



HAL
open science

Lawrence-Krammer-Paris representations under graph automorphisms.

Anatole Castella

► **To cite this version:**

Anatole Castella. Lawrence-Krammer-Paris representations under graph automorphisms.. 2008. hal-00261496v1

HAL Id: hal-00261496

<https://hal.science/hal-00261496v1>

Preprint submitted on 7 Mar 2008 (v1), last revised 3 Apr 2008 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

LAWRENCE-KRAMMER-PARIS REPRESENTATIONS UNDER GRAPH AUTOMORPHISMS

ANATOLE CASTELLA

ABSTRACT. Lawrence-Krammer-Paris representations (LKP for short) are the first examples of faithful linear representations of the Artin-Tits monoids of small type (and hence of the Artin-Tits groups of spherical and small type). If the construction is essentially unique for the spherical and small types, the same is not clear for a spherical and non-small type. Another important open question is to ask if there exists an analogue of this construction for the non-small types.

The aim of this paper is to classify all the LKP-representations for the *affine* and small types, and to generalize the construction of [Digne, *On the linearity of Artin Braid groups*. J. Algebra **268**, (2003) 39-57] in order to provide “LKP-like” faithful linear representations of any Artin-Tits monoid that appears as the submonoid of fixed points of an Artin-Tits monoid of small type under the action of graph automorphisms.

INTRODUCTION

In the early 2000’s, Krammer defined by explicit formulae a linear representation of braid groups, which is known to first appear in the work of Lawrence [18], and showed that this representation is faithful [16, 17] (see also [1]). This construction and the proof of the faithfulness have been generalized by Cohen and Wales, and independently by Digne, to the Artin-Tits groups of spherical and small type [8, 12], and then to all the Artin-Tits monoids of small type by Paris [21]. Such an LKP-representation (LKP for Lawrence-Krammer-Paris) is of degree the number of positive roots in the associated root system, and the matrices of the representation have nice properties regarding the combinatorics of this set of positive roots (see section 1 below).

The construction is essentially unique in the spherical and small type cases : the LKP-representations for a given type form an infinite “one parameter”-family (see [8, 12] or section 2.2.1 below). It is not clear whether the same holds in a non-spherical and small type case and, when the type *has a triangle* (*i.e.* a subgraph of affine type \tilde{A}_2), even the existence of the construction is not established (see [21]).

Natural questions on this subject are then

- (i) what about the existence and unicity of the construction in general,
- (ii) what can be done for Artin-Tits monoids and groups of non-small type ?

A first answer to the second question is provided by [12], where is defined an “LKP-like” faithful linear representation for the Artin-Tits group of type B_n , F_4 or G_2 , using the fact that it appears as the subgroup of fixed points, under the action of graph automorphisms, of an Artin-Tits group of type A_{2n-1} , E_6 or D_4 respectively.

Date: March 7, 2008.

The aim of this paper is to go further on those two questions.

In section 2.2.2, we characterize all the LKP-representations in the affine and small type cases : for a given type, they form an infinite “infinite parameters”-family. Since this also holds for the type \tilde{A}_2 , we thus get the first examples of LKP-representations for a Coxeter graph of small type with a triangle.

In section 3, we generalize the construction and the faithfulness result of [12] to any Artin-Tits monoid that appears as the submonoid of fixed points, under the action of graph automorphisms, of an Artin-Tits monoid of small type. Note that our proof of faithfulness is different from the one of [12] as it does not use any case-by-case consideration. Moreover, applied to the Coxeter graphs A_{2n} and D_{n+1} , our result provides two new faithful representations for the Artin-tits group of type B_n . We then explicit the formulae of the induced representation when the graph automorphism involved is of order two. As a consequence, we show that the three linear representations of an Artin-tits group of type B_n just mentioned are pairwise non-equivalent.

1. PRELIMINARIES

Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a *Coxeter matrix*, i.e. with $m_{i,j} = m_{j,i} \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ and $m_{i,j} = 1 \Leftrightarrow i = j$. We will always assume in this paper that I is finite ; this condition could be removed at a cost of some refinements in certain statements below (see [6, Ch. 11] for some of them), which are left to the reader.

As usual, we encode the data of Γ by its *Coxeter graph*, i.e. the graph with vertex set I , an edge between the vertices i and j if $m_{i,j} \geq 3$, and a label $m_{i,j}$ on that edge when $m_{i,j} \geq 4$. In the remainder of the paper, we will identify a Coxeter matrix with its Coxeter graph.

For sake of brevity in this paper, when considering a monoid homomorphism $\rho : M \rightarrow M'$, we will often denote by ρ_b the image $\rho(b)$ of a given $b \in M$ by ρ .

1.1. Coxeter groups and Artin-Tits monoids and groups.

We denote by $W = W_\Gamma$ (resp. $B = B_\Gamma$, resp. $B^+ = B_\Gamma^+$) the *Coxeter group* (resp. *Artin-Tits group*, resp. *Artin-Tits monoid*) associated with Γ :

$$\begin{aligned} W &= \langle s_i, i \in I \mid \underbrace{s_i s_j s_i \cdots}_{m_{i,j} \text{ terms}} = \underbrace{s_j s_i s_j \cdots}_{m_{i,j} \text{ terms}} \text{ if } m_{i,j} \neq \infty, \text{ and } s_i^2 = 1 \rangle, \\ B &= \langle s_i, i \in I \mid \underbrace{s_i s_j s_i \cdots}_{m_{i,j} \text{ terms}} = \underbrace{s_j s_i s_j \cdots}_{m_{i,j} \text{ terms}} \text{ if } m_{i,j} \neq \infty \rangle, \\ B^+ &= \langle s_i, i \in I \mid \underbrace{s_i s_j s_i \cdots}_{m_{i,j} \text{ terms}} = \underbrace{s_j s_i s_j \cdots}_{m_{i,j} \text{ terms}} \text{ if } m_{i,j} \neq \infty \rangle^+. \end{aligned}$$

Note that there is no ambiguity in writing with the same symbols the generators of B and of B^+ since the canonical morphism $\mu : B^+ \rightarrow B$, given by the universal properties of the presentations, is injective [21], so B^+ can be identified with the submonoid of B generated by the $s_i, i \in I$.

We denote by \preceq the (left) *divisibility* in the monoid B^+ , i.e. for $b, b' \in B^+$, we write $b' \preceq b$ if there exists $b'' \in B^+$ such that $b = b'b''$; this leads to the natural notions of (left) gcd's and (right) lcm's in B^+ . We denote by ℓ the length function on B^+ relatively to its generating set $\{s_i \mid i \in I\}$.

Let J be a subset of I . We denote by

- $\Gamma_J = (m_{i,j})_{i,j \in J}$ the submatrix of Γ of index set J ,
- $W_J = \langle s_j, j \in J \rangle$ the subgroup of W generated by the $s_j, j \in J$,
- $B_J = \langle s_j, j \in J \rangle$ the subgroup of B generated by the $s_j, j \in J$,
- $B_J^+ = \langle s_j, j \in J \rangle$ the submonoid of B^+ generated by the $s_j, j \in J$.

It is known that W_J , (resp. B_J , resp. B_J^+) is the Coxeter group (resp. Artin-Tits group, resp. Artin-Tits monoid) associated with Γ_J (see [2, Ch. IV, n° 1.8, Thm. 2] for the Coxeter case, [22, Ch. II, Thm. 4.13] for the Artin-Tits group case, the Artin-Tits monoid case being obvious).

We say that J and Γ_J are *spherical* if W_J is finite, or, equivalently, if the elements $s_j, j \in J$, have a common (right) multiple in B^+ . In that case, the elements $s_j, j \in J$, have a unique (right) lcm in B^+ , denoted by Δ_J and called *the Garside element* of B_J^+ ; and the group B_J is the group of (left) fractions of B_J^+ , i.e. every $b \in B_J$ can be written $b = b'^{-1}b''$ with $b', b'' \in B_J^+$ (see [3, Props. 4.1, 5.5 and Thm. 5.6]).

For $b \in B^+$, we set $I(b) = \{i \in I \mid s_i \preceq b\}$. In view of what has just been said, $I(b)$ is a spherical subset of I .

Let us conclude this section by the following easy, but fundamental, lemma :

Lemma 1. *Consider a monoid homomorphism $\psi : B^+ \rightarrow G$, where G is a group. Then ψ extends to a group homomorphism $\psi_{gr} : B \rightarrow G$ such that $\psi = \psi_{gr} \circ \mu$.*

Moreover if Γ is spherical and if ψ is injective, then ψ_{gr} is injective.

Proof. The universal property of B gives the first part. For the second, take $b \in B$ such that $\psi_{gr}(b) = 1$ and consider a decomposition $b = b'^{-1}b''$ with $b', b'' \in B^+$. We get $\psi(b') = \psi_{gr}(b') = \psi_{gr}(b'') = \psi(b'')$, whence $b' = b''$ by injectivity of ψ and hence $b = 1$. \square

Note that if one is able to construct an injective morphism $\psi : B^+ \rightarrow G$ where G is a group, then one gets that the canonical morphism μ is injective; this is the idea of [21]. In this paper, we will be interested in linear representations $\psi : B^+ \rightarrow GL(V)$, hence proving their faithfulness will prove at the same time the faithfulness of the corresponding group representation ψ_{gr} when Γ is spherical.

1.2. Standard root systems.

Details on the notions introduced here can be found in [11].

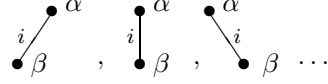
Let $E = \bigoplus_{i \in I} \mathbb{R}\alpha_i$ be a \mathbb{R} -vector space with basis $(\alpha_i)_{i \in I}$ indexed by I . We endow E with a symmetric bilinear form $(\cdot | \cdot) = (\cdot | \cdot)_\Gamma$ given on the basis $(\alpha_i)_{i \in I}$ by $(\alpha_i | \alpha_j) = -2 \cos(\frac{\pi}{m_{i,j}})$. The Coxeter group $W = W_\Gamma$ acts on E via $s_i(\beta) = \beta - (\beta | \alpha_i)\alpha_i$.

The (*standard*) *root system* associated with Γ is by definition the set $\Phi = \Phi_\Gamma = \{w(\alpha_i) \mid w \in W, i \in I\}$. It is well-known that $\Phi = \Phi^+ \sqcup \Phi^-$, where $\Phi^+ = \Phi \cap (\bigoplus_{i \in I} \mathbb{R}^+\alpha_i)$ and $\Phi^- = -\Phi^+$.

We will always represent a subset Ψ of Φ^+ by a graph with vertex set Ψ and an edge labelled i between two vertices α and β if $\alpha = s_i(\beta)$. For example, the situation where β is fixed by s_i will be drawn by a loop $\beta \bullet \circlearrowleft i$.

Such a graph is naturally \mathbb{N} -graded via the *depth* function on Φ^+ , where the depth of a root $\alpha \in \Phi^+$ is by definition $dp(\alpha) = \min\{l(w) \mid w \in W, w(\alpha) \in \Phi^-\}$.

Contrary to what suggests this terminology, in all the graphs that we will draw, we chose to place a root of great depth *above* a root of small depth ; so drawings like the following ones (with α *above* β), will all mean that $\beta = s_i(\alpha)$ (or equivalently $\alpha = s_i(\beta)$) and $\text{dp}(\alpha) > \text{dp}(\beta)$:



Lemma 2. *Let $i \in I$ and $\alpha \in \Phi^+ \setminus \{\alpha_i\}$. Then*

$$\text{dp}(s_i(\alpha)) = \begin{cases} \text{dp}(\alpha) - 1 & \text{if } (\alpha|\alpha_i) > 0, \\ \text{dp}(\alpha) & \text{if } (\alpha|\alpha_i) = 0, \\ \text{dp}(\alpha) + 1 & \text{if } (\alpha|\alpha_i) < 0, \end{cases}$$

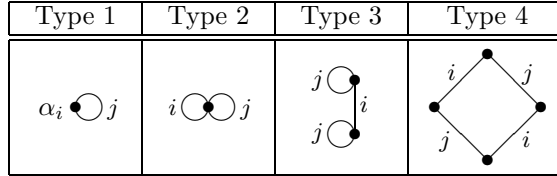
Proof. This is [4, Lem 1.7]. □

In the remainder of the paper, we will often consider subsets of Φ^+ of the form $\{w(\alpha) \mid w \in W_{\{i,j\}}\} \cap \Phi^+$, for $\alpha \in \Phi^+$ and $i, j \in I$ with $m_{i,j} = 2$ or 3 , so the following definition and remark will be useful :

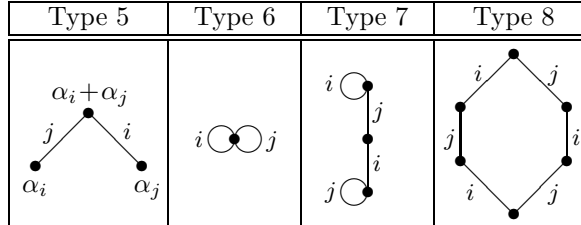
Definition 3. Let $\alpha \in \Phi^+$ and $J \subseteq I$. We call J -*mesh* of α , or simply *mesh*, the set $[\alpha]_J := \{w(\alpha) \mid w \in W_J\} \cap \Phi^+$. This terminology is inspired by personal communications with Hée.

Remark 4. Let $\alpha \in \Phi^+$ and $i, j \in I$ with $m_{i,j} = 2$ or 3 . Then, up to exchanging i and j , the graph of the mesh $[\alpha]_{\{i,j\}}$ is one of the following :

- if $m_{i,j} = 2$:



- if $m_{i,j} = 3$:



Let J be a subset of I . We denote by Φ_J the subset $\{w(\alpha_j) \mid w \in W_J, j \in J\}$ of Φ . It is clear that Φ_J is the root system associated with Γ_J in $\bigoplus_{j \in J} \mathbb{R}\alpha_j$.

1.3. Graph automorphisms.

We call *automorphism* of Γ every permutation g of I such that $m_{g(i),g(j)} = m_{i,j}$ for all $i, j \in I$, and we denote by $\text{Aut}(\Gamma)$ the group the constitute.

Any automorphism of Γ clearly acts by automorphisms on W , B and B^+ by permuting the corresponding generating set. If G is a subgroup of $\text{Aut}(\Gamma)$, we denote by W^G , B^G and $(B^+)^G$ the corresponding subset of fixed points under the action of the elements of G .

It is known that W^G (resp. $(B^+)^G$) is a Coxeter group (resp. Artin-Tits monoid) associated with a certain Coxeter graph Γ' easily deduced from Γ , and the analogue holds for B^G when Γ is spherical, or more generally of *FC-type* (see [13, 20] for the Coxeter case, [19, 9, 10, 7] for the Artin-Tits case). Note that the standard generator of $(B^+)^G$ are the Garside elements Δ_J of B_J^+ , for J running through the spherical orbits of I under G .

Similarly, any automorphism g of Γ acts by a linear automorphism on $E = \bigoplus_{i \in I} \mathbb{R}\alpha_i$ by permuting the basis $(\alpha_i)_{i \in I}$. This action stabilizes Φ and Φ^+ , and the induced action on those sets is given by $w(\alpha_i) \mapsto (g(w))(\alpha_{g(i)})$.

2. LKP-REPRESENTATIONS IN THE SMALL TYPE CASE

From now on, we assume that $\Gamma = (m_{i,j})_{i,j \in I}$ is a Coxeter matrix of small type, *i.e.* with $m_{i,j} \in \{1, 2, 3\}$ for all $i, j \in I$.

We denote by $\mathbb{Q}[x, y]$ the ring of polynomials in two indeterminates over \mathbb{Q} , and by $\mathbb{K} = \mathbb{Q}(x, y)$ its field of fractions. Let $V = \bigoplus_{\alpha \in \Phi^+} \mathbb{K}e_\alpha$ be a \mathbb{K} -vector space with basis $(e_\alpha)_{\alpha \in \Phi^+}$ indexed by Φ^+ .

Definition 5. We call *LKP-representation* every representation $\psi : B^+ \rightarrow \text{GL}(V)$ sending the generator s_i on a linear map ψ_{s_i} whose matrix in the basis $(e_\alpha)_{\alpha \in \Phi^+}$, properly arranged, is of the form

$$\begin{pmatrix} e_{\alpha_i} & \cdots \\ * & * & \cdots & * \\ 0 & & & \\ \vdots & M_i & & \\ 0 & & & \end{pmatrix}, \text{ where } \begin{cases} \text{- the line } \alpha_i \text{ is of the form } (xT_{i,\alpha})_{\alpha \in \Phi^+}, \text{ with} \\ \quad T_{i,\alpha} \in \mathbb{Q}[y], \\ \text{- the matrix } M_i \text{ is block diagonal, with blocks} \\ \quad \begin{cases} \begin{pmatrix} e_\alpha \\ (1) \end{pmatrix} & \text{if } \alpha \circlearrowright i, \\ \begin{pmatrix} e_\beta & e_\alpha \\ 1-y & y \\ 1 & 0 \end{pmatrix} & \text{if } \begin{matrix} \alpha \\ \beta \end{matrix} \downarrow i. \end{cases} \end{cases}$$

We call *LKP-family* every suitable family $(T_{i,\alpha})_{(i,\alpha) \in I \times \Phi^+} \in \mathbb{Q}[y]^{I \times \Phi^+}$, that is, every family of polynomials that makes those maps invertible and satisfying the relations $\psi_{s_i}\psi_{s_j} = \psi_{s_j}\psi_{s_i}$ (resp. $\psi_{s_i}\psi_{s_j}\psi_{s_i} = \psi_{s_j}\psi_{s_i}\psi_{s_j}$) if $m_{i,j} = 2$ (resp. 3).

Remark 6. We follow here the definition of [21]. The other authors who have worked on the subject make different choices for the matrix M_i . For example the one of [12] is the transpose of our M_i , whereas the one of [8] is symmetric.

We will discuss the existence of LKP-representations, that is of LKP-families, in the next section. For the moment, let us just note that such a linear map ψ_{s_i} is invertible if and only if $T_{i,\alpha_i} \neq 0$, in which case its inverse is given by the matrix :

$$\begin{pmatrix} e_{\alpha_i} & \cdots \\ * & * & \cdots & * \\ 0 & & & \\ \vdots & & M'_i & \\ 0 & & & \end{pmatrix}, \text{ where } \left\{ \begin{array}{l} \text{- the line } \alpha_i \text{ is of the form } (F_{i,\alpha})_{\alpha \in \Phi^+}, \text{ with} \\ \quad \left\{ \begin{array}{ll} F_{i,\alpha} = \frac{1}{xT_{i,\alpha_i}} & \text{if } \alpha = \alpha_i, \\ F_{i,\alpha} = -\frac{T_{i,\alpha}}{T_{i,\alpha_i}} & \text{if } \alpha \circlearrowleft i, \\ \left\{ \begin{array}{ll} F_{i,\alpha} = -\frac{yT_{i,\beta} + (y-1)T_{i,\alpha}}{yT_{i,\alpha_i}} & \text{if } \begin{array}{c} \alpha \\ \bullet \\ \beta \end{array} \Big| i, \\ F_{i,\beta} = -\frac{T_{i,\alpha}}{yT_{i,\alpha_i}} & \end{array} \right. \\ \text{- the matrix } M'_i = M_i^{-1} \text{ is block diagonal, with} \\ \text{blocks } \left\{ \begin{array}{ll} \begin{pmatrix} e_\alpha \\ (1) \end{pmatrix} & \text{if } \alpha \circlearrowleft i, \\ \frac{1}{y} \begin{pmatrix} e_\beta & e_\alpha \\ 0 & y \\ 1 & y-1 \end{pmatrix} & \text{if } \begin{array}{c} \alpha \\ \bullet \\ \beta \end{array} \Big| i. \end{array} \right. \end{array} \right. \end{array} \right.$$

2.1. Faithfulness of an LKP-representation.

The key argument in [1, 16, 17, 8, 12, 21] is that an LKP-representation ψ is faithful. More precisely, if we fix the indeterminate y to some real value $y_0 \in]0, 1[$ that is not a zero of some T_{i,α_i} , $i \in I$, then we get a representation $\psi_0 : B^+ \rightarrow \text{GL}(V_0)$, where $V_0 = \bigoplus_{\alpha \in \Phi^+} \mathbb{R}(x)e_\alpha$, and one can show that ψ_0 is faithful.

We sketch here the (much easier) proof of this fact obtained by Hée in [14]. It does not involve any consideration on closed sets of positive roots, nor on the maximal simple (left) divisor of an element of B^+ .

Lemma 7 ([14, Prop. 1]). *Let $\rho : B^+ \rightarrow M$ be a monoid homomorphism satisfying $\rho(b) = \rho(b') \Rightarrow I(b) = I(b')$ for all $b, b' \in B^+$. If M is left cancellative (for example if M is a group), then ρ is injective.*

We denote by $\text{Bin}(\Omega)$ the monoid of binary relations on a set Ω , where the product RR' of two binary relations R and R' is defined on Ω by $\beta RR'\alpha \Leftrightarrow \exists \gamma \in \Omega$ such that $\beta R\gamma$ and $\gamma R'\alpha$.

For $R \in \text{Bin}(\Omega)$ and $\Psi \subseteq \Omega$, we denote by $R(\Psi)$ the set $\{\beta \in \Omega \mid \exists \alpha \in \Psi, \beta R\alpha\}$. It is easily seen that for a family $(\Psi_\lambda)_{\lambda \in \Lambda}$ of subsets of Ω , we get $R(\bigcup_{\lambda \in \Lambda} \Psi_\lambda) = \bigcup_{\lambda \in \Lambda} R(\Psi_\lambda)$ (this remark will be useful in the proof of our theorem 28 below).

Definition 8 ([14, 4.3]). By choice of y_0 , the coefficients of the matrix of $\psi_0(\mathbf{s}_i)$ in the basis $(e_\alpha)_{\alpha \in \Phi^+}$ are real polynomials in x with non-negative constant terms. This property naturally extends to the matrix of $\psi_0(b)$ for an arbitrary $b \in B^+$.

For $b \in B^+$, we denote by R_b the binary relation on Φ^+ defined by $\beta R_b\alpha$ if and only if the coefficient (β, α) of the matrix of $\psi_0(b)$ in the basis $(e_\alpha)_{\alpha \in \Phi^+}$ is non-zero modulo x .

Lemma 9 ([14, 4.3 and 4.4]). *The map $B^+ \rightarrow \text{Bin}(\Phi^+)$, $b \mapsto R_b$, is a monoid homomorphism which satisfies the following properties :*

- (i) $\alpha_i \notin R_{\mathbf{s}_i}(\Phi^+)$,
- (ii) if $i \neq j$, then $\alpha_i R_{\mathbf{s}_j} \alpha_i$,
- (iii) if $m_{i,j} = 3$, then $\alpha_i R_{\mathbf{s}_j} R_{\mathbf{s}_i} \alpha_j$.

Lemma 10 ([14, Prop. 2]). *Let $B^+ \rightarrow \text{Bin}(\Omega)$, $b \mapsto R_b$, be a monoid homomorphism, and let $(\alpha_i)_{i \in I}$ be a family of elements of Ω such that*

- (i) $\alpha_i \notin R_{\mathbf{s}_i}(\Omega)$,
- (ii) if $i \neq j$, then $\alpha_i R_{\mathbf{s}_j} \alpha_i$,

(iii) if $m_{i,j} = 3$, then $\alpha_i R_{\mathbf{s}_j} R_{\mathbf{s}_i} \alpha_j$.

Then for every $b \in B^+$, we have $\mathbf{s}_i \preceq b \Leftrightarrow \alpha_i \notin R_b(\Omega)$. In particular, for $b, b' \in B^+$, we get $R_b(\Omega) = R_{b'}(\Omega) \Rightarrow I(b) = I(b')$.

A combination of the three previous lemmas easily gives the following :

Theorem 11. *The representation ψ_0 (and hence ψ) is faithful.*

2.2. On LKP-families.

Fix an arbitrary family of polynomials $(T_{i,\alpha})_{(i,\alpha) \in I \times \Phi^+}$, and consider the linear maps $\psi_{\mathbf{s}_i}$, $i \in I$, as in definition 5 above.

Proposition 12. *The map $\mathbf{s}_i \mapsto \psi_{\mathbf{s}_i}$ extends to a monoid homomorphism $\psi : B^+ \rightarrow \text{End}(V)$ if and only if the relations listed in the table below hold among the polynomials $T_{i,\alpha}$, $(i,\alpha) \in I \times \Phi^+$. (Note that the relations of cases (6) and (8) must hold whether $(\alpha_i | \alpha)$ is positive, zero or negative.)*

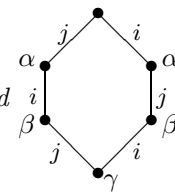



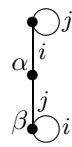

n°	Relations among the $T_{i,\alpha}$'s	Configuration of roots
(1)	$T_{i,\alpha_j} = 0$	if $i \neq j$
(2)	$T_{i,\alpha_i} = T_{j,\alpha_j}$	if $m_{i,j} = 3$
(3) (4)	$T_{i,\alpha} = T_{j,\alpha'}$ $T_{i,\beta} + (1-y)T_{i,\gamma} = T_{j,\beta'} + (1-y)T_{j,\gamma}$	if $m_{i,j} = 3$ and 
(5)	$T_{i,\alpha} = T_{j,\alpha}$	if $m_{i,j} = 3$ and 
(6)	$T_{i,\alpha} = yT_{i,\beta}$	if $m_{i,j} = 2$ and 
(7)	$T_{i,\alpha} = (y-1)T_{i,\alpha_i}$	if $m_{i,j} = 3$ and $\alpha = \alpha_i + \alpha_j$
(8)	$T_{i,\alpha} = (y-1)T_{i,\beta} + yT_{j,\gamma}$	if $m_{i,j} = 3$ and 
(9)	$T_{i,\alpha} = (y-1)T_{i,\beta} + T_{j,\beta}$	if $m_{i,j} = 3$ and 
(10)	$T_{i,\alpha} = yT_{j,\beta}$	if $m_{i,j} = 3$ and 

TABLE 1. Relations for an LKP-family (apart $T_{i,\alpha_i} \neq 0$, $i \in I$).

Proof. This result is essentially contained in [21, proofs of lemmas 3.6 and 3.7]. The map $s_i \mapsto \psi_{s_i}$ extends to a monoid homomorphism $B^+ \rightarrow \text{End}(V)$ if and only if for every $i, j \in I$ with $m_{i,j} = 2$ (resp. 3), then $\psi_{s_i}\psi_{s_j} = \psi_{s_j}\psi_{s_i}$ (resp. $\psi_{s_i}\psi_{s_j}\psi_{s_i} = \psi_{s_j}\psi_{s_i}\psi_{s_j}$).

It is an easy (but rather long) exercise to compute the image, in the basis $(e_\beta)_{\beta \in \Phi^+}$, of a given e_α by $\psi_{s_i}\psi_{s_j}$ (resp. $\psi_{s_i}\psi_{s_j}\psi_{s_i}$) when $m_{i,j} = 2$ (resp. 3). The only roots involved in this computation are the elements of the $\{i, j\}$ -mesh $[\alpha]_{\{i,j\}}$ of α and the roots α_i and α_j (resp. α_i, α_j and $\alpha_i + \alpha_j$) if $m_{i,j} = 2$ (resp. 3), and the result naturally depend on the graph of $[\alpha]_{\{i,j\}}$ (see lemma 4) and on the place of α in this graph. The coefficients in this basis $(e_\beta)_{\beta \in \Phi^+}$ are elements of $\mathbb{Q}[x, y] = (\mathbb{Q}[y])[x]$ of degree in x less or equal to three (recall that the polynomials $T_{i,\beta}$ are elements of $\mathbb{Q}[y]$).

The prescribed relations then naturally appear when one requires equality, coefficient wise in the basis $(e_\beta)_{\beta \in \Phi^+}$, between $\psi_{s_i}\psi_{s_j}(e_\alpha)$ and $\psi_{s_j}\psi_{s_i}(e_\alpha)$ (resp. $\psi_{s_i}\psi_{s_j}\psi_{s_i}(e_\alpha)$ and $\psi_{s_j}\psi_{s_i}\psi_{s_j}(e_\alpha)$) if $m_{i,j} = 2$ (resp. 3). \square

Note that these relations are of two kinds : relations (1) to (5) give equalities between polynomials associated with roots of the same depth, whereas relations (6) to (10) express a polynomial $T_{i,\alpha}$ in terms of a linear combination of some $T_{j,\beta}$'s with $\text{dp}(\beta) < \text{dp}(\alpha)$. In fact, relation (5) can be deleted from the list (this generalizes the analogous observation of [8], and [21, Lem. 3.5]), this is proved in lemma 14 below.

Lemma 13. *Let $i, j, k \in I$ be such that $m_{i,j} = m_{j,k} = m_{k,i} = 3$. Then for every $\alpha \in \Phi^+$, we have $(\alpha_i|\alpha) + (\alpha_j|\alpha) + (\alpha_k|\alpha) \leq 0$.*

Proof. It is easy to see that $(\alpha_i|\beta) + (\alpha_j|\beta) + (\alpha_k|\beta)$ is constant for β running through the mesh $[\alpha]_{\{i,j,k\}}$, hence it is enough to prove the desired property for an arbitrary element of $[\alpha]_{\{i,j,k\}}$. Consider an element β of minimal depth in $\Phi^+ \cap [\alpha]_{\{i,j,k\}}$. If $\beta \notin \{\alpha_i, \alpha_j, \alpha_k\}$ then the coefficients $(\alpha_i|\beta)$, $(\alpha_j|\beta)$ and $(\alpha_k|\beta)$ are necessarily non-positive (see lemma 2), and if $\beta \in \{\alpha_i, \alpha_j, \alpha_k\}$ then we clearly have $(\alpha_i|\beta) + (\alpha_j|\beta) + (\alpha_k|\beta) = 0$, whence the result. \square

Lemma 14. *Relation (5) of table 1 is implied by relations (1), (6), (8) and (10).*

Proof. Fix $i, j \in I$ with $m_{i,j} = 3$ and $\alpha \in \Phi^+$ such that $(\alpha_i|\alpha) = (\alpha_j|\alpha) = 0$. Let us show by induction on $\text{dp}(\alpha)$ that $T_{i,\alpha} = T_{j,\alpha}$, using only relations (1), (6), (8) and (10). If $\text{dp}(\alpha) = 1$, i.e. if $\alpha = \alpha_k$ for some $k \in I$, then $k \neq i, j$ and this is a consequence of (1). So assume that $\text{dp}(\alpha) \geq 2$ and fix $k \in I$ such that $(\alpha_k|\alpha) > 0$; we set $\beta = s_k(\alpha) \in \Phi^+$.

If $m_{i,k} = m_{j,k} = 2$, then we get $(\alpha_i|\beta) = (\alpha_j|\beta) = 0$, whence $T_{i,\beta} = T_{j,\beta}$ by induction and hence $T_{i,\alpha} = T_{j,\alpha}$ by (6). If $m_{i,k} = 2$ and $m_{j,k} = 3$, then the graph of $[\alpha]_{\{i,j,k\}}$ is the following :

$$\begin{array}{c}
 \begin{array}{c}
 \alpha \\
 \circlearrowleft \\
 i \quad j \\
 | \\
 k \\
 | \\
 \beta \\
 | \\
 j \\
 | \\
 \gamma \\
 | \\
 i \\
 | \\
 \delta \\
 \circlearrowleft \\
 j \quad k
 \end{array}
 , \text{ whence }
 \begin{array}{l}
 T_{i,\alpha} = yT_{i,\beta} \\
 \quad = y(y-1)T_{i,\gamma} + y^2T_{j,\delta} \\
 T_{j,\alpha} = (y-1)T_{j,\beta} + yT_{k,\gamma} \\
 \quad = y(y-1)T_{i,\gamma} + y^2T_{k,\delta} \\
 T_{j,\delta} = T_{k,\delta}
 \end{array}
 \begin{array}{l}
 \text{by (6)} \\
 \text{by (8)} \\
 \text{by (8)} \\
 \text{by (10) and (6)} \\
 \text{by induction}
 \end{array}
 , \text{ hence } T_{i,\alpha} = T_{j,\alpha}.
 \end{array}$$

Thanks to lemma 13, we cannot have $m_{i,k} = m_{j,k} = 3$ in that situation, so we are done (up to exchanging i and j). \square

Remark 15. Some further remarks on the equations of table 1.

- They are linear, *i.e.* the solutions of those equations form a submodule of the $\mathbb{Q}[y]$ -module $\mathbb{Q}[y]^{I \times \Phi^+}$. In particular, if $(T_{i,\alpha})_{(i,\alpha) \in I \times \Phi^+}$ is a solution and if $P \in \mathbb{Q}[y]$, then $(PT_{i,\alpha})_{(i,\alpha) \in I \times \Phi^+}$ is still a solution.
- If $\Gamma_1, \dots, \Gamma_p$ are the connected components of Γ , with vertex set I_1, \dots, I_p respectively, then $\Phi = \Phi_{I_1} \sqcup \dots \sqcup \Phi_{I_p}$ and relations (1) and (6) imply that $T_{i,\alpha} = 0$ for every $(i,\alpha) \in I_m \times \Phi_{I_n}^+$ whenever $m \neq n$. As a consequence, any LKP-representation ψ of B_Γ^+ is the direct sum of the induced LKP-representations ψ_n of $B_{\Gamma_n}^+$.

Hence when considering LKP-representations (or LKP-families), there is no loss of generality in assuming that Γ is connected, in which case the polynomials T_{i,α_i} , $i \in I$, are all equal by relation (2). Recall however that in order to get invertible maps, we are interested in solutions with $T_{i,\alpha_i} \neq 0$ for $i \in I$.

2.2.1. The spherical case.

We assume here that Γ is of type *ADE*.

In that case, there is no mesh of type 8 (see lemma 4) in Φ^+ . Hence every $T_{i,\alpha}$ with $\text{dp}(\alpha) \geq 2$ can be expressed as a linear combination of some $T_{j,\beta}$'s with $\text{dp}(\beta) < \text{dp}(\alpha)$ (and coefficients in $\mathbb{Z}[y]$), via some relation (6)–(10).

As a consequence, we get that an LKP-family $(T_{i,\alpha})_{(i,\alpha) \in I \times \Phi^+}$ is entirely determined by the common value $T \in \mathbb{Q}[x, y]$ of the polynomials T_{i,α_i} , $i \in I$. In fact, there exists an LKP-family for every choice of $T \in \mathbb{Q}[y] \setminus \{0\}$ *cf.* [8, 12]. The idea is to define it inductively, with basis step $T_{i,\alpha_i} = T$ (one could chose $T = 1$ by linearity) and $T_{i,\alpha_j} = 0$ if $i \neq j$, and inductive step one of the suitable relations (6)–(10) to define $T_{i,\alpha}$ (with $\text{dp}(\alpha) \geq 2$) in terms of $T_{j,\beta}$ with $\text{dp}(\beta) < \text{dp}(\alpha)$. Proving that the obtained family is indeed an LKP-family amounts to proving that the definition of $T_{i,\alpha}$ does not depend on the choice of the suitable relation chosen in the inductive step.

Moreover, any LKP-family satisfies the following properties (*cf.* [8, Cor. 3.3]) :

- $T_{i,\alpha} = 0$ if $i \notin \text{Supp}(\alpha)$,
- $T_{i,\alpha} = y^{\text{dp}(\alpha)-2}(y-1)T$ if $\text{dp}(\alpha) \geq 2$ and $(\alpha_i|\alpha) > 0$,
- $\deg(T_{i,\alpha}) = \deg(T) + \text{dp}(\alpha) - 1$, and $T_{i,\alpha} \in (y-1)T\mathbb{Z}[y]$ if $\alpha \neq \alpha_i$.

This construction has been generalized in [21] to an arbitrary Coxeter matrix of small type with no triangle. We recall that construction in section 2.2.3 below.

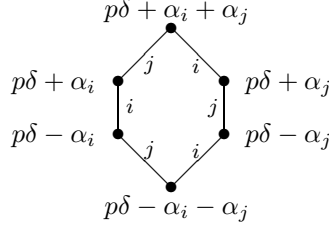
2.2.2. The affine case.

Let $\Gamma = (m_{i,j})_{0 \leq i,j \leq n}$ be a Coxeter matrix of type \tilde{A}_n ($n \geq 2$), \tilde{D}_n ($n \geq 4$) or \tilde{E}_n ($n = 6, 7, 8$) and let $\Gamma_0 = (m_{i,j})_{1 \leq i,j \leq n}$ be the corresponding spherical Coxeter matrix. Let Φ (resp. Φ_0) be the root system associated with Γ (resp. Γ_0) in $E = \oplus_{i=0}^n \mathbb{R}\alpha_i$ (resp. $E_0 = \oplus_{i=1}^n \mathbb{R}\alpha_i$) and let δ be the first positive imaginary root of Φ . Then we have the following decomposition (see [15]) :

$$\Phi = \bigsqcup_{p \in \mathbb{Z}} (\Phi_0 + p\delta) \text{ and } \Phi^+ = \Phi_0^+ \bigsqcup \left(\bigsqcup_{p \in \mathbb{N}_{\geq 1}} (\Phi_0 + p\delta) \right).$$

We thus get the following remark :

Remark 16. The only meshes of type 8 in Φ^+ (see lemma 4) are the following ones, for $p \geq 1$ and $m_{i,j} = 3$:



As a consequence, it is easy to see that, for a given $(i, \alpha) \in \llbracket 0, n \rrbracket \times \Phi^+$ with $\text{dp}(\alpha) \geq 2$, then either $\alpha = p\delta \pm \alpha_i$ for some $p \geq 1$, or the pair (i, α) appears in at least one of the configurations (6)–(10) of table 1, and for every configuration where it appears and every pair (j, β) involved in the right-hand side of the corresponding relation, then $\beta \neq q\delta - \alpha_j$ for every $q \geq 1$.

Let $\mathcal{T} = (T_{i,\alpha})_{(i,\alpha) \in \llbracket 0, n \rrbracket \times \Phi^+}$ be a family of elements of $\mathbb{Q}[y]$.

Lemma 17. Fix $(i, p) \in \llbracket 0, n \rrbracket \times \mathbb{N}_{\geq 1}$ and assume that the relations of table 1 are satisfied by the polynomials of \mathcal{T} if the roots involved are of depth (strictly) smaller than $\text{dp}(p\delta - \alpha_i)$. Then the polynomial $yT_{i,p\delta - \alpha_i - \alpha_j} + T_{j,p\delta - \alpha_i - \alpha_j}$ does not depend on $j \in \llbracket 0, n \rrbracket$ such that $m_{i,j} = 3$.

Proof. Assume that $j, k \in \llbracket 0, n \rrbracket$ are such that $m_{i,j} = m_{i,k} = 3$.

If $m_{j,k} = 2$, then the result follows from relations (6) and (9) : indeed, we get

$$\begin{aligned} T_{i,p\delta - \alpha_i - \alpha_j} &= (y - 1)T_{i,p\delta - \alpha_i - \alpha_j - \alpha_k} + T_{k,p\delta - \alpha_i - \alpha_j - \alpha_k} && \text{by (9),} \\ T_{i,p\delta - \alpha_i - \alpha_k} &= (y - 1)T_{i,p\delta - \alpha_i - \alpha_j - \alpha_k} + T_{j,p\delta - \alpha_i - \alpha_j - \alpha_k} && \text{by (9),} \\ T_{j,p\delta - \alpha_i - \alpha_j} &= yT_{j,p\delta - \alpha_i - \alpha_j - \alpha_k} && \text{by (6),} \\ T_{k,p\delta - \alpha_i - \alpha_k} &= yT_{k,p\delta - \alpha_i - \alpha_j - \alpha_k} && \text{by (6).} \end{aligned}$$

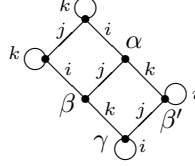
If $m_{j,k} = 3$, then $\Gamma = \tilde{A}_2$, $\{i, j, k\} = \{0, 1, 2\}$ and $p\delta - \alpha_i - \alpha_j = (p - 1)\delta + \alpha_k$. In that case we can prove more, namely that the value of $T_{l,(p-1)\delta + \alpha_m}$ does not depend on the pair $(l, m) \in \{0, 1, 2\}^2$ such that $l \neq m$. This is clear if $p = 1$ (by relation (1)), so let us assume that $p \geq 2$. One can first prove the same statement for the polynomials $T_{l,q\delta - \alpha_m}$ with $1 \leq q \leq p - 1$ by induction on q , thanks to relations (3) and (8) (the case $q = 1$ is given by relations (7) and (2)), and then prove this statement for the polynomials $T_{l,q\delta + \alpha_m}$ with $0 \leq q \leq p - 1$ by induction on q , thanks to relation (8) and the intermediate result (the case $q = 0$ is given by relation (1)). \square

Lemma 18. Consider $(i, \alpha) \in \llbracket 0, n \rrbracket \times \Phi^+$, with $\alpha \neq p\delta \pm \alpha_i$ for every $p \in \mathbb{N}$, and assume that the relations of table 1 are satisfied by the polynomials of \mathcal{T} if the roots involved are of depth (strictly) smaller than $\text{dp}(\alpha)$. Then the polynomial defined by the right-hand side of a relation (6)–(10) corresponding to a configuration where (i, α) appears does not depend on the configuration.

Proof. Let us assume that (i, α) appears in two of the configurations (6)–(10) and let us denote by j (resp. k) the index distinct from i involved in the first (resp. the second) of those two configurations. Note that, in that situation, then $\Gamma \neq \tilde{A}_2$ and there are only ten possible cases up to exchanging j and k (use remark 16) : two configurations (6) with $m_{j,k} = 2$ or $m_{j,k} = 3$, configurations (6) and (8) with $m_{j,k} = 2$, configurations (6) and (9) with $m_{j,k} = 2$ or $m_{j,k} = 3$, configurations (6)

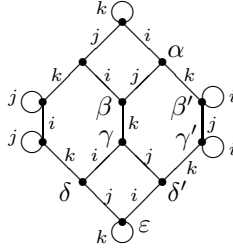
and (10) with $m_{j,k} = 2$ or $m_{j,k} = 3$, two configurations (8) with $m_{j,k} = 2$, two configurations (9) with $m_{j,k} = 2$, two configurations (10) with $m_{j,k} = 2$.

For example in the case (6) and (9) with $m_{j,k} = 2$, the graph of $[\alpha]_{\{i,j,k\}}$ is



$$\text{whence } \begin{cases} yT_{i,\beta} &= y(y-1)T_{i,\gamma} + yT_{k,\gamma} & \text{by (9), and} \\ (y-1)T_{i,\beta'} + T_{k,\beta'} &= y(y-1)T_{i,\gamma} + yT_{k,\gamma} & \text{by (6).} \end{cases}$$

And in the case (6) and (9) with $m_{j,k} = 3$, the graph of $[\alpha]_{\{i,j,k\}}$ is



$$\text{whence } \begin{cases} yT_{i,\beta} &= y(y-1)T_{i,\gamma} + y^2T_{k,\delta} & \text{by (8),} \\ &= y^2(y-1)T_{i,\delta'} + y^2(y-1)T_{k,\epsilon} + y^2T_{j,\epsilon} & \text{by (6) and (9),} \\ (y-1)T_{i,\beta'} + T_{k,\beta'} &= y(y-1)T_{i,\gamma'} + (y-1)T_{k,\gamma'} + yT_{j,\delta'} & \text{by (6) and (8),} \\ &= y(y-1)^2T_{i,\delta'} + y^2(y-1)T_{k,\epsilon} & \text{by (8),} \\ &\quad + y(y-1)T_{i,\delta'} + y^2T_{j,\epsilon} & \text{(10) and (6),} \\ &= y^2(y-1)T_{i,\delta'} + y^2(y-1)T_{k,\epsilon} + y^2T_{j,\epsilon}. \end{cases}$$

The eight remaining cases are similar and left to the reader. \square

Proposition 19. *Let us assume that \mathcal{T} is an LKP-family. Then \mathcal{T} is entirely determined by the family of polynomials $(T_q)_{q \in \mathbb{N}}$, where*

- (i) $T_{2p} = T_{i,p\delta+\alpha_i}$ for every $p \in \mathbb{N}$ and $i \in \llbracket 0, n \rrbracket$,
- (ii) $T_{2p-1} = T_{i,p\delta-\alpha_i} - yT_{i,p\delta-\alpha_i-\alpha_j} - T_{j,p\delta-\alpha_i-\alpha_j}$ for every $p \in \mathbb{N}_{\geq 1}$ and $i, j \in \llbracket 0, n \rrbracket$ such that $m_{i,j} = 3$.

Proof. Relations (2) or (3) and the fact Γ is connected prove that the polynomial $T_{i,p\delta+\alpha_i}$ is independent of $i \in I$, whence (i). To see (ii), we have to prove that the polynomial $T_{i,j} = T_{i,p\delta-\alpha_i} - yT_{i,p\delta-\alpha_i-\alpha_j} - T_{j,p\delta-\alpha_i-\alpha_j}$ is independent of $(i, j) \in \llbracket 0, n \rrbracket^2$ such that $m_{i,j} = 3$. Relation (4) shows that $T_{i,j} = T_{j,i}$ if $m_{i,j} = 3$, and since Γ is connected, it is then sufficient to see that $T_{i,j} = T_{i,k}$ if j and k are such that $m_{i,j} = m_{i,k} = 3$. This last equality is given by lemma 17.

Now thanks to remark 16 above, an easy induction on $\text{dp}(\alpha)$ shows that, for every $(i, \alpha) \in \llbracket 0, n \rrbracket \times \Phi^+$, either $\alpha = p\delta - \alpha_i$ for some $p \in \mathbb{N}_{\geq 1}$, or $T_{i,\alpha}$ is a linear combination of some $T_{j,p\delta+\alpha_j} = T_{2p}$'s. Hence the LKP-family \mathcal{T} is entirely determined by the polynomials T_{2p} , $p \in \mathbb{N}$, and $T_{j,p\delta-\alpha_j}$, $(j, p) \in \llbracket 0, n \rrbracket \times \mathbb{N}_{\geq 1}$. But then by (ii), a polynomial $T_{j,p\delta-\alpha_j}$ is entirely determined by the polynomials T_{2p} , $p \in \mathbb{N}$, and T_{2p-1} , whence the result. \square

The family $(T_q)_{q \in \mathbb{N}}$ can in fact be chosen arbitrarily (with $T_0 \neq 0$) :

Proposition 20. Let $(T_q)_{q \in \mathbb{N}}$ be a family of elements of $\mathbb{Q}[y]$ with $T_0 \neq 0$, and assume that \mathcal{T} is constructed by induction as follows :

- *Basis step* : $T_{i,\alpha_j} = 0$ for $i \neq j$, and $T_{i,p\delta+\alpha_i} = T_{2p}$ for $i \in \llbracket 0, n \rrbracket$ and $p \in \mathbb{N}$.
- *Inductive step* : consider $(i, \alpha) \in \llbracket 0, n \rrbracket \times \Phi^+$ that is not handled by the basis step (hence $\text{dp}(\alpha) \geq 2$), and such that all the $T_{j,\beta}$ for $0 \leq j \leq n$ and $\text{dp}(\beta) < \text{dp}(\alpha)$ are constructed.

- (i) If $\alpha = p\delta - \alpha_i$, then set $T_{i,\alpha} = yT_{i,\alpha-\alpha_j} + T_{j,\alpha-\alpha_j} + T_{2p-1}$ for some j such that $m_{i,j} = 3$,
- (ii) If not, then (i, α) appears in (at least) one of the configurations (6)–(10) ; define $T_{i,\alpha}$ via the corresponding relation.

Then \mathcal{T} is an LKP-family.

Proof. We show by induction on $m \in \mathbb{N}$ that the relations of table 1 that involve only roots of depth smaller than (or equal to) m are satisfied by the polynomials of \mathcal{T} . If $m = 0$, the only relations to consider are relations (1) and (2), which are satisfied by construction of the basis step. Relation (3) is also satisfied for arbitrary depths by construction of the basis step.

Assume that we know the result for some $m \in \mathbb{N}$ and consider a relation that involves a root of depth $m + 1$ and no root of higher depth. If it is a relation of type (4), then lemma 17 and inductive step (i) show that this relation is satisfied by the polynomials of \mathcal{T} involved. If it is a relation of type (6)–(10), then lemma 18 and inductive step (ii) show that this relation is satisfied by the polynomials of \mathcal{T} involved. Whence the result. \square

Example 21. Let us assume that $\Gamma = \tilde{A}_n$.

It is known that each $\alpha \in \Phi^+$ can uniquely be written as $\alpha = p\delta + \sum_{k=j}^{j+\ell} \alpha_{\bar{k}}$, with $p \in \mathbb{N}$, $j \in \llbracket 0, n \rrbracket$, $\ell \in \llbracket 0, n-1 \rrbracket$, and \bar{k} the rest of k modulo $n+1$. We then call *domain* (resp. *interior*, resp. *boundary*) of α the set $\bar{\alpha} = \{\bar{k} \mid j \leq k \leq j+\ell\}$ (resp. $\alpha^\circ = \{\bar{k} \mid j+1 \leq k \leq j+\ell-1\}$, resp. $\partial\alpha = \bar{\alpha} \setminus \alpha^\circ = \{\bar{j}, \bar{j}+\ell\}$).

If \mathcal{T} is an LKP-family, then one can check, by induction on $\text{dp}(\alpha)$, that the polynomial $T_{i,\alpha}$ is equal to :

- T_{2p} if $\alpha = p\delta + \alpha_i$ (i.e. $i \in \partial\alpha$ and $\ell = 0$),
- $(y-1)y^{\ell-1} \sum_{q=0}^p y^{q(n-1)} T_{2(p-q)}$ if $i \in \partial\alpha$ and $\ell \geq 1$,
- $(y-1)^2 y^{\ell-1} \sum_{q=1}^p qy^{q(n-1)} T_{2(p-q)}$ if $i \notin \bar{\alpha}$ and $\ell \leq n-2$,
- $(y+1)(y-1)^2 y^{n-3} \sum_{q=1}^p qy^{q(n-1)} T_{2(p-q)} + T_{2p-1}$ if $\alpha = p\delta - \alpha_i$ (i.e. $i \notin \bar{\alpha}$ and $\ell = n-1$),
- $(y-1)^2 y^{\ell-2} \sum_{q=0}^p (q+1)y^{q(n-1)} T_{2(p-q)}$ if $i \in \alpha^\circ$.

2.2.3. The construction of Paris.

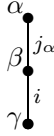

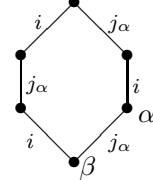
A big part of [21] is devoted to a uniform construction of an LKP-family for an arbitrary Coxeter matrix of small type $\Gamma = (m_{i,j})_{i,j \in I}$, at least when it has no triangle, i.e. no subset $\{i, j, k\} \subseteq I$ with $m_{i,j} = m_{j,k} = m_{k,i} = 3$.

Definition 22 ([21]). For every $\alpha \in \Phi^+$ with $\text{dp}(\alpha) \geq 2$, fix an element $j_\alpha \in I$ such that $(\alpha|\alpha_{j_\alpha}) > 0$. Define $T_{i,\alpha}$ by induction on $\text{dp}(\alpha)$, as follows :

Basis step : fix a non-trivial polynomial $T \in y\mathbb{Q}[y]$ and set

Case	Value of $T_{i,\alpha}$	Condition
(C1)	T	if $\alpha = \alpha_i$
(C2)	0	if $\alpha = \alpha_j$ for $j \neq i$
(C3)	$y^{\text{dp}(\alpha)-2}(y-1)T$	if $\text{dp}(\alpha) \geq 2$ and $(\alpha \alpha_i) > 0$

Inductive step : if $\text{dp}(\alpha) \geq 2$ and $(\alpha|\alpha_i) \leq 0$ — hence $i \neq j_\alpha$ — then set

Case	Value of $T_{i,\alpha}$	Condition
(C4)	$yT_{i,\beta}$	if $m_{i,j_\alpha} = 2$
(C5)	$(y-1)T_{i,\beta} + yT_{j_\alpha,\gamma}$	if $m_{i,j_\alpha} = 3$ and 
(C6)	$(y-1)T_{i,\beta} + T_{j_\alpha,\beta}$	if $m_{i,j_\alpha} = 3$ and 
(C7)	$yT_{i,\beta} + T_{j_\alpha,\beta} + y^{\text{dp}(\beta)-2}(1-y)T$	if $m_{i,j} = 3$ and 

Note that cases (C4) and (C5) occur whether $(\alpha_i|\alpha)$ is zero or negative. Case (C3) is a generalization of what happens when Γ is spherical, but is no longer a consequence of the relations of table 1 in general. Similarly, the formula of case (C7) is not implied by the relations of table 1.

In [21], Paris chooses $T = y^2$, this simplifies the formulae of cases (C3) and (C7). The assumption $T \in y\mathbb{Q}[y]$ is needed for the formula (C7) to define a polynomial (and not a fraction) when $\text{dp}(\alpha) = 2$; this only occurs if $\alpha = \alpha_{j_\alpha} + \alpha_k$ where $\Gamma_{\{i,j_\alpha,k\}}$ is a triangle, so if Γ has no triangle, then $T \neq 0$ can be chosen arbitrarily.

Proposition 23 ([21, lemmas 3.3 and 3.4]). *Assume that Γ has no triangle. Then the $T_{i,\alpha}$'s of definition 22 above do not depend on the choice of the j_α 's.*

Corollary 24 ([21, lemmas 3.5, 3.6 and 3.7]). *Assume that Γ has no triangle. Then the $T_{i,\alpha}$'s of definition 22 above form an LKP-family.*

Note that the proofs of [21, lemmas 3.5, 3.6 and 3.7] really show that the family of polynomials of definition 22 is an LKP-family as soon as it does not depend of the choice of the j_α 's. Indeed, the last case that should be considered in the proof of [21, lemma 3.5] to deal with Coxeter matrices of small type with triangle is in fact impossible (see the proof of lemma 14).

3. ACTION OF GRAPH AUTOMORPHISMS

Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix of small type, let G be a subgroup of $\text{Aut}(\Gamma)$ and let $\psi : B^+ \rightarrow \text{GL}(V)$ be an LKP-representation of $B^+ = B_\Gamma^+$.

Recall that the group G naturally acts on B^+ and on Φ^+ (see section 1.3). The action of G on Φ^+ induces an action of G on V by permutation of the basis $(e_\alpha)_{\alpha \in \Phi^+}$. We denote by $(B^+)^G$ (resp. V^G) the submonoid (resp. subspace) of fixed points of B^+ (resp. of V) under the action of G .

Lemma 25. *Assume that $T_{i,\alpha} = T_{g(i),g(\alpha)}$ for every $(i, \alpha, g) \in I \times \Phi^+ \times G$. Then for every $b \in (B^+)^G$, the linear maps ψ_b and ψ_b^{-1} stabilize V^G . Hence ψ induces a linear representation*

$$\psi^G : (B^+)^G \rightarrow \text{GL}(V^G), \quad b \mapsto \psi_b|_{V^G}.$$

Proof. It is sufficient to prove that, for every $(b, v, g) \in B^+ \times V \times G$, one has $g(\psi_b(v)) = \psi_{g(b)}(g(v))$ and $g(\psi_b^{-1}(v)) = \psi_{g(b)}^{-1}(g(v))$. By linearity and induction on $\ell(b)$, proving this property reduces to proving that, for every $(i, \alpha, g) \in I \times \Phi^+ \times G$, one has $g(\psi_{s_i}(e_\alpha)) = \psi_{s_{g(i)}}(e_{g(\alpha)})$ and $g(\psi_{s_i}^{-1}(e_\alpha)) = \psi_{s_{g(i)}}^{-1}(e_{g(\alpha)})$. It is easily seen, on the formulae that define the maps ψ_{s_i} and $\psi_{s_i}^{-1}$ for $i \in I$, that these equalities occur if and only if $T_{i,\alpha} = T_{g(i),g(\alpha)}$ for every $(i, \alpha, g) \in I \times \Phi^+ \times G$. \square

The assumption of the previous lemma is not always satisfied : for example if i and $g(i)$ are not in the same connected component of Γ , then T_{i,α_i} and $T_{g(i),\alpha_{g(i)}}$ can be chosen to be distinct (see remark 15 above). I do not know if this assumption is always satisfied when Γ is connected, but we have the following partial result :

Lemma 26. *Let $\mathcal{T} = (T_{i,\alpha})_{(i,\alpha) \in I \times \Phi^+}$ be an LKP-family.*

- (i) *If Γ is spherical and irreducible (i.e. of type ADE), then $T_{i,\alpha} = T_{g(i),g(\alpha)}$ for every $(i, \alpha, g) \in I \times \Phi^+ \times \text{Aut}(\Gamma)$.*
- (ii) *If Γ is affine (i.e. of type $\tilde{A}\tilde{D}\tilde{E}$), then $T_{i,\alpha} = T_{g(i),g(\alpha)}$ for every $(i, \alpha, g) \in I \times \Phi^+ \times \text{Aut}(\Gamma)$.*
- (iii) *If \mathcal{T} is the family constructed in definition 22 and if it does not depend on the choice of the j_α 's (in particular if Γ has no triangle), then $T_{i,\alpha} = T_{g(i),g(\alpha)}$ for every $(i, \alpha, g) \in I \times \Phi^+ \times \text{Aut}(\Gamma)$.*

Proof. The three points (note that the first one is a consequence of the third one) are easy to see by induction on $\text{dp}(\alpha)$, using the inductive construction of \mathcal{T} (and the corresponding independence results for the inductive steps) and the fact that the action of $\text{Aut}(\Gamma)$ on $E = \bigoplus_{i \in I} \mathbb{R}\alpha_i$ respects the bilinear form $(\cdot | \cdot)_\Gamma$. \square

We denote by Φ^+/G the set of orbits of Φ^+ under G and, for every $\Theta \in \Phi^+/G$, we set $e_\Theta = \sum_{\alpha \in \Theta} e_\alpha$. The family $(e_\Theta)_{\Theta \in \Phi^+/G}$ is a basis of V^G .

3.1. Faithfulness of the induced representation ψ^G .

In this section, we assume that the condition $T_{i,\alpha} = T_{g(i),g(\alpha)}$ for every $(i, \alpha, g) \in I \times \Phi^+ \times G$ is satisfied, and we prove our main theorem, namely that the induced representation ψ^G of lemma 25 is faithful.

In fact, like in the small type case, we prove more : if we fix the indeterminate y to some real value $y_0 \in]0; 1[$ that is not a zero of some T_{i,α_i} , $i \in I$, then we get a representation $\psi_0^G : (B^+)^G \rightarrow \text{GL}(V_0^G)$, where $V_0^G = \bigoplus_{\Theta \in \Phi^+/G} \mathbb{R}(x)e_\Theta$ and

$\psi_0^G(b) = \psi_0(b)|_{V_0^G}$ (with the notations of section 2.1), and we prove that ψ_0^G is faithful. The proof follows the lines of the one of section 2.1.

Lemma 27. *Let $\rho : (B^+)^G \rightarrow M$ be a monoid homomorphism satisfying $\rho(b) = \rho(b') \Rightarrow I(b) = I(b')$ for all $b, b' \in (B^+)^G$. If M is left cancellative (for example if M is a group), then ρ is injective.*

Proof. The proof is adapted from the one of lemma 7. Let $b, b' \in (B^+)^G$ be such that $\rho(b) = \rho(b')$. We prove by induction on $\ell(b)$ that $b = b'$. If $\ell(b) = 0$, i.e. if $b = 1$, then $I(b) = I(b') = \emptyset$, hence $b' = 1$ and we are done.

If $\ell(b) > 0$, fix $i \in I(b) = I(b')$. Since the action of G on B^+ respects the divisibility and since b is fixed by G , the orbit J of i under G is included in $I(b) = I(b')$, but then J is spherical and there exist $b_1, b'_1 \in B^+$ such that $b = \Delta_J b_1$ and $b' = \Delta_J b'_1$. Since J is an orbit of I under G , the element Δ_J is fixed by G and hence so are b_1 and b'_1 , so we get $\rho(\Delta_J)\rho(b_1) = \rho(\Delta_J)\rho(b'_1)$ in M , whence $\rho(b_1) = \rho(b'_1)$ by cancellation, therefore $b_1 = b'_1$ by induction and finally $b = b'$. \square

Theorem 28. *The representation ψ_0^G (and hence ψ^G) is faithful.*

Proof. Let $b, b' \in (B^+)^G$ be such that $\psi_0^G(b) = \psi_0^G(b')$. Thanks to lemma 27, it suffices to show that $I(b) = I(b')$. Since $\psi_0(b)$ and $\psi_0(b')$ coincide on V^G , we get in particular $(\psi_0)_b(e_\Theta) = (\psi_0)_{b'}(e_\Theta)$ for every $\Theta \in \Phi^+/G$.

With the notations of definition 8, let us consider the set $R_b(\Theta)$. Since the coefficients of the matrix of $\psi_0(b)$ in the basis $(e_\alpha)_{\alpha \in \Phi^+}$ are polynomials in x with non-negative constant terms, the set $R_b(\Theta)$ is precisely the set of those indices $\beta \in \Phi^+$ for which the coefficient of e_β in the decomposition of $(\psi_0)_b(e_\Theta)$ in the basis $(e_\alpha)_{\alpha \in \Phi^+}$ is non-zero modulo x .

The same occurs for b' , and hence $(\psi_0)_b(e_\Theta) = (\psi_0)_{b'}(e_\Theta)$ implies $R_b(\Theta) = R_{b'}(\Theta)$. Since we have $\Phi^+ = \bigcup_{\Theta \in \Phi^+/G} \Theta$, we thus get $R_b(\Phi^+) = \bigcup_{\Theta \in \Phi^+/G} R_b(\Theta) = \bigcup_{\Theta \in \Phi^+/G} R_{b'}(\Theta) = R_{b'}(\Phi^+)$, and hence $I(b) = I(b')$ by lemma 10. \square

We also call LKP-representations the faithful representations ψ^G thus obtained. The following section will justify this definition (at least when $|G| = 2$): the matrices of the representation in the basis $(e_\Theta)_{\Theta \in \Phi^+/G}$ are very similar to the ones of the small type case.

3.2. Formulae when $|G| = 2$.

Recall that $(B^+)^G$ is an Artin-Tits monoid generated by the elements Δ_J , for J running through the spherical orbits of I under G (see section 1.3). We compute below the matrix of $\psi_{\Delta_J}^G$ in the basis $(e_\Theta)_{\Theta \in \Phi^+/G}$ of V^G for such an orbit J , when $|G| = 2$.

The computation uses the formulae defining the representation ψ and the assumption that $T_{i,\alpha} = T_{g(i),g(\alpha)}$ for every $(i, \alpha, g) \in I \times \Phi^+ \times G$. It depends on the cardinality of the chosen orbit J , and on the configuration of the graph of $\{w(\alpha) \mid w \in W_J, \alpha \in \Theta\} \cap \Phi^+$ (union of the J -meshes of the elements of Θ).

Note that $\Theta_J := \{\alpha_i \mid i \in J\}$ is an orbit of Φ^+ under G .

3.2.1. Case $J = \{i\}$.

The matrix of $\psi_{\Delta_J}^G = \psi_{\mathbf{s}_i}^G$ in the basis $(e_\Theta)_{\Theta \in \Phi^+/G}$ is of the form

$$\begin{pmatrix} e_{\Theta_J} & \cdots \\ * & * & \cdots & * \\ 0 & & & \\ \vdots & M_J & & \\ 0 & & & \end{pmatrix}, \text{ where } \begin{cases} - \text{ the line } \Theta_J \text{ is of the form } (xT_{J,\Theta})_{\Theta \in \Phi^+/G}, \text{ with} \\ T_{J,\Theta} = \begin{cases} T_{i,\alpha} & \text{if } \Theta = \{\alpha\}, \\ 2T_{i,\alpha} & \text{if } \Theta = \{\alpha, \alpha'\}, \alpha \neq \alpha', \end{cases} \\ - \text{ the matrix } M_J \text{ is block diagonal, with blocks :} \end{cases}$$

$$\begin{cases} \begin{pmatrix} e_{\Theta} \\ (1) \end{pmatrix} & \text{if } \Theta \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \text{ or } \Theta \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}, \\ \begin{pmatrix} e_{\Theta_1} & e_{\Theta_2} \\ (1-y & y \\ 1 & 0) \end{pmatrix} & \text{if } \Theta_2 \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \\ & \text{or } \Theta_1 \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}. \end{cases}$$

3.2.2. Case $J = \{i, j\}$ with $m_{i,j} = 2$.

The matrix of $\psi_{\Delta_J}^G = \psi_{\mathbf{s}_i \mathbf{s}_j}^G$ in the basis $(e_{\Theta})_{\Theta \in \Phi^+/G}$ is of the form

$$\begin{pmatrix} e_{\Theta_J} & \cdots \\ * & * & \cdots & * \\ 0 & & & \\ \vdots & M_J & & \\ 0 & & & \end{pmatrix}, \text{ where } \begin{cases} - \text{ the line } \Theta_J \text{ is of the form } (xT_{J,\Theta})_{\Theta \in \Phi^+/G}, \text{ with} \\ T_{J,\Theta} = \begin{cases} T_{i,\alpha} & \text{if } \Theta = \{\alpha\}, \\ T_{i,\alpha} + T_{j,\alpha} & \text{if } \Theta = \{\alpha, \alpha'\}, \alpha \neq \alpha', \end{cases} \\ - \text{ the matrix } M_J \text{ is block diagonal, with blocks :} \end{cases}$$

$$\begin{cases} \begin{pmatrix} e_{\Theta} \\ (1) \end{pmatrix} & \text{if } \Theta \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \text{ or } \Theta \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}, \\ \begin{pmatrix} e_{\Theta_1} & e_{\Theta_2} \\ (1-y & y \\ 1 & 0) \end{pmatrix} & \text{if } \Theta_2 \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \\ & \Theta_1 \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}, \\ \begin{pmatrix} e_{\Theta_1} & e_{\Theta_2} & e_{\Theta_3} \\ ((1-y)^2 & 2y(1-y) & y^2) \\ (1-y & y & 0) \\ 1 & 0 & 0) \end{pmatrix} & \text{if } \Theta_3 \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \\ & \Theta_2 \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}, \\ \Theta_1 \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}, \\ \begin{pmatrix} e_{\Theta_1} & e_{\Theta_2} & e_{\Theta_3} & e_{\Theta_4} \\ ((1-y)^2 & y(1-y) & y(1-y) & y^2) \\ (1-y & 0 & y & 0) \\ (1-y & y & 0 & 0) \\ 1 & 0 & 0 & 0) \end{pmatrix} & \text{if } \Theta_4 \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \\ & \Theta_2 \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \Theta_3 \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}. \\ & \Theta_1 \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}. \end{cases}$$


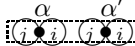
3.2.3. Case $J = \{i, j\}$ with $m_{i,j} = 3$.

In that case, $\alpha_i + \alpha_j$ is a (positive) root fixed by G . The matrix of $\psi_{\Delta_J}^G = \psi_{\mathbf{s}_i \mathbf{s}_j \mathbf{s}_i}^G$ in the basis $(e_{\Theta})_{\Theta \in \Phi^+/G}$ is of the form

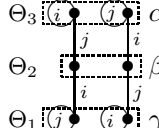
$${}^{e_{\Theta_J} e_{\{\alpha_i + \alpha_j\}}} \begin{pmatrix} * & 0 & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & & & \\ \vdots & \vdots & & M_J & \\ 0 & 0 & & & \end{pmatrix}, \text{ where } \begin{cases} - \text{ the line } \Theta_J \text{ (resp. } \{\alpha_i + \alpha_j\}) \text{ is of the form} \\ \quad (xT_{J,\Theta})_{\Theta \in \Phi^+/G} \text{ (resp. } (xU_{J,\Theta})_{\Theta \in \Phi^+/G}), \text{ with} \\ \quad \text{the } T_{J,\Theta}\text{'s and } U_{J,\Theta}\text{'s computed below,} \\ - \text{ the matrix } M_J \text{ is block diagonal, with blocks} \\ \quad \text{described below.} \end{cases}$$

Case 1. $\Theta = \{\alpha_i, \alpha_j\}$ or $\Theta = \{\alpha_i + \alpha_j\}$

Value of $T_{J,\Theta}$	Value of $U_{J,\Theta}$	Value of Θ
yT_{i,α_i}	0	$\Theta = \{\alpha_i, \alpha_j\}$
0	yT_{i,α_i}	$\Theta = \{\alpha_i + \alpha_j\}$

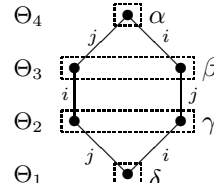
Case 2. Configuration Θ  or Θ 

Value of $T_{J,\Theta}$	Value of $U_{J,\Theta}$	Block in M_J
$\text{Card}(\Theta)T_{i,\alpha}$	$\text{Card}(\Theta)T_{i,\alpha}$	$\begin{matrix} e_{\Theta} \\ (1) \end{matrix}$

Case 3. Configuration Θ_2 

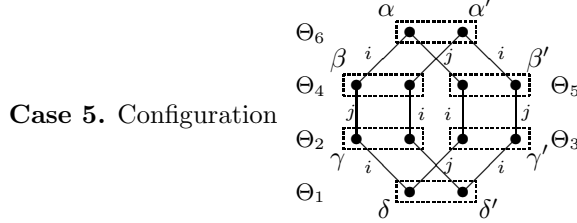
Values of $T_{J,\cdot}$	Values of $U_{J,\cdot}$
$T_{J,\Theta_1} = yT_{i,\gamma} + (2-y)T_{j,\gamma}$	$U_{J,\Theta_1} = 2T_{j,\gamma}$
$T_{J,\Theta_2} = T_{i,\alpha} + y(1-y)T_{i,\gamma} + yT_{j,\gamma}$	$U_{J,\Theta_2} = 2yT_{i,\gamma}$
$T_{J,\Theta_3} = T_{i,\alpha} + y^2T_{i,\gamma}$	$U_{J,\Theta_3} = 2T_{i,\alpha}$

Block in M_J : $\begin{pmatrix} e_{\Theta_1} & e_{\Theta_2} & e_{\Theta_3} \\ 1-y & y(1-y) & y^2 \\ 1-y & y & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Case 4. Configuration Θ_4 

Values of $T_{J,\cdot}$	Values of $U_{J,\cdot}$
$T_{J,\Theta_1} = (1-y)T_{j,\gamma} + (1-y+y^2)T_{i,\delta}$	$U_{J,\Theta_1} = T_{j,\gamma} + (1-y)T_{j,\delta}$
$T_{J,\Theta_2} = y(T_{i,\gamma} + T_{j,\gamma} + 2(1-y)T_{i,\delta})$	$U_{J,\Theta_2} = 2yT_{i,\delta}$
$T_{J,\Theta_3} = y(T_{j,\beta} + (1-y)T_{i,\gamma} + yT_{i,\delta})$	$U_{J,\Theta_3} = 2yT_{i,\gamma}$
$T_{J,\Theta_4} = y^2T_{i,\gamma}$	$U_{J,\Theta_4} = yT_{j,\beta}$

$$\text{Block in } M_J : \begin{pmatrix} (1-y)(1-y+y^2) & 2y(1-y)^2 & 2y^2(1-y) & y^3 \\ (1-y)^2 & 2y(1-y) & y^2 & 0 \\ 1-y & y & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$



Values of $T_{J,\cdot}$	Values of $U_{J,\cdot}$
$T_{J,\Theta_1} = 2(1-y)(T_{j,\gamma} + (1-y)T_{j,\delta}) + y(T_{i,\delta} + T_{i,\delta'})$	$U_{J,\Theta_1} = 2(T_{j,\gamma} + (1-y)T_{j,\delta})$
$T_{J,\Theta_2} = y(T_{i,\gamma} + T_{j,\gamma} + 2(1-y)T_{j,\delta})$	$U_{J,\Theta_2} = 2yT_{j,\delta}$
$T_{J,\Theta_3} = y(T_{i,\gamma'} + T_{j,\gamma'} + 2(1-y)T_{j,\delta'})$	$U_{J,\Theta_3} = 2yT_{j,\delta'}$
$T_{J,\Theta_4} = y(T_{j,\beta} + (1-y)T_{i,\gamma} + yT_{j,\delta})$	$U_{J,\Theta_4} = 2yT_{i,\gamma}$
$T_{J,\Theta_5} = y(T_{j,\beta'} + (1-y)T_{i,\gamma'} + yT_{j,\delta'})$	$U_{J,\Theta_5} = 2yT_{i,\gamma'}$
$T_{J,\Theta_6} = y^2(T_{i,\gamma} + T_{i,\gamma'})$	$U_{J,\Theta_6} = 2yT_{j,\beta}$

Block in M_J :

$$\begin{pmatrix} (1-y)(1-y+y^2) & y(1-y)^2 & y(1-y)^2 & y^2(1-y) & y^2(1-y) & y^3 \\ (1-y)^2 & y(1-y) & y(1-y) & 0 & y^2 & 0 \\ (1-y)^2 & y(1-y) & y(1-y) & y^2 & 0 & 0 \\ 1-y & 0 & y & 0 & 0 & 0 \\ 1-y & y & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3.2.4. Consequence for the LKP-representations of type B.

Fix $n \in \mathbb{N}_{\geq 3}$. Let us consider the Coxeter matrices Γ_1 , Γ_2 and Γ_3 of type A_{2n-1} , A_{2n} and D_{n+1} respectively (and vertex set I_1 , I_2 and I_3 respectively), and three LKP-representations $\psi_1 : B_{\Gamma_1}^+ \rightarrow \text{GL}(V_1)$, $\psi_2 : B_{\Gamma_2}^+ \rightarrow \text{GL}(V_2)$ and $\psi_3 : B_{\Gamma_3}^+ \rightarrow \text{GL}(V_3)$ associated with those matrices. Each of these representations ψ_k for $1 \leq k \leq 3$ is characterized by a certain polynomial $T_k \in \mathbb{Q}[y] \setminus \{0\}$ which is the common value of the T_{i,α_i} for $i \in I_k$ (see section 2.2.1).

Let us consider the groups of graph automorphisms $G_1 = \text{Aut}(\Gamma_1)$, $G_2 = \text{Aut}(\Gamma_2)$, and G_3 a subgroup of order two of $\text{Aut}(\Gamma_3)$ (resp. $G_3 = \text{Aut}(\Gamma_3)$) when $n = 3$ (resp. $n > 3$). Then the submonoid $(B_{\Gamma_k}^+)^{G_k}$, for $1 \leq k \leq 3$, is an Artin-Tits monoid of type B_n (see [9, 7]).

By lemmas 25 and 26, we get three linear representations $\psi_1^{G_1}$, $\psi_2^{G_2}$ and $\psi_3^{G_3}$ for the Artin-Tits monoid of type B_n . Note that $\psi_1^{G_1}$ is essentially the representation considered in [12].

The aim of this section is to show that those three representations are pairwise non-equivalent, where two linear representations ψ and ψ' of a monoid M , on

vector spaces V and V' respectively, are said to be equivalent if there exists a linear isomorphism $f : V \rightarrow V'$ such that, for every $b \in M$, $\psi'_b = f \circ \psi_b \circ f^{-1}$.

Lemma 29. *We label by $1, 2, \dots, n$, the vertices of B_n , in such a way that the vertex n is the terminal vertex of the edge labeled 4. We denote by $\Delta_1, \dots, \Delta_n$ the corresponding standard generators of $B_{B_n}^+$. Then $\det((\psi_k^{G_k})_{\Delta_i})$, for $1 \leq k \leq 3$ and $1 \leq i \leq n$, is given by the following table :*

	$1 \leq i < n$	$i = n$
$k = 1$	$-xy^{2n-1}T_1$	$(-1)^{n-1}xy^{n-1}T_1$
$k = 2$	$xy^{2n}T_2$	$(-1)^{n-1}y^{3n-1}(xT_2)^2$
$k = 3$	$-xy^{2n-3}T_3$	$(-1)^{n-1}xy^{3(n-1)}T_3$

Proof. Considering the formulae of sections 3.2.1, 3.2.2 and 3.2.3, the determinant of $(\psi_k^{G_k})_{\Delta_i}$ is of the form $xT_k \det(M)$ (resp. $(xyT_k)^2 \det(M)$) if $(i, k) \neq (n, 2)$ (resp. $(i, k) = (n, 2)$), where M is a block diagonal matrix, with blocks of determinant $1, -y, -y^3$ or y^4 (resp. $1, -y^3, y^6$ or $-y^9$) depending on the configuration of the corresponding orbit in $\Phi_{\Gamma_k}^+$. The result then follows from a computation of the number of occurrences of each configuration in the three root systems considered here. \square

Proposition 30. *The linear representations $\psi_1^{G_1}, \psi_2^{G_2}$ and $\psi_3^{G_3}$ are pairwise non-equivalent.*

Proof. This is trivial for $\psi_2^{G_2}$ since it is a representation of degree $\text{Card}(\Phi_{\Gamma_2}^+/G_2) = n(n+1)$ whereas the two others are of degree $\text{Card}(\Phi_{\Gamma_1}^+/G_1) = \text{Card}(\Phi_{\Gamma_3}^+/G_3) = n^2$.

Anyway, the result is a direct consequence of the previous lemma, since it shows that, for a given $1 \leq i < n$, we cannot have at the same time $\det((\psi_k^{G_k})_{\Delta_i}) = \det((\psi_l^{G_l})_{\Delta_i})$ and $\det((\psi_k^{G_k})_{\Delta_n}) = \det((\psi_l^{G_l})_{\Delta_n})$ for $k \neq l$. \square

Note that the same holds for the Artin-Tits *group* of type B_n by virtue of lemma 1.

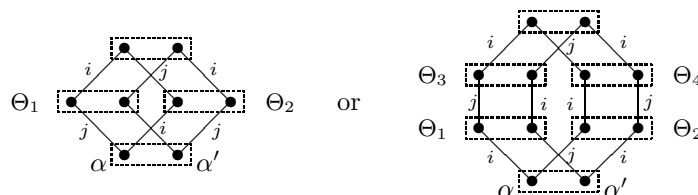
3.3. Final remark on Φ^+/G .

Let us denote by Γ' the type of W^G and $(B^+)^G$.

When Γ is of spherical type ADE , it is possible to increase the resemblance between the LKP-representation ψ^G and the (small type) LKP-representation ψ , by indexing the basis $(e_\Theta)_{\Theta \in \Phi^+/G}$ of V^G with the set of positive roots of a *finite crystallographic* root system (*i.e.* a root system in the sense of [2, Ch. VI]) of Weyl group $W_{\Gamma'}$, via the bijection $\Theta \mapsto \alpha_\Theta$, where $\alpha_\Theta = \frac{1}{\text{Card}(\Theta)} \sum_{\alpha \in \Theta} \alpha$ (see for example [5, Ch. 13] for justifications).

For example if $\Gamma = A_{2n-1}, A_{2n}$ or D_{n+1} and $G = \text{Aut}(\Gamma)$ (or a subgroup of order 2 of $\text{Aut}(\Gamma)$ for D_4), we get a finite crystallographic root system of Dynkin type C_n, BC_n or B_n respectively. This change of index set has been used by Digne in [12] for his proof of faithfulness of ψ^G , in the particular cases $\Gamma = A_{2n-1}, E_6$ or D_4 and $G = \text{Aut}(\Gamma)$.

But this change of index set is not possible in general. Indeed, the map $\Theta \mapsto \alpha_\Theta$ is not necessarily injective if Γ is not spherical : for example when $|G| = 2$, then for the following configurations of orbits



we get $\alpha_{\Theta_1} = \alpha_{\Theta_2}$ and $\alpha_{\Theta_3} = \alpha_{\Theta_4}$ as soon as $(\alpha_i|\alpha) = (\alpha_j|\alpha)$.

Note that the first of those counterexamples occurs for example in a root system of type \tilde{A}_{2n-1} ($n \geq 2$) with G generated by the “half turn”, and the second (which does not occur in the affine cases in view of remark 16 above) occurs for example in the root system associated with the Coxeter graph $1 \begin{smallmatrix} 4 \\ \square \\ 2 \end{smallmatrix} 3$ with $G = \langle (1\ 3)(2\ 4) \rangle$, $\{i, j\} = \{1, 3\}$ and $\{\alpha, \alpha'\} = \{\alpha_2, \alpha_4\}$.

REFERENCES

- [1] S. BIGELOW. *Braid groups are linear*. J. Amer. Math. Soc. **14** (2001), 471–486.
- [2] N. BOURBAKI. *Groupes et Algèbres de Lie, Chapitres IV-VI*. Hermann, Paris, 1968.
- [3] E. BRIESKORN, K. SAITO. *Artin-Gruppen und Coxeter-Gruppen*. Invent. Math. **17** (1972) 245–271.
- [4] B. BRINK, R. B. HOWLETT. *A finiteness property and an automatic structure for Coxeter groups*. Math. Ann. **296** (1993), 179–190.
- [5] R.W. CARTER. *Simple groups of Lie type*. Wiley (1972).
- [6] A. CASTELLA. *Automorphismes et admissibilité dans les groupes de Coxeter et les monoïdes d’Artin-Tits*. PhD Thesis, Orsay (2006).
- [7] A. CASTELLA. *Admissible submonoids of Artin-Tits monoids*. J. Pure Appl. Algebra (2007), doi:10.1016/j.jpaa.2007.10.010.
- [8] A. M. COHEN, D. B. WALES. *Linearity of Artin groups of finite type*. Israel. J. Math. **131** (2002), 101–123.
- [9] J. CRISP. *Symmetrical subgroups of Artin groups*. Adv. in Math. **152** (2000) 159–177.
- [10] J. CRISP. *Erratum to “Symmetrical subgroups of Artin groups”*. Adv. in Math. **179** (2003) 318–320.
- [11] V. V. DEODHAR. *On the root system of a Coxeter group*. Comm. Algebra **10** (1982), 611–630.
- [12] F. DIGNE. *On the linearity of Artin Braid groups*. J. Algebra **268**, (2003) 39–57.
- [13] J.-Y. HÉE. *Systèmes de racines sur un anneau commutatif totalement ordonné*. Geom. Dedic. **37** (1991), 65–102.
- [14] J.-Y. HÉE. *Une démonstration simple de la fidélité de la représentation de Lawrence-Krammer-Paris*. Preprint.
- [15] V. G. KAC. *Infinite dimensional Lie algebras*. Cambridge University Press (1990).
- [16] D. KRAMMER. *The braid group B_4 is linear*. Invent. Math. **142** (2000), 451–486.
- [17] D. KRAMMER. *Braid groups are linear*. Ann. of Math. **155** (2002), 131–156.
- [18] R. J. LAWRENCE. *Homological representations of the Hecke algebra*. Commun. Math. Phys. **135** (1990), 141–191.
- [19] J. MICHEL. *A note on words in braid monoids*. J. Algebra **215** (1999), 366–377.
- [20] B. MÜHLHERR. *Coxeter groups in Coxeter groups*. Finite Geometry and Combinatorics, Cambridge University Press (1993), 277–287.
- [21] L. PARIS. *Braid monoids inject in their groups*. Comment. Math. Helv. **77** (2002), 609–637.
- [22] H. VAN DER LEK. *The homotopy type of complex hyperplane complements*. PhD Thesis, Nijmegen (1983).