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EXAMPLES OF SMOOTH MAPS WITH FINITELY MANY CRITICAL POINTS IN DIMENSIONS $(4, 3)$, $(8, 5)$ AND $(16, 9)$

LOUIS FUNAR, CORNEL PINTEA, AND PING ZHANG

ABSTRACT. We consider manifolds M^{2n} which admit smooth maps into a connected sum of $S^1 \times S^n$ with only finitely many critical points, for $n \in \{2, 4, 8\}$, and compute the minimal number of critical points.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let $\varphi(M^m, N^n)$ denote the minimal number of critical points of smooth maps between the manifolds M^m and N^n . When superscripts are specified they denote the dimension of the respective manifolds. We are interested below in the case when $m \geq n \geq 2$ and the manifolds are compact. The main problem concerning φ is to characterize those pairs of manifolds for which it is finite non-zero and then to compute its value (see [1]).

In [1] the authors found that, in small codimension $0 \leq m - n - 1 \leq 3$, if $\varphi(M^m, N^{n+1})$ is finite then $\varphi(M^m, N^{n+1}) \in \{0, 1\}$, except for the exceptional pairs of dimensions $(m, n + 1) \in \{(2, 2), (4, 3), (4, 2), (5, 2), (6, 3), (8, 5)\}$. Moreover, under the finiteness hypothesis, $\varphi(M, N) = 1$ if and only if M is the connected sum of a smooth fibration over N with an exotic sphere and not a fibration itself. There are two essential ingredients in this result. First, there are local obstructions to the existence of isolated singularities, namely the germs of smooth maps $\mathbb{R}^m \rightarrow \mathbb{R}^n$ having an isolated singularity at origin are actually locally topologically equivalent to a projection. Thus, these maps are topological fibrations. Second, singular points located in a disk cluster together.

The simplest exceptional case is that of (pairs of) surfaces, which is completely understood by elementary means (see [2] for explicit computations). Very little is known for the other exceptional and generic (i.e. $m - n - 1 \geq 4$) cases and even the case of pairs of spheres is unsettled yet. In particular, it is not known whether φ is bounded in terms only of the dimensions, in general.

The aim of this note is to find non-trivial examples in dimensions $(4, 3)$, $(8, 5)$ and $(16, 9)$ inspired by the early work of Antonelli ([3]). The smooth maps considered in [3] are so-called Montgomery-Samelson fibrations with finitely many singularities where several fibers are pinched to points. According to [9] these maps should be locally topologically equivalent to a cone over the Hopf fibration, in a neighborhood of a critical point.

The main ingredient of our approach is the existence of global obstructions of topological nature to the clustering of genuine critical points in these dimensions. This situation seems rather exceptional and it permits us to obtain the precise value of φ using only basic algebraic topology.

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Our computations show that φ can take arbitrarily large even values. Thus the behaviour of φ is qualitatively different from what it was seen before in [1].

Theorem 1.1. *Let $n \in \{2, 4, 8\}$, $e \geq c \geq 0$, with $c \neq 1$, and Σ^{2n} be a homotopy $2n$ -sphere. Then*

$$\varphi(\Sigma^{2n} \#_e S^n \times S^n \#_c S^1 \times S^{2n-1}, \#_c S^1 \times S^n) = 2e - 2c + 2$$

Here $\#_c S^1 \times S^n = S^{n+1}$ if $c = 0$ and $\#_e S^n \times S^n \#_c S^1 \times S^{2n-1} = S^{2n}$ if $e = c = 0$.

Remark 1.1. For $n \in \{4, 8\}$ the closed orientable simply connected $2n$ -manifolds having finite $\varphi(M^{2n}, S^{n+1}) > 1$ are of the form $\Sigma^{2n} \#_r S^n \times S^n$, where Σ^{2n} is a homotopy $2n$ -sphere. It appears that the singularities arising this way are confined in pairs since $\varphi(\Sigma^{2n} \#_r S^n \times S^n, S^{n+1}) = 2r + 2$ is always even.

The structure of the proof of the Theorem is as follows. We prove Proposition 2.1 which yields a lower bound for the number of critical values derived from topological obstructions of algebraic nature. The existence of a non-trivial lower bound is not obvious since one might imagine that several singularities could combine into a single more complicated singularity. The next step taken in section 3 is to construct explicit smooth maps with any even number of singularities. This follows by taking fiber sums of elementary blocks of such maps coming naturally from Hopf fibrations. Then Corollary 3.1 concludes the proof.

Remark 1.2. Observe that $S^1 \times S^{2n-1}$ fibers over $S^1 \times S^n$, so that the formula in the Theorem is still valid for $e = 0$ and $c = 1$. However, we do not know how to evaluate φ when $e \leq c - 1$. The present methods do not work for $e \geq c = 1$ either.

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2. A LOWER BOUND FOR THE NUMBER OF CRITICAL VALUES

Proposition 2.1. *For any dimension $n \geq 2$ and non-negative integers e and c , with $c \neq 1$ we have*

$$\varphi(\Sigma^{2n} \#_e S^n \times S^n \#_c S^1 \times S^{2n-1}, \#_c S^1 \times S^n) \geq 2e - 2c + 2$$

Here $\#_c S^1 \times S^n = S^{n+1}$ if $c = 0$ and $\#_e S^n \times S^n \#_c S^1 \times S^{2n-1} = S^{2n}$ if $e = c = 0$ and Σ^{2n} denotes a homotopy $2n$ -sphere.

We will prove, more generally, the following:

Proposition 2.2. *Let M^{2n} and N^{n+1} be closed orientable manifolds and $n \geq 2$. Assume that $\pi_1(M) \cong \pi_1(N)$ is a free group $\mathbb{F}(c)$ on c generators, $c \neq 1$ (with $\mathbb{F}(0) = 0$) and that $\pi_j(M) = \pi_j(N) = 0$, for $2 \leq j \leq n - 1$. Then $\varphi(M, N) \geq \beta_n(M) - 2c + 2$, where β_k denotes the k -th Betti number.*

Proof. Let $B = B(f)$ denote the set of critical values of a smooth map $f : M \rightarrow N$. We will prove that the cardinality $|B|$ of $B(f)$ satisfies $|B| \geq \beta_n(M) - 2c + 2$, which will imply our

claim. Set $V = f^{-1}(B(f)) \subset M$. We can assume that f has finitely many critical points, since otherwise the claim of Proposition 2.2 would be obviously verified.

The following two Lemmas do not depend on the homotopy assumptions of Proposition 2.2.

Lemma 2.1. *If A is a nonempty finite subset of a connected closed manifold N^{n+1} , then $\beta_n(N \setminus A) = \beta_n(N) + |A| - 1$.*

Proof. Clear from the homology exact sequence of the pair $(N, N \setminus A)$. \square

Lemma 2.2. *If M^{n+q+1} and N^{n+1} are smooth manifolds and $f : M \rightarrow N$ is a smooth map with finitely many critical points, then the inclusions $M \setminus V \hookrightarrow M$ and $N \setminus B \hookrightarrow N$ are n -connected.*

Proof. This is obvious for $N \setminus B \hookrightarrow N$. It remains to prove that $\pi_k(M, M \setminus V) \cong 0$ for $k \leq n$. Take $\alpha : (D^k, S^{k-1}) \rightarrow (M, M \setminus V)$ to be an arbitrary smooth map of pairs. Since the critical set $C(f)$ of f is finite and contained in V , there exists a small homotopy of α relative to the boundary such that the image $\alpha(D^k)$ avoids $C(f)$. By compactness there exists a neighborhood U of $C(f)$ consisting of disjoint balls centered at the critical points such that $\alpha(D^k) \subset M \setminus U$. We can arrange by a small isotopy that V becomes transversal to ∂U .

Observe further that $V \setminus U$ consists of regular points of f and thus it is a properly embedded sub-manifold of $M \setminus U$. General transversality arguments show that α can be made transverse to $V \setminus U$ by a small homotopy. By dimension counting this means that $\alpha(D^k) \subset M \setminus U$ is disjoint from V and thus the class of α in $\pi_k(M, M \setminus V)$ vanishes. \square

The restriction of f to $M \setminus V$ is a proper submersion and hence, according to Ehresmann, a locally trivial smooth fibration over $N \setminus B$ with compact smooth fiber F^{n-1} (see [4]).

Lemma 2.3. *Assume that $c \neq 1$. Then the generic fiber F is homotopy equivalent to the $(n-1)$ -sphere.*

Proof. Consider the last terms of the homotopy exact sequence of this fibration:

$$\rightarrow \pi_1(M \setminus V) \xrightarrow{f_*} \pi_1(N \setminus B) \xrightarrow{p} \pi_0(F) \rightarrow \pi_0(M \setminus V) \rightarrow \pi_0(N \setminus B)$$

From Lemma 2.2 $M \setminus V$ and $N \setminus B$ are connected and $\pi_1(M \setminus V) \cong \pi_1(N \setminus B) \cong \mathbb{F}(c)$. The claim is obvious for $c = 0$. Let then $c \geq 2$. If F has $d \geq 2$ connected components then the kernel $\ker p$ of p is a finite index proper subgroup of the free non-abelian group $\mathbb{F}(c)$. Thus, by the Nielsen-Schreier theorem, $\ker p$ is a free group of rank $d(c-1) + 1$, where d is the number of components of F , and hence of rank larger than c . On the other hand, by exactness of the sequence above, $\ker p$ is also the image of f_* and thus it is a group of rank at most c . This contradiction shows that F is connected.

If $n = 2$ then F is a circle, as claimed.

Let now $n > 2$. We obtained above that f_* is surjective. Since finitely generated free groups are Hopfian any surjective homomorphism $\mathbb{F}(c) \rightarrow \mathbb{F}(c)$ is also injective. Since $\pi_2(N \setminus B) \cong \pi_2(N) = 0$ and f_* is injective we derive that $\pi_1(F) = 0$. The remaining terms of the homotopy exact sequence of the fibration and Lemma 2.2 show then that $\pi_j(F) = 0$ for $2 \leq j \leq n-1$. Thus F is a homotopy sphere. \square

Lemma 2.4. (1) *We have $H_1(N \setminus B) \cong \mathbb{Z}^c$, $H_n(N \setminus B) = \mathbb{Z}^{|B|+c-1}$ and $H_{n+1}(N \setminus B) = 0$.*

(2) *If $n > 2$ then $H_{n-1}(M \setminus V) = 0$.*

(3) *The homomorphism $H_n(M \setminus V) \rightarrow H_n(M)$ induced by the inclusion map is surjective.*

Proof. The first two assertions are consequences of Lemma 2.1, Lemma 2.2 and standard algebraic topology. For instance, $H_1(N \setminus B) \cong H_1(N) = \mathbb{Z}^c$. The last claim follows from Lemma 2.2 and the long exact sequence in homology of the pair $(M, M \setminus V)$. \square

Lemma 2.5. *If $c \neq 1$ then the rank of $H_n(M \setminus V)$ is $2c + |B| - 2$.*

Proof. The Gysin sequence of this spherical fibration reads:

$$\rightarrow H_m(M \setminus V) \rightarrow H_m(N \setminus B) \rightarrow H_{m-n}(N \setminus B) \rightarrow H_{m-1}(M \setminus V) \rightarrow$$

Consider the exact subsequence

$$H_{n+1}(N \setminus B) \rightarrow H_1(N \setminus B) \rightarrow H_n(M \setminus V) \rightarrow H_n(N \setminus B) \rightarrow H_0(N \setminus B) \rightarrow H_{n-1}(M \setminus V)$$

If $n > 2$ then the first and the last terms vanish.

The Euler characteristic of this subsequence is zero by exactness and thus the rank of $H_n(M \setminus V)$ is $2c + |B| - 2$ by Lemma 2.4.

When $n = 2$, we can complete the exact sequence above by adding one more term to its right, namely $H_1(M \setminus B) \xrightarrow{f_*} H_1(N \setminus B)$. However, f_* is actually the map induced in homology by the isomorphism $f_* : \pi_1(M) \rightarrow \pi_1(N)$ and thus an isomorphism itself. The argument with the Euler characteristic can be applied again and yields the claimed result. \square

From Lemma 2.5 and Lemma 2.4 (3) we derive that

$$2c + |B| - 2 \geq \beta_n(M)$$

and the proposition is proved. \square

Corollary 2.1. If M^{2n} is a smooth $(n-1)$ -connected closed manifold, then

$$\varphi(M, \Sigma^{n+1}) \geq \beta_n(M) + 2,$$

where Σ^{n+1} is a homotopy sphere.

Remark 2.1. The present approach does not work for $c = 1$. In fact, fibers might have several connected components, each one being a homotopy sphere. Without an upper bound of their number the Leray-Serre spectral sequence leads only to a trivial lower bound for the number of critical values.

3. FIBER SUMS OF SUSPENSIONS OF HOPF FIBRATIONS

Recall from [1] that $\varphi(S^{2n}, S^{n+1}) = 2$ if $n = 2, 4$ or 8 . This is realized by taking suspensions of both spaces in the Hopf fibration $h : S^{2n-1} \rightarrow S^n$, where $n = 2, 4$ or 8 , and then smoothing the new map at both ends. The extension $H : S^{2n} \rightarrow S^{n+1}$ has precisely two critical points. This is also the basic example of a Montgomery-Samelson fibration with finitely many singularities, as considered in [3].

Our aim is to define fiber sums of Hopf fibrations leading to other examples of pairs of manifolds with finite φ . Identify S^{n+1} (and respectively S^{2n}) with the suspension of S^n (respectively S^{2n-1}) and thus equip it with the coordinates (x, t) , where $|x|^2 + t^2 = 1$, and $t \in [-1, 1]$. We call the coordinate t the height of the respective point. The suspension H is then given by:

$$H(x, t) = \left(|x|h\left(\frac{x}{|x|}\right), t \right)$$

Pick up a number of points $x_1, x_2, \dots, x_k \in S^{n+1}$ and their small enough disk neighborhoods $x_i \in D_i \subset S^{n+1}$, such that:

- (1) the projections of D_i on the height coordinate axis are disjoint;
- (2) the D_i 's do not contain the two poles, i.e. their projections on the height axis are contained in the open interval $(-1, 1)$.

Let A_k be the manifold with boundary obtained by deleting from S^{n+1} of the interiors of the disks D_i , for $1 \leq i \leq k$. Let also B_k denote the preimage $H^{-1}(A_k) \subset S^{2n}$ by the suspended Hopf map. Since H restricts to a trivial fibration over the disks D_i it follows that B_k is a manifold, each one of its boundary component being diffeomorphic to $S^{n-1} \times S^n$. Moreover, the boundary components are endowed with a natural trivialization induced from D_i .

Let now Γ be a finite connected graph. To each vertex v of valence k we associate a block $(B_v, A_v, H|_{B_v})$ which will be denoted $(B_k, A_k, H|_{B_k})$ when we want to emphasize the dependence on the number of boundary components. Each boundary component of A_v or B_v corresponds to an edge incident to the vertex v . We define the fiber sum along Γ as the following triple $(B_\Gamma, A_\Gamma, H_\Gamma)$:

- (1) A_Γ is the result of gluing the manifolds with boundary A_v , associated to the vertices v of Γ , by identifying, for each edge e joining the vertices v and w (which might coincide) the pair of boundary components in A_v and A_w corresponding to the edge e ;
- (2) B_Γ is the result of gluing the manifolds with boundary B_v , associated to the vertices v of Γ , by identifying, for each edge e joining the vertices v and w (which might coincide) the boundary components in B_v and B_w corresponding to the pair of boundary components in A_Γ associated to e ;
- (3) As the boundary components are identified the natural trivializations of the boundary components of B_v agree in pairs. Thus the maps H_v induce a well-defined map $H_\Gamma : B_\Gamma \rightarrow A_\Gamma$.

Proposition 3.1. *The map $H_\Gamma : B_\Gamma \rightarrow A_\Gamma$ has $2m$ critical points, where m is the number of vertices of Γ .*

Proof. Clear, by construction. □

We say that Γ has c independent cycles if the rank of $H_1(\Gamma)$ is c . This is equivalent to ask that Γ becomes a tree only after removal of at least c edges. Moreover, $c = e - m + 1$ where e denotes the number of edges.

Proposition 3.2. *If Γ has e edges and c cycles, i.e. $e - c + 1$ vertices, then B_Γ is diffeomorphic to $\sharp_e S^n \times S^n \sharp_c S^1 \times S^{2n-1}$, while A_Γ is diffeomorphic to $\sharp_c S^1 \times S^n$. For $c = 0$ the manifold $\sharp_c S^1 \times S^n$ is actually S^{n+1} .*

Proof. The sub-blocks A_k are diffeomorphic to the connected sum of k copies of disks D^{n+1} out of their boundaries. When gluing together two such distinct sub-blocks (due to the presence of an edge in Γ joining the respective vertices) the respective pair of disks leads to a sphere factor S^{n+1} . When gluing all sub-blocks in the pattern of the graph Γ the only non-trivial contribution comes from the cycles. Each cycle of Γ introduces a 1-handle. Thus the manifold A_Γ is $\sharp_c S^1 \times S^n$.

The sub-blocks B_k are diffeomorphic to the connected sum of k copies of the product $S^n \times D^n$ out of their boundaries. In fact, one obtains B_k by deleting out k copies of $H^{-1}(D_i)$; each $H^{-1}(D_i)$ is a tubular neighborhood of the (generic) fiber of H and thus diffeomorphic to $S^{n-1} \times D^{n+1}$.

When $k = 1$ the generic fiber of H is an S^{n-1} embedded in S^{2n} . Although the fiber of the Hopf fibration is knotted in S^{2n-1} (e.g. it is a trefoil knot when $n = 2$), its image in the suspension sphere S^{2n} is unknotted. This is an immediate consequence of Haefliger's classification of smooth embeddings. According to [5], any smooth embedding of S^k in S^m is unknotted, i.e. isotopic to the boundary of a standard ball in the meta-stable range $k < \frac{2}{3}m - 1$.

This implies that the complement of a regular neighborhood of the fiber is diffeomorphic to the complement of a standard sphere and thus to $S^n \times D^n$.

When $k \geq 2$ we remark that the fibers over the points $x_i \in D_i$ lie at different heights and thus they are contained in disjoint slice spheres of the suspension S^{2n} . This implies that these fibers are unlinked i.e. isotopic to the boundary of a set of disjoint standard balls. Thus the complement of a regular neighborhood of their union is diffeomorphic to the connected sum of their individual complements, and therefore to the connected sum of k copies of the product $S^n \times D^n$ out of their boundaries.

When gluing together two such sub-blocks associated to adjacent vertices in Γ the respective pair of $S^n \times D^n$'s yields a factor $S^n \times S^n$, because the gluing map respects the product structure. When gluing all sub-blocks in the pattern of the graph Γ the only additional non-trivial contribution comes from the cycles. Each cycle of Γ introduces an extra 1-handle. Thus the manifold B_Γ is $\#_e S^n \times S^n \#_c S^1 \times S^{2n-1}$. \square

Remark 3.1. There are some small variations of the method above. When $n = 8$ let us decorate each edge a of the graph Γ by an element $\psi_a \in \Gamma_8$. Here Γ_8 denotes the group of (pseudo-) isotopy classes of diffeomorphisms of S^7 . When gluing the sub-blocks of A_Γ along the boundary 7-spheres corresponding to the edge a we make a twist by the element ψ_a . Further, for the gluings along the boundary components $S^7 \times S^8$ of the sub-blocks of B_Γ use the product class $\psi_a \times \mathbf{1}$. This will produce some twisted pairs of manifolds, corresponding to adjoining exotic spheres to our former examples.

Other examples can be obtained by adjoining an exotic sphere to $\#_e S^n \times S^n \#_c S^1 \times S^{2n-1}$ in dimensions $2n = 8$ or 16 , or a homotopy 4-sphere when $2n = 4$. Recall that the group Θ^k of homotopy k -spheres is $\Theta^8 = \Theta^{16} = \mathbb{Z}/2\mathbb{Z}$. This can be realized by removing a small disk centered at a critical point and gluing it back differently, when $n \neq 2$, and respectively gluing back a homotopy 4-disk, when $n = 2$. In this way, using the theorem of Huebsch and Morse for $n = 2$ (see [6] and also [1] where this argument is carried out in detail) we still have a function with $2e - 2c + 2$ critical points and Proposition 2.1 implies that

Corollary 3.1. If $e \geq c \geq 0$, $c \neq 1$ and Σ is a homotopy sphere then:

$$\varphi(\Sigma^{2n} \#_e S^n \times S^n \#_c S^1 \times S^{2n-1}, \#_c S^1 \times S^n) \geq 2e - 2c + 2$$

with equality when $n \in \{2, 4, 8\}$.

4. EXAMPLES WITH $\varphi = 1$

The result of [1] show that if $\varphi(M^m, N^{n+1})$ is finite non-zero (small codimension non-exceptional dimensions) then $\varphi(M^m, N^{n+1}) = 1$ and M^m should be diffeomorphic to $\Sigma^m \# \widehat{N}$, where Σ^m is an exotic sphere such that M^m is not fibered over N . Actually this construction might produce non-trivial examples in any codimension.

Proposition 4.1. *If Σ^m is an exotic sphere and $\widehat{N} \rightarrow N$ a fibration then $\varphi(\Sigma^m \# \widehat{N}, N) \in \{0, 1\}$.*

Proof. We obtain $\Sigma^m \# \widehat{N}$ from \widehat{N} by excising a ball D^{n+1} and gluing it (or a homotopy 4-disk when $m = 4$) back by means of a suitable diffeomorphism h of its boundary. By a classical result of Huebsch and Morse ([6]), there exists a smooth homeomorphism $\Sigma^m \# \widehat{N} \rightarrow \widehat{N}$ which has only one critical point located in the ball D^{n+1} . This provides a smooth map $\Sigma^m \# \widehat{N} \rightarrow N$ with one critical point. Notice however that $\Sigma^m \# \widehat{N}$ might still be fibered over N , as it is the case when $\widehat{N} \rightarrow N$ is the Hopf fibration $S^7 \rightarrow S^4$. \square

Remark 4.1. Notice however that $\Sigma^m \sharp \widehat{N}$ might still be fibered over N , although not diffeomorphic to \widehat{N} . This is so when $\widehat{N} \rightarrow N$ is the Hopf fibration $S^7 \rightarrow S^4$ and $\Sigma^7 \sharp \widehat{N}$ is a Milnor exotic sphere, namely a S^3 -fibration over S^4 with Euler class ± 1 .

Remark 4.2. The manifold $M^m = \Sigma^m \sharp S^{m-n-1} \times S^{n+1}$ is not diffeomorphic to $S^{m-n-1} \times S^{n+1}$ if Σ^m is an exotic sphere (see [8]). Thus, the Proposition above yields effectively examples with $\varphi = 1$.

If Σ^8 is the exotic 8-sphere which generates the group $\Theta^8 = \mathbb{Z}/2\mathbb{Z}$ then $\varphi(\Sigma^8 \sharp S^3 \times S^5, S^5) = 1$. In fact $M^8 = \Sigma^8 \sharp S^3 \times S^5$ is homeomorphic but not diffeomorphic to $S^3 \times S^5$. Assume that M^8 fibers over S^5 . Then the fiber is a homotopy 3-sphere and hence S^3 , by the Poincaré Conjecture. Isomorphism classes of S^3 -fibrations over S^5 are classified by elements of $\pi_4(SO(4)) \cong \mathbb{Z}/2\mathbb{Z}$. Thus there exists only one non-trivial S^3 -fibration W^8 . According to ([7], p.164) W^8 is the only S^3 -fibration over S^5 without cross-sections and thus it has not the homotopy type of $S^3 \times S^5$. In particular, W^8 is not homeomorphic to M^8 and thus M^8 cannot smoothly fiber over S^5 .

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