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Stein's method and exact Berry-Esséen asymptotics for functionals of Gaussian fields

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Abstract: We show how to detect optimal Berry-Esséen bounds in the normal approximation of functionals of Gaussian fields. Our techniques are based on a combination of Malliavin calculus, Stein's method and the method of moments and cumulants, and provide *de facto* local (one term) Edgeworth expansions. The findings of the present paper represent a further refinement of the main results proved in Nourdin and Peccati (2009b). Among several examples, we discuss three crucial applications: (i) to Toeplitz quadratic functionals of continuous-time stationary processes (extending results by Ginovyan (1994) and Ginovyan and Sahakyan (2007)), (ii) to "exploding" quadratic functionals of a Brownian sheet, and (iii) to a continuous-time version of the Breuer-Major CLT for functionals of a fractional Brownian motion.

Key words: Berry-Esséen bounds; Breuer-Major CLT; Brownian sheet; Fractional Brownian motion; Local Edgeworth expansions; Malliavin calculus; Multiple stochastic integrals; Normal approximation; Optimal rates; Quadratic functionals; Stein's method; Toeplitz quadratic forms.

2000 Mathematics Subject Classification: 60F05; 60G15; 60H05; 60H07.

1 Introduction

Let $\{F_n : n \geqslant 1\}$ be a sequence of zero-mean real-valued random variables, and consider a standard Gaussian variable $N \sim \mathscr{N}(0,1)$. Assume that each F_n is a functional of an infinite-dimensional Gaussian field and suppose that, as $n \to \infty$,

$$F_n \xrightarrow{\text{Law}} N. \tag{1.1}$$

In the paper [20], we demonstrated that one can naturally combine *Malliavin calculus* (see *e.g.* [13, 21]) with *Stein's method* (see *e.g.* [4, 29, 33, 34]) in order to obtain explicit bounds of the type

$$d(F_n, N) \leqslant \varphi(n), \quad n \geqslant 1, \tag{1.2}$$

where $d(F_n, N)$ stands for some appropriate distance (for example, the Kolmogorov distance or the total variation distance) between the laws of F_n and N, and $\varphi(n)$ is some positive sequence converging to zero. The aim of the present work is to develop several techniques, allowing to asses the *optimality* of the bound $\varphi(n)$ appearing in (1.2), for a given sequence

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$\{F_n\}$. Formally, one says that the bound $\varphi(n)$ is *optimal* for the sequence $\{F_n\}$ and the distance d, whenever there exists a constant $c \in (0,1)$ (independent of n) such that, for n sufficiently large,

$$c < d(F_n, N)/\varphi(n) \leqslant 1. \tag{1.3}$$

We shall establish relations such as (1.3) by pushing one step further the Malliavin-type approach to Stein's method initiated in [20]. In particular, the findings of this paper represent a new substantial refinement of the central limit theorems (CLTs) for functionals of Gaussian fields previously proved in [22, 23, 25, 26]. Once again, our techniques do not require that the random variables $\{F_n\}$ have the specific form of partial sums. Indeed, we will see in Sections 4–6 below that our results yield optimal Berry-Esséen type bounds for CLTs involving objects as diverse as: (i) Toeplitz quadratic functionals of continuous-time stationary processes, (ii) quadratic functionals of a Brownian motion or of a Brownian sheet indexed by a compact set of $\mathbb{R}^d$ ($d \geqslant 2$), and (iii) polynomial functionals constructed from a fractional Brownian motion.

Note that, in the subsequent sections, we shall focus uniquely on the normal approximation of random variables with respect to the *Kolmogorov distance*. This distance is defined as

$$d_{\mathrm{Kol}}(X,Y) = \sup_{z\in\mathbb{R}} |P(X \leqslant z) - P(Y \leqslant z)| \tag{1.4}$$

for any pair of random variables X and Y. It will become clear later on that many results of the present paper extend almost verbatim to alternate distances, such as the Wasserstein or the total variation distances, between laws of real-valued random variables.

Our basic approach can be described as follows. Fix $z \in \mathbb{R}$, and consider the *Stein equation*

$$\mathbf{1}_{(-\infty,z]}(x) - \Phi(z) = f'(x) - xf(x), \quad x \in \mathbb{R}, \tag{1.5}$$

where, here and for the rest of the paper, we use the standard notation $\Phi(z) = P(N \leqslant z)$ ($N \sim \mathscr{N}(0,1)$) and $\mathbf{1}_A$ stands for the indicator of a set A. It is well-known that, for every fixed z, equation (1.5) admits a solution f_z such that $\|f_z\|_\infty \leqslant \sqrt{2\pi}/4$ and $\|f_z'\|_\infty \leqslant 1$ (see e.g. [4, Lemma 2.2] or formulae (2.20)–(2.21) below). Now suppose that the elements of the sequence $\{F_n\}$ appearing in (1.1) are functionals of some Gaussian field, say X, and assume that each F_n is differentiable in the sense of Malliavin calculus (see Section 2.1 for details). Denote by DF_n the Malliavin derivative of F_n and write L^{-1} for the *pseudo-inverse of the Ornstein-Uhlenbeck generator* (see again Section 2.1). Recall that DF_n is a random element with values in an appropriate Hilbert space $\mathfrak{H}$. In [20, Section 3], we proved and applied the following relations, that are direct consequences of the fact that f_z solves (1.5) on the one hand, and of the celebrated *integration by parts formula* of Malliavin calculus on the other hand: for every $z \in \mathbb{R}$,

$$P(F_n \leqslant z) - \Phi(z) = E[f_z'(F_n) - F_n f_z(F_n)] = E[f_z'(F_n)(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}})]. \tag{1.6}$$

By using (1.4), by applying the Cauchy-Schwarz inequality to the RHS of (1.6) and by using the fact that f_z' is bounded by 1, one immediately obtains that

$$d_{\mathrm{Kol}}(F_n, N) \leqslant \sqrt{E[(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}})^2]}. \tag{1.7}$$

The starting point of [20] was that, in several crucial cases (for instance, when each F_n is a multiple Wiener-It integral of a fixed order), the upper bound

$$\varphi(n) := \sqrt{E[(1-\langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}})^2]}, \quad n \geqslant 1, \tag{1.8}$$

is such that: (i) the quantity $\varphi(n)$ can be explicitly computed (for instance in terms of *contraction operators*), (ii) $\varphi(n) \to 0$ as $n \to \infty$, and (iii) $\varphi(n)$ is directly related to quantities playing a fundamental role in the CLTs for functionals of Gaussian fields proved in [22, 23, 25, 26]. The aim of the present paper is to establish conditions on the sequence $\{F_n\}$ ensuring that the ratios

$$\frac{E[f_z'(F_n)(1-\langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}})]}{\varphi(n)}, \quad n \geqslant 1, \tag{1.9}$$

involving (1.8) and the RHS of (1.6), converge to a nonzero limit for all z outside some finite set. Such a result yields immediately the existence of a constant c, verifying (1.3) for $d = d_{\mathrm{Kol}}$. We will show that a very effective way to prove the convergence of the quantities appearing in (1.9) is to characterize the joint convergence in distribution of the random vectors

$$\left(F_n, \frac{1-\langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}}}{\varphi(n)}\right), \quad n \geqslant 1, \tag{1.10}$$

towards a two-dimensional Gaussian vector with non-zero covariance. The applications presented in Sections 4–6 will show that this specific convergence takes place in several crucial situations, involving for instance quadratic or polynomial functionals of stationary Gaussian processes. We will see that, in order to prove a CLT for the vector appearing in (1.10), a useful tool is the multi-dimensional version of the CLT for multiple stochastic integrals proved in [26]. Also, it is interesting to note that, if each F_n in (1.1) is a double stochastic integral, then our conditions can be expressed exclusively in terms of the second, third, fourth and eighth cumulants associated with the sequence $\{F_n\}$ — see Section 3.3 below.

The rest of the paper is organized as follows. Section 2 deals with preliminaries about Malliavin calculus, Stein's method and related topics. Section 3 contains our main results, with special attention devoted to random variables belonging to the second Wiener chaos of a Gaussian field. In Section 4 we develop an application to Toeplitz quadratic functionals of stationary continuous-time Gaussian processes, thus extending and refining some results by Ginovyan [7] and Ginovyan and Sahakyan [8]. Section 5 is devoted to quadratic functionals of Brownian motion and of the Brownian sheet, whereas Section 6 focuses on a continuous-time version of the Breuer-Major CLT for processes subordinated to a fractional Brownian motion.

2 Preliminaries

2.1 Gaussian fields and Malliavin calculus

We shall now provide a short description of the tools of Malliavin calculus that will be needed in the forthcoming sections. The reader is referred to the monographs [13] and [21] for any unexplained notion or result.

Let $\mathfrak{H}$ be a real separable Hilbert space. We denote by $X = \{X(h) : h \in \mathfrak{H}\}$ an *isonormal Gaussian process* over $\mathfrak{H}$. By definition, X is a centered Gaussian family indexed by the elements of $\mathfrak{H}$ and such that, for every $h, g \in \mathfrak{H}$,

$$E\big[X(h)X(g)\big] = \langle h, g\rangle_{\mathfrak{H}}. \tag{2.11}$$

In what follows, we shall use the notation $\mathrm{L}^2(X) = \mathrm{L}^2(\Omega, \sigma(X), P)$. For every $q \geqslant 1$, we write $\mathfrak{H}^{\otimes q}$ to indicate the qth tensor power of $\mathfrak{H}$; the symbol $\mathfrak{H}^{\odot q}$ stands for the qth *symmetric* tensor power of $\mathfrak{H}$, equipped with the norm $\sqrt{q!}\|\cdot\|_{\mathfrak{H}^{\otimes q}}$. We denote by I_q the isometry between $\mathfrak{H}^{\odot q}$ and the qth Wiener chaos of X. It is well-known (see again [21, Ch. 1] or [13]) that any random variable F belonging to $\mathrm{L}^2(X)$ admits the *chaotic expansion*:

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{2.12}$$

where $I_0(f_0) := E[F]$, the series converges in L^2 and the kernels $f_q \in \mathfrak{H}^{\odot q}$, $q \geqslant 1$, are uniquely determined by F. In the particular case where $\mathfrak{H} = \mathrm{L}^2(A, \mathscr{A}, \mu)$, where $(A, \mathscr{A})$ is a measurable space and μ is a σ-finite and non-atomic measure, one has that $\mathfrak{H}^{\odot q} = \mathrm{L}^2_s(A^q, \mathscr{A}^{\otimes q}, \mu^{\otimes q})$ is the space of symmetric and square integrable functions on A^q. Moreover, for every $f \in \mathfrak{H}^{\odot q}$, $I_q(f)$ coincides with the multiple Wiener-Itô integral (of order q) of f with respect to X (see [21, Ch. 1]). It is well-known that a random variable of the type $I_q(f)$, $f \in \mathfrak{H}^{\odot q}$, has finite moments of all orders (see e.g. [13, Ch. VI]). Moreover, any non-zero finite sum of multiple stochastic integrals has a law which is absolutely continuous with respect to Lebesgue measure (see e.g. Shigekawa [32] for a proof of this fact; see [21, Ch. 1] or [30] for a connection between multiple Wiener-It integrals and Hermite polynomials on the real line). For every $q \geqslant 0$, we write J_q to indicate the orthogonal projection operator on the qth Wiener chaos associated with X, so that, if $F \in \mathrm{L}^2(\sigma(X))$ is as in (2.12), then $J_q F = I_q(f_q)$ for every $q \geqslant 0$.

Let $\{e_k,\, k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \ldots, p \wedge q$, the rth contraction of f and g is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined as

$$f \otimes_r g = \sum_{i_1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \ldots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \ldots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.13}$$

In the particular case where $\mathfrak{H} = \mathrm{L}^2(A, \mathscr{A}, \mu)$ (with μ non-atomic), one has that

$$f \otimes_r g = \int_{A^r} f(t_1, \ldots, t_{p-r}, s_1, \ldots, s_r)\, g(t_{p-r+1}, \ldots, t_{p+q-2r}, s_1, \ldots, s_r) d\mu(s_1) \ldots d\mu(s_r).$$

Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $p = q$, $f \otimes_p g = \langle f, g\rangle_{\mathfrak{H}^{\otimes p}}$. Note that, in general (and except for trivial cases), the contraction $f \otimes_r g$ is *not* a symmetric element of $\mathfrak{H}^{\otimes(p+q-2r)}$. The canonical symmetrization of $f \otimes_r g$ is written $f \widetilde{\otimes}_r g$. We also have the following multiplication formula: if $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then

$$I_p(f) I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g). \tag{2.14}$$

Let $\mathscr{S}$ be the set of all smooth cylindrical random variables of the form

$$F = g\big(X(\phi_1), \ldots, X(\phi_n)\big),$$

where $n \geqslant 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support and $\phi_i \in \mathfrak{H}$. The Malliavin derivative of F with respect to X is the element of $\mathrm{L}^2(\Omega, \mathfrak{H})$ defined as

$$DF \;=\; \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\big(X(\phi_1), \ldots, X(\phi_n)\big)\phi_i.$$

Also, $DX(h) = h$ for every $h \in \mathfrak{H}$. By iteration, one can define the mth derivative $D^m F$ (which is an element of $\mathrm{L}^2(\Omega, \mathfrak{H}^{\otimes m})$) for every $m \geqslant 2$. As usual, for $m \geqslant 1$, $\mathbb{D}^{m,2}$ denotes the closure of $\mathscr{S}$ with respect to the norm $\| \cdot \|_{m,2}$, defined by the relation

$$\|F\|_{m,2}^2 \;=\; E\left[F^2\right] + \sum_{i=1}^{m} E\big[\|D^i F\|_{\mathfrak{H}^{\otimes i}}^2\big].$$

Note that, if F is equal to a finite sum of multiple Wiener-It integrals, then $F \in \mathbb{D}^{m,2}$ for every $m \geqslant 1$. The Malliavin derivative D verifies the following *chain rule*: if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is in $\mathscr{C}_b^1$ (that is, the collection of bounded continuously differentiable functions with a bounded derivative) and if $\{F_i\}_{i=1,\ldots,n}$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F_1, \ldots, F_n) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F_1, \ldots, F_n) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F_1, \ldots, F_n) DF_i.$$

Observe that the previous formula still holds when φ is a Lipschitz function and the law of $(F_1, \ldots, F_n)$ has a density with respect to the Lebesgue measure on $\mathbb{R}^n$ (see e.g. Proposition 1.2.3 in [21]). We denote by δ the adjoint of the operator D, also called the *divergence operator*. A random element $u \in \mathrm{L}^2(\Omega, \mathfrak{H})$ belongs to the domain of δ, noted $\mathrm{Dom}\delta$, if, and only if, it verifies

$$|E\langle DF, u\rangle_{\mathfrak{H}}| \leqslant c_u \, \|F\|_{\mathrm{L}^2} \quad \text{for any } F \in \mathscr{S},$$

where c_u is a constant depending uniquely on u. If $u \in \mathrm{Dom}\delta$, then the random variable $\delta(u)$ is defined by the duality relationship (i.e., the "integration by parts formula"):

$$E(F\delta(u)) = E\langle DF, u\rangle_{\mathfrak{H}}, \tag{2.15}$$

which holds for every $F \in \mathbb{D}^{1,2}$.

The operator L, acting on square integrable random variables of the type (2.12), is defined through the projection operators $\{J_q\}_{q \geqslant 0}$ as $L = \sum_{q=0}^{\infty} -qJ_q$, and is called the *infinitesimal generator of the Ornstein-Uhlenbeck semigroup*. It verifies the following crucial property: a random variable F is an element of $\mathrm{Dom}L$ $(= \mathbb{D}^{2,2})$ if, and only if, $F \in \mathrm{Dom}\delta D$ (i.e. $F \in \mathbb{D}^{1,2}$ and $DF \in \mathrm{Dom}\delta$), and in this case: $\delta DF = -LF$. Note that a random variable F as in (2.12) is in $\mathbb{D}^{1,2}$ (resp. $\mathbb{D}^{2,2}$) if, and only if,

$$\sum_{q=1}^{\infty} q\|f_q\|_{\mathfrak{H}^{\odot q}}^2 < \infty \quad (\text{resp. } \sum_{q=1}^{\infty} q^2\|f_q\|_{\mathfrak{H}^{\odot q}}^2 < \infty),$$

and also $E\big[\|DF\|_{\mathfrak{H}}^2\big] = \sum_{q \geqslant 1} q\|f_q\|_{\mathfrak{H}^{\odot q}}^2$. If $\mathfrak{H} = \mathrm{L}^2(A, \mathscr{A}, \mu)$ (with μ non-atomic), then the derivative of a random variable F as in (2.12) can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_a F = \sum_{q=1}^{\infty} qI_{q-1}\big(f_q(\cdot, a)\big), \quad a \in A. \tag{2.16}$$

We also define the operator L^{-1}, which is the *pseudo-inverse* of L, as follows: for every $F \in \mathrm{L}^2(X)$, we set $L^{-1}F = \sum_{q \geqslant 1} \frac{1}{q} J_q(F)$. Note that L^{-1} is an operator with values in $\mathbb{D}^{2,2}$ and that $LL^{-1}F = F - E(F)$ for all $F \in \mathrm{L}^2(X)$.

The following lemma generalizes Lemma 2.1 in [19].

Lemma 2.1 *Let $F \in \mathbb{D}^{1,2}$ be such that $E(F) = 0$. Suppose that, for some integer $s \geqslant 0$, $E|F|^{s+2} < \infty$. Then,*

$$E\big(F^s \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\big) = \frac{1}{s+1} E\big(F^{s+2}\big). \tag{2.17}$$

Proof. Since $L^{-1}F \in \mathbb{D}^{2,2}$, we can write:

$$\begin{aligned} &E\big(F^s \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\big) = \frac{1}{s+1} E\big(\langle D(F^{s+1}), D(-L^{-1}F)\rangle_{\mathfrak{H}}\big) \\ &= -\frac{1}{s+1} E\big(\delta DL^{-1}F \times F^{s+1}\big) \quad \text{(by integration by parts (2.15))} \\ &= \frac{1}{s+1} E\big(F^{s+2}\big) \quad \text{(by the relation } -\delta DL^{-1}F = F\text{)}. \end{aligned}$$

□

Remark 2.2 If $F = I_q(f)$, for some $q \geqslant 2$ and $f \in \mathfrak{H}^{\odot q}$, then

$$\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}} = \langle DI_q(f), -DL^{-1}I_q(f)\rangle_{\mathfrak{H}} = \frac{1}{q}\|DI_q(f)\|^2_{\mathfrak{H}}, \tag{2.18}$$

so that (2.17) yields that, for every integer $s \geqslant 1$,

$$E\big(I_q(f)^s \|DI_q(f)\|^2_{\mathfrak{H}}\big) = \frac{q}{s+1} E\big(I_q(f)^{s+2}\big). \tag{2.19}$$

2.2 Stein's method and normal approximation on a Gaussian space

We start by recalling that, for every fixed $z \in \mathbb{R}$, the function

$$f_z(x) = \mathrm{e}^{x^2/2} \int_{-\infty}^{x} [\mathbf{1}_{(-\infty,z]}(a) - \Phi(z)]\mathrm{e}^{-a^2/2} da \tag{2.20}$$

$$= \begin{cases} \sqrt{2\pi}\mathrm{e}^{x^2/2}\Phi(x)(1-\Phi(z)) & \text{if } x \leqslant z, \\ \sqrt{2\pi}\mathrm{e}^{x^2/2}\Phi(z)(1-\Phi(x)) & \text{if } x > z, \end{cases} \tag{2.21}$$

is a solution to the Stein's equation (1.5), verifying moreover $\|f_z\|_\infty \leqslant \sqrt{2\pi}/4$ and $\|f_z'\|_\infty \leqslant 1$.

The following lemma will play a crucial role in the sequel, see also (1.6). Its content is the starting point of [20].

Lemma 2.3 *Let $F \in \mathbb{D}^{1,2}$ have zero mean. Assume moreover that Z has an absolutely continuous law with respect to the Lebesgue measure. Then, for every $z \in \mathbb{R}$,*

$$P(F \leqslant z) - \Phi(z) = E[f_z'(F)(1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}})].$$

Proof. Fix $z \in \mathbb{R}$. Since f_z solves the Stein's equation (1.5), we have $P(F \leqslant z) - \Phi(z) = E[f_z'(F) - Ff_z(F)]$. Now, observe that one can write $F = LL^{-1}F = -\delta DL^{-1}F$. By using the integration by parts formula (2.15) and the fact that $Df_z(F) = f_z'(F)DF$ (note that, for this formula to hold with f_z only Lipschitz, one needs F to have an absolutely continuous law, see Section 2.1), we deduce

$$E[Ff_z(F)] = E[-\delta DL^{-1}F\, f_z(F)] = E[\langle Df_z(F), -DL^{-1}F\rangle_{\mathfrak{H}}] = E[f_z'(F)\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}].$$

It follows that $E[f_z'(F) - Ff_z(F)] = E[f_z'(F)(1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}})]$ and the proof of the lemma is done.

□

As an application, we get the following result proved in [20] (we reproduce the proof here for sake of completeness).

Theorem 2.4 *Let $F \in \mathbb{D}^{1,2}$ have zero mean, and $N \sim \mathscr{N}(0,1)$. Then,*

$$d_{\mathrm{Kol}}(F,N) \leqslant \sqrt{E[(1-\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}})^2]}. \tag{2.22}$$

If $F = I_q(f)$, for some $q \geqslant 2$ and $f \in \mathfrak{H}^{\odot q}$, then $\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}} = q^{-1}\|DF\|^2_{\mathfrak{H}}$, and therefore

$$d_{\mathrm{Kol}}(F,N) \leqslant \sqrt{E[(1-q^{-1}\|DF\|^2_{\mathfrak{H}})^2]}. \tag{2.23}$$

Proof. If f is a bounded continuously differentiable function such that $\|f'\|_\infty \leqslant 1$ then, using the same arguments than in the proof of Lemma 2.3 (here, since f belongs to $\mathscr{C}^1_b$, observe that we do not need to assume that the law of F is absolutely continuous), we have

$$\big|E[f'(F) - Ff(F)]\big| = \big|E[f'(F)(1-\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}})]\big| \leqslant E|1-\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}|.$$

In fact, the inequality $\big|E[f'(F) - Ff(F)]\big| \leqslant E|1-\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}|$ continues to hold with $f = f_z$ (which is bounded and Lipschitz with Lipschitz constant less than one), as we see it easily by convoluting f_z by an approximation of the identity. Hence Lemma 2.3, combined with Cauchy-Schwarz inequality, imply the desired conclusion. □

Remark 2.5 In general, the bound appearing on the RHS of (2.22) may be infinite. Indeed, the fact that $F \in \mathbb{D}^{1,2}$ only implies that $\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}} \in \mathrm{L}^1(\Omega)$. By using twice the Cauchy-Schwarz inequality, one sees that a sufficient condition, in order to have that $\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}} \in \mathrm{L}^2(\Omega)$, is that $\|DF\|_{\mathfrak{H}}$ and $\|DL^{-1}F\|_{\mathfrak{H}}$ belong to $\mathrm{L}^4(\Omega)$. Note also that, if F is equal to a finite sum of multiple integrals (for instance, F is a polynomial functional of X), then the random variable $\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}$ is also a finite sum of multiple integrals, and therefore has finite moments of all orders. In particular, for $F = I_q(f)$ the RHS of (2.23) is always finite.

The bounds appearing in Theorem 2.4 should be compared with the forthcoming Theorem 2.6, dealing with CLTs on a single Wiener chaos (Part A) and on a fixed sum of Wiener chaoses (Part B).

Theorem 2.6 (see [22, 23, 25, 26]) *Fix $q \geqslant 2$ and let the sequence $F_n = I_q(f_n)$, $n \geqslant 1$, where $\{f_n\} \subset \mathfrak{H}^{\odot q}$, be such that $E[F_n^2] \to 1$ as $n \to \infty$.*
(A) *The following four conditions are equivalent as $n \to \infty$:*

(i) $F_n \xrightarrow{\text{Law}} N \sim \mathscr{N}(0,1)$;

(ii) $E(F_n^4) \to 3$;

(iii) $\|f_n \otimes_j f_n\|_{\mathfrak{H}^{\otimes 2(q-j)}} \to 0$, *for every* $j = 1, ..., q-1$;

(iv) $1 - q^{-1}\|DF_n\|^2_{\mathfrak{H}} \to 0$ *in* L^2.

(B) *Assume that either one of conditions (i)–(iv) of Part A is satisfied. Let the sequence G_n, $n \geqslant 1$, have the form*

$$G_n = \sum_{p=1}^{M} I_p(g_n^{(p)}), \quad n \geqslant 1,$$

for some $M \geqslant 1$ (independent of n) and some kernels $g_n^{(p)} \in \mathfrak{H}^{\odot p}$ ($p = 1, ..., M$, $n \geqslant 1$). Suppose that, as $n \to \infty$,

$$E(G_n^2) = \sum_{p=1}^{M} p! \|g_n^{(p)}\|^2_{\mathfrak{H}^{\otimes p}} \longrightarrow c^2 > 0 \quad \textit{and} \quad \|g_n^{(p)} \otimes_j g_n^{(p)}\|_{\mathfrak{H}^{\otimes 2(p-j)}} \longrightarrow 0,$$

for every $p = 1, ..., M$ and every $j = 1, ..., p-1$. If the sequence of covariances $E(F_n G_n)$ converges to a finite limit, say $\rho \in \mathbb{R}$, then (F_n, G_n) converges in distribution to a two-dimensional Gaussian vector (N_1, N_2) such that $E(N_1^2) = 1$, $E(N_2^2) = c^2$ and $E(N_1 N_2) = \rho$.

The equivalence between points (i)–(iii) in Part A of the previous statement has been first proved in [23] by means of stochastic calculus techniques; the fact that condition (iv) is also necessary and sufficient has been proved in [22]. Part B (whose proof is straightforward and omitted) is a consequence of the main results established in [25, 26]. Note that, in Part B of the previous statement, we may allow some of the kernels $g_n^{(p)}$ to be equal to zero. See [19] and [20, Section 3.3] for some extensions of Theorems 2.4 and 2.6 to the framework of non-central limit theorems.

Remark on notation. In what follows, given two numerical sequences $\{a_n\}$ and $\{b_n\}$, the symbol $a_n \sim b_n$ means that $\lim a_n/b_n = 1$, whereas $a_n \asymp b_n$ means that the ratio a_n/b_n converges to a non-zero finite limit.

2.3 A useful computation

We shall denote by $\{H_q : q \geqslant 0\}$ the class of *Hermite polynomials*, defined as: $H_0 \equiv 1$ and, for $q \geqslant 1$,

$$H_q(z) = (-1)^q \mathrm{e}^{z^2/2} \frac{d^q}{dz^q} \mathrm{e}^{-z^2/2}, \quad z \in \mathbb{R}; \tag{2.24}$$

for instance, $H_1(z) = z$, $H_2(z) = z^2 - 1$, and so on. Note that the definition of the class $\{H_q\}$ implies immediately the recurrence relation

$$\frac{d}{dz} H_q(z) \mathrm{e}^{-z^2/2} = -H_{q+1}(z) \mathrm{e}^{-z^2/2}, \tag{2.25}$$

yielding that the Hermite polynomials are related to the derivatives of $\Phi(z) = P(N \leqslant z)$ ($N \sim \mathscr{N}(0,1)$), written $\Phi^{(q)}(z)$ ($q = 1, 2, ...$), by the formula

$$\Phi^{(q)}(z) = (-1)^{q-1} H_{q-1}(z) \frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}}. \tag{2.26}$$

We also have, for any $q \geqslant 1$:

$$\frac{d}{dz} H_q(z) = q\, H_{q-1}(z). \tag{2.27}$$

Now denote by f_z the solution to the Stein equation (1.5) given in formulae (2.20)–(2.21). The following result, connecting f_z with the Hermite polynomials and the derivatives of Φ, will be used in Section 3.

Proposition 2.7 *For every $q \geqslant 1$ and every $z \in \mathbb{R}$,*

$$\int_{-\infty}^{+\infty} f_z'(x) H_q(x) \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}} dx = \frac{1}{q+2} H_{q+1}(z) \frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}} = \frac{1}{q+2}(-1)^{q+1}\Phi^{(q+2)}(z). \tag{2.28}$$

Proof. By integrating by parts and by exploiting relations (2.21) and (2.25), one obtains that

$$\begin{aligned}\int_{-\infty}^{+\infty} f_z'(x) H_q(x) \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}} dx &= \int_{-\infty}^{+\infty} f_z(x) H_{q+1}(x) \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} H_{q+1}(x) \left(\int_{-\infty}^{x} \big(\mathbf{1}_{(-\infty,z]}(a) - \Phi(z)\big) \mathrm{e}^{-a^2/2} da \right) dx \end{aligned} \tag{2.29}$$

By integrating by parts, using $H_{q+1} = \frac{1}{q+2} H_{q+2}'$ (see (2.27)) and in view of (2.25), one easily proves that

$$\begin{aligned} & \int_{-\infty}^{+\infty} H_{q+1}(x) \left(\int_{-\infty}^{x} \big(\mathbf{1}_{(-\infty,z]}(a) - \Phi(z)\big) \mathrm{e}^{-a^2/2} da \right) dx \\ = & -\frac{1}{q+2} \int_{-\infty}^{+\infty} H_{q+2}(x) \big(\mathbf{1}_{(-\infty,z]}(x) - \Phi(z)\big) \mathrm{e}^{-x^2/2} dx \\ = & -\frac{1}{q+2} \left(\int_{-\infty}^{z} H_{q+2}(x) \mathrm{e}^{-x^2/2} dx - \Phi(z) \int_{-\infty}^{+\infty} H_{q+2}(x) \mathrm{e}^{-x^2/2} dx \right) = \frac{1}{q+2} H_{q+1}(z). \end{aligned}$$

By plugging this expression into (2.29), we immediately deduce the desired conclusion. □

For instance, by specializing formula (2.28) to the case $q = 1$ one obtains, for $N \sim \mathcal{N}(0,1)$:

$$E[f_z'(N) \times N] = \frac{1}{3}(z^2 - 1) \frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}} = \frac{1}{3}\Phi^{(3)}(z). \tag{2.30}$$

3 Main results

3.1 Two general statements

We start by studying the case of a general sequence of Malliavin derivable functionals.

Theorem 3.1 *Let F_n, $n \geqslant 1$, be a sequence of centered and square-integrable functionals of some isonormal Gaussian process $X = \{X(h) : h \in \mathfrak{H}\}$, such that $E(F_n^2) \longrightarrow 1$ as $n \to \infty$. Suppose that the following three conditions hold:*

(i) for every n, one has that $F_n \in \mathbb{D}^{1,2}$ and F_n has an absolutely continuous law (with respect to the Lebesgue measure);

(ii) the quantity $\varphi(n) = \sqrt{E\big[(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}})^2\big]}$ (as defined in (1.8)) is such that: (a) $\varphi(n)$ is finite for every n, (b) as $n \to \infty$, $\varphi(n)$ converges to zero, and (c) there exists $m \geqslant 1$ such that $\varphi(n) > 0$ for $n \geqslant m$;

(iii) as $n \to \infty$, the two-dimensional vector $\left(F_n, \frac{1-\langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}}}{\varphi(n)}\right)$ (as defined in formula (1.10)) converges in distribution to a centered two-dimensional Gaussian vector (N_1, N_2), such that $E(N_1^2) = E(N_2^2) = 1$ and $E(N_1 N_2) = \rho$;

Then, the upper bound $d_{\mathrm{Kol}}(F_n, N) \leqslant \varphi(n)$ holds. Moreover, for every $z \in \mathbb{R}$:

$$\varphi(n)^{-1}[P(F_n \leqslant z) - \Phi(z)] \underset{n\to\infty}{\longrightarrow} \frac{\rho}{3}(z^2-1)\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}} = \frac{\rho}{3}\Phi^{(3)}(z). \tag{3.31}$$

As a consequence, if $\rho \neq 0$ there exists a constant $c \in (0,1)$, as well as an integer $n_0 \geqslant 1$, such that relation (1.3) holds for $d = d_{\mathrm{Kol}}$ and for every $n \geqslant n_0$.

Proof. Fix $z \in \mathbb{R}$. From assumption (i) and Lemma 2.3, recall that

$$\varphi(n)^{-1}[P(F_n \leqslant z) - \Phi(z)] = E[f_z'(F_n)\varphi(n)^{-1}(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}})].$$

The facts that f_z' is bounded by 1 on the one hand, and that $\varphi(n)^{-1}(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}})$ has variance 1 on the other hand, imply that the sequence

$$f_z'(F_n)\varphi(n)^{-1}(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}}), \quad n \geqslant 1,$$

is uniformly integrable. Now deduce from (2.20) that $x \to f_z'(x)$ is continuous at every $x \neq z$. This yields that, as $n \to \infty$ and due to assumption (iii),

$$E[f_z'(F_n)\varphi(n)^{-1}(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}})] \longrightarrow E(f_z'(N_1)N_2) = \rho \times E(f_z'(N_1)N_1).$$

Relation (3.31) now follows from formula (2.30). If in addition $\rho \neq 0$, one can obtain the lower bound (1.3), by using the elementary relation

$$|P(F_n \leqslant 0) - \Phi(0)| \leqslant d_{\mathrm{Kol}}(F_n, N).$$

□

Remark 3.2 Plainly, the conclusion of Theorem 3.1 still holds when n is replaced by some continuous parameter. The same remark holds for the forthcoming results of this section.

The next Proposition connects our results with one-term Edgeworth expansions. Note that, in the following statement, we assume that $E(F_n) = 0$ and $E(F_n^2) = 1$, so that the first term in the (formal) Edgeworth expansion of $P(F_n \leqslant z) - \Phi(z)$ coincides with $-(3!)^{-1}E(F_n^3)\Phi^{(3)}(z)$. For an introduction to Edgeworth expansions, the reader is referred e.g. to McCullagh [18, Chapter 3] or Hall [12, Chapter 2]. See also Rotar [31] for another application of Stein's method to Edgeworth expansions.

Proposition 3.3 (One-term Edgeworth expansions) *Let F_n, $n \geqslant 1$, be a sequence of centered and square-integrable functionals of the isonormal Gaussian process $X = \{X(h) : h \in \mathfrak{H}\}$, such that $E(F_n^2) = 1$. Suppose that conditions (i)–(iii) of Theorem 3.1 are satisfied, and also that*

(a) for every n, one has that $E|F_n|^3 < \infty$;

(b) there exists $\varepsilon > 0$ such that $\sup_{n\geqslant 1} E|F_n|^{2+\varepsilon} < \infty$.

Then, as $n \to \infty$,

$$\frac{1}{2\varphi(n)} E(F_n^3) \longrightarrow -\rho, \tag{3.32}$$

and, for every $z \in \mathbb{R}$, one has the following one-term local Edgeworth expansion

$$P(F_n \leqslant z) - \Phi(z) + \frac{1}{3!} E(F_n^3)\Phi^{(3)}(z) = o_z(\varphi(n)), \tag{3.33}$$

where $o_z(\varphi(n))$ indicates a numerical sequence (depending on z) such that $\varphi(n)^{-1} o_z(\varphi(n)) \to 0$, as $n \to \infty$.

Remark 3.4 Of course, relation (3.33) is interesting only when $\rho \neq 0$. Indeed, in this case one has that, thanks to Theorem 3.1, $P(F_n \leqslant z) - \Phi(z) \asymp \varphi(n)$ (the symbol $\asymp$ means asymptotic equivalence), so that, for a fixed z, the addition of $\frac{1}{3!} E(F_n^3)\Phi^{(3)}(z)$ actually increases the rate of convergence to zero.

Proof of Proposition 3.3. Since assumption (a) is in order and $E(F_n) = 0$, one can deduce from Lemma 2.1, in the case $s = 1$, that

$$E\left(F_n \times \frac{1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}}}{\varphi(n)}\right) = -\frac{1}{2\varphi(n)} E(F_n^3).$$

Assumption (b) combined with the fact that $\varphi(n)^{-1}(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}})$ has variance 1 immediately yields that there exists $\delta > 0$ such that

$$\sup_{n\geqslant 1} E\big|F_n \times \varphi(n)^{-1}(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}})\big|^{1+\delta} < \infty.$$

In particular, the sequence $\big\{F_n \times \varphi(n)^{-1}(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}}) : n \geqslant 1\big\}$ is uniformly integrable. Therefore, since assumption (iii) in the statement of Theorem 3.1 is in order, one deduce that, as $n \to \infty$,

$$\frac{1}{2\varphi(n)} E(F_n^3) \longrightarrow -E(N_1 N_2) = -\rho.$$

As a consequence,

$$\begin{aligned}
&\varphi(n)^{-1}\left|P(F_n \leqslant z) - \Phi(z) + \frac{1}{3!} E(F_n^3)\Phi^{(3)}(z)\right| \\
&\leqslant \left|\frac{P(F_n \leqslant z) - \Phi(z)}{\varphi(n)} - \frac{\rho}{3}\Phi^{(3)}(z)\right| + \frac{|\Phi^{(3)}(z)|}{3}\left|\frac{1}{2\varphi(n)} E(F_n^3) + \rho\right|,
\end{aligned}$$

and the conclusion follows from Theorem 3.1. □

Remark 3.5 By inspection of the proof of Proposition 3.3, one sees that Assumption (b) in the statement may be as well replaced by the following weaker condition: (b′) *the sequence*

$$F_n \times \varphi(n)^{-1}(1 - \langle DF_n, -DL^{-1}F_n\rangle_{\mathfrak{H}}), \quad n \geqslant 1,$$

is uniformly integrable.

3.2 Multiple integrals

The following statement specializes the content of the previous subsection to multiple integrals with respect to some isonormal Gaussian process $X = \{X(h) : h \in \mathfrak{H}\}$. Recall that a non-zero finite sum of multiple integrals of arbitrary orders is always an element of $\mathbb{D}^{1,2}$, and also that its law admits a density with respect to Lebesgue measure (this implies that assumption (i) in the statement of Theorem 3.1 is automatically satisfied in this section), see Shigekawa [32].

Proposition 3.6 *Let $q \geqslant 2$ be a fixed integer, and let the sequence F_n, $n \geqslant 1$, have the form $F_n = I_q(f_n)$, where, for $n \geqslant 1$, $f_n \in \mathfrak{H}^{\odot q}$. Suppose that $E(F_n^2) = q!\|f_n\|^2_{\mathfrak{H}^{\otimes q}} \to 1$. Then, the quantity $\varphi(n)$ appearing in formula (1.8) is such that*

$$\begin{aligned} \varphi(n)^2 &= E[(1 - q^{-1}\|DF_n\|^2_{\mathfrak{H}})^2] && (3.34)\\ &= (1 - q!\|f_n\|^2_{\mathfrak{H}^{\otimes q}})^2 && (3.35)\\ &+ q^2 \sum_{r=1}^{q-1} (2q-2r)!(r-1)!^2 \binom{q-1}{r-1}^4 \|f_n \widetilde{\otimes}_r f_n\|^2_{\mathfrak{H}^{\otimes 2(q-r)}}. \end{aligned}$$

Now suppose that, as $n \to \infty$,

$$\|f_n \otimes_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}} \to 0, \tag{3.36}$$

for every $r = 1, ..., q-1$, and also

$$\frac{1 - q!\|f_n\|^2_{\mathfrak{H}^{\otimes q}}}{\varphi(n)} \longrightarrow 0. \tag{3.37}$$

Then, assumption (ii) in the statement of Theorem 3.1 is satisfied, and a set of sufficient conditions, implying that assumption (iii) in the same Theorem holds, are the following relations (3.38)–(3.39): as $n \to \infty$,

$$\varphi(n)^{-2}\|(f_n \widetilde{\otimes}_r f_n) \otimes_l (f_n \widetilde{\otimes}_r f_n)\|_{\mathfrak{H}^{\otimes 2(2(q-r)-l)}} \to 0, \tag{3.38}$$

for every $r = 1, ..., q-1$ and every $l = 1, ..., 2(q-r)-1$, and, if q is even,

$$-qq!(q/2-1)!\binom{q-1}{q/2-1}^2 \varphi(n)^{-1}\langle f_n, f_n \widetilde{\otimes}_{q/2} f_n\rangle_{\mathfrak{H}^{\otimes q}} \longrightarrow \rho. \tag{3.39}$$

If q is odd and (3.38) holds, then assumption (ii) in Theorem 3.1 holds with $\rho = 0$.

Proof. Formulae (3.34)–(3.35) are a consequence of [20, Proposition 3.2]. The fact that (3.36) implies $\varphi(n) \longrightarrow 0$ is immediate (recall that $\|f_n \otimes_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}} \geqslant \|f_n \widetilde{\otimes}_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}}$). According again to [20, formula (3.42)], one has that

$$\frac{1 - q^{-1}\|DI_q(f_n)\|^2_{\mathfrak{H}}}{\varphi(n)} = \frac{1 - q!\,\|f_n\|^2_{\mathfrak{H}^{\otimes q}}}{\varphi(n)} - q\sum_{r=1}^{q-1}(r-1)!\binom{q-1}{r-1}^2 I_{2(q-r)}\left(\frac{f_n \widetilde{\otimes}_r f_n}{\varphi(n)}\right). \tag{3.40}$$

Finally, the fact that (3.38) and (3.39) (for q even) imply that assumption (iii) in Theorem 3.1 is satisfied, is a consequence of representation (3.40) and Part B of Theorem 2.6, in the case

$$G_n = -q\sum_{r=1}^{q-1}(r-1)!\binom{q-1}{r-1}^2 I_{2(q-r)}\left(\frac{f_n \widetilde{\otimes}_r f_n}{\varphi(n)}\right),$$

and $c^2 = 1$, by taking into account the fact that, for q even,

$$E(F_n G_n) = -qq!(q/2-1)!\binom{q-1}{q/2-1}^2 \varphi(n)^{-1}\langle f_n, f_n \widetilde{\otimes}_{q/2} f_n\rangle_{\mathfrak{H}^{\otimes q}},$$

whereas $E(F_n G_n) = 0$ for q odd. □

Remark 3.7 Observe that, due to Part A of Theorem 2.6, condition (3.36) is actually necessary and sufficient to have $\varphi(n) \longrightarrow 0$. Moreover, if conditions (3.36)–(3.39) are satisfied, then the usual properties of finite sums of multiple integrals (see e.g. [13, Chapter VI]) imply that assumptions (a)–(b) in the statement of Proposition 3.3 are automatically met, so that Proposition 3.6 provides indeed one-term local Edgeworth expansions.

3.3 Second Wiener chaos

In this subsection, we focus on random variables in the second Wiener chaos associated with an isonormal Gaussian process $X = \{X(h) : h \in \mathfrak{H}\}$, that is, random variables of the type $F = I_2(f)$, where $f \in \mathfrak{H}^{\odot 2}$. To every kernel $f \in \mathfrak{H}^{\odot 2}$ we associate two objects: (**I**) the Hilbert-Schmidt operator

$$H_f : \mathfrak{H} \to \mathfrak{H};\ g \to f \otimes_1 g, \tag{3.41}$$

where the contraction $f \otimes_1 g$ is defined according to (2.13), and (**II**) the sequence of kernels $\{f \otimes_1^{(p)} f : p \geqslant 1\} \subset \mathfrak{H}^{\odot 2}$, defined as follows: $f \otimes_1^{(1)} f = f$, and, for $p \geqslant 2$,

$$f \otimes_1^{(p)} f = \left(f \otimes_1^{(p-1)} f\right) \otimes_1 f. \tag{3.42}$$

We write $\{\lambda_{f,j}\}_{j\geqslant 1}$ to indicate the eigenvalues of H_f. Now, for $p \geqslant 1$, denote by $\kappa_p(I_2(f))$ the pth cumulant of $I_2(f)$. The following relation, giving an explicit expression for the cumulants of $I_2(f)$, is well-known (see e.g. [6] for a proof): one has that $\kappa_1(I_2(f)) = E(I_2(f)) = 0$, and, for $p \geqslant 2$,

$$\begin{aligned} \kappa_p(I_2(f)) &= 2^{p-1}(p-1)! \times \mathbf{Tr}(H_f^p) \\ &= 2^{p-1}(p-1)! \times \langle f \otimes_1^{(p-1)} f, f\rangle_{\mathfrak{H}^{\otimes 2}} = 2^{p-1}(p-1)! \times \sum_{j=1}^{\infty} \lambda_{f,j}^p, \end{aligned} \tag{3.43}$$

where $\mathbf{Tr}(H_f^p)$ stands for the trace of the pth power of H_f.

Proposition 3.8 *Let $F_n = I_2(f_n)$, $n \geqslant 1$, be such that $f_n \in \mathfrak{H}^{\odot 2}$, and write $\kappa_p^{(n)} = \kappa_p(F_n)$, $p \geqslant 1$. Assume that $\kappa_2^{(n)} = E(F_n^2) \longrightarrow 1$ as $n \to \infty$. Then, as $n \to \infty$, $F_n \xrightarrow{\text{Law}} N \sim \mathscr{N}(0,1)$ if, and only if, $\kappa_4^{(n)} \longrightarrow 0$. In this case, we have moreover*

$$d_{\text{Kol}}(F_n, N) \leqslant \sqrt{\frac{\kappa_4^{(n)}}{6} + (\kappa_2^{(n)} - 1)^2}. \tag{3.44}$$

If, in addition, we have, as $n \to \infty$,

$$\frac{\kappa_2^{(n)} - 1}{\frac{\kappa_4^{(n)}}{6} + (\kappa_2^{(n)} - 1)^2} \longrightarrow 0, \tag{3.45}$$

$$\frac{\kappa_3^{(n)}}{\sqrt{\frac{\kappa_4^{(n)}}{6} + (\kappa_2^{(n)} - 1)^2}} \longrightarrow \alpha \quad \textit{and} \quad \frac{\kappa_8^{(n)}}{\left(\frac{\kappa_4^{(n)}}{6} + (\kappa_2^{(n)} - 1)^2\right)^2} \longrightarrow 0, \tag{3.46}$$

then

$$\frac{P(F_n \leqslant z) - \Phi(z)}{\sqrt{\frac{\kappa_4^{(n)}}{6} + (\kappa_2^{(n)} - 1)^2}} \longrightarrow \frac{\alpha}{3!} \frac{1}{\sqrt{2\pi}} \left(1 - z^2\right) e^{-\frac{z^2}{2}}, \quad \textit{as } n \to \infty. \tag{3.47}$$

In particular, if $\alpha \neq 0$, there exists $c \in (0,1)$ and $n_0 \geqslant 1$ such that, for any $n \geqslant n_0$,

$$\sup_{z \in \mathbb{R}} |P(F_n \leqslant z) - \Phi(z)| \geqslant c \sqrt{\frac{\kappa_4^{(n)}}{6} + (\kappa_2^{(n)} - 1)^2}. \tag{3.48}$$

Remark 3.9 1. If $E(F_n^2) = \kappa_2^{(n)} = 1$, then condition (3.45) becomes immaterial, and the denominators appearing in formula (3.46) involve uniquely $\kappa_4^{(n)}$.

2. By combining (3.46) with (3.47), we have that, as $n \to \infty$,

$$P(F_n \leqslant z) - \Phi(z) \sim \frac{\kappa_3^{(n)}}{3!\sqrt{2\pi}} \left(1 - z^2\right) e^{-\frac{z^2}{2}},$$

whenever $z \neq \pm 1$ and $\alpha \neq 0$.

Proof of Proposition 3.8. First, since $E(F_n) = 0$, we have $\kappa_4^{(n)} = E(F_n^4) - 3\,E(F_n^2)^2$. Thus, the equivalence between $\kappa_4^{(n)} \longrightarrow 0$ and $F_n \xrightarrow{\text{Law}} \mathscr{N}(0,1)$ is a direct consequence of Part A of Theorem 2.6. Now observe that

$$\frac{1}{2}\|DF_n\|^2 - 1 = 2\,I_2(f_n \otimes_1 f_n) + E(F_n^2) - 1 = 2\,I_2(f_n \otimes_1 f_n) + \kappa_2^{(n)} - 1.$$

In particular

$$\mathrm{Var}\left(\frac{1}{2}\|DZ_n\|^2 - 1\right) = 8\,\|f_n \otimes_1 f_n\|^2_{\mathfrak{H}^{\otimes 2}} + \left(\kappa_2^{(n)} - 1\right)^2 = \frac{\kappa_4^{(n)}}{6} + (\kappa_2^{(n)} - 1)^2,$$

where we have used (3.43) in the case $p = 4$ (note that $\langle f \otimes_1^{(3)} f, f\rangle_{\mathfrak{H}^{\otimes 2}} = \|f \otimes_1 f\|^2_{\mathfrak{H}^{\otimes 2}}$). This implies that the quantity $\varphi(n)$ appearing in (1.7) equals indeed $\sqrt{\kappa_4^{(n)}/6 + (\kappa_2^{(n)} - 1)^2}$. To conclude the proof, it is sufficient to apply Proposition 3.6 in the case $q = 2$, by observing that

$$\frac{1 - \kappa_2^{(n)}}{\sqrt{\frac{\kappa_4^{(n)}}{6} + (\kappa_2^{(n)} - 1)^2}} = \frac{1 - 2\|f_n\|^2_{\mathfrak{H}^{\otimes 2}}}{\varphi(n)},$$

and also, by using (3.43), respectively, in the case $p = 3$ and $p = 8$,

$$\frac{\kappa_3^{(n)}}{\sqrt{\frac{\kappa_4^{(n)}}{6} + (\kappa_2^{(n)} - 1)^2}} = \frac{8\langle f_n, f_n \otimes_1 f_n\rangle_{\mathfrak{H}^{\otimes 2}}}{\varphi(n)}$$

and

$$\begin{aligned}\frac{\kappa_8^{(n)}}{\left(\frac{\kappa_4^{(n)}}{6} + (\kappa_2^{(n)} - 1)^2\right)^2} &= 2^7 7! \times \frac{\langle f \otimes_1^{(7)} f, f\rangle_{\mathfrak{H}^{\otimes 2}}}{\varphi(n)^4} \\ &= 2^7 7! \times \frac{\|(f_n \otimes_1 f_n) \otimes_1 (f_n \otimes_1 f_n)\|^2_{\mathfrak{H}^{\otimes 2}}}{\varphi(n)^4}.\end{aligned}$$

□

4 Toeplitz quadratic functionals of continuous-time stationary processes

In this section, we apply our results to establish (possibly optimal) Berry-Esséen bounds in CLTs involving quadratic functionals of continuous-time stationary Gaussian processes. Our results represent a substantial refinement of the CLTs proved in the papers by Ginovyan [7] and Ginovyan and Sahakyan [8], where the authors have extended to a continuous-time setting the discrete-time results by Avram [1], Fox and Taqqu [6] and Giraitis and Surgailis [10]. In the discrete-time case, Berry-Esséen type bounds for CLTs involving special quadratic functionals of stationary Gaussian processes are obtained in [35], and Edgeworth expansions are studied e.g. in [17]. However, to our knowledge, the results proved in this section are the first (exact) Berry-Esséen bounds ever proved in the continuous-time case. Observe that it is not clear whether one can deduce bounds in continuous-time, by using the discrete-time findings of [17] and [35]. We refer the reader to [2] and [11] (and the references therein) for CLTs and one-term Edgeworth expansions concerning quadratic functionals of general discrete-time processes.

Let $X = (X_t)_{t\in\mathbb{R}}$ be a centered real-valued Gaussian process with spectral density $f : \mathbb{R} \to \mathbb{R}$. This means that, for every $u, t \in \mathbb{R}$, one has

$$E(X_u X_{u+t}) := r(t) = \int_{-\infty}^{+\infty} e^{i\lambda t} f(\lambda) d\lambda, \quad t \in \mathbb{R},$$

where $r : \mathbb{R} \to \mathbb{R}$ is the *covariance function* of X. We stress that the density f is necessarily an even function. For $T > 0$, let $Q_T = \iint_{[0,T]^2} \widehat{g}(t-s) X_t X_s dt ds$ where

$$\widehat{g}(t) = \int_{-\infty}^{+\infty} e^{i\lambda t} g(\lambda) d\lambda, \quad t \in \mathbb{R},$$

is the Fourier transform of some integrable *even* function $g : \mathbb{R} \to \mathbb{R}$. The random variable Q_T is customarily called the *Toeplitz quadratic functional* of X, associated with g and T. We also set

$$\tilde{Q}_T = \frac{Q_T - E(Q_T)}{\sqrt{T}} \quad \text{and} \quad \check{Q}_T = \frac{\tilde{Q}_T}{\sigma(T)}$$

with $\sigma(T)^2 = \mathrm{Var}(\tilde{Q}_T)$. The cumulants of $\tilde{Q}_T$ and $\check{Q}_T$ are denoted, respectively, by

$$\tilde{\kappa}_j^{(T)} = \kappa_j(\tilde{Q}_T) \quad \text{and} \quad \check{\kappa}_j^{(T)} = \check{\kappa}_j(\tilde{Q}_T), \quad j \geqslant 1.$$

Given $T > 0$ and $\psi \in \mathrm{L}^1(\mathbb{R})$, we denote by $B_T(\psi)$ the *truncated Toeplitz operator* associated with ψ and T, acting on a square-integrable function u as follows:

$$B_T(\psi)(u)(\lambda) = \int_0^T u(x)\widehat{\psi}(\lambda - x)dx, \quad \lambda \in \mathbb{R},$$

where $\widehat{\psi}$ is the Fourier transform of ψ. Given $\psi, \gamma \in \mathrm{L}^1(\mathbb{R})$, we denote by $B_T(\psi)B_T(\gamma)$ the product of the two operators $B_T(\psi)$ and $B_T(\gamma)$; also, $[B_T(\psi)B_T(\gamma)]^j$, $j \geqslant 1$, is the jth power of $B_T(\psi)B_T(\gamma)$; the symbol $\mathbf{Tr}(U)$ indicates the trace of an operator U.

The following statement collects some of the results proved in [7, 8].

Theorem 4.1 (See [7, 8]) *1. For every $j \geqslant 1$, the jth cumulant of $\tilde{Q}_T$ is given by*

$$\tilde{\kappa}_j^{(T)} = \begin{cases} 0 & \text{if } j = 1 \\ T^{-j/2}\, 2^{j-1}\, (j-1)!\, \mathbf{Tr}[B_T(f)B_T(g)]^j & \text{if } j \geqslant 2. \end{cases}$$

2. Assume that $f \in \mathrm{L}^p(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$ $(p \geq 1)$, that $g \in \mathrm{L}^q(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$ $(q \geq 1)$, and that $\frac{1}{p} + \frac{1}{q} \leqslant \frac{1}{j}$. Then

$$\tilde{\kappa}_j^{(T)} \underset{T\to\infty}{\sim} T^{1-j/2} \times 2^{j-1}(j-1)!(2\pi)^{2j-1} \int_{-\infty}^{+\infty} f(x)^j g(x)^j dx.$$

3. If $\frac{1}{p} + \frac{1}{q} \leqslant \frac{1}{2}$, then

$$\sigma^2(T) = \tilde{\kappa}_2^{(T)} \underset{T\to\infty}{\longrightarrow} \sigma^2(\infty) := 16\pi^3 \int_{-\infty}^{+\infty} f^2(x)g^2(x)dx,$$

and $\check{Q}_T \overset{\text{Law}}{\longrightarrow} Z \sim \mathscr{N}(0,1)$ as $T \to \infty$.

The next statement shows that one can apply Proposition 3.8 in order to obtain Berry-Esséen bounds for the CLT appearing at Point 3 of Theorem 4.1. Observe that, since the variance of $\check{Q}_T$ is equal to 1 by construction, to establish an upper bound we need to control uniquely the fourth cumulant of $\check{Q}_T$: this will be done by using Point 2 of Theorem 4.1 and by assuming that $\frac{1}{p} + \frac{1}{q} \leqslant \frac{1}{4}$. On the other hand, to prove lower bounds one needs to have a precise estimate of the asymptotic behaviour of the eighth cumulant of $\check{Q}_T$: in view again of Point 2 of Theorem 4.1, this requires that $\frac{1}{p} + \frac{1}{q} \leqslant \frac{1}{8}$.

Theorem 4.2 *Assume that $f \in \mathrm{L}^p(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$ $(p \geqslant 1)$ and that $g \in \mathrm{L}^q(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$ $(q \geqslant 1)$. Let $\Phi(z) = P(N \leqslant z)$, where $N \sim \mathscr{N}(0,1)$.*

1. If $\frac{1}{p} + \frac{1}{q} \leqslant \frac{1}{4}$, there exists $C = C(f,g) > 0$ such that, for all $T > 0$, we have

$$\sup_{z\in\mathbb{R}} \left| P(\check{Q}_T \leqslant z) - \Phi(z) \right| \leqslant \frac{C}{\sqrt{T}}.$$

2. If $\frac{1}{p}+\frac{1}{q}\leqslant\frac{1}{8}$ *and*

$$\int_{-\infty}^{+\infty} f^3(x)g^3(x)dx \neq 0,$$

there exists $c=c(f,g)>0$ *and* $T_0=T_0(f,g)$ *such that* $T\geqslant T_0$ *implies*

$$\sup_{z\in\mathbb{R}}\big|P(\check{Q}_T\leqslant z)-\Phi(z)\big|\geqslant\frac{c}{\sqrt{T}}.$$

More precisely, for any $z\in\mathbb{R}$*, we have*

$$\sqrt{T}\big(P(\check{Q}_T\leqslant z)-\Phi(z)\big)\underset{T\to\infty}{\longrightarrow}\sqrt{\frac{2}{3}}\frac{\int_{-\infty}^{+\infty}f^3(x)g^3(x)dx}{(\int_{-\infty}^{+\infty}f^2(x)g^2(x)dx)^{3/2}}\left(1-z^2\right)e^{-\frac{z^2}{2}}. \quad (4.49)$$

Proof. It is a standard result that each random variable $\check{Q}_T$ can be represented as a double Wiener-It integral with respect to X. It follows that the statement can be proved by means of Proposition 3.8. Now, whenever $\frac{1}{p}+\frac{1}{q}\leqslant\frac{1}{j}$, one easily obtains from Points 2 and 3 in Theorem 4.1 that

$$\check{\kappa}_j^{(T)}\underset{T\to\infty}{\sim}T^{1-\frac{j}{2}}\frac{2^{j-1}\,(j-1)!\,(2\pi)^{2j-1}}{(16\pi^3)^{\frac{j}{2}}}\frac{\int_{-\infty}^{+\infty}f^j(x)g^j(x)dx}{\left(\int_{-\infty}^{+\infty}f^2(x)g^2(x)dx\right)^{\frac{j}{2}}}, \quad (4.50)$$

and the desired conclusion is then obtained by a direct application of Proposition 3.8. In particular, Point 1 in the statement is immediately deduced from the fact that $\frac{1}{p}+\frac{1}{q}\leqslant\frac{1}{4}$, from relation (4.50) and from the bound (3.44), with $\check{\kappa}_4^{(T)}$ replacing $\kappa_4^{(n)}$ (observe that $\check{\kappa}_2^{(T)}=1$ by construction). On the other hand, Point 2 is a consequence of the fact that, if $\frac{1}{p}+\frac{1}{q}\leqslant\frac{1}{8}$, then (4.50) implies that condition (3.46) is met. The exact value of the constant appearing on the RHS of (4.49) is deduced from elementary simplifications. □

5 Exploding quadratic functionals of a Brownian sheet

In this section, we apply our results to the study of some quadratic functionals of a standard Brownian sheet on $[0,1]^d$ ($d\geqslant 1$), noted $\mathbf{W}=\{\mathbf{W}(t_1,...,t_d):(t_1,...,t_d)\in[0,1]^d\}$. We recall that $\mathbf{W}$ is a centered Gaussian process such that, for every $(t_1,...,t_d)$, $(u_1,...,u_d)\in[0,1]^d$,

$$E[\mathbf{W}(t_1,...,t_d)\mathbf{W}(u_1,...,u_d)]=\prod_{i=1,...,d}\min(u_i,t_i),$$

so that, if $d=1$, the process $\mathbf{W}$ is indeed a standard Brownian motion on $[0,1]$. It is easily proved that, for every $d\geqslant 1$, the Gaussian space generated by $\mathbf{W}$ can be identified with an isonormal Gaussian process of the type $X=\{X(f):f\in\mathrm{L}^2([0,1]^d,d\lambda)\}$, where $d\lambda$ indicates the restriction of Lebesgue measure on $[0,1]^d$. It is also well-known that the trajectories of $\mathbf{W}$ enjoy the following remarkable property:

$$\int_{[0,1]^d}\left(\frac{\mathbf{W}(t_1,...,t_d)}{t_1\cdots t_d}\right)^2dt_1\cdots dt_d=+\infty,\quad \text{a.s.-}P. \quad (5.51)$$

Relation (5.51) is a consequence of the scaling properties of $\mathbf{W}$ and of the well-known *Jeulin's Lemma* (see [14, Lemma 1, p. 44] or [24]). In the case $d = 1$, the study of phenomena such as (5.51) arose at the end of the seventies, in connection with the theory of enlargement of filtrations (see [14, 15]); see also [16] for some relations with non-canonical representations of Gaussian processes.

Now denote, for every $\varepsilon > 0$,

$$B_\varepsilon^d = \left\{ \int_{[\varepsilon,1]^d} \left(\frac{\mathbf{W}(t_1, ..., t_d)}{t_1 \cdots t_d} \right)^2 dt_1 \cdots dt_d \right\} - \left(\log \frac{1}{\varepsilon} \right)^d ,$$

and observe that B_ε^d is a centered random variable with moments of all orders. The CLT stated in the forthcoming Proposition gives some insights into the "rate of explosion around zero" of the random function

$$(t_1, ..., t_d) \to \left(\frac{\mathbf{W}(t_1, ..., t_d)}{t_1 \cdots t_d} \right)^2 .$$

Proposition 5.1 *For every $d \geqslant 1$, as $\varepsilon \to 0$,*

$$\widetilde{B}_\varepsilon^d := (4 \log 1/\varepsilon)^{-d/2} \times B_\varepsilon^d \xrightarrow{\text{Law}} N \sim \mathscr{N}(0, 1). \tag{5.52}$$

Proposition 5.1 has been established in [27] (for the case $d = 1$), [5] (for the case $d = 2$) and [23] (for the case $d > 2$). See [27, 28] for an application of the CLT (5.52) (in the case $d = 1$) to the study of Brownian local times. See [5] for some applications to conditioned bivariate Gaussian processes and to statistical tests of independence. The next result, which is obtained by means of the techniques developed in this paper, gives an exact description (in terms of the Kolmogorov distance) of the rate of convergence of $\widetilde{B}_\varepsilon^d$ towards a Gaussian random variable.

Proposition 5.2 *For every $d \geqslant 1$, there exist constants $0 < c(d) < C(d) < +\infty$ and $0 < \eta(d) < 1$, depending uniquely on d, such that, for every $\varepsilon > 0$,*

$$d_{\text{Kol}}[\widetilde{B}_\varepsilon^d, N] \leqslant C(d)(\log 1/\varepsilon)^{-d/2}$$

and, for $\varepsilon < \eta(d)$,

$$d_{\text{Kol}}[\widetilde{B}_\varepsilon^d, N] \geqslant c(d)(\log 1/\varepsilon)^{-d/2}.$$

Proof. We denote by

$$\widetilde{\kappa}_j(d, \varepsilon), \quad j = 1, 2, ...,$$

the sequence of the cumulants of the random variable $\widetilde{B}_\varepsilon^d$. We deal separately with the cases $d = 1$ and $d \geqslant 2$.

(Case $d = 1$) In this case, $\mathbf{W}$ is a standard Brownian motion on $[0, 1]$, so that $\widetilde{B}_\varepsilon^1$ takes the form $\widetilde{B}_\varepsilon^1 = I_2(f_\varepsilon)$, where I_2 is the double Wiener-It integral with respect to $\mathbf{W}$, and

$$f_\varepsilon(x, y) = (4 \log 1/\varepsilon)^{-1/2} \left[(x \vee y \vee \varepsilon)^{-1} - 1 \right]. \tag{5.53}$$

Lengthy (but standard) computations yield the following estimates: as $\varepsilon \to 0$

$$\begin{aligned} \widetilde{\kappa}_2(1, \varepsilon) &\longrightarrow 1, \\ \widetilde{\kappa}_j(1, \varepsilon) &\asymp \left(\log \frac{1}{\varepsilon} \right)^{1-j/2}, \quad j \geqslant 3. \end{aligned}$$

The conclusion now follows from Proposition 3.8.
(Case $d \geqslant 2$) In this case, $\widetilde{B}^d_\varepsilon$ has the form $\widetilde{B}^d_\varepsilon = I_2(f^d_\varepsilon)$, with

$$f^d_\varepsilon(x_1, ..., x_d; y_1, ..., y_d) = (4 \log 1/\varepsilon)^{-d/2} \prod_{j=1}^{d} [(x_j \vee y_j \vee \varepsilon)^{-1} - 1]. \tag{5.54}$$

By using (3.43), one sees that the following relation holds

$$(2^{j-1}(j-1)!)^{-1} \times \widetilde{\kappa}_j(d, \varepsilon) = [(2^{j-1}(j-1)!)^{-1} \times \widetilde{\kappa}_j(1, \varepsilon)]^d,$$

so that the conclusion derives once again from Proposition 3.8. □

6 Exact asymptotics in the Breuer-Major CLT

Let B be a fractional Brownian motion (fBm) with Hurst index $H \in (0, \frac{1}{2})$, that is, $\{B_x : x \geqslant 0\}$ is a centered Gaussian process with covariance given by

$$E(B_x B_y) = \frac{1}{2}\big(x^{2H} + y^{2H} - |x-y|^{2H}\big), \quad x, y \geqslant 0.$$

It is well-known that, for every choice of the parameter $H \in (0, \frac{1}{2})$, the Gaussian space generated by B can be identified with an isonormal Gaussian process of the type $X = \{X(h) : h \in \mathfrak{H}\}$, where the real and separable Hilbert space $\mathfrak{H}$ is defined as follows: (i) denote by $\mathscr{E}$ the set of all $\mathbb{R}$-valued step functions on $\mathbb{R}_+$, (ii) define $\mathfrak{H}$ as the Hilbert space obtained by closing $\mathscr{E}$ with respect to the scalar product

$$\big\langle \mathbf{1}_{[0,x]}, \mathbf{1}_{[0,y]} \big\rangle_{\mathfrak{H}} = E(B_x B_y).$$

Such a construction implies in particular that $B_x = X(\mathbf{1}_{[0,x]})$. The reader is referred e.g. to [21] for more details on fBm, including crucial connections with fractional operators. We also define $\rho(\cdot)$ to be the covariance function associated with the stationary process $x \mapsto B_{x+1} - B_x$, that is,

$$\rho(x) := E[(B_{t+1} - B_t)(B_{t+x+1} - B_{t+x})] = \frac{1}{2}\big(|x+1|^{2H} + |x-1|^{2H} - 2|x|^{2H}\big), \quad x \in \mathbb{R}.$$

Now fix an *even* integer $q \geqslant 2$ and set

$$Z_T := \frac{1}{\sigma(T)\sqrt{T}} \int_0^T H_q(B_{u+1} - B_u) du, \quad T > 0,$$

where H_q is the qth Hermite polynomial defined in (2.24), and where

$$\sigma(T) := \sqrt{\mathrm{Var}\left(\frac{1}{\sqrt{T}} \int_0^T H_q(B_{u+1} - B_u) du\right)} = \sqrt{\frac{q!}{T} \int_{[0,T]^2} \rho^q(u-v) du dv}.$$

Observe that each Z_T can be represented as a multiple Wiener-It integral of order q, and also that

$$\sigma^2(T) \underset{T \to \infty}{\longrightarrow} \sigma^2(\infty) := q! \int_{-\infty}^{+\infty} \rho^q(x) dx < +\infty.$$

According e.g. to the main results in [3] or [9], one always has the following CLT

$$Z_T \xrightarrow[T\to\infty]{\text{Law}} Z \sim \mathscr{N}(0,1)$$

(which also holds for odd values of q). The forthcoming Theorem 6.1 shows that the techniques of this paper may be used to deduce an exact asymptotic relation (as $T \to \infty$) for the difference $P(Z_T \leqslant z) - \Phi(z)$, where $\Phi(z) = P(N \leqslant z)$ ($N \sim \mathscr{N}(0,1)$). We stress that the main results of this section deal with the case of a generic Hermite polynomial of even order $q \geqslant 2$, implying that our techniques provide explicit results even outside the framework of *quadratic* functionals, as the ones analyzed in Sections 4 and 5. In what follows, we use the notation

$$\begin{aligned}\widehat{\sigma}^2(\infty) \quad := \quad & \frac{q^2}{\sigma^4(\infty)} \sum_{s=1}^{q-1} (s-1)!^2 \binom{q-1}{s-1}^4 (2q-2s)! \\ & \times \int_{\mathbb{R}^3} \rho^s(x_1)\rho^s(x_2)\rho^{q-s}(x_3)\rho^{q-s}(x_2+x_3-x_1)dx_1dx_2dx_3,\end{aligned} \tag{6.55}$$

and

$$\widehat{\gamma}(\infty) = -\frac{q!(\frac{q}{2})!\binom{q}{q/2}^2}{2\,\sigma^3(\infty)} \int_{\mathbb{R}^2} \rho^{\frac{q}{2}}(x)\rho^{\frac{q}{2}}(y)\rho^{\frac{q}{2}}(x-y)dxdy. \tag{6.56}$$

Theorem 6.1 *There exists a constant $C > 0$ such that*

$$d_{\text{Kol}}(Z_T, N) = \sup_{z\in\mathbb{R}} |P(Z_T \leqslant z) - \Phi(z)| \leqslant \frac{C}{\sqrt{T}}. \tag{6.57}$$

Moreover, for any fixed $z \in \mathbb{R}$, we have

$$\sqrt{T}\big(P(Z_T \leqslant z) - \Phi(z)\big) \underset{T\to\infty}{\longrightarrow} \frac{\widehat{\gamma}(\infty)}{3}(z^2-1)\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}} \tag{6.58}$$

Proof. The proof is divided into three steps.

Step 1. Let us first prove the following convergence:

$$\sqrt{T}\left(\frac{1}{q}\|DZ_T\|^2_{\mathfrak{H}} - 1\right) \xrightarrow[T\to\infty]{\text{Law}} \mathscr{N}\big(0, \widehat{\sigma}^2(\infty)\big), \tag{6.59}$$

where $\widehat{\sigma}^2(\infty)$ is given by (6.55). Note that, once (6.59) is proven to be true, one deduces immediately that, as $T \to \infty$,

$$\text{Var}\left(\frac{1}{q}\|DZ_T\|^2_{\mathfrak{H}} - 1\right) \sim \frac{\widehat{\sigma}^2(\infty)}{T},$$

so that (6.57) follows from Theorem 2.4. Now, to prove that (6.59) holds, start by using the well-known relation between Hermite polynomials and multiple integrals to write

$$H_q(B_{u+1} - B_u) = I_q(\mathbf{1}^{\otimes q}_{[u,u+1]}).$$

As a consequence, we have

$$DZ_T = \frac{q}{\sigma(T)\sqrt{T}} \int_0^T I_{q-1}(\mathbf{1}_{[u,u+1]}^{\otimes q-1})\mathbf{1}_{[u,u+1]} du.$$

Thus, by an appropriate use of the multiplication formula (2.14), one has that

$$\begin{aligned}
\|DZ_T\|_{\mathfrak{H}}^2 &= \frac{q^2}{\sigma^2(T)T} \int_{[0,T]^2} \rho(u-v) I_{q-1}(\mathbf{1}_{[u,u+1]}^{\otimes q-1}) I_{q-1}(\mathbf{1}_{[v,v+1]}^{\otimes q-1}) du dv \\
&= \frac{q^2}{\sigma^2(T)T} \int_{[0,T]^2} \sum_{r=0}^{q-1} r! \binom{q-1}{r}^2 I_{2q-2-2r}(\mathbf{1}_{[u,u+1]}^{\otimes q-1-r} \otimes \mathbf{1}_{[v,v+1]}^{\otimes q-1-r}) \rho^{r+1}(u-v) du dv \\
&= \frac{q^2}{\sigma^2(T)T} \sum_{s=1}^{q} (s-1)! \binom{q-1}{s-1}^2 \int_{[0,T]^2} I_{2q-2s}(\mathbf{1}_{[u,u+1]}^{\otimes q-s} \otimes \mathbf{1}_{[v,v+1]}^{\otimes q-s}) \rho^s(u-v) du dv,
\end{aligned}$$

yielding

$$\frac{1}{q}\|DZ_T\|_{\mathfrak{H}}^2 - 1 = \frac{q}{\sigma^2(T)T} \sum_{s=1}^{q-1} (s-1)! \binom{q-1}{s-1}^2 \int_{[0,T]^2} I_{2q-2s}(\mathbf{1}_{[u,u+1]}^{\otimes q-s} \otimes \mathbf{1}_{[v,v+1]}^{\otimes q-s}) \rho^s(u-v) du dv.$$

We shall first prove that, for every $s \in \{1, \ldots, q-1\}$,

$$\frac{1}{\sqrt{T}} \int_{[0,T]^2} I_{2q-2s}\big(\mathbf{1}_{[u,u+1]}^{\otimes(q-s)} \otimes \mathbf{1}_{[v,v+1]}^{\otimes(q-s)}\big) \rho^s(u-v) du dv \xrightarrow[T\to\infty]{\text{Law}} \mathscr{N}\big(0, \widehat{\sigma}_s^2(\infty)\big) \tag{6.60}$$

where

$$\widehat{\sigma}_s^2(\infty) := (2q-2s)! \int_{\mathbb{R}^3} \rho^s(x_1)\rho^s(x_2)\rho^{q-s}(x_3)\rho^{q-s}(x_2+x_3-x_1) dx_1 dx_2 dx_3.$$

Fix $s \in \{1, \ldots, q-1\}$. Observe first that

$$\begin{aligned}
\widehat{\sigma}_s^2(T) &:= \mathrm{Var}\left(\frac{1}{\sqrt{T}} \int_{[0,T]^2} I_{2q-2s}\big(\mathbf{1}_{[u,u+1]}^{\otimes(q-s)} \otimes \mathbf{1}_{[v,v+1]}^{\otimes(q-s)}\big) \rho^s(u-v) du dv \right) \\
&= \frac{(2q-2s)!}{T} \int_{[0,T]^4} \rho^s(u-v)\rho^s(w-z)\rho^{q-s}(u-w)\rho^{q-s}(v-z) du dv dw dz \\
&\xrightarrow[T\to\infty]{} \widehat{\sigma}_s^2(\infty)
\end{aligned}$$

so that (6.60) holds if, and only if, the following convergence takes place:

$$Q_T^{(s)} := \frac{1}{\widehat{\sigma}_s(T)\sqrt{T}} \int_{[0,T]^2} I_{2q-2s}\big(\mathbf{1}_{[u,u+1]}^{\otimes(q-s)} \otimes \mathbf{1}_{[v,v+1]}^{\otimes(q-s)}\big) \rho^s(u-v) du dv \xrightarrow[T\to\infty]{\text{Law}} \mathscr{N}(0,1). \tag{6.61}$$

We have

$$DQ_T^{(s)} = \frac{2q-2s}{\widehat{\sigma}_s(T)\sqrt{T}} \int_{[0,T]^2} \rho^s(u-v) I_{2q-2s-1}\big(\mathbf{1}_{[u,u+1]}^{\otimes(q-s-1)} \otimes \mathbf{1}_{[v,v+1]}^{\otimes(q-s)}\big) \mathbf{1}_{[u,u+1]} du dv.$$

Thus $\|DQ_T^{(s)}\|_{\mathfrak{H}}^2$ is given by

$$
\begin{aligned}
& \frac{(2q-2s)^2}{\widehat{\sigma}_s^2(T)\,T}\int_{[0,T]^4}\rho^s(u_1-u_2)\rho^s(u_3-u_4)\rho(u_1-u_3) \\
& \times I_{2q-2s-1}\big(\mathbf{1}_{[u_1,u_1+1]}^{\otimes(q-s-1)}\otimes\mathbf{1}_{[u_2,u_2+1]}^{\otimes(q-s)}\big)I_{2q-2s-1}\big(\mathbf{1}_{[u_3,u_3+1]}^{\otimes(q-s-1)}\otimes\mathbf{1}_{[u_4,u_4+1]}^{\otimes(q-s)}\big)du_1\dots du_4 \\
= \; & \frac{(2q-2s)^2}{\widehat{\sigma}_s^2(T)\,T}\int_{[0,T]^4}\rho^s(u_1-u_2)\rho^s(u_3-u_4)\rho(u_1-u_3) \\
& \times\left(\sum_{t=0}^{q-s}t!\binom{2q-2s-1}{t}^2 I_{4q-4s-2-2t}\big(\mathbf{1}_{[u_1,u_1+1]}^{\otimes(q-s-1)}\otimes\mathbf{1}_{[u_3,u_3+1]}^{\otimes(q-s-1)}\otimes\mathbf{1}_{[u_2,u_2+1]}^{\otimes(q-s-t)}\otimes\mathbf{1}_{[u_4,u_4+1]}^{\otimes(q-s-t)}\big)\right. \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\times\,\rho^t(u_2-u_4)\right)du_1\dots du_4 \\
+ \; & \frac{(2q-2s)^2}{\widehat{\sigma}_s^2(T)\,T}\int_{[0,T]^4}\rho^s(u_1-u_2)\rho^s(u_3-u_4)\rho(u_1-u_3) \\
& \times\left(\sum_{t=q-s+1}^{2q-2s-1}t!\binom{2q-2s-1}{t}^2 I_{4q-4s-2-2t}\big(\mathbf{1}_{[u_1,u_1+1]}^{\otimes(2q-2s-1-t)}\otimes\mathbf{1}_{[u_3,u_3+1]}^{\otimes(2q-2s-1-t)}\big)\right. \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\times\,\rho^{q-s}(u_2-u_4)\rho^{t-q+s}(u_1-u_3)\right)du_1\dots du_4.
\end{aligned}
$$

Consequently, $\frac{1}{2q-2s}\|DQ_T^{(s)}\|_{\mathfrak{H}}^2-1$ is given by

$$
\begin{aligned}
& \frac{2q-2s}{\widehat{\sigma}_s^2(T)\,T}\int_{[0,T]^4}\rho^s(u_1-u_2)\rho^s(u_3-u_4)\rho(u_1-u_3)\left(\sum_{t=1}^{q-s+1}(t-1)!\binom{2q-2s-1}{t-1}^2\right. \\
& \left.\times\, I_{4q-4s-2t}\big(\mathbf{1}_{[u_1,u_1+1]}^{\otimes(q-s-1)}\otimes\mathbf{1}_{[u_3,u_3+1]}^{\otimes(q-s-1)}\otimes\mathbf{1}_{[u_2,u_2+1]}^{\otimes(q-s+1-t)}\otimes\mathbf{1}_{[u_4,u_4+1]}^{\otimes(q-s+1-t)}\big)\rho^{t+1}(u_2-u_4)\right)du_1\dots du_4 \\
+ \; & \frac{2q-2s}{\widehat{\sigma}_s^2(T)\,T}\int_{[0,T]^4}\rho^s(u_1-u_2)\rho^s(u_3-u_4)\rho(u_1-u_3)\left(\sum_{t=q-s+2}^{2q-2s-1}(t-1)!\binom{2q-2s-1}{t-1}^2\right. \\
& \left.\times\, I_{4q-4s-2t}\big(\mathbf{1}_{[u_1,u_1+1]}^{\otimes(2q-2s-t)}\otimes\mathbf{1}_{[u_3,u_3+1]}^{\otimes(2q-2s-t)}\big)\rho^{q-s}(u_2-u_4)\rho^{t-q+s-1}(u_1-u_3)\right)du_1\dots du_4.
\end{aligned}
$$

For a fixed t such that $1\leqslant t\leqslant q-s+1$, we have that

$$
\begin{aligned}
& E\left|\frac{1}{T}\int_{[0,T]^4}\rho^s(u_1-u_2)\rho^s(u_3-u_4)\rho(u_1-u_3)\rho^{t+1}(u_2-u_4)\right. \\
& \qquad\left.\times I_{4q-4s-2t}\big(\mathbf{1}_{[u_1,u_1+1]}^{\otimes(q-s-1)}\otimes\mathbf{1}_{[u_3,u_3+1]}^{\otimes(q-s-1)}\otimes\mathbf{1}_{[u_2,u_2+1]}^{\otimes(q-s+1-t)}\otimes\mathbf{1}_{[u_4,u_4+1]}^{\otimes(q-s+1-t)}\big)du_1\dots du_4\right|^2 \\
= \; & \frac{1}{T^2}\int_{[0,T]^8}\rho^s(u_1-u_2)\rho^s(u_3-u_4)\rho(u_1-u_3)\rho^t(u_2-u_4)\rho^s(u_5-u_6)\rho^s(u_7-u_8)\times \\
& \qquad\times\rho(u_5-u_7)\rho^t(u_6-u_8)\rho^{q-s-1}(u_1-u_5)\rho^{q-s-1}(u_3-u_7)\times \\
& \qquad\qquad\times\rho^{q-s+1-t}(u_2-u_6)\rho^{q-s+1-t}(u_4-u_8)du_1\dots du_8 \\
\underset{T\to\infty}{\sim} \; & \frac{1}{T}\int_{\mathbb{R}^7}\rho^s(x_1)\rho^s(x_2)\rho(x_3)\rho^t(x_2+x_3-x_1)\rho^s(x_4)\rho^s(x_5)\rho(x_6)\rho^t(x_5+x_6-x_4)\rho^{q-s-1}(x_7) \\
& \times\rho^{q-s-1}(x_6+x_7-x_3)\rho^{q-s+1-t}(x_4+x_7-x_1)\times \\
& \qquad\times\rho^{q-s+1-t}(x_5+x_6+x_7-x_2-x_3)dx_1\dots dx_7
\end{aligned}
$$

tends to zero as $T \to \infty$. Similarly, we can prove, for a fixed t such that $q-s+2 \leqslant t \leqslant 2q-2s-1$, that

$$E\left|\frac{1}{T}\int_{[0,T]^4}\rho^s(u_1-u_2)\rho^s(u_3-u_4)\rho^{t-q+s}(u_1-u_3)\rho^{q-s}(u_2-u_4)\right.$$
$$\left.\times I_{4q-4s-2t}\big(\mathbf{1}_{[u_1,u_1+1]}^{\otimes(2q-2s-t)}\otimes\mathbf{1}_{[u_3,u_3+1]}^{\otimes(2q-2s-t)}\big)du_1\ldots du_4\right|^2$$

tends to zero as $T \to \infty$. Thanks to the main result in [22], the last relation implies that, for each s, the converge (6.61) holds, and therefore (6.60) is verified. Finally, by combining (6.60) with the results in [25] and [26], we obtain (6.59). Indeed, by using the orthogonality and isometric properties of multiple stochastic integrals, we can write

$$\begin{aligned}
&\mathrm{Var}\left(\sqrt{T}\left(\frac{1}{q}\|DZ_T\|^2_{\mathfrak{H}}-1\right)\right)\\
=\;&\frac{q^2}{\sigma^4(T)T}\sum_{s=1}^{q-1}(s-1)!^2\binom{q-1}{s-1}^4(2q-2s)!\\
&\left\langle\int_{[0,T]^2}\mathbf{1}_{[u,u+1]}^{\otimes(q-s)}\otimes\mathbf{1}_{[v,v+1]}^{\otimes(q-s)}\rho^s(u-v)dudv,\int_{[0,T]^2}\mathbf{1}_{[w,w+1]}^{\otimes(q-s)}\otimes\mathbf{1}_{[z,z+1]}^{\otimes(q-s)}\rho^s(w-z)dwdz\right\rangle_{\mathfrak{H}^{\otimes(2q-2s)}}\\
=\;&\frac{q^2}{\sigma^4(T)T}\sum_{s=1}^{q-1}(s-1)!^2\binom{q-1}{s-1}^4(2q-2s)!\\
&\qquad\times\int_{[0,T]^4}\rho^s(u-v)\rho^s(w-z)\rho^{q-s}(u-w)\rho^{q-s}(v-z)dudvdwdz.
\end{aligned}$$

from which we easily deduce that $\mathrm{Var}\left(\sqrt{T}\left(\frac{1}{q}\|DZ_T\|^2_{\mathfrak{H}}-1\right)\right)\underset{T\to\infty}{\longrightarrow}\widehat{\sigma}^2(\infty)$.

Step 2. Let us prove the following convergence:

$$\left(Z_T,\sqrt{T}\big(\frac{1}{q}\|DZ_T\|^2_{\mathfrak{H}}-1\big)\right)\xrightarrow[T\to\infty]{\text{Law}}(U,V) \tag{6.62}$$

with (U,V) a centered Gaussian vector such that $E(U^2)=1$,

$$E(V^2)=\widehat{\sigma}^2(\infty)\quad\text{and}\quad E(UV)=-\widehat{\gamma}(\infty)=\frac{q!(\frac{q}{2})!\binom{q}{q/2}^2}{2\,\sigma^3(\infty)}\int_{\mathbb{R}^2}\rho^{\frac{q}{2}}(x)\rho^{\frac{q}{2}}(y)\rho^{\frac{q}{2}}(x-y)dxdy,$$

Observe that we already know that $Z_T\overset{\text{Law}}{\to}U$ and also that (6.59) is verified. Note also that we have proved (6.59) by first decomposing $\sqrt{T}\big(q^{-1}\|DZ_T\|^2_{\mathfrak{H}}-1\big)$ into a finite sum of multiple integrals, and then by showing that each multiple integral satisfies an appropriate CLT. As a consequence, according to Part B of Theorem 2.6 (with G_n replaced by $\sqrt{T}\big(q^{-1}\|DZ_T\|^2_{\mathfrak{H}}-1\big)$) it is sufficient to show the following convergence:

$$E\left(Z_T\times\sqrt{T}\big(\frac{1}{q}\|DZ_T\|^2_{\mathfrak{H}}-1\big)\right)\underset{T\to\infty}{\longrightarrow}\frac{q!(\frac{q}{2})!\binom{q}{q/2}^2}{2\,\sigma^3(\infty)}\int_{\mathbb{R}^2}\rho^{\frac{q}{2}}(x)\rho^{\frac{q}{2}}(y)\rho^{\frac{q}{2}}(x-y)dxdy. \tag{6.63}$$

By the orthogonality of multiple stochastic integrals, we can write

$$
\begin{aligned}
& E\left(Z_T \times \sqrt{T}\big(\frac{1}{q}\|DZ_T\|^2_{\mathfrak{H}} - 1\big)\right) \\
= \quad & \frac{q}{\sigma^3(T)T}\left(\frac{q}{2}-1\right)!\binom{q-1}{\frac{q}{2}-1}^2 \\
& \qquad \times \int_{[0,T]^3} \rho^{\frac{q}{2}}(u-v)\, E\left(I_q\big(\mathbf{1}^{\otimes q}_{[w,w+1]}\big) I_q\big(\mathbf{1}^{\otimes \frac{q}{2}}_{[u,u+1]} \otimes \mathbf{1}^{\otimes \frac{q}{2}}_{[v,v+1]}\big)\right) dudvdw \\
= \quad & \frac{q\,q!}{\sigma^3(T)T}\left(\frac{q}{2}-1\right)!\binom{q-1}{\frac{q}{2}-1}^2 \int_{[0,T]^3} \rho^{\frac{q}{2}}(u-v)\rho^{\frac{q}{2}}(u-w)\rho^{\frac{q}{2}}(w-v)dudvdw \\
\underset{T\to\infty}{\to} \quad & \frac{q\,q!}{\sigma^3(\infty)}\left(\frac{q}{2}-1\right)!\binom{q-1}{\frac{q}{2}-1}^2 \int_{\mathbb{R}^2} \rho^{\frac{q}{2}}(x)\rho^{\frac{q}{2}}(y)\rho^{\frac{q}{2}}(x-y)dxdy \\
= \quad & \frac{q!(\frac{q}{2})!\binom{q}{q/2}^2}{2\,\sigma^3(\infty)} \int_{\mathbb{R}^2} \rho^{\frac{q}{2}}(x)\rho^{\frac{q}{2}}(y)\rho^{\frac{q}{2}}(x-y)dxdy.
\end{aligned}
$$

Step 3. Step 1 and convergence (6.62) imply that, as $T \to \infty$,

$$\varphi(T) \sim \frac{\widehat{\sigma}(\infty)}{\sqrt{T}},$$

where $\varphi(T) = \mathrm{Var}\left(1 - q^{-1}\|DZ_T\|^2_{\mathfrak{H}} - 1\right)$, and

$$\left(Z_T, \frac{1 - \frac{1}{q}\|DZ_T\|^2_{\mathfrak{H}}}{\varphi(T)}\right) \xrightarrow[T\to\infty]{\text{Law}} (U, \widehat{\sigma}(\infty)^{-1}V).$$

As a consequence, one can apply Theorem 3.1 in the case $\rho = \frac{\widehat{\gamma}(\infty)}{\widehat{\sigma}(\infty)}$ (the remaining assumptions are easily verified), yielding that

$$\varphi(T)^{-1}\big(P(Z_T \leqslant z) - \Phi(z)\big) \underset{T\to\infty}{\longrightarrow} \frac{\widehat{\gamma}(\infty)}{3\widehat{\sigma}(\infty)}(z^2-1)\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}},$$

from which the conclusion follows. □

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References

[1] F. Avram (1988). On bilinear forms in Gaussian random variables and Toeplitz matrices. *Probab. Theory Related Fields* **79**, 37-45.

[2] R.J. Bhansalia, L. Giraitis and P.S. Kokoszka (2007). Approximations and limit theory for quadratic forms of linear processes. *Stochastic Process. Appl.*, **117**, 71-95.

[3] P. Breuer and P. Major (1983). Central limit theorems for nonlinear functionals of Gaussian fields. *J. Multivariate Anal.* **13** (3), 425-441.

[4] L. Chen and Q.-M. Shao (2005). Stein's method for normal approximation. In: *An introduction to Stein's method*, 1-59. Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap. **4**, Singapore Univ. Press, Singapore, 2005.

[5] P. Deheuvels, G. Peccati and M. Yor (2006). On quadratic functionals of the Brownian sheet and related processes. *Stochastic Process. Appl.* **116**, 493-538.

[6] R. Fox and M.S. Taqqu (1987). Central limit theorems for quadratic forms in random variables having long-range dependence. *Probab. Theory Related Fields* **74**, 213-240.

[7] M.S. Ginovyan (1994). On Toeplitz type quadratic functionals in Gaussian stationary process. *Probab. Theory Related Fields* **100**, 395-406.

[8] M.S. Ginovyan and A.A. Sahakyan (2007). Limit theorems for Toeplitz quadratic functionals of continuous-time stationary processes. *Probab. Theory Related Fields* **138**, no. 3-4, 551-579.

[9] L. Giraitis and D. Surgailis (1985). CLT and other limit theorems for functionals of Gaussian processes. *Z. Wahrsch. verw. Geb.* **70**, 191-212.

[10] L. Giraitis and D. Surgailis (1990). A central limit theorem for quadratic forms in strongly dependent linear variables and its application to asymptotical normality of Whittle's estimate. *Probab. Theory Related Fields* **86**, 87-104.

[11] F. Gtze, A.N. Tikhomirov and V.A. Yurchenko (2007). Asymptotic expansion in the central limit theorem for quadratic forms. *Journal of Mathematical Sciences* **147**(4), 6891-6911.

[12] P. Hall (1992). *Bootstrap and Edgeworth Expansions.* Springer-Verlag, Berlin Heidelberg New York.

[13] S. Janson (1997). *Gaussian Hilbert Spaces.* Cambridge University Press, Cambridge.

[14] Th. Jeulin (1980). *Semimartingales et Grossissement d'une Filtration*, Lecture Notes in Mathematics **833**. Springer-Verlag, Berlin Heidelberg New York.

[15] Th. Jeulin and M. Yor (1979). Inégalité de Hardy, semi-martingales et faux-amis. In: *Séminaire de Probabilités XIII*, Lecture Notes in Mathematics **721**, Springer-Verlag, Berlin Heidelberg New York, pp. 332-359.

[16] Th. Jeulin and M. Yor (1992). Une dcomposition non-canonique du drap brownien. *Sminaire de Probabilits, XXVI*, 322–347, Lecture Notes in Mathematics **1526**, Springer, Berlin.

[17] O. Lieberman, J. Rousseau and D.M. Zucker (2001). Valid Edgeworth expansion for the sample autocorrelation function under long dependence. *Econometric Theory* **17**, 257-275.

[18] P. McCullagh (1987). *Tensor Methods in Statistics.* Chapman and Hall. London.

[19] I. Nourdin and G. Peccati (2009a). Non-central convergence of multiple integrals. To appear in: *Ann. Probab.*

[20] I. Nourdin and G. Peccati (2009b). Stein's method on Wiener chaos. To appear in: *Probab. Theory Related Fields.*

[21] D. Nualart (2006). *The Malliavin calculus and related topics.* Springer-Verlag, Berlin, 2nd edition.

[22] D. Nualart and S. Ortiz-Latorre (2008). Central limit theorems for multiple stochastic integrals and Malliavin calculus. *Stochastic Process. Appl.* **118** (4), 614-628.

[23] D. Nualart and G. Peccati (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33** (1), 177-193.

[24] G. Peccati (2001). On the convergence of multiple random integrals. *Studia Sc. Mat. Hungarica*, **37**, 429-470.

[25] G. Peccati (2007). Gaussian approximations of multiple integrals. *Electron. Comm. Prob* **12**, 350-364 (electronic).

[26] G. Peccati and C.A. Tudor (2005). Gaussian limits for vector-valued multiple stochastic integrals. In: *Séminaire de Probabilités XXXVIII* , 247-262. Lecture Notes in Mathematics **1857**, Springer-Verlag, Berlin.

[27] G. Peccati et M. Yor (2004a). Hardy's inequality in $L^2([0,1])$ and principal values of Brownian local times. *Asymptotic Methods in Stochastics*, AMS, Fields Institute Communications Series, 49-74.

[28] G. Peccati et M. Yor (2004b). Four limit theorems for quadratic functionals of Brownian motion and Brownian bridge. *Asymptotic Methods in Stochastics*, AMS, Fields Institute Communication Series, 75-87.

[29] G. Reinert (2005). Three general approaches to Stein's method. In: *An introduction to Stein's method*, 183-221. Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap. **4**, Singapore Univ. Press, Singapore.

[30] G.C. Rota and T.C. Wallstrom (1997) Stochastic integrals: a combinatorial approach. *Ann. Probab.* **25**(3), 1257–1283

[31] V. Rotar (2005). Stein's method, Edgeworth's expansions and a formula of Barbour. In: *In Stein's method and applications*, 59-84. Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap. **5**, Singapore Univ. Press, Singapore.

[32] I. Shigekawa (1978). Absolute continuity of probability laws of Wiener functionals. *Proc. Japan Acad. Ser. A Math. Sci* **54** (8), 230-233.

[33] Ch. Stein (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In: *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Vol. II: Probability theory*, 583-602. Univ. California Press, Berkeley, Calif..

[34] Ch. Stein (1986). *Approximate computation of expectations.* Institute of Mathematical Statistics Lecture Notes – Monograph Series, **7**. Institute of Mathematical Statistics, Hayward, CA.

[35] M. Taniguchi (1986). Berry-Esséen theorems for quadratic forms of Gaussian stationary processes. *Probab. Theory Related Fields* **72**, 185-194.