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Characterizing path graphs by forbidden induced subgraphs

Benjamin Lévêque, Frédéric Maffray, Myriam Preissmann

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Abstract

A graph is a path graph if it is the intersection graph of a family of subpaths of a tree. In 1970, Renz asked for a characterization of path graph by forbidden induced subgraphs. Here we answer this question by listing all graphs that are not path graphs and are minimal with this property.

1 Introduction

A hole is a chordless cycle of length at least four. A graph is chordal (or triangulated) if it contains no holes as an induced subgraph. Gavril [3] proved that a graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree.

An *interval graph* is the intersection graph of a family of intervals; equivalently, it is the intersection graph of a family of subpaths of a path. An *asteroidal triple* in a graph G is a set of three non adjacent vertices such that for any two of them, there exists a path between them in G that does not intersect the neighborhood of the third. Lekkerkerker and Boland [5] proved that a graph is an interval graph if and only if it is chordal and contains no asteroidal triple. They derive from this result the list of minimal forbidden subgraphs for interval graphs.

Another interesting class is the class of path graphs. A graph is a *path graph* if it is the intersection graph of a family of subpaths of a tree. Clearly, the class of path graphs is included in the class of chordal graphs and contains the class of interval graphs. Several characterizations of path graphs have been given [4, 6, 7] but no characterization by forbidden subgraphs is known, whereas such results exists for intersection graphs of subpaths of a paths (interval graphs [3]) and subtrees of a tree (chordal graphs [5]).

In 1970, Renz [7] gave two examples of graphs that are chordal and not path graphs and are minimal with this property. He also asked for a complete list of such graphs. In this paper we answer this question and obtain a characterization of path graphs by forbidden induced subgraphs.

2 Special simplicial vertices in chordal graphs

A vertex is *simplicial* if its neighborhood is a clique.

Let G be a chordal graph, and let $\mathcal{Q}(G)$ be the set of all maximal cliques of G . A *separator* in a graph G is a set S of vertices of G such that there exist two vertices of G that lie in two different components of $G \setminus S$ and S is minimal with this property. Let $\mathcal{S}(G)$ be the set of separators of G . Given a simplicial vertex v , let $Q_v = N(v) \cup \{v\}$ and $S_v = Q_v \cap N(V \setminus Q_v)$. Since v is simplicial, $Q_v \in \mathcal{Q}$ (remark that S_v is not necessarily in \mathcal{S}).

A *clique tree* T of G is a tree whose vertices are in \mathcal{Q} and such that for each vertex v of G , the set of cliques that contain v induces a subtree of T . A graph is chordal if and only if it has a clique tree representation.

For a clique tree T , the label of an edge QQ' of T is $S_{QQ'} = Q \cap Q'$. For all edges QQ' , we have $S_{QQ'} \in \mathcal{S}$. The number of times an element S of \mathcal{S} appears as a label of an edge is constant for every clique-tree.

Given $X \subseteq \mathcal{Q}$, let $G(X)$ be the subgraph of G induced by all the vertices that appears in members of X . If T is a clique tree of G , then $T[X]$ is the subtree of T of minimum size whose vertices contains X . Note that if $|X| = 2$, then $T[X]$ is a path. For every vertex a , set $T^a = T[\{Q \in \mathcal{Q} \mid a \in Q\}]$.

Given a subtree T' of a clique-tree T of G . Let $\mathcal{Q}(T')$ be the set of vertices of T' and $\mathcal{S}(T')$ be the set of separators of $G(\mathcal{Q}(T'))$.

The following lemma is clear.

Lemma 1 *A vertex is simplicial if and only if it does not belong to any separator.*

Dirac [2] proved that a chordal graph that is not a clique contains two non adjacent simplicial vertices. We need to generalize this theorem to the following. Let us say that a simplicial vertex v is *special* if S_v is (inclusionwise) maximal in \mathcal{S} .

Theorem 1 *In a chordal graph that is not a clique, there exist two non adjacent special simplicial vertices.*

Proof. We prove the lemma by induction on $|\mathcal{Q}|$. By the hypothesis, G is not a clique, so $|\mathcal{Q}| \geq 2$. If $|\mathcal{Q}| = 2$, then \mathcal{S} has only one element S that is maximal. Let $\mathcal{Q} = \{Q, Q'\}$, $v \in Q \setminus Q'$ and $v' \in Q' \setminus Q$. Then v and v' are simplicial and $S_v = S_{v'} = S$ is a maximal set of \mathcal{S} . Suppose now that $|\mathcal{Q}| \geq 3$.

Case 1: \mathcal{S} has only one maximal element S . Let Q, Q' be two maximal cliques such that $Q \cap Q' = S$. Let $v \in Q \setminus Q'$ and $v' \in Q' \setminus Q$. The set S is the only maximal separator and it does not contain v or v' . So v and v' do not belong to any element of \mathcal{S} , and they are simplicial by Lemma 1 and $S_v = S_{v'} = S$.

Case 2: \mathcal{S} has two distinct maximal elements S, S' . Let T be a clique tree of G . Let $Q_1, Q_2, Q'_1, Q'_2 \in \mathcal{Q}$ bz such that $S = S_{Q_1 Q_2}$, $S' = S_{Q'_1 Q'_2}$, and Q_2, Q_1, Q'_1, Q'_2 appear in this order along the path $T[\{Q_2, Q_1, Q'_1, Q'_2\}]$ (maybe $Q_1 = Q'_1$).

Let Z be the subtree of $T \setminus Q_1$ that contains Q_2 plus the vertex Q_1 and the edge $Q_1 Q_2$. The subtree Z does not contain Q'_2 , so $G(\mathcal{Q}(Z))$ has strictly fewer maximal cliques than G . By the induction hypothesis, let v, v'' be two non adjacent vertices of $G(\mathcal{Q}(Z))$ such that $S_v, S_{v''}$ are maximal elements of $\mathcal{S}(Z)$. At most one of v, v'' is in Q_1 since they are not adjacent. Suppose v is not in Q_1 .

We claim that v is a simplicial vertex of G with S_v a maximal element of \mathcal{S} . Vertex v does not belong to any element of $\mathcal{S}(Z)$. If it belongs to an element of $\mathcal{S} \setminus \mathcal{S}(Z)$, then it will also belong to $S \in \mathcal{S}(Z)$, a contradiction. So v does not belong to any element of \mathcal{S} and it is a simplicial vertex of G by Lemma 1. The set S_v is a maximal element of $\mathcal{S}(Z)$. If it is not a maximal element of \mathcal{S} , then it will be included in $S \in \mathcal{S}(Z)$, a contradiction. So v is a special simplicial vertex of G .

Let Z' be the subtree of $T \setminus Q'_1$ that contains Q'_2 plus the vertex Q'_1 and the edge $Q'_1 Q'_2$. Just like with v , we can find a simplicial vertex v' of $G(\mathcal{Q}(Z'))$ not in Q'_1 that is a simplicial vertex of G with $S_{v'}$ being a maximal element of \mathcal{S} .

Vertices v, v' are not adjacent as there are separated by Q_1 . \square

Algorithms LexBFS [8] and MCS [10] are linear time algorithms that were developed to find a simplicial vertex in a chordal graph. Most of the time the simplicial vertex that is found is not special. For example, on the graph with vertices a, b, c, d, e, f and edges $ab, cb, ca, da, db, ea, ec, af$, both algorithms LexBFS and MCS will always end on the simplicial vertex f , which is not special, if they start with one of d, b, c, e .

The proof of Theorem 1 can be turned into a polynomial time algorithm to find a special simplicial vertex in a chordal graph. We do not know how to find such a vertex in linear time.

3 Forbidden induced subgraphs

Figures 1, 2, 3, 4 and 5 give a list of minimal forbidden subgraphs for path graphs. In this section we prove that they are not path graphs and are minimal with this property. In the next sections, we prove that they are the only minimal forbidden induced subgraphs for path graphs.

Each graph in Figure 2 is obtained by adding a universal vertex to some minimal forbidden subgraph for interval graphs. Graphs $F_{10}(n)_{n \geq 8}$ are also forbidden in interval graphs. Graphs F_6 and $F_{10}(8)$ are from Renz [7, Figures 1 and 5].

The following lemma is clear.

Lemma 2 *In a path graph the neighborhood of every vertex is an interval graph.*

Lekkerkerker and Boland [5] proved that a graph is an interval graph if and only if it is chordal and contains no asteroidal triple. So we can deduce the following corollaries.

Corollary 1 *In a path graph the neighborhood of every vertex contains no asteroidal triple.*

Corollary 2 F_1, \dots, F_5 are not path graphs.

One could have hoped (as we did initially) that a chordal graph in which the neighborhood of every vertex contains no asteroidal triple is a path graph; but this is not true, as shown by graphs F_6, \dots, F_{15} .

Theorem 2 F_0, \dots, F_{15} are minimally not path graphs.

Proof. We give here only a brief outline of the proof and leave the details to the reader. Suppose that F_i is a path graph for any $i = 0, \dots, 15$. Try to build an intersection model for F_i , and realize that it is impossible. Then, for each vertex x , observe that $F_i \setminus x$ is a path graph. \square

Let \mathcal{P} be the class of graphs that do not contain any of F_0, \dots, F_{15} as an induced subgraph. We will prove that graphs in \mathcal{P} are exactly path graphs.

4 Co-special simplicial vertices

A *clique path tree* T of G is a clique tree of G such that, for each vertex v of G , the subtree induced by the set of cliques that contain v is a path. Clearly, a graph is a path graph if and only if it has a clique path tree.

A simplicial vertex v is *co-special* if S_v is a minimal element of \mathcal{S} and $G \setminus S_v$ has exactly two components (note that in that case S_v appears on exactly one label of any path tree of G).

Lemma 3 Let G be a minimally not path graph. Then either G is one of F_{11}, \dots, F_{15} or every simplicial vertex of G is co-special.

Proof. Suppose on the contrary that G is minimally not path graph, different from F_{11}, \dots, F_{15} , and there is a simplicial vertex v of G that is not co-special. All simplicial vertices of F_0, \dots, F_{10} are co-special, so G is not any of these graphs; moreover it does not contain any of them strictly (for otherwise G would not be minimally not path graph). So G is in \mathcal{P} .

Let T_0 be a clique path tree of $G(\mathcal{Q} \setminus Q_v)$. Let $Q'_v \in \mathcal{Q} \setminus Q_v$ be such that $S_v \subseteq Q'_v$. If $Q'_v = S_v$, then we can add v to Q'_v to obtain a clique path tree of G , a contradiction. So $Q'_v \neq S_v$ and $S_v \in \mathcal{S}$.

Let T' be the maximal subtree of T_0 that contains Q'_v and such that no label of the edges of T_0 is included in S_v . Vertex v is not co-special, so in T_0 there is an edge whose label is included in S_v , and so T' has strictly fewer vertices than T_0 . So $G(\mathcal{Q}(T') \cup \{Q_v\})$ is a path graph. Let T be a clique path tree of this graph.

We claim that Q_v is a leaf of T . If not, then there are at least two labels of T that are included in S_v , which contradicts the definition of T' (the number of times a label appears in a clique tree is constant).

Let T_1, \dots, T_l be the subtrees of $T_0 \setminus T'$ ($l \geq 1$). For $1 \leq i \leq l$, let $Q_i Q'_i$ be the edge between T_i and T' with $Q_i \in T_i$ and $Q'_i \in T'$ (remark that Q_1, \dots, Q_l are disjoint but maybe Q'_v, Q'_1, \dots, Q'_l are not). Let $S_i = Q_i \cap Q'_i$ and $v_i \in Q_i \setminus Q'_i$. Let $\mathcal{H} = (V_{\mathcal{H}}, E_{\mathcal{H}})$ be the intersection graph of S_1, \dots, S_l , that is, $V_{\mathcal{H}} = \{S_1, \dots, S_l\}$ and $E_{\mathcal{H}} = \{S_i S_j \mid S_i \cap S_j \neq \emptyset\}$. For $1 \leq i \leq l$, let $\mathcal{R}_i = \{S \in \mathcal{S}(T') \mid S_i \cap S \neq \emptyset \text{ and } S_i \setminus S \neq \emptyset\}$. Let $X = \{S_i \mid \mathcal{R}_i \neq \emptyset\}$.

Claim 1 \mathcal{H} contains no odd cycle.

Proof. Suppose on the contrary that $S_{i_1} \cdots S_{i_p} S_{i_1}$ is a cycle of \mathcal{H} of odd length $p = 2r + 1$ ($r \geq 1$). Let $I_j = S_{i_j} \cap S_{i_{j+1}}$, with $S_{i_{p+1}} = S_{i_1}$. Suppose that for some $j \neq k$ we have $I_j \cap I_k \neq \emptyset$; then there are at least three distinct cliques in $Q_{i_j}, Q_{i_{j+1}}, Q_{i_k}, Q_{i_{k+1}}$ with a common vertex, which contradicts the fact that T_0 is a clique path tree as these three cliques are not on a path of T_0 . For $1 \leq j \leq p$, let $s_j \in I_j$. By the preceding remark, the s_j 's are pairwise distinct. By the definition of T' , we have $S_j \subseteq S_v$ for each $1 \leq j \leq p$, so the s_j 's are all in Q_v and Q'_v . Let $v' \in Q'_v \setminus Q_v$. Let us consider the subgraph induced by $v, v', v_{i_1}, \dots, v_{i_p}, s_1, \dots, s_p$. Each of the non-adjacent vertices v and v' is adjacent to all of the clique formed by the s_j 's. Each vertex v_j is adjacent to s_{j-1} and s_j (with $s_0 = s_p$) and not to v . Vertex v' can have 0, 1 or 2 neighbors among the v_j 's. If v has 0 neighbor, then $v, v', v_{i_1}, \dots, v_{i_p}, s_1, \dots, s_p$ induce $F_{11}(4r + 4)_{r \geq 1}$. If v has 1 neighbor, then $v, v', v_{i_1}, \dots, v_{i_p}, s_1, \dots, s_p$ induce $F_{12}(4r + 4)_{r \geq 1}$. If v has 2 consecutive neighbors $v_{i_j}, v_{i_{j+1}}$, then $v, v', v_{i_j}, v_{i_{j+1}}, s_{j-1}, s_j, s_{j+1}$ induce F_2 . If v has 2 non-consecutive neighbors v_{i_j}, v_{i_k} , then we can assume that $1 \leq j < j+1 < k \leq p$ and $k - j$ is odd, $k - j = 2s + 1$ with $s \geq 1$, and then $v, v', v_{i_j}, \dots, v_{i_k}, s_j, \dots, s_{k-1}$ induce $F_{14}(4s + 5)_{s \geq 1}$, in all cases we obtain a contradiction. \square

Claim 2 \mathcal{H} contains no odd path between two vertices in X .

Proof. Suppose on the contrary that $S_{i_1} \cdots S_{i_p}$ is an odd path of \mathcal{H} between two vertices S_{i_1}, S_{i_p} in X , and assume that its length $p = 2k$ ($k \geq 1$) is minimal with this property. By the minimality, all interior vertices S_{i_j} ($1 < j < p$) are not in X . For $1 \leq j < p$, let $s_j = S_{i_j} \cap S_{i_{j+1}}$. As in the preceding claim, the s_j 's are pairwise distinct and lie in Q_v and Q'_v . Let P be the subpath $T(\{Q'_{i_1}, Q'_{i_2}\})$. If $p \neq 2$, then S_{i_2} is not in X , so $Q'_{i_3} = Q'_{i_1}$; then S_{i_3} is not in X , so $Q'_{i_4} = Q'_{i_2}$, and so on. So, the two extremities of P are $Q'_{i_1} = Q'_{i_3} = \dots = Q'_{i_{p-1}}$ and $Q'_{i_2} = Q'_{i_4} = \dots = Q'_{i_p}$.

Set S_{i_1} is in X , so we can choose $R_1 \in \mathcal{R}_{i_1}$ such that R_1 is the label of an edge $Q_{i_0}Q'_{i_0}$ of T' , with $Q'_{i_0} \in P$ and $Q_{i_0} \notin P$. Let $s_0 \in S_{i_1} \cap R_1$ and $v_{i_0} \in Q_{i_0} \setminus Q'_{i_0}$. Vertex s_0 belongs to Q_v and Q'_v by the definition of T' and is distinct from s_1, \dots, s_{p-1} because T' is a clique path tree. Vertex v_{i_0} is not adjacent to any of s_1, \dots, s_{p-1} because T' is a clique path tree. Similarly, define $R_p \in \mathcal{R}_{i_p}$ such that R_p is the label of an edge $Q_{i_{p+1}}Q'_{i_{p+1}}$ of T' , with $Q'_{i_{p+1}} \in P$ and $Q_{i_{p+1}} \notin P$. Let $s_p \in S_{i_p} \cap R_p$ and $v_{i_{p+1}} \in Q_{i_{p+1}} \setminus Q'_{i_{p+1}}$. Vertex s_p belongs to Q_v and Q'_v and is distinct from s_0, \dots, s_{p-1} . Vertex $v_{i_{p+1}}$ is not adjacent to any of s_1, \dots, s_{p-1} . As T' is a clique path tree, vertices s_p and s_0 are distinct, Q'_v lies between Q_{i_0}, Q'_{i_1} and between $Q_{i_{p+1}}, Q'_{i_p}$ along P . Cliques $Q_{i_0}, Q_{i_{p+1}}$ are not necessarily disjoint as for $Q'_v, Q'_{i_0}, Q'_{i_{p+1}}$.

Case 1 : $Q_{i_0} = Q_{i_{p+1}}$. Then $Q'_{i_0} = Q'_{i_{p+1}} = Q'$ and we can assume that $v_{i_0} = v_{i_{p+1}}$. By the definition of T' , there exists $r \in R_1 \setminus S_v$. Vertex r is distinct from s_0, s_p as it is not adjacent to v . Vertex r is adjacent to v_{i_0}, s_0, \dots, s_p and to no other vertex among $v, v_{i_1}, \dots, v_{i_p}$. So $v, r, v_{i_0}, \dots, v_{i_p}, s_0, \dots, s_p$ induce $\mathcal{F}_{13}(4k + 4)_{k \geq 1}$, a contradiction.

Case 2 : $Q_{i_0} \neq Q_{i_{p+1}}$. Then v_{i_0} and $v_{i_{p+1}}$ are distinct. We can choose vertices x_1, \dots, x_r , ($r \geq 1$) not in S_v and on the labels of $T'[\{Q_{i_0}, Q_{i_{p+1}}\}]$ such that they

form a chordless path $v_{i_0}-x_1-\dots-x_r-v_{i_{p+1}}$. These vertices are distinct from the previous ones, there are adjacent to s_0, \dots, s_p and not to $v, v_{i_1}, \dots, v_{i_p}$. If $r = 1$, then $v, v_{i_0}, \dots, v_{i_{p+1}}, s_0, \dots, s_p, x_1$ induce $\mathcal{F}_{14}(4k+5)_{k \geq 1}$. If $r = 2$, then $v, v_{i_0}, \dots, v_{i_{p+1}}, s_0, \dots, s_p, x_1, x_2$ induce $\mathcal{F}_{15}(4k+6)_{k \geq 1}$. If $r \geq 3$, then $v, v_{i_0}, v_{i_{p+1}}, s_0, s_p, x_1, \dots, x_r$ induce $\mathcal{F}_{10}(r+5)_{r \geq 3}$. A contradiction. \square

By the preceding claims, \mathcal{H} is a bipartite graph $(A, B, E_{\mathcal{H}})$ such that $X \subseteq A$.

We claim that all the subtrees T_i can be linked to T to get a clique path tree of G . For each $S_i \in A$, we add an edge $Q_v Q_i$ between T and T_i . This creates no illegal branching because A is a stable set of \mathcal{H} and Q_v is a leaf of T . For each $S_i \in B$, let $Q_i'' \in \mathcal{Q}(T)$ be such that $Q_i'' \cap S_i \neq \emptyset$ and the length of $T[\{Q_v, Q_i''\}]$ is maximal. For each $S_i \in B$, we have $\mathcal{R}_i = \emptyset$ so $S_i \subseteq Q_i''$ and we can add an edge $Q_i'' Q_i$ between T and T_i . This creates no illegal branching because B is a stable set of \mathcal{H} and by the definition of Q_i'' . Thus we obtain a clique path tree of G , a contradiction. \square

5 Characterization of path graphs

In this section we prove the main theorem, that is, the class of path graphs is exactly \mathcal{P} . We could not find a characterization similar to the one found by Lekkerkerker and Boland [5] for interval graphs (“an interval graph is a chordal graph with no asteroidal triple”). We know that in a path graph, the neighborhood of every vertex contains no asteroidal triple but the converse is not true. So we prove directly that a graph that does not contain any of the excluded subgraphs is a path graph.

Lemma 4 *In a graph in \mathcal{P} , the neighborhood of every vertex does not contain an asteroidal triple.*

Proof. It suffice to check that when a universal vertex is added to a minimal forbidden induced subgraph for interval graphs ([5]), then one obtains a graph that contains one of F_0, \dots, F_5, F_{10} . The easy details are left to the reader. \square

Given three non adjacent vertices a, b, c , we say that a is the *middle* of b, c if every path between b and c contains a vertex from $N(a)$. If a, b, c is not an asteroidal triple, then at least one of them is the middle of the others.

Let us say that a vertex x is *complete* to a set S of vertices if x is adjacent to every vertex in S .

Lemma 5 *In a chordal graph G with clique tree T , a vertex a is the middle of b, c if and only if for all cliques Q_b and Q_c such that $b \in Q_b$ and $c \in Q_c$, there is an edge of the path $T[\{Q_b, Q_c\}]$ such that a is complete to its label.*

Proof. Suppose that a is the middle of b, c . Let Q_b and Q_c , such that $b \in Q_b$ and $c \in Q_c$, and suppose there is no edges of $T[\{Q_b, Q_c\}]$ such that a is complete to its label. For each edge on $T[\{Q_b, Q_c\}]$, one can select a vertex that is not adjacent to a . Then

the set of selected vertices forms a path from b to c that uses no vertex from $N(a)$, a contradiction.

Suppose now that for all cliques Q_b and Q_c such that $b \in Q_b$ and $c \in Q_c$, there is an edge of the path $T[\{Q_b, Q_c\}]$ such that a is complete to its label. Suppose that there exists a path $x_0 \cdots x_r$, with $b = x_0$ and $c = x_r$ and none of the x_i 's is in $N(a)$. We can assume that this path is chordless. Then, for $1 \leq i \leq r$, let Q_i be a maximal clique containing x_{i-1}, x_i . Then Q_1, \dots, Q_r appear in this order along a subpath of T . On each $T[\{Q_i, Q_{i+1}\}]$ ($1 \leq i \leq r-1$), vertex a is not adjacent to x_i , so a is not complete to any label of $T[\{Q_1, \dots, Q_r\}]$, but Q_1 contains b and Q_r contains c , a contradiction. \square

Now we are ready to prove the main theorem. Part of the proof has been done in the previous section. Lemma 3 deals with the case where there exists a simplicial vertex that is the middle of two other vertices; now we have to look at the case where all simplicial vertices are not the middle of any pair of vertices.

Theorem 3 *A chordal graph is a path graph if and only if it does not contain any of F_0, \dots, F_{16} as an induced subgraph.*

Proof. By Theorem 2, a path graph is in \mathcal{P} . Suppose now that there exists a minimally not path graph G in \mathcal{P} . Graph G is chordal. By Theorem 1, there is a special simplicial vertex q of G . By Lemma 3, q is co-special. Let $Q = Q_v$ and $S_Q = S_v \in \mathcal{S}$. Let T_0 be a clique path tree of $G(\mathcal{Q} \setminus Q)$. Let $Q' \in \mathcal{Q} \setminus Q$ be such that $S_Q \subseteq Q'$. We add the edge QQ' to T_0 to obtain a clique tree T'_0 of G .

For each clique $L \in \mathcal{Q} \setminus \{Q, Q'\}$, let L' be the neighbor of L along $T_0[\{L, Q'\}]$. Let $S_L = L \cap L'$. Let \mathcal{S}_L be the set of labels of edges incident to L in T_0 . Let \mathcal{L} be the set of cliques of $\mathcal{Q} \setminus \{Q, Q'\}$ such that no element of $\mathcal{S}_L \setminus S_L$ contains S_L .

For each clique $L \in \mathcal{L}$, we define a subtree T_L of T'_0 , where T_L is the biggest subtree of T'_0 that contains Q' and for which no label is included in S_L . Subtree T_L contains Q as q is co-special, and so $S_Q \not\subseteq S_L$.

Claim 1 $L' \in T_L$.

Proof. Suppose on the contrary that $L' \notin T_L$. Then there exist $\overline{L}, \overline{L}'$ such that $L, L', \overline{L}, \overline{L}'$ appear in this order along $T_0[\{L, Q'\}]$, $\overline{L}\overline{L}'$ is an edge of T_0 , and $\overline{L} \notin T_L, \overline{L}' \in T_L$ (maybe $L' = \overline{L}$ and $\overline{L}' = Q'$). Let $\overline{S}_L = \overline{L} \cap \overline{L}'$. By the definition of T_L , we have $\overline{S}_L \subseteq S_L$. When we remove the edges $LL', \overline{L}\overline{L}'$ from T'_0 , there remains three connected subtrees. Let T_1 be the subtree containing L , T_2 be the subtree containing L', \overline{L} , and T_3 be the subtree containing \overline{L}', Q', Q . Let T_4 be the tree formed by T_1, T_3 plus the edge $L\overline{L}'$. Then T_4 is a clique tree of $G(\mathcal{Q}(T_4))$. The set $\mathcal{Q}(T_4)$ contains strictly fewer maximal cliques than \mathcal{Q} , so let T_5 be a clique path tree of $G(\mathcal{Q}(T_4))$.

We claim that there is an edge of T_5 that is incident to L and that has \overline{S}_L as a label. On the clique tree T_4 , the label \overline{S}_L is on the edge $L\overline{L}'$, so it is also a label of T_5 . So there is an edge with label \overline{S}'_L , incident to L such that $\overline{S}_L \subseteq \overline{S}'_L \subseteq L$. Suppose that $\overline{S}_L \subsetneq \overline{S}'_L$. Then there is an edge of T_1 or T_3 with label \overline{S}'_L . No label of T_1 can be \overline{S}'_L by

the definition of \mathcal{L} . All the labels of T_3 that are included in L are also included in \overline{S}_L , so no label of T_3 can be \overline{S}'_L . So $\overline{S}_L = \overline{S}'_L$.

Let $L\overline{L}''$ be an edge of T_5 incident to L with label \overline{S}_L (maybe $\overline{L}'' = \overline{L}'$). We can remove this edge from T_5 and replace it by the subtree T_2 and edges LL' , $\overline{L}\overline{L}''$. Thus we obtain a clique path tree of G , a contradiction. \square

Let \mathcal{L}^* be the subset of \mathcal{L} such that T_L is a strict subtree of $T'_0 \setminus L$. Let A be the set of vertices a of Q such that Q' is a vertex of T'_0 that is not a leaf. Then A is not empty, for otherwise T'_0 would be a clique path tree of G . For each $a \in A$, the leaves of T'_0 are in \mathcal{L} and we claim that at least one of them is in \mathcal{L}^* . Let $a \in A$ and let L_1, L_2 be the leaves of T'_0 . For $i = 1, 2$, let $l_i \in L_i \setminus S_{L_i}$. The three vertices q, l_1, l_2 are adjacent to a so they do not form an asteroidal triple by Lemma 4, so one of them is the middle of the others. Vertex q cannot be the middle of l_1, l_2 , for otherwise by Lemma 5 there would be an edge of $T_0[\{L_1, L_2\}]$ with a label included in S_Q , contradicting that q is co-special. So one of l_1, l_2 is the middle of the others (maybe both). By symmetry we can assume that l_1 is the middle of q, l_2 . So there is an edge of $T'_0[\{Q, L_2\}]$ with a label included in S_{L_1} . So T_{L_1} is a strict subtree of $T'_0 \setminus L_1$ and so $L_1 \in \mathcal{L}^*$. So \mathcal{L}^* is not empty.

We choose $L \in \mathcal{L}^*$ such that the subtree T_L is maximal. Let $S_{Q'}$ be the label of the edge of $T_0[\{L, Q'\}]$ that is incident to Q' . Vertex q is special and co-special, so there exists s_Q in $S_Q \setminus S_{Q'}$, and we have $s_Q \notin S_L$. We add the edge LL' to T_L to obtain a clique tree T'_L of $G(\mathcal{Q}(T_L) \cup \{L\})$. The subtree T'_L is a strict subtree of T'_0 , so we can consider a clique path tree T of $G(\mathcal{Q}(T'_L))$. We claim that L is a leaf of T . If not, then there are at least two labels of T that are included in S_L , which contradicts the definition of T_L .

We define $\mathcal{U}, \mathcal{V}, \mathcal{W}$ as follows :

$$\mathcal{U} = \{U \in \mathcal{Q} \setminus \mathcal{Q}(T'_L) \mid UL \text{ is an edge of } T_0\}$$

$$\mathcal{V} = \{V \in \mathcal{Q} \setminus \mathcal{Q}(T'_L) \mid \exists V' \in \mathcal{Q}(T_L), \text{ s.t. } VV' \text{ is an edge of } T_0 \text{ and } S_{VV'} \cap S_Q = \emptyset\}$$

$$\mathcal{W} = \{W \in \mathcal{Q} \setminus \mathcal{Q}(T'_L) \mid \exists W' \in \mathcal{Q}(T_L), \text{ s.t. } WW' \text{ is an edge of } T_0 \text{ and } S_{WW'} \cap S_Q \neq \emptyset\}$$

For each $U \in \mathcal{U} \cup \mathcal{V} \cup \mathcal{W}$, let T_U be the connected component of $T'_0 \setminus T'_L$ that contains U .

We claim that the S_V 's, with $V \in \mathcal{V} \cup \mathcal{W}$, are pairwise disjoint. For if they are not disjoint, then there is a vertex in $S_V \cap S_{V'}$ with V, V' in $\mathcal{V} \cup \mathcal{W}$. But then $S_V, S_{V'}$ are included in S_L , so this vertex is on three labels of T_0 that are not on a path, contradicting that T_0 is a clique path tree.

Let $\mathcal{U}_1 = \{U \in \mathcal{U} \mid \exists W \in \mathcal{W} \text{ such that } U \cap W \neq \emptyset\}$.

Claim 2 *There exists $U \in \mathcal{U}_1$, such that $S_U \setminus Q' \neq \emptyset$.*

Proof. We define $\mathcal{U}_{p \geq 1}, \mathcal{V}_{p \geq 0}$ as follows. Let $V_0 = W$ and for $p \geq 1$:

$$\mathcal{U}_p = \{U \in \mathcal{U} \setminus (\mathcal{U}_1 \cup \dots \cup \mathcal{U}_{p-1}) \mid \exists V \in \mathcal{V}_{p-1} \text{ such that } U \cap V \neq \emptyset\}$$

$$\mathcal{V}_p = \{V \in \mathcal{V} \setminus (\mathcal{V}_1 \cup \dots \cup \mathcal{V}_{p-1}) \mid \exists U \in \mathcal{U}_{p-1} \text{ such that } V \cap U \neq \emptyset\}$$

Let k be the smallest $k \geq 1$ such that there exists $U \in \mathcal{U}_k$ with $S_U \setminus Q' \neq \emptyset$. Let $k = \infty$ if it does not exist.

Suppose by contradiction that $k > 1$. For all $1 \leq p \leq k - 1$ and all $U \in \mathcal{U}_p$, we have $S_U \subseteq Q'$ and we define $U'' \in \mathcal{Q}(T)$ such that $U'' \cap S_U \neq \emptyset$ and the length of $T[\{L, U''\}]$ is maximal.

Suppose that there exists $U_p \in \mathcal{U}_p$, $1 \leq p \leq k - 1$, such that $S_{U_p} \not\subseteq U_p''$, and let p be minimal with this property. Let $V_0, \dots, V_{p-1}, U_1, \dots, U_p$ be such that $V_i \in \mathcal{V}_i$, $U_i \in \mathcal{U}_i$, $V_{i-1} \cap U_i \neq \emptyset$, and $U_i \cap V_i \neq \emptyset$. Let $u_i \in U_i \setminus S_{U_i}$, let $v_i \in V_i \setminus S_{V_i}$. Let x_1, \dots, x_r be such that $x_1 \in V_0 \cap U_1$, $x_2 \in U_1 \cap V_1$, \dots , $x_r \in V_{p-1} \cap U_p$ with $r = 2p - 1$. By the definition of \mathcal{V} , none of x_2, \dots, x_r is in Q . Let $x_0 \in V_0 \cap Q$ (maybe $x_0 = x_1$). As $x_0 \in S_{V_0}$, vertex x_0 is also in L . None of U_2, \dots, U_p can contain x_0 by the definition of \mathcal{U}_1 . Let Z be a clique of T_L such that $Z' \in T_0^{x_0}$, $S_{U_p} \subseteq Z'$, $S_{U_p} \cap Z \neq \emptyset$ and $S_{U_p} \setminus Z \neq \emptyset$ (such a Z exists because of U_p''). Let $z \in Z \setminus Z'$. Vertex Q' is on $T[\{L, Z'\}]$ as $S_{U_p} \subseteq Q'$. We choose vertices y_1, \dots, y_s on the labels of $T_0'[Z, Q]$ such that none of them is in S_L and $z-y_1-\dots-y_s-q$ is a chordless path.

If $Z \in T_0^{x_0}$, then let $b \in S_{U_p} \setminus Z$. As q is special and co-special, we have $S_Q \not\subseteq S_Z$, so let $c \in S_Q \setminus S_Z$. Then z, l, q form an asteroidal triple, with paths $z-y_1-\dots-y_s-q$ and $l-b-c-q$, and they lie in the neighborhood of x_0 , a contradiction. So $Z \notin T_0^{x_0}$. Let $x_{r+1} \in Z \cap U_p$. If $x_{r+1} \in Q$, then z, l, q form an asteroidal triple, with paths $z-y_1-\dots-y_s-q$ and $l-x_0-q$, and they lie in the neighborhood of x_{r+1} , a contradiction; so $x_{r+1} \notin Q$. The S_{U_i} 's are all included in Q' and so in S_L too. They are pairwise disjoint, for otherwise T_0 is not a clique path tree. Let $l \in L \setminus S_L$. Vertex l is not in any of the S_{U_i} 's, and l is adjacent to x_0, \dots, x_{r+1} but none of $u_1, \dots, u_p, v_0, \dots, v_{p-1}, y_1, \dots, y_s, z, q$.

Suppose that one of x_1, x_0 is in $V_0 \cap U_1 \cap Q$. Then we can assume that $x_0 = x_1$. Let $L_1, L_2 \in \mathcal{L}$ be the leaves of $T_0^{x_0}$ with $U_1 \in T_0[\{L_1, Q'\}]$ and $V_0 \in T_0[\{L_2, Q'\}]$. Every edge of T_L is not included in S_L and so is not included in S_{L_1} . So T_{L_1} contains T_L . If $L_1 \in \mathcal{L}^*$, then $T_{L_1} = T_L$ by the maximality of T_L . But then L_1' is not in T_{L_1} , which contradicts Claim 1. So $L_1 \notin \mathcal{L}^*$ and $L_2 \in \mathcal{L}^*$. Every edge of T_L is not included in S_L and so is not included in S_{V_0} and also not in S_{L_2} . So T_{L_2} contains T_L . Vertex $x_{r+1} \notin S_{V_0}$, so $x_{r+1} \notin S_{L_2}$, so $S_L \not\subseteq S_{L_2}$, so T_{L_2} contains L , which contradicts the maximality of T_L . So $x_0 \neq x_1$, $x_0 \notin U_1$, $x_1 \notin Q$.

If $s = 1$, then $u_1, \dots, u_p, v_0, \dots, v_{p-1}, x_0, \dots, x_{r+1}, y_1, q, z, l$ induce $F_{14}(4p + 5)_{p \geq 1}$. If $s = 2$, then $u_1, \dots, u_p, v_0, \dots, v_{p-1}, x_0, \dots, x_{r+1}, y_1, y_2, q, z, l$ induce $F_{15}(4p + 6)_{p \geq 1}$. If $s \geq 3$, then $x_0, x_{r+1}, y_1, \dots, y_s, q, z, l$ induce $F_{10}(s + 5)_{s \geq 3}$. A contradiction.

Therefore we have $S_U \subseteq U''$ for every $U \in \mathcal{U}_p$, $1 \leq p \leq k - 1$.

Suppose that k is infinite. Then, the U_i 's are pairwise disjoint, for otherwise T_0 is not a clique path tree as $S_{U_i} \subseteq Q'$. For each $V \in \bigcup_{p \geq 0} \mathcal{V}_p$, we add the edge VL between T_V and T . For each $U \in \bigcup_{p \geq 1} \mathcal{U}_p$, we add the edge UU'' between T_U and T . For each $U \in \mathcal{U} \setminus (\bigcup_{p \geq 1} \mathcal{U}_p)$, we add the edge UL between T_U and T . For each $V \in \mathcal{V} \setminus (\bigcup_{p \geq 1} \mathcal{V}_p)$, we define $V'' \in \mathcal{Q}(T)$ such that $V'' \cap S_V \neq \emptyset$ and the length of $T[\{L, V''\}]$ is maximal. By the definition of \mathcal{V} , we have $S_V \cap Q = \emptyset$, so $V'' \neq Q$, so V'' is a vertex of T_L on $T_0[L, V]$ so it contains all S_V as $S_V \subseteq S_L$. Then we can add the edge VV'' between T_V and T to obtain a clique path tree of G , a contradiction.

So k is finite and ≥ 2 .

As before, let $V_0, \dots, V_{k-1}, U_1, \dots, U_k$ be such that $V_i \in \mathcal{V}_i$, $U_i \in \mathcal{U}_i$, $V_{i-1} \cap U_i \neq \emptyset$, and $U_i \cap V_i \neq \emptyset$. Let $u_i \in U_i \setminus S_{U_i}$, let $v_i \in V_i \setminus S_{V_i}$. Let x_1, \dots, x_r be such that $x_1 \in V_0 \cap U_1$, $x_2 \in U_1 \cap V_1$, \dots , $x_r \in V_{k-1} \cap U_k$ with $r = 2k - 1$. None of x_2, \dots, x_r is in Q . Let $x_0 \in V_0 \cap Q$.

Suppose that one of x_1, x_0 is in $V_0 \cap U_1 \cap Q$. Then we can assume that $x_0 = x_1$. Let $L_1, L_2 \in \mathcal{L}$ be the leaves of $T_0^{x_0}$ with $U_1 \in T_0[\{L_1, Q'\}]$ and $V_0 \in T_0[\{L_2, Q'\}]$. As before, we have $L_2 \in \mathcal{L}^*$. Every edge of T_L is not included in S_L so it is also not in S_{V_0} and not in S_{L_2} . So T_{L_2} contains T_L . We have $k \geq 2$, so U_2 exists and there is a vertex $x_2 \in U_1 \cap V_1$. Vertex x_2 is not in S_{V_0} , so it is not in S_{L_2} , so $S_L \not\subseteq S_{L_2}$, so T_{L_2} contains L , which contradicts the maximality of T_L . So $x_0 \neq x_1$, $x_0 \notin U_1$, $x_1 \notin Q$.

Let $s_{U_k} \in S_{U_k} \setminus Q'$. Vertex s_{U_k} is not adjacent to any of $q, s_Q, v_0, \dots, v_{k-1}$ because $s_{U_k} \notin Q'$, and by the minimality of k , vertex s_{U_k} is not adjacent to u_1, \dots, u_{k-1} . Then $u_1, \dots, u_k, v_0, \dots, v_{k-1}, x_0, \dots, x_r, s_{U_k}, s_Q, q$ induce $F_{16}(4k+3)_{k \geq 2}$. A contradiction. \square

Let $U \in \mathcal{U}_1$ be such that $S_U \setminus Q' \neq \emptyset$. Let $s_U \in S_U \setminus Q'$. Vertex s_U is not adjacent to s_Q . Let $u \in U \setminus S_U$. Let $W \in \mathcal{W}$ be such that $U \cap W \neq \emptyset$. Let $w \in W \setminus S_W$.

Claim 3 $S_W = S_L$.

Proof. Suppose $S_W \neq S_L$. Then $S_W \subsetneq S_L$. By the definition of \mathcal{W} , there exists $a \in W \cap Q$ and so $a \in L$.

Suppose $S_W \subseteq S_U$. Then a is in U . Let $L_1, L_2 \in \mathcal{L}$ be the leaves of T_0^a with $U \in T_0[\{L_1, Q'\}]$ and $W \in T_0[\{L_2, Q'\}]$. Every edge of T_L is not included in S_L so it is also not in S_{L_1} . By the definition of \mathcal{L} , the set S_L is not included in S_{L_1} . So T_{L_1} is strictly greater than T_L . So $L_1 \notin \mathcal{L}^*$, so $L_2 \in \mathcal{L}^*$. Every edge of T_L is not included in S_L , so it is also not in S_W and not in S_{L_2} . The same goes for S_L by the hypothesis. So T_{L_2} is strictly greater than T_L , which contradicts the definition of L . So $S_W \not\subseteq S_U$. Let b be a vertex of $S_W \setminus S_U$.

Vertex s_U is in $S_U \setminus Q'$, so $S_U \not\subseteq S_W$. Every labels of the edges of T_L is not included in S_L , so it is also not in S_W . So we can choose vertices x_1, \dots, x_r on the labels of $T_0'[\{U, Q\}]$ such that none of the x_i 's is in S_W , $x_1 \in U$, $x_r \in Q$ and $x_1 \cdots x_r$ is a chordless path.

Then $w-b-s_Q-q$ is a path from w to q that avoids $N(u)$; and $u-x_1 \cdots x_r-q$ is a path from u to q that avoids $N(w)$. So q, u, w form an asteroidal triple. By Lemma 4, we have $Q \cap U \cap V = \emptyset$. So $a \notin U$.

Let $c \in U \cap W$ (by the definition of \mathcal{U}_1). Vertex c is not in Q . Let $r = 1$. Then x_1 is different from s_U and s_Q , and $q, u, w, a, c, s_Q, s_U, x_1$ induce F_8 . Let $r = 2$. If x_1 is adjacent to s_Q , then $q, u, w, a, c, s_Q, s_U, x_1$ induce F_9 ; and if x_1 is not adjacent to s_Q , then $q, u, w, a, c, s_Q, x_1, x_2$ induce F_9 . If $r \geq 3$, then $q, u, w, a, c, x_1, \dots, x_r$ induce $F_{10}(r+5)_{r \geq 3}$. In all cases we obtain a contradiction. \square

Claim 4 $W \in \mathcal{L}^*$

Proof. Suppose that $W \notin \mathcal{L}^*$. Let $a \in W \cap Q$, we have $a \in L$. Let $L_1, L_2 \in \mathcal{L}$ be the leaves of T_0^a , with $L \in T_0[\{L_1, Q'\}]$ and $W \in T_0[\{L_2, Q'\}]$. Let $K \in T_0[\{L_2, W\}] \cap \mathcal{L}$ be such that the length of $T_0[\{K, W\}]$ is minimal. If $W \in \mathcal{L}$, then $T_W = T_L$ and $W \in \mathcal{L}^*$, a contradiction. So $W \notin \mathcal{L}$, so $W \neq K$. The edges of T_L are not included in S_L , so they are also not in S_W and not in S_K . So T_K contains T_L . If $K \in \mathcal{L}^*$, then $T_K = T_L$ by the maximality of T_L , which contradicts Claim 1; so $K \notin \mathcal{L}^*$. So $T_K = T_0' \setminus K$ and the labels of $T_0' \setminus K$ are not included in S_K , so $S_W \not\subseteq S_K$. Let X be the edge of $T_0[\{W, K\}]$ such that X' contains S_W and X does not (maybe $X' = W$, $X = K$). The set S_X contains a but not all of S_X' . So no element of $S_{X'} \setminus S_X$ contains $S_{X'}$. So $X' \in \mathcal{L}$, a contradiction to the definition of K . \square

By claim 4, we have $W \in \mathcal{L}^*$. Then $T_W = T_L$ is also maximal and what we have proved for L can be done for W . By Claim 2 we know that there exists $X \notin T_L$ such that XW is an edge of T_0 with $S_X \cap S_W \neq \emptyset$ and $S_X \setminus Q' \neq \emptyset$. Let $x \in X \setminus W$. Let $s_X \in S_X \setminus Q'$. Vertex s_X is not in S_W , for otherwise it would also be in S_L and in Q' . Vertex s_Q is not in S_L , so not in S_W . So s_Q, s_X are not adjacent. We distinguish between two cases.

Case 1: $U \cap X = \emptyset$. Let $a \in U \cap W$, so $a \notin X$. Suppose $a \notin Q$. If there exists $b \in X \cap Q$ then b is also in L and $q, u, x, s_Q, s_U, s_X, a, b$ induces F_6 , a contradiction. So there is no vertex in $X \cap Q$. Let $c \in W \cap Q$, we have $c \in L$ and $c \notin X$. Let $d \in X \cap S_W$, we have $d \notin Q$. If c is adjacent to U , then $q, u, x, s_Q, s_U, s_X, c, d$ induces F_6 , if c misses u , then $q, u, x, s_Q, s_U, s_X, a, c, d$ induce F_7 , a contradiction. So $a \in Q$. Let $e \in X \cap S_W$. If $e \notin Q$ then $q, u, x, s_Q, s_U, s_X, a, e$ induce F_6 , a contradiction, so $e \in Q$. Let $f \in S_W \setminus S_Q$ (as q is special and co-special); maybe f is in U or X , but not in both. Then $q, u, x, s_U, s_X, a, e, f$ induce F_9 or $F_{10}(8)$, according to whether f is adjacent to none or exactly one of u, x , a contradiction.

Case 2: $U \cap X \neq \emptyset$. Suppose $U \cap X \cap Q \neq \emptyset$. Let $z \in U \cap X \cap Q$. Let $L_1, L_2 \in \mathcal{L}$ be the leaves of T_0^z . Let $i \in \{1, 2\}$ be such that $L_i \in \mathcal{L}^*$. The edges of T_L are not included in $S_L = S_W$, thus also not in S_{L_i} . So T_{L_i} contains T_L , so $T_{L_i} = T_L$ by maximality of T_L . But this contradicts Claim 1. So $U \cap X \cap Q = \emptyset$.

Let $a \in U \cap X$. Vertex a is not in Q . Let $b \in W \cap Q$. If $b \notin X \cup U$, then $q, u, x, s_Q, s_U, s_X, a, b$ induce F_6 , a contradiction. If $b \in X$, then $b \notin U$. Let $c \in S_W \setminus S_X$. If $c \in U \setminus Q$, then $q, u, x, s_Q, s_U, s_X, b, c$ induce F_6 . If $c \in Q \setminus U$, then $q, u, x, s_Q, s_U, s_X, a, c$ induce F_6 . If $c \in U \cap Q$, then $q, u, x, s_Q, s_U, s_X, a, b, c$ induce F_8 . If $c \notin U \cup Q$, then $q, u, x, s_Q, s_U, a, b, c$ induce $F_{10}(8)$. A contradiction. If $b \in U$, then $b \notin X$. Let $d \in S_L \setminus S_U$. If $d \in X \setminus Q$, then $q, u, x, s_Q, s_U, s_X, b, d$ induce F_6 . If $d \in Q \setminus X$, then $q, u, x, s_Q, s_U, s_X, a, d$ induce F_6 . If $d \in U \cap Q$, then $q, u, x, s_Q, s_U, s_X, a, b, d$ induce F_8 . If $d \notin U \cup Q$, then $q, u, x, s_Q, s_U, a, b, d$ induce $F_{10}(8)$. A contradiction.

This ends the proof of Theorem 3. \square

6 Recognition algorithm

Gavril [4] and Schäffer [9] gave polynomial time algorithms to recognize path graphs. The characterization that we give in this paper suggests a new recognition algorithm, which takes any graph G as input and either builds a clique path tree for G or finds one of F_0, \dots, F_{16} . We have not analyzed the exact complexity of such a method but it will give a new polynomial algorithm to recognize path graphs.

There are linear time recognition algorithms for interval graphs [1] and triangulated graphs [8] but surprisingly not for path graphs. One can hope that the work presented here will be helpful in the search for a linear time recognition algorithm for path graphs.

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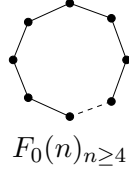


Figure 1: Forbidden subgraphs with no simplicial vertices

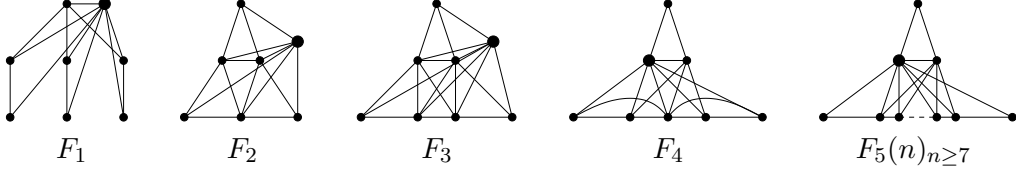


Figure 2: Forbidden subgraphs with a universal vertex

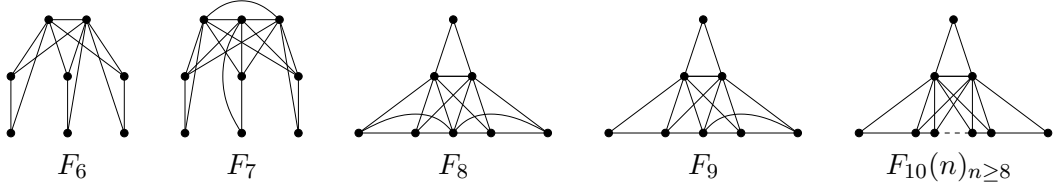


Figure 3: Forbidden subgraphs with no universal vertex and exactly three simplicial vertices

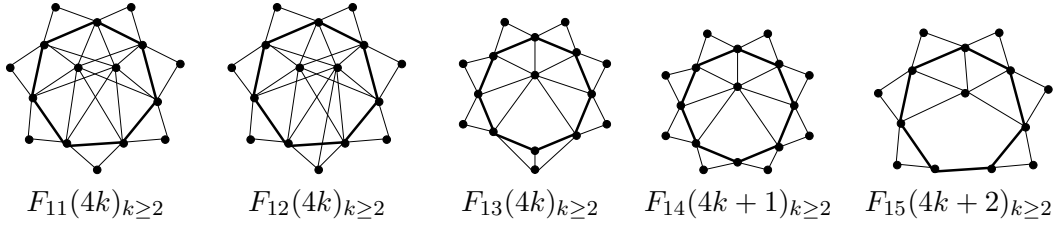


Figure 4: Forbidden subgraphs with at least one simplicial vertex that is not co-special. (bold edges form a clique)

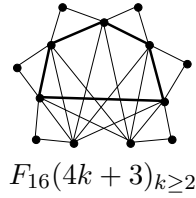


Figure 5: Forbidden subgraphs with ≥ 4 simplicial vertices that are all co-special. (bold edges form a clique)