bdb^*	0	0	0	0	$\frac{-4}{q^4-1}$	$\frac{-2}{q^2-1}$	$\frac{4q^2}{q^4-1}$

1) Clearly the spectral action depends on q: for instance,

$$\mathcal{S}(\mathcal{D}_{a^*da}, \Phi, \Lambda) \;=\; 2\,\Phi_3\,\Lambda^3 - 8\,\Phi_2\,\Lambda^2 + \tfrac{q^2+15}{2(1-q^2)}\,\Phi_1\,\Lambda^1 + \tfrac{11q^4+36q^2+13}{3(q^4-1)}\,\Phi(0).$$

2) Moreover, for $B := a\,\delta a^*$ and $A := B + B^*$, we get since $B \simeq B^* \mod \mathcal{R}$,

$$⨍ A^p |\mathcal{D}|^{-k} = 2^p ⨍ B^p |\mathcal{D}|^{-k}, \quad 1 \le p \le k \le 3\,. \tag{66}$$

Thus the spectral action of the selfadjoint one-form $\mathbb{A} := ada^* + (ada^*)^*$ is

$$\mathcal{S}(\mathcal{D}_{\mathbb{A}}, \Phi, \Lambda) \;=\; 2\,\Phi_3\,\Lambda^3 + 16\,\Phi_2\,\Lambda^2 + \tfrac{q^2-33}{2(1-q^2)}\,\Phi_1\,\Lambda^1 + \tfrac{122q^4+168q^2-2}{3(q^4-1)}\,\Phi(0).$$

3) When $B_n := (bb^*)^n\, b\,\delta b^*$, then by Lemma (5.15), $B_n \simeq B_n^*$, so for $A_n := B_n + B_n^*$, the equation (66) is still valid and $⨍ B_n^p\,|\mathcal{D}|^{-k}$ are all zero but $⨍ B_n\,|\mathcal{D}|^{-1} = \frac{2}{1-q^{2n+2}}$ and $⨍ B_n^2\,|\mathcal{D}|^{-2} = \frac{4}{1-q^{4n+4}}$, so

$$\mathcal{S}(\mathcal{D}_{\mathbb{A}_n}, \Phi, \Lambda) \;=\; 2\,\Phi_3\,\Lambda^3 - \tfrac{1}{2}\,\Phi_1\Lambda^1 + \tfrac{8}{1+q^{2n+2}}\,\Phi(0). \tag{67}$$

Remark that this spectral action still exists as $q \to 1$!
Note however that the symmetrization process (66) is not true in general, for instance if $B := a\,\delta b$ and $A := B + B^*$, then $⨍ A^2|\mathcal{D}|^{-1} = \frac{8(q^4-q^2-1)}{(1-q^4)^2}$ while $⨍ B^2|\mathcal{D}|^{-1} = 0$ or $⨍[B,\, B^*]|\mathcal{D}|^{-1} = \frac{4}{1-q^4}$.
4) The spectral action can be also independent of q: for instance, if $\mathbb{A} = \frac{1}{1-q^2}\,\xi(\mathcal{D})$ is the q-dependent selfadjoint one-form given in (58), then,

$$\mathcal{S}(\mathcal{D}_{\mathbb{A}}, \Phi, \Lambda) \;=\; 2\,\Phi_3\,\Lambda^3 - 8\,\Phi_2\,\Lambda^2 + \tfrac{15}{2}\,\Phi_1\,\Lambda^1 - \tfrac{13}{3}.$$

7 The commutative sphere $\mathbb{S}^3$

Since $SU(2) \simeq \mathbb{S}^3$, we get a concrete spinorial representation of the algebra $\mathcal{A} := C^\infty(\mathbb{S}^3)$ on the same Hilbert space $\mathcal{H}$ and same Dirac operator $\mathcal{D}$ with (10) where $q = 1$ which means that q-numbers are trivial: $[\alpha] = \alpha$. So

$$\begin{aligned}
\pi(a)\,|j\mu n\rangle\!\rangle &:= \alpha^+_{j\mu n}\,|j^+\mu^+n^+\rangle\!\rangle + \alpha^-_{j\mu n}\,|j^-\mu^+n^+\rangle\!\rangle,\\
\pi(b)\,|j\mu n\rangle\!\rangle &:= \beta^+_{j\mu n}\,|j^+\mu^+n^-\rangle\!\rangle + \beta^-_{j\mu n}\,|j^-\mu^+n^-\rangle\!\rangle,\\
\pi(a^*)\,|j\mu n\rangle\!\rangle &:= \tilde\alpha^+_{j\mu n}\,|j^+\mu^-n^-\rangle\!\rangle + \tilde\alpha^-_{j\mu n}\,|j^-\mu^-n^-\rangle\!\rangle,\\
\pi(b^*)\,|j\mu n\rangle\!\rangle &:= \tilde\beta^+_{j\mu n}\,|j^+\mu^-n^+\rangle\!\rangle + \tilde\beta^-_{j\mu n}\,|j^-\mu^-n^+\rangle\!\rangle
\end{aligned} \tag{68}$$

where

$$\alpha^+_{j\mu n} := \sqrt{j+\mu+1}\begin{pmatrix} \frac{\sqrt{j+n+3/2}}{2j+2} & 0 \\ \frac{\sqrt{j-n+1/2}}{(2j+1)(2j+2} & \frac{\sqrt{j+n+1/2]}}{2j+1} \end{pmatrix},$$

$$\alpha^-_{j\mu n} := \sqrt{j-\mu}\begin{pmatrix} \frac{\sqrt{j-n+1/2}}{2j+1} & -\frac{\sqrt{j+n+1/2}}{2j(2j+1)} \\ 0 & \frac{\sqrt{j-n-1/2}}{2j} \end{pmatrix},$$

$$\beta^+_{j\mu n} := \sqrt{j+\mu+1}\begin{pmatrix} \frac{\sqrt{j-n+3/2}}{2j+2} & 0 \\ -\frac{\sqrt{j+n+1/2}}{(2j+1)(2j+2)} & \frac{\sqrt{j-n+1/2}}{2j+1} \end{pmatrix},$$

$$\beta^-_{j\mu n} := \sqrt{j-\mu}\begin{pmatrix} -\frac{\sqrt{j+n+1/2}}{2j+1} & -\frac{\sqrt{j-n+1/2}}{2j(2j+1} \\ 0 & -\frac{\sqrt{j+n-1/2}}{2j} \end{pmatrix}$$

with $\tilde\alpha^\pm_{j\mu n} := (\alpha^\mp_{j^\pm\mu^-n^-})^*$, $\tilde\beta^\pm_{j\mu n} := (\beta^\mp_{j^\pm\mu^-n^+})^*$.
Note that the representation on the vectors $v^j_{m,l}$ is not as convenient as in (11).
One can check that the generators $\pi(a)$, $\pi(b)$ and their adjoint commute and that $[x, [\mathcal{D}', y]] = 0$ for any x, $y \in \mathcal{A}$.

7.1 Translation of Dirac operator

In general the Dirac operator is defined in a more symmetric way than that we did. So, although not absolutely necessary here, we define for the interested reader the unbounded self-adjoint translated operator $\mathcal{D}'$ on $\mathcal{H}$ by the constant λ as

$$\mathcal{D}' := \mathcal{D} + \lambda.$$

For instance, this gives for $\lambda = -\frac{1}{2}$ in the case of $\mathbb{S}^3$, see [32], $\mathcal{D}'\, v^j_{m,l} = (2j+1)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) v^j_{m,l}$ so $v^j_{m,l}$ is an eigenvector of $|\mathcal{D}'|$.
As the following lemma shows, the computation of noncommutative integrals involving $\mathcal{D}$ can be reduced to the computation of certain noncommutative integrals involving $\mathcal{D}'$:

Lemma 7.1. *If $⨍' T := \mathrm{Res}_{s=0}\,\mathrm{Tr}\left(T|\mathcal{D}'|^{-s}\right)$, then for any 1-form A on a spectral triple of dimension n,*

$$⨍ A\,|\mathcal{D}|^{-(n-2)} = ⨍' A\,|\mathcal{D}'|^{-(n-2)} + \lambda\,(n-2) ⨍' A\,\mathcal{D}'|\mathcal{D}'|^{-n} + \lambda^2\,\tfrac{(n-1)(n-2)}{2} ⨍' A\,|\mathcal{D}'|^{-n},$$
$$⨍ A\,\mathcal{D}^{-(n-2)} = ⨍' A\,\mathcal{D}'^{-(n-2)} + \lambda\,(n-2) ⨍' A\,\mathcal{D}'^{-(n-1)} + \lambda^2\,\tfrac{(n-1)(n-2)}{2} ⨍' A\,\mathcal{D}'^{-n}.$$

Proof. Recall from [22, Proposition 4.8] that for any pseudodifferential operator P,

$$⨍ P|\mathcal{D}|^{-r} = \mathrm{Res}_{s=0}\,\mathrm{Tr}\left(P|\mathcal{D}|^{-r}|\mathcal{D}'|^{-s}\right).$$

Moreover, by [22, Lemma 4.3], for any $s \in \mathbb{C}$ and $N \in \mathbb{N}^*$

$$|\mathcal{D}|^{-s} = |\mathcal{D}'|^{-s} + \sum_{p=1}^{N} K_{p,s}\, Y^p\, |\mathcal{D}'|^{-s} \mod OP^{-N-1-\Re(s)} \tag{69}$$

where $Y = \sum_{k=1}^{N} \frac{(-1)^{k+1}}{k}(-2\lambda\mathcal{D}' + \lambda^2)^k \mathcal{D}'^{-2k} \mod OP^{-N-1}$ and $K_{p,s}$ are complex numbers that can be explicitly computed. Precisely, we find $K_{p,s} = (-\frac{s}{2})^p\, V(p)$ where $V(p)$ is the volume of the p-simplex. Since the spectral dimension is n, we work modulo $OP^{-(n+1)}$, and for $s = n-2$, we get from (69): $|\mathcal{D}|^{-(n-2)} = |\mathcal{D}'|^{-(n-2)} + \lambda(n-2)\mathcal{D}'|\mathcal{D}'|^{-n} + \lambda^2\frac{(n-1)(n-2)}{2}|\mathcal{D}'|^{-n} \mod OP^{-(n+1)}$.
As a consequence, we have for $P \in OP^0$ (the OP^0 spaces are the same for $\mathcal{D}$ or $\mathcal{D}'$),

$$⨍ P|\mathcal{D}|^{-(n-2)} = ⨍' P|\mathcal{D}'|^{-(n-2)} + \lambda(n-2) ⨍' P\mathcal{D}'|\mathcal{D}'|^{-n} + \lambda^2\tfrac{(n-1)(n-2)}{2} ⨍' P|\mathcal{D}'|^{-n}.$$

Since A and AF are in OP^0, we get both formulae. □

7.2 Tadpole and spectral action on $\mathbb{S}^3$

We consider now the commutative spectral triple $(C^\infty(\mathbb{S}^3),\, \mathcal{H},\, \mathcal{D})$. It is 1-summable since $\langle\!\langle j\mu n\, s \,|\, [F, \pi(x)]\, |j\mu n\, s\rangle\!\rangle = 0$ when $x = a, a^*, b, b^*$ for any $j,\ \mu,\ n,\ s = \uparrow, \downarrow$.
All integrals of above lemma are zero for $\mathbb{S}^3$:

Proposition 7.2. *There is no tadpole of any order on the commutative real spectral triple $(C^\infty(\mathbb{S}^3),\, \mathcal{H},\, \mathcal{D})$. In fact, for any one-form or δ-one-form A, $⨍ A\mathcal{D}^{-p} = 0$ for $p \in \{1, 2, 3\}$.*

Proof. We first want to prove $⨍ A\mathcal{D}^{-p} = 0$ for $p \in \{1,2,3\}$ and any one-form A. Since the representation is real, that is any matrix elements of the generators are real, so must be the trace of $A\mathcal{D}^{-p}$. Hence $⨍ A\mathcal{D}^{-p} = ⨍ A^*\mathcal{D}^{-p}$.
The reality operator J introduced in (29) satisfies, when $q = 1$, the commutative relation $JxJ^{-1} = x^*$ for $x \in \mathcal{A}$. Thus $JAJ^{-1} = -A^*$, so $⨍ A\mathcal{D}^{-p} = ⨍ J\,(A^*\mathcal{D}^{-p})\,J^{-1} = -⨍ A^*\mathcal{D}^{-p}$ and $⨍ A\mathcal{D}^{-p} = 0$. Similar proof for the δ-form AF. □

For any selfadjoint one-form A, $\mathcal{D}_A := \mathcal{D} + \widetilde{A} = \mathcal{D}$. Thus, the spectral action for the real spectral triple $\big(C^\infty(\mathbb{S}^3),\, \mathcal{H},\, \mathcal{D}\big)$ for $\mathcal{D}_A$ is trivialized by

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) \;=\; 2\,\Phi_3\,\Lambda^3 - \tfrac{1}{2}\,\Phi_1\,\Lambda^1 + \mathcal{O}(\Lambda^{-1}). \tag{70}$$

But it is more natural to compare with the spectral action of $\mathcal{D}+A$. This is obtained respectively from Lemma 4.18 and general heat kernel approach [27]:

$$\mathcal{S}(\mathcal{D} + A, \Phi, \Lambda) \;=\; 2\,\Phi_3\,\Lambda^3 + ⨍ |\mathcal{D} + A|^{-1}\,\Phi_1\,\Lambda^1 + \mathcal{O}(\Lambda^{-1})$$

since all terms of (2) in Λ^{n-k} are zero for k odd and $\zeta_{\mathcal{D}+A}(0) = 0$ when n is odd: as a verification, $⨍ |\mathcal{D} + A|^{-2}$ is zero according to [22, Lemma 4.10], Lemmas 4.18 and Proposition 7.2. Similarly, $\zeta_{\mathcal{D}+A}(0) = 0$ because in (3), all terms with k odd are zero (same proof as in Proposition 7.2) but for k even, it is not that easy to show that $⨍ A\mathcal{D}^{-1}A\mathcal{D}^{-1} = 0$.
Moreover, the curvature term does not depend on A:

Lemma 7.3. *For any one-form A on a commutative spectral triple of dimension n based on a compact Riemannian spin^c manifold without boundary, we have*

$$⨍ |\mathcal{D} + A|^{-(n-2)} = ⨍ |\mathcal{D}|^{-(n-2)}. \tag{71}$$

Proof. Follows from [23, first formula page 511] with $\rho := A = A^*$, $N(\rho) = \rho$ (the constraint $J\rho J^{-1} = \pm\rho$ is not used.)
One can also use [15, Proposition 1.149]. □

From [22, Lemma 4.10] $⨍ |\mathcal{D} + A|^{-(n-2)} = ⨍ |\mathcal{D}|^{-(n-2)} + \frac{n(n-2)}{4}\, ⨍ (AF)^2|\mathcal{D}|^{-3} + \frac{(n-2)^2}{4}\, ⨍ A^2|\mathcal{D}|^{-3}$ using $X := A\mathcal{D} + \mathcal{D}A + A^2$ and $[|\mathcal{D}|, A] \in OP^0$, but again, it is not that easy to show that the last two terms cancelled: for instance here, for $B = b[\mathcal{D}, b^*]$, we obtain by direct computation (using the easiest translated Dirac operator $\mathcal{D}'$)

$$\begin{aligned}\mathrm{Tr}\left(B^2|\mathcal{D}'|^{-3-s}\right) &= \mathrm{Tr}\left((B^*)^2|\mathcal{D}'|^{-3-s}\right) = \tfrac{1}{2}\,\mathrm{Tr}\left(BB^*|\mathcal{D}'|^{-3-s}\right) = \tfrac{1}{2}\,\mathrm{Tr}\left(B^*B|\mathcal{D}'|^{-3-s}\right)\\ &= \tfrac{4}{3}\sum_{2j\in\mathbb{N}} \tfrac{j+1}{(2j+1)^{2+s}}\,,\end{aligned}$$

so $⨍ B^2|\mathcal{D}'|^{-3} = \frac{2}{3}$. Similarly, one checks that $⨍ (BF)^2|\mathcal{D}|^{-3} = \frac{1}{2}⨍ BFB^*F|\mathcal{D}|^{-3} = -\frac{2}{9}$. Thus if $A := B + B^*$, $⨍ A^2|\mathcal{D}|^{-3} = ⨍ A^2|\mathcal{D}'|^{-3} = 4$ and $⨍ (AF)^2|\mathcal{D}|^{-3} = -\frac{4}{3}$ which yields to (71).
Thus for any one-form A on the 3-sphere,

$$\mathcal{S}(\mathcal{D} + A, \Phi, \Lambda) \;=\; 2\,\Phi_3\,\Lambda^3 - \tfrac{1}{2}\,\Phi_1\,\Lambda^1 + \mathcal{O}(\Lambda^{-1}, A)$$

which as (70) is not identical to (67) which contains a nonzero constant term Λ^0 for $q = 1$.

8 Conclusion

We computed in this paper the full spectral action on the $SU_q(2)$ spectral triple of [21] with the reality operator J (notice the change of definition for pseudodifferential operators.) The dimension spectrum being a finite set, there is only a finite number of terms in the spectral action expansion. The tadpole hypothesis is not satisfied on $SU_q(2)$. We saw that that the action depends on q and the limit $q \to 1$ does not exist automatically. When it exists, such limit does not lead to the associated action on the commutative sphere $\mathbb{S}^3$. The sign F of the Dirac operator has special properties: first, it commutes modulo $OP^{-\infty}$ with elements of the algebra, and second, it can be seen as a one-form, giving terms independent of q in the spectral action. Here, we were interested in the computation of the spectral action of a quantum group. Naturally, it would be interesting to investigate other related cases like the Podleś spheres [17, 19] or the Euclidean quantum spheres [20, 35], especially the 4-sphere [18].

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