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A Method to Compute the Transition Function of a Piecewise Deterministic Markov Process with Application to Reliability

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Abstract

We study the time evolution of an increasing stochastic process governed by a first order stochastic differential system. This defines a particular piecewise deterministic Markov process (PDMP). We consider a Markov renewal process (MRP) associated to the PDMP and its Markov renewal equation (MRE) which is solved in order to obtain a closed-form solution of the transition function of the PDMP. It is then applied in the framework of survival analysis to evaluate the reliability function of a given system. We give a numerical illustration and we compare this analytical solution with the Monte-Carlo estimator.

Key words: Piecewise deterministic Markov Process, Markov Renewal Process, Transition function, Reliability, Survival Analysis

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1 Introduction

In the literature of stochastic differential systems, a large part is dedicated to those involving diffusion type processes, that lead to the theory of stochastic differential equations (see, e.g., Øksendal, 2003). Another class of stochastic models has arisen to describe the random evolution of processes that do not involve diffusion type motion, but rather the mixture of deterministic motions and random jumps. Such an idea has been suggested by various authors, yet Davis provided a major contribution to this approach : he gave the underlying theory for the class of stochastic models called *piecewise deterministic Markov processes* (PDMP) in Davis (1984), farther developed in Davis (1993) and Jacobsen (2006). In Koroliuk and Limnios (2005) the limit theory is studied, in a functional setting. Lapeyre and Pardoux (2003) worked with *transport processes*, which are a special case of PDMP, in order to give a stochastic interpretation of the transport equations used in physics for the modeling of the motion of particles. This family of processes are nowadays much used in several applications, e.g., reliability analysis (Devooght and Smidts, 1996) or insurance (Embrechts et al., 2001). In Chiquet and Limnios (2006); Chiquet et al. (to appear), we took advantage of a PDMP for the modeling of degradation mechanisms arising in a structure subject to random environmental effects.

The main contribution of this paper is to provide a closed-form solution for the transition function of a PDMP defined through a first order differential equation. For this purpose, we associate to this PDMP a *Markov renewal process* (MRP), thus we will refer to results from this theory (see e.g. Limnios and Oprıçan, 2001).

The outline is the following: in section 2, we give the model settings and standard notation for MRP. We then build and solve a *Markov renewal equation* (MRE) for the transition function of the PDMP. In section 3, an application for survival analysis is studied, where the numerical computation of the reliability is detailed. We compare the result obtained through the direct resolution of the MRE with the empirical estimator obtained with the Monte-Carlo method.

2 Transition function of the PDMP

Let $(Z_t, t \in \mathbb{R}_+)$ be a real-valued stochastic process starting almost surely from $z > 0$ and governed by the following first order differential system

$$\dot{Z}_t = C(Z_t, X_t), \quad Z_0 = z, \quad (1)$$

with the following assumptions:

A. 1 *The process $(X_t, t \in \mathbb{R}_+)$ is an irreducible Markov process with a countable state space E , an initial distribution $\alpha_i = \mathbb{P}(X_0 = i)$, and a matrix generator $\mathbf{A} = (a_{ij})_{i,j \in E}$ such that $a_{ij} \geq 0$, for all $i \neq j$, and $a_{ii} = -a_i = -\sum_{k \in E, i \neq k} a_{ik}$.*

A. 2 *The function $C : \mathbb{R}_+ \times E \rightarrow \mathbb{R}_+$ is measurable, strictly positive and Lipschitz w.r.t. the first argument, i.e., there is a function $f : E \rightarrow \mathbb{R}$, for $x, y \in \mathbb{R}_+$ and $i \in E$, such that $|C(x, i) - C(y, i)| \leq f(i) |x - y|$.*

Each path $Z_t(\omega)$ is built in a piecewise manner according to the function C and a path $X_t(\omega)$ of the jump Markov process. That is, for $t \in \mathbb{R}_+$, the Cauchy problem

$$\dot{Z}_t(\omega) = C(Z_t(\omega), i), \quad Z_0(\omega) = z, \quad i \in E, \quad (2)$$

has a unique solution built on the successive intervals $[S_n(\omega), S_{n+1}(\omega))$, where $(S_n, n \in \mathbb{N})$ is a random sequence describing the jump times of X_t . For any $t < S_1(\omega)$, we denote by $\varphi_{z,i}(t)$ the solution of (2), where $X_0(\omega) = i$, i.e., $\varphi_{z,i}(t)$ is the solution before the first jump time of X_t , conditionally to the starting value of $(Z_0, X_0) = (z, i)$. We also assume that Z_0 and X_0 are independent. In the sequel, we focus on the transition function P , defined by

$$P_{ij}(z, B, t) := \mathbb{P}_{z,i}(Z_t \in B, X_t = j), \quad i, j \in E, B \in \mathcal{B}, \quad (3)$$

where B is a subset of \mathcal{B} , the Borel σ -field of \mathbb{R}_+ and $\mathbb{P}_{z,i}(\cdot) := \mathbb{P}(\cdot | Z_0 = z, X_0 = i)$.

A Markov process is a special MRP, thus we may associate to (Z_t, X_t) the *extended* MRP $(\zeta_n, Y_n, S_n, n \in \mathbb{N})$ such as

$$\zeta_n = Z_{S_n}, \quad Y_n = X_{S_n}, \quad n \in \mathbb{N}.$$

The process (Y_n, S_n) is a standard MRP, while (ζ_n, Y_n, S_n) is an extended one. In the homogeneous case the associated *semi-Markov kernel* Q is defined, for $t > 0$, by

$$Q_{ij}(z, B, t) := \mathbb{P}_{z,i}(\zeta_1 \in B, Y_1 = j, S_1 - S_0 \leq t). \quad (4)$$

The Stieltjes-convolution is denoted by “*”, hence, the convolution of Q with a measurable function ϕ on the space $\mathbb{R}_+ \times E$ is

$$(Q * \phi)_{ij}(z, B, t) = \sum_{k \in E} \int_{\mathbb{R}_+} \int_0^t Q_{ik}(z, dy, ds) \phi_{kj}(y, B, t - s),$$

for $i, j \in E, z > 0$ and $B \in \mathcal{B}$. In the same way, the successive n -fold convolutions of the semi-Markov kernel are defined recursively. For $n = 0, 1$,

$$Q_{ij}^{(0)}(z, B, t) = \mathbb{1}_{\{i=j\}} \mathbb{1}_B(z) \mathbb{1}_{\mathbb{R}_+}(t), \quad Q_{ij}^{(1)}(z, B, t) = Q_{ij}(z, B, t),$$

where $\mathbb{1}_B(x)$ is the indicator function, i.e., $\mathbb{1}_B(x) = 1$ if $x \in B$, 0 otherwise. For $n \geq 2$, the n -fold convolution turns to

$$\begin{aligned} Q_{ij}^{(n)}(z, B, t) &:= (Q * Q^{(n-1)})_{ij}(z, B, t) \\ &= \sum_{k \in E} \int_{\mathbb{R}_+} \int_0^t Q_{ik}(z, dy, ds) Q_{kj}^{(n-1)}(y, B, t - s). \end{aligned} \quad (5)$$

The Markov renewal function Ψ , which plays a central role, is defined by

$$\Psi_{ij}(z, B, t) = \sum_{n \geq 0} Q_{ij}^{(n)}(z, B, t). \quad (6)$$

In the case at hand, we have (ζ_n, Y_n, S_n) a *normal* MRP, that is, $\Psi_{ij}(z, B, t) < \infty$ for any fixed $t > 0, z > 0, B \in \mathcal{B}$ and $i, j \in E$.

A MRE has the following form

$$\Theta_{ij}(z, B, t) = g_{ij}(z, B, t) + (Q * \Theta)_{ij}(z, B, t), \quad (7)$$

where g is a known function defined on $\mathbb{R}_+ \times E \times \mathbb{R}_+$ and Θ is the unknown function.

The solution (see e.g Koroliuk and Limnios, 2005) is given by

$$\Theta_{ij}(z, B, t) = (\Psi * g)_{ij}(z, B, t). \quad (8)$$

Let us now take advantage of Markov renewal theory to build a solvable MRE for P . For this purpose, we first need to calculate Q . This is done in the following Lemma.

Lemma 1 *The semi-Markov kernel Q of the MRP (ζ_n, Y_n, S_n) verifies, for $i \neq j$,*

$$Q_{ij}(z, B, dt) = a_{ij}e^{-a_i t} \delta_{\varphi_{z,i}(t)}(B) dt, \quad (9)$$

where $\delta_x(B)$ is the Dirac distribution, equal to 1 if $x \in B$, 0 otherwise.

PROOF. Assuming that $S_0 = 0$, and conditioning on definition (4), we get

$$Q_{ij}(z, B, dt) = \mathbb{P}_{z,i}(Y_1 = j, S_1 \in dt) \mathbb{P}_{z,i}(\zeta_1 \in B | Y_1 = j, S_1 = t).$$

First, since S_1 and Y_1 are independent, we have $\mathbb{P}_{z,i}(Y_1 = j, S_1 \in dt) = a_{ij}e^{-a_i t} dt$ from usual results of Markov theory. Second, Z_t is fully characterized by $\varphi_{z,i}(t)$ before the first jump time S_1 , thus $\mathbb{P}_{z,i}(\zeta_1 \in B | X_1 = j, S_1 = t) = \mathbb{P}_{z,i}(Z_t \in B) = \delta_{\varphi_{z,i}(t)}(B)$. Indeed, the probability $\mathbb{P}_{z,i}(Z_t \in B)$ is zero everywhere, excepted for the time points where B is reached. We hence get the expected result.

Proposition 2 *The transition function P of (Z_t, X_t) is governed by the MRE*

$$P_{ij}(z, B, t) = g_{ij}(z, B, t) + (Q * P)_{ij}(z, B, t),$$

whose solution is $P_{ij}(z, B, t) = (\Psi * g)_{ij}(z, B, t)$ and where

$$g_{ij}(z, B, t) = e^{-a_i t} \mathbf{1}_{\{i=j\}} \mathbf{1}_B(\varphi_{z,i}(t)). \quad (10)$$

PROOF. It is convenient to make appear S_1 in (3). Hence,

$$P_{ij}(z, B, t) = \underbrace{\mathbb{P}_{z,i}(Z_t \in B, X_t = j, S_1 > t)}_{P_1} + \underbrace{\mathbb{P}_{z,i}(Z_t \in B, X_t = j, S_1 \leq t)}_{P_2}.$$

Before the first jump of X_t , $i = j$ and Z_t evolves according to $\varphi_{z,i}(t)$. Thus, $P_1 = e^{-a(i)t} \mathbf{1}_B(\varphi_t(z, i)) \mathbf{1}_{\{i=j\}}$. From Total Probability Theorem, it holds for P_2 that

$$P_2 = \sum_{\substack{k \in E \\ k \neq i}} \int_0^t \mathbb{P}_{z,i}(Z_t \in B, X_t = j | Y_1 = k, S_1 = s) \mathbb{P}_{z,i}(Y_1 = k, S_1 \in ds).$$

As long as $\mathbb{P}_{z,i}(Y_1 = k, S_1 \in ds) = a_{ik}e^{-a_i s} ds$, and noticing that $\mathbb{P}_{z,i}(Z_t \in B, X_t = j | Y_1 = k, S_1 = s) = P_{kj}(\varphi_{z,i}(s), B, t - s)$, then P_2 is fully characterized. Finally,

$$P_{ij}(z, B, t) = e^{-a_i t} \mathbf{1}_{\{i=j\}} \mathbf{1}_B(\varphi_{z,i}(t)) + \sum_{\substack{k \in E \\ k \neq i}} a_{ik} \int_0^t e^{-a(i)s} P_{kj}(\varphi_{z,i}(s), B, t - s) ds,$$

which may be written, with Q given by (9) and g given by (10), as

$$P_{ij}(z, B, t) = g_{ij}(z, B, t) + \sum_{k \in E} \int_{\mathbb{R}_+} \int_0^t Q_{ik}(z, dy, ds) P_{kj}(y, B, t - s).$$

This last equation is of the general form (7), whose solution is given by (8).

3 Application to reliability

As an application, let us now study system (1) under assumptions **A.1**, **A.2** and E a finite state space. The system remains reliable until Z_t reaches a critical threshold $\Delta > z > 0$ at random time τ , which is the failure time, with a cumulative distribution function F_τ . A schematic view of the parallel evolution of both components Z_t and X_t can be seen on Figure 1, where three typical paths of the PDMP are plotted. Of course, since, $C > 0$, Z_t is an increasing process. Looking towards reliability

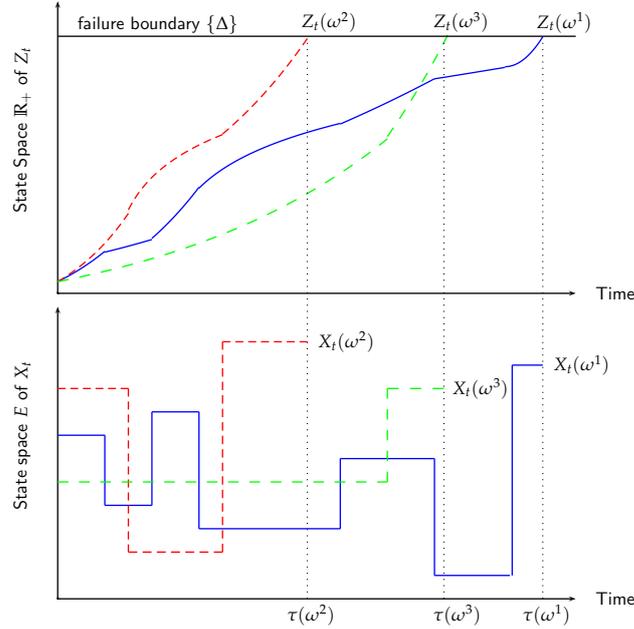


Fig. 1. Schematic view of the PDMP – three paths of (Z_t, X_t)

analysis, we put $U = [z, \Delta)$ and $D = [\Delta, \infty)$ the respective sets of good states and failure states for Z_t . Due to the continuous, increasing evolution of Z_t , it necessarily

passes through Δ . The reliability function R is defined as follows:

$$R(t) = \mathbb{P}((Z_t, X_t) \in U \times E) = \sum_{i,j \in E} \alpha_i P_{ij}(z, U, t) = 1 - F_\tau(t).$$

Through Proposition 2, P is known. Hence R (as well as F_τ) is fully characterized:

$$R(t) = \sum_{i,j \in E} \alpha_i \times (\Psi * g)_{ij}(z, U, t). \quad (11)$$

Note that the kernel Q can be calculated at a given time point $t > 0$ for the Borel subset U , by integrating (9). As a matter of fact, it holds that

$$Q_{ij}(z, U, t) = a_{ij} \int_0^t e^{-a_i s} \mathbb{1}_U(\varphi_{z,i}(s)) ds = \frac{a_{ij}}{a_i} \left(1 - e^{-a_i \min(t, t_{z,i}(\Delta))} \right),$$

where $t_{z,i}(\Delta) = \inf \{ t \geq 0 : \varphi_{z,i}(t) \geq \Delta \}$.

3.1 Numerical implementation

The numerical calculation of R successively requires the kernel Q , the n -fold convolutions $Q^{(n)}$ for each $n \geq 0$, the Markov renewal function Ψ built upon the $Q^{(n)}$ and the transition function P , by a convolution between g and Ψ . Since convolution products are time consuming, any simplification would mean a great time saving. By (9), the n -fold convolution (5) of Q turns to

$$Q_{ij}^{(n)}(z, B, t) = \sum_{\substack{k \in E \\ k \neq i}} a_{ik} \int_0^t e^{-a_i s} Q_{kj}^{(n-1)}(\varphi_{z,i}(s), B, t-s) ds, \quad (12)$$

hence removing the integral on \mathbb{R}_+ , thanks to the Dirac distribution. Since our main interest is the reliability, we compute P just for the subset $B \equiv U$, that is,

$$P_{ij}(z, U, t) = \int_U \int_0^t \Psi_{ij}(z, dy, ds) e^{-a_j(t-s)} \mathbb{1}_U(\varphi_{y,j}(t-s)). \quad (13)$$

Indeed, the sum on E has been removed thanks to the structure of g . Furthermore, the integration on $y \in \mathbb{R}_+$ is limited on U since $\mathbb{1}_U(\varphi_{y,j}(t-s))$ is zero elsewhere.

Theses functions have to be properly discretized to achieve the numerical computation. In the following, a function with an upper-index “#” means its discretized

version. This discretization must be operated on both intervals $U = [z, \Delta)$ and $[0, t]$, thus we set two numerical partitions $\{z = y_0 < y_1 < \dots < y_\ell < \dots < y_L = \Delta^-\}$ and $\{0 = t_0 < t_1 < \dots < t_m < \dots < t_M = t\}$. Both L and M , being the respective numbers of discretization steps for $[z, \Delta)$ and $[0, t]$, have to be sufficiently large. When $L, M \rightarrow \infty$ each numerical function tends to the associated "true" one. For instance, when $L, M \rightarrow \infty$, then $Q^\# \rightarrow Q$ uniformly w.r.t a given matrix norm, by example, $\|Q\| = \max_{i,j} Q_{ij}(z, y, t)$ with t, z, B fixed. Hence, the numerical version of (13) is

$$P_{ij}^\#(z, U, t) = \sum_{y_\ell \in [z, \Delta)} \sum_{t_m \in (0, t]} \Delta_{yt} \Psi_{ij}^\#(z, y_\ell, t_m) e^{-a_j(t-t_m)} \mathbf{1}_{\varphi_{y_\ell, j}(t-t_m)}(U),$$

where $\Delta_{yt} \Psi_{ij}^\#(z, y_\ell, t_m)$ is the only unknown, which stands for the numerical evaluation of $\Psi(z, dy, ds)$ in (13). By the definition (6), it can be evaluated through

$$\Delta_{yt} \Psi_{ij}^\#(z, y_\ell, t_m) = \sum_{n \geq 0} \Delta_{yt} Q^{\#(n)}(z, y_\ell, t_m). \quad (14)$$

The derivative $\Delta_{yt} Q^{\#(n)}$ is estimated by finite differences on y and t :

$$\begin{aligned} \Delta_{yt} Q^{\#(n)}(z, y_\ell, t_m) &= [Q^{\#(n)}(z, y_\ell, t_m) - Q^{\#(n)}(z, y_{\ell-1}, t_m)] \\ &\quad - [Q^{\#(n)}(z, y_\ell, t_{m-1}) - Q^{\#(n)}(z, y_{\ell-1}, t_{m-1})]. \end{aligned} \quad (15)$$

Each element $Q^{\#(n)}$ in (15) is obtained by the numerical version of (12):

$$Q_{ij}^{\#(n)}(z, y_\ell, t_m) = \sum_{\substack{k \in E \\ k \neq i}} a_{ik} \sum_{t_m \in (0, t]} e^{-a_i t_m} Q_{kj}^{\#(n-1)}(\varphi_{z, i}(t_m), y_\ell, t - t_m) \Delta t_m,$$

with $\Delta t_m = t_m - t_{m-1}$, the time-step discretization. Finally, we point out that the sum on the n -fold convolutions of the kernel in (14) is truncated from the rank n^* , provided that $\|Q^{\#(n^*)}\| < \varepsilon$. We put ε a small real number, chosen closed to the machine precision. Note that the integer n^* is finite since $Q_{ij}^{\#(n)}(z, y, t) \xrightarrow[n \rightarrow \infty]{} 0$ for a normal MRP with fixed values of $i, j \in E, t > 0, z > 0$ and $y \in [z, \Delta]$.

3.2 Numerical results

To check the validity of our results, let us study the process Z_t governed by

$$\dot{Z}_t = aZ_t \times X_t, \quad Z_0 = z, \quad (16)$$

with $a = 0.0075$, $z = 10$, $\Delta = 50$. The process X_t is a three-state jump Markov process with $E = \{1, 2, 3\}$, a matrix generator and an initial law given by

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 & 0 \\ 0.027 & -0.03 & 0.003 \\ 0.01 & 0 & -0.01 \end{bmatrix}, \quad \alpha = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

The coupled process (Z_t, X_t) defines a PDMP whose transition function and reliability function can be computed through Proposition 2 and (11), based upon the numerical implementation described above for which we put $M = L = 100$ points of discretization. Another way to compute the reliability is the Monte-Carlo method, which consists in simulating a large number of paths of Z_t and counting when the state $\{\Delta\}$ is reached or not. We use the empirical estimator applied on $K = 10000$ paths $(Z_t^k)_{k=1, \dots, K}$ simulated through Monte-Carlo techniques, that is $\widehat{R}(t) = \frac{1}{K} \sum_{k=1}^K \mathbb{1}_{\{Z_t^k < \Delta\}}$. This estimator is compared with the direct calculus of R through the Markov renewal argument developed in this paper. Results can be found on Figure 2 where the Monte-Carlo estimator is used as a reference to check the validity both of the theoretical results and the numerical implementation. One can see that the reliability curves obtained with the two methods are quite close.

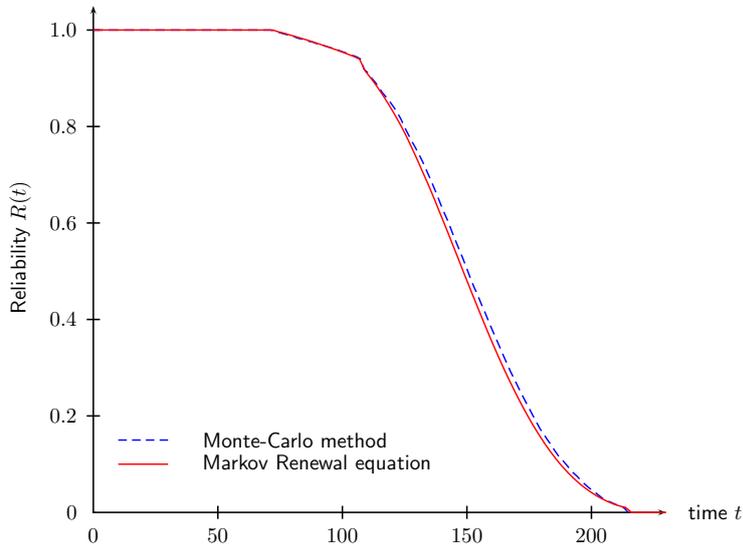


Fig. 2. Reliability estimation with two different methods

4 Concluding remarks

Because models based on system (1) are encountered in the literature for many different fields of applications, we think that the reliability results of this paper may be useful for engineering studies, such as structural mechanics (see, e.g., Chiquet et al., to appear). Yet, to be more suitable to real phenomena, the case of PDMP which are not only monotone should be considered, as further developments.

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