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Abstract

The goal of this paper is to describe in mathematical terms the effect on the ocean circulation of a random stationary wind stress at the surface of the ocean. In order to avoid singular behaviour, non-resonance hypotheses are introduced, which ensure that the time frequencies of the wind-stress are different from that of the Earth rotation. We prove a convergence result for a three-dimensional Navier-Stokes-Coriolis system in a bounded domain, in the asymptotic of fast rotation and vanishing vertical viscosity, and we exhibit some random and stationary boundary layer profiles. At last, an average equation is derived for the limit system in the case of the non-resonant torus.

Key words. Rotating fluids, Oceanic circulation, Stationary setting, Boundary layer.

AMS subjects classification. 76U05, 35B40, 86A05, 35R60.

1 Introduction

The goal of this paper is to study mathematically a problem arising in ocean dynamics, namely the behaviour of ocean currents under stimulation by the wind. Following the books by Pedlosky [16, 17] and Gill [8], the velocity of the fluid in the ocean, denoted by u, is described by the incompressible Navier-Stokes equations in three dimensions, in rotating coordinates, with Coriolis force

$$\begin{aligned} \rho(\partial_t u + u\cdot\nabla u + 2\Omega e\wedge u) - A_h\Delta_h u - A_v\partial_z^2 u &= \nabla p, \quad t>0,\ (x,y,z)\in U(t)\subset\mathbb{R}^3, \\ \mathrm{div} u &= 0. \end{aligned}$$

In the above equation, A_h and A_v are respectively the horizontal and vertical turbulent viscosities, p is the pressure inside the fluid, ρ is the homogeneous and constant density, and Ωe is the rotation vector of the Earth ($\Omega>0$ and e is a unitary vector, parallel to the pole axis, oriented from South to North). $U(t)$ is an open set in $\mathbb{R}^3$, depending on the time variable t: indeed, the interface between the ocean and the atmosphere may be moving, and is described in general by a free surface $z=h(t)$.

In order to focus on the influence of the wind, let us now make a series of crude modeling hypotheses on the boundary conditions: first, we assume that the lateral boundaries of the ocean are flat, and that the velocity u satisfies periodic boundary conditions in the horizontal variable. We also neglect the fluctuations of the free surface, namely, we assume that $h(t)\equiv aD$, with a, D positive constants. This approximation, although highly unrealistic, is justified by the fact that the behaviour of the fluid around the surface is in general very turbulent. Hence, as emphasized in [5], only a modelization is tractable and meaningful. Let us also mention that the justification of this rigid lid approximation starting from a free surface is open from a mathematical point of view. At last, we assume that the bottom of the ocean is flat; the case of a nonflat bottom has already been investigated by several authors, and we refer to [5, 7, 14] for more details regarding that point.

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As a consequence, we assume that $U(t) = [0, a_1 L) \times [0, a_2 L) \times [0, aD]$, where $L > 0$ is the typical horizontal lengthscale, and u satisfies the following boundary conditions

$$
\begin{aligned}
&u \text{ is periodic in the horizontal variable with period } [0, a_1 L) \times [0, a_2 L),\\
&u_{|z=0} = 0 \quad \text{(no slip condition at the bottom of the ocean)},\\
&\partial_z u_{h|z=aD} = A_0 \sigma \quad \text{(influence of the wind)},\\
&u_{3|z=aD} = 0 \quad \text{(no flux condition at the surface)}.
\end{aligned}
$$

Let us now reduce the problem by scaling arguments. First, we neglect the effect of the horizontal component of the rotation vector e, which is classical in a geophysical framework (see [3]). Furthermore, we assume that the motion occurs at midlatitudes (far from the equator), and on a "small" geographical zone, meaning $L \ll R_0$, where R is the earth radius. In this setting, it is legitimate to use the so-called *f-plane approximation* (see [6]), and to neglect the fluctuations of the quantity $e_3 \cdot e$ with respect to the latitude. In rescaled variables, the equation becomes

$$
\partial_t u^{\varepsilon,\nu} + u^{\varepsilon,\nu} \cdot \nabla u^{\varepsilon,\nu} + \frac{1}{\varepsilon} e_3 \wedge u^{\varepsilon,\nu} - \eta \Delta_h u^{\varepsilon,\nu} - \nu \partial_z^2 u^{\varepsilon,\nu} + \nabla p = 0, \tag{1.1}
$$

where

$$
\varepsilon := \frac{U}{2L\Omega},\ \eta := \frac{A_h}{\rho u L},\ \nu := \frac{L A_v}{\rho U D^2},
$$

and U is the typical horizontal relative velocity of the fluid. We are interested in the limit

$$
\nu \ll 1, \quad \varepsilon \ll 1, \quad \eta \sim 1.
$$

Such a scaling of parameters seems convenient for instance for the mesoscale eddies that have been observed in western Atlantic (see [16]). One has indeed

$$
U \sim 5\,\mathrm{cm} \cdot \mathrm{s}^{-1}, \quad L \sim 100\,\mathrm{km}, \quad D \sim 4\,\mathrm{km} \text{ and } \Omega \sim 10^{-4}\mathrm{s}^{-1}
$$

which leads to $\varepsilon \sim 5 \times 10^{-3}$. Possible values for the turbulent viscosities given in [16] are

$$
A_h \sim 10^7\,\mathrm{cm}^2 \cdot \mathrm{s}^{-1} \text{ and } A_z \sim 10\,\mathrm{cm}^2 \cdot \mathrm{s}^{-1}
$$

so that $\nu = 10^{-3}$. Moreover, the amplitude of the wind stress at the surface of the ocean may be very large; thus we set

$$
\beta := \frac{A_0 S_0 D}{U},
$$

where S_0 is the amplitude of the wind velocity, and we study the limit $\beta \to \infty$. Equation (1.1) is now supplemented with the boundary conditions

$$
\begin{aligned}
u^{\varepsilon,\nu}_{|z=0} &= 0,\\
\partial_z u^{\varepsilon,\nu}_{h|z=a} &= \beta \sigma^\varepsilon,\\
u^{\varepsilon,\nu}_{3|z=a} &= 0,
\end{aligned} \tag{1.2}
$$

and with periodic boundary conditions in the horizontal variable x_h. In the rest of the paper, we set $\mathbb{T}^2 := [0, a_1) \times 0, a_2)$. The assumptions on the wind-stress σ^ε will be made clear later on.

1.1 General results on rotating fluids

Let us now explain heuristically what is the expected form of $u^{\varepsilon,\nu}$ at the limit. Assume for instance that $\nu = \varepsilon$ and that the family $u^{\varepsilon,\nu}$ admits a two-scale limit in time, say u^0, as $\varepsilon \to 0$; we thereby mean that

$$
\begin{aligned}
&\forall \phi \in \mathcal{C}_0^\infty([0,\infty) \times [0,\infty) \times \mathbb{T}^2 \times [0,a])\\
&\lim_{\varepsilon \to 0} \int_0^\infty \int_{\mathbb{T}^2 \times [0,a]} u^{\varepsilon,\nu}(t, x_h, z) \phi\left(t, \frac{t}{\varepsilon}, x_h, z\right)\, dt\, dx_h\, dz\\
= &\int_0^\infty \int_0^\infty \int_{\mathbb{T}^2 \times [0,a]} u^0(t, \tau, x_h, z) \phi\left(t, \tau, x_h, z\right)\, dt\, d\tau\, dx_h\, dz
\end{aligned}
$$

Rigorous definitions and properties of two-scale convergence can be found in the paper by G. Allaire [1].

Then, assuming that $u^{\varepsilon,\nu}$ is bounded in a "good" functional space, we may pass to the two-scale limit in (1.1), which yields

$$\left\{\begin{array}{l} \partial_\tau u^0 + e_3 \wedge u^0 = 0, \\ \mathrm{div} u^0 = 0, \\ u^0_{3|z=0} = u^0_{3|z=a} = 0. \end{array}\right. \tag{1.3}$$

Hence we introduce the vector space

$$\mathcal{H} := \left\{ u \in L^2(\mathbb{T}^2 \times [0,a])^3,\ \mathrm{div} u = 0,\ u_{3|z=0} = u_{3|z=a} = 0 \right\}.$$

We denote by $\mathbb{P}$ the orthogonal projection on $\mathcal{H}$ in $L^2(\mathbb{T}^2 \times [0,a])^3$, and we set $L := \mathbb{P}(e_3 \wedge \cdot)$. Notice that $\mathbb{P}$ differs from the Leray projector in general, because of the no-flux conditions at the bottom and the surface of the fluid. It is known (see for instance [3]) that there exists a hilbertian basis $(N_k)_{k\in\mathbb{Z}^3\setminus\{0\}}$ of $\mathcal{H}$ such that for all k,

$$\mathbb{P}(e_3 \wedge N_k) = i\lambda_k N_k \text{ with } \lambda_k = -\frac{k'_3}{|k'|},$$

where $k' = (2\pi k_1/a_1, 2\pi k_2/a_2, \pi k_3/a)$. The vector N_k is given by

$$N_k(x_h, z) = e^{ik'_h \cdot x_h} \begin{pmatrix} \cos(k'_3 z) n_1(k) \\ \cos(k'_3 z) n_2(k) \\ \sin(k'_3 z) n_3(k) \end{pmatrix}$$

where

$$\left\{\begin{array}{ll} n_1(k) = \dfrac{1}{\sqrt{a_1 a_2 a}|k'_h|}(ik'_2 + k'_1\lambda_k) & \\ n_2(k) = \dfrac{1}{\sqrt{a_1 a_2 a}|k'_h|}(-ik'_1 + k'_2\lambda_k) & \text{if } k_h \neq 0, \\ n_3(k) = i\dfrac{|k'_h|}{\sqrt{a_1 a_2 a}|k'|} & \end{array}\right.$$

and

$$\left\{\begin{array}{l} n_1(k) = \dfrac{\mathrm{sgn}(k_3)}{\sqrt{a_1 a_2 a}} \\ n_2(k) = \dfrac{i}{\sqrt{a_1 a_2 a}} \\ n_3(k) = 0 \end{array}\right. \quad \text{else.}$$

We infer from equation (1.3) that $u^0(t,\tau) \in \mathcal{H}$ almost everywhere, and that there exists a function u^0_L such that

$$u^0 = \exp(-\tau L) u^0_L = \sum_k e^{-i\lambda_k \tau} \left\langle N_k, u^0_L \right\rangle N_k.$$

Consequently the main effect of the Coriolis operator L is to create waves, propagating at frequencies of order ε^{-1}. The goal is now to identify the function u^0_L, which in general depends on the slow time variable t. This is achieved thanks to filtering methods, developed independently by S. Schochet in [19] and E. Grenier in [9]. Precisely, setting

$$u^{\varepsilon,\nu}_L = \exp\left(\frac{t}{\varepsilon} L\right) u^{\varepsilon,\nu}$$

it is proved in [3, 14] in the case of Dirichlet boundary conditions at $z = 0$ and $z = a$ that $u^{\varepsilon,\nu}_L$ converges strongly in $L^2_{\mathrm{loc}}([0,\infty) \times \mathbb{T}^2 \times [0,a])$ towards a function u^0_L. Moreover, the function u^0_L satisfies a nonlinear equation of the type

$$\partial_t u^0_L + \bar{Q}(u^0_L, u^0_L) - \Delta_h u^0_L = \bar{S}, \tag{1.4}$$

where the quadratic term $\bar{Q}(u^0_L, u^0_L)$ corresponds to the filtering of oscillations in the non-linear term $u^{\varepsilon,\nu} \cdot \nabla u^{\varepsilon,\nu}$, and the source term $\bar{S}$ to the filtering of oscillations in lower order terms in $u^{\varepsilon,\nu}$. The

quadratic term $\bar{Q}$ is defined as follows (see [3], Proposition 6.1 and [14]): for $w_1, w_2 \in \mathcal{H} \cap H^1(\mathbb{T}^2 \times [0,a])$,

$$\begin{aligned} 2\bar{Q}(w_1,w_2) &:= w - \lim_{\varepsilon\to 0}\left[\exp\left(\frac{t}{\varepsilon}L\right)\mathbb{P}\left(\exp\left(-\frac{t}{\varepsilon}L\right)w_1\cdot\nabla\exp\left(-\frac{t}{\varepsilon}L\right)w_2\right)\right. \\ &\qquad \left.\exp\left(\frac{t}{\varepsilon}L\right)\mathbb{P}\left(\exp\left(-\frac{t}{\varepsilon}L\right)w_2\cdot\nabla\exp\left(-\frac{t}{\varepsilon}L\right)w_1\right)\right] \\ &= \sum_{m\in\mathbb{Z}^3}\sum_{(k,l)\in\mathcal{K}_m}\langle N_k,w_1\rangle\langle N_k,w_2\rangle\,\alpha_{k,l,m}N_l, \end{aligned} \tag{1.5}$$

where the resonant set $\mathcal{K}_m$ is defined for $m \in \mathbb{Z}^3 \setminus \{0,\}$ by

$$\mathcal{K}_m := \left\{(k,l)\in\mathbb{Z}^6,\ \begin{array}{l} k_h + l_h = m_h, \\ \lambda_k + \lambda_l = \lambda_m \end{array} \text{ and } \exists\eta\in\{-1,1\}^2,\ \eta_1 k_3 + \eta_2 l_3 = m_3\right\}$$

and the coefficient $\alpha_{k,l,m}$ by

$$\alpha_{k,l,m} = \langle N_m, (N_k\cdot\nabla)N_l\rangle + \langle N_m, (N_l\cdot\nabla)N_k\rangle\,.$$

In order that the equation on u_L^0 is defined unambiguously, the value of the source term $\bar{S}$ has to be specified. In the present case, we have

$$\bar{S} = -\sqrt{\frac{\nu}{\varepsilon}}S_B(u_L^0) - \nu\beta S_T(\sigma),$$

where $S_B : \mathcal{H} \to \mathcal{H}$ is a linear continuous non-negative operator (see [3, 4, 14]) recalled in formula (4.9) below, and $S_T(\sigma)$ depends on the time oscillations in the wind-stress σ. Thus, in the next paragraph, we precise the assumptions on the wind-stress σ^ε, and we define the source term S_T. In the above formula and throughout the article, the subscripts B and T refer to top and bottom, respectively.

1.2 Definition of the limit equation

Let us first introduce the hypotheses on the time-dependance of the wind velocity σ^ε. Since the Coriolis operator generates oscillations at frequencies of order ε^{-1}, it seems natural to consider functions σ^ε which depend on the fast time variable t/ε. The case where this dependance is periodic, of almost periodic, has been investigated by N. Masmoudi in [14] in the non-resonant case, that is, when the frequencies of the wind-stress are different from ± 1. The results of [14] were then extended by the author and Laure Saint-Raymond in [4]. In fact, it is proved in [4] that when the wind-stress oscillates with the same frequency as the rotation of the Earth (i.e. ± 1), the typical size of the boundary layers is much larger than the one of the classical Ekman layers. Moreover, a resonant forcing overall destabilizes the whole fluid for large times. Here, we wish to avoid these singular behaviours, and thus to consider a more general non-resonant setting.

Let $(E, \mathcal{F}, m_0)$ be a probability space, and let $(\theta_\tau)_{\tau\in\mathbb{R}}$ be a measure preserving group transformation acting on E. We assume that the function σ^ε can be written

$$\sigma^\varepsilon(t,x_h) = \sigma\left(t,\frac{t}{\varepsilon},x_h;\omega\right), \quad t>0,\ x_h\in\mathbb{T}^2,\ \omega\in E,$$

and that the function σ is stationary, i.e.

$$\sigma(t,\tau+s,x_h;\omega) = \sigma(t,\tau,x_h;\theta_s\omega)$$

almost everywhere.

The periodic setting can be embedded the stationary (ergodic) setting in the following way (see [15]): take $E = \mathbb{R}/\mathbb{Z} \simeq [0,1)$, and let m_0 be the Lebesgue measure on E. Define the group transformation $(\theta_\tau)_{\tau\in\mathbb{R}}$ by

$$\theta_\tau s = s + \tau \mod \mathbb{Z} \quad \forall(\tau,s)\in\mathbb{R}\times E.$$

Then it is easily checked that θ_τ preserves the measure m_0 for all $\tau \in \mathbb{R}$. Thus the periodic setting is a particular case of the stationary setting; the almost periodic setting can also be embedded in the stationary setting, but the construction is more involved, and we refer the interested reader to [15].

The interest of the stationary setting, in addition of its generalization of the almost periodic one, lies in the introduction of some randomness in equation (1.1). Hence, we also expect to recover a random function in the limit $\varepsilon, \nu \to 0$. In fact, we will prove rigorously a strong convergence result of this kind; additionally, we will characterize the average behaviour of $u^{\varepsilon,\nu}$ in the limit. Thus, one of the secondary goals of this paper is to derive some averaging techniques adapted to highly rotating fluids, which may be of interest in the framework of a mathematical theory of weak turbulence.

Since the function σ is not an almost periodic function, we now introduce a notion of approximate spectral decomposition of σ. For $\alpha > 0$, let

$$\hat{\sigma}_\alpha(\lambda) := \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-\alpha|\tau|) e^{-i\lambda\tau} \sigma(\tau)\, d\tau, \tag{1.6}$$

and define the family of functions $(\sigma_\alpha)_{\alpha>0}$ by the formula

$$\sigma_\alpha(\tau) := \int_{\mathbb{R}} \exp(-\alpha|\lambda|) e^{i\lambda\tau} \hat{\sigma}_\alpha(\lambda)\, d\lambda. \tag{1.7}$$

It is proved in the Appendix A (see Lemma 1) that the family $(\sigma_\alpha)_{\alpha>0}$ converges towards σ, as $\alpha \to 0$, in $L^\infty_{\mathrm{loc}}([0,\infty) \times [0,\infty), L^\infty(E, L^2(\mathbb{T}^2)))$. We assume that there exists $s > 4$ such that the following non-resonance hypotheses hold:

(H1) For all $\alpha > 0$, $T > 0$, $\hat{\sigma}_\alpha \in L^\infty([0,T] \times E, L^1(\mathbb{R}_\lambda, H^s(\mathbb{T}^2)))$, and

$$\forall T > 0, \quad \sup_{\alpha>0} ||\hat{\sigma}_\alpha||_{L^\infty([0,T]\times E, L^1(\mathbb{R}, H^s(\mathbb{T}^2)))} < +\infty.$$

(H2) There exist neighbourhoods $V_\pm$ of ± 1, independent of $\alpha > 0$, such that

$$\forall T > 0, \quad \lim_{\alpha\to 0} \sup_{\lambda \in V_+ \cup V_-} \|\hat{\sigma}_\alpha(\lambda)\|_{L^\infty([0,T]\times E, H^s(\mathbb{T}^2))} = 0.$$

We refer to Remark 1.2 below for some details about the meaning of hypotheses **(H1)-(H2)** for almost periodic functions. Let un now explain how random oscillations are filtered:

Proposition 1. *Let $\phi \in L^\infty(\mathbb{R}_\tau, L^2(E))$ be stationary, and let $\lambda \in \mathbb{R}$. Then the family*

$$\phi^\lambda_\theta : \omega \in E \mapsto \frac{1}{\theta} \int_0^\theta \phi(\tau, \omega) e^{-i\lambda\tau} d\tau, \quad \theta > 0$$

converges, almost surely and in $L^2(E)$, towards a function denoted by $\mathcal{E}_\lambda[\phi] \in L^2(E)$ as $\theta \to \infty$. Moreover, $\mathcal{E}_\lambda[\phi]$ satisfies the equality

$$\mathcal{E}_\lambda[\phi](\theta_\tau \omega) = \mathcal{E}_\lambda[\phi](\omega) e^{i\lambda\tau}$$

almost surely in ω, for all $\tau \in \mathbb{R}$.

Additionally, if σ satisfies **(H1)-(H2)**, *then*

$$\mathcal{E}_\lambda[\sigma] = 0 \tag{1.8}$$

for λ in a neighbourhood of ± 1.

Proposition 1 is proved in Appendix B, except property (1.8), which will be proved in the course of the proof page 28.

With the above definition of $\mathcal{E}_\lambda$, the source term S_T is defined by

$$S_T(\sigma)(t) = \frac{1}{2\sqrt{aa_1a_2}} \sum_{k\in\mathbb{Z}^3} \sum_{\pm} \mathbf{1}_{k_h \neq 0} \frac{(-1)^{k_3}}{|k'_h|} \left(\lambda_k k'_h - i(k'_h)^\perp\right) \cdot \mathcal{E}_{-\lambda_k}\left[\hat{\sigma}(t,\cdot,k_h)\right] N_k,$$

where

$$\hat{\sigma}(t,\tau,k_h;\omega)=\frac{1}{\sqrt{a_1a_2}}\int_{\mathbb{T}^2}\sigma(t,x_h;\omega)e^{-ik_h'\cdot x_h}\,dx_h.$$

Notice that $S_T(\sigma)$ is a random function in general, and is well-defined in $L^\infty_{\rm loc}([0,\infty)\times E,L^2(\mathbb{T}^2\times[0,a))$ thanks to **(H1)-(H2)** provided $\sigma\in L^\infty([0,T]\times[0,\infty)\times E,H^{1/2}(\mathbb{T}^2))$ for all $T>0$.

• We now state an existence result for the limit system, based on the analysis in [3]. To that end, we introduce the anisotropic Sobolev spaces $H^{s,s'}$ by

$$H^{s,s'}:=\left\{u\in L^2(\mathbb{T}^2\times[0,a_3]),\ \forall\alpha\in\mathbb{N}^3,\ |\alpha_h|\le s,|\alpha_3|\le s',\ \nabla_h^{\alpha_h}\partial_z^{\alpha_3}u\in L^2\right\}.$$

Then the following result holds:

Proposition 2. *Let $\nu,\varepsilon,\beta>0$ be arbitrary.*
Let $u_0\in\mathcal{H}\cap H^{0,1}$, and let $\sigma\in L^\infty_{loc}([0,\infty)_t,L^\infty([0,\infty)_\tau\times E,H^{3/2}(\mathbb{T}^2))$.
Assume that the hypotheses **(H1)-(H2)** *hold.*
Then $S_T(\sigma)\in L^\infty_{loc}([0,\infty)_t,L^\infty(E,H^{0,1}))$, and consequently, the equation

$$\begin{aligned}&\partial_t w+\bar{Q}(w,w)-\Delta_h w+\sqrt{\frac{\nu}{\varepsilon}}S_B(w)+\nu\beta S_T(\sigma)=0,\\&w_{|t=0}=u_0\end{aligned}\tag{1.9}$$

has a unique solution $w\in L^\infty(E,\mathcal{C}([0,\infty),\mathcal{H}\cap H^{0,1}))$ such that $\nabla_h u$ belongs to $L^\infty(E,L^2_{loc}([0,\infty),H^{0,1}))$.

Remark 1.1. (i) Notice that the function w is random in general because of the source term S_T.
(ii) In [3], Proposition 2 is proved for $S_T=0$ (see Proposition 6.5 p. 145). As stressed by the authors, the result is non trivial since the system (1.9) is similar to a three-dimensional Navier-Stokes equation, with a vanishing vertical viscosity. The proof relies on two arguments: first, a careful analysis of the structure of the quadratic term $\bar{Q}$ shows that the limit equation is in fact close to a two-dimensional one. Second, the divergence-free property enables one to recover estimates on the vertical derivatives on the third component of the velocity field, and thus to bypass the difficulties due to the lack of smoothing in the vertical direction.

In fact, the proof of Proposition 2 can easily be adapted from the one of Proposition 6.5 in [3], and is therefore left to the reader. The method remains exactly the same, the only difference being the presence of the source term S_T in the energy estimates. This does not rise any particular difficulty, thanks to the assumptions on σ.

1.3 Convergence result

Theorem 1. *Assume that $\nu=\mathcal{O}(\varepsilon)$, and that $\sqrt{\varepsilon\nu}\beta=\mathcal{O}(1)$.*

Let $\sigma\in L^\infty_{loc}([0,\infty)_t,L^\infty([0,\infty)_\tau\times E,H^{3/2}(\mathbb{T}^2))$ such that **(H1)-(H2)** *are satisfied, and $\partial_\tau\sigma\in H^1(\mathbb{T}^2,L^\infty([0,\infty)_t\times[0,\infty)_\tau\times E))$.*

Let $u^{\varepsilon,\nu}\in L^\infty(E,\mathcal{C}([0,\infty),L^2)\cap L^2_{loc}([0,\infty),H^1))$ be a weak solution of (1.1), supplemented with the conditions (1.2) and the initial data $u^{\varepsilon,\nu}_{|t=0}=u_0\in\mathcal{H}\times H^{0,1}$. Let w be the solution of (1.9). Then for all $T>0$,

$$u^{\varepsilon,\nu}-\exp\left(-\frac{t}{\varepsilon}L\right)w\to 0$$

in $L^2([0,T]\times E,H^{1,0})\cap L^\infty([0,T],L^2(E\times\mathbb{T}^2\times[0,a]))$.

In the case of the nonresonant torus (see (1.11) below), it is likely that the hypothesis $\nu=\mathcal{O}(\varepsilon)$ can be relaxed. Indeed, in this case, the equation on w decouples between a nonlinear equation on the vertical average of w on the one hand, and a linear equation on the vertical modes of w on the other (see paragraph 1.4 below, together with Section 6). Moreover, it can be proved that the purely horizontal modes of w decay exponentially in time at a rate $\exp(-\sqrt{\nu/\varepsilon}t)$, and the rate of decay does not depend on the particular horizontal mode considered. Thus, in this particular case, the regime $\nu\gg\varepsilon$ may be investigated, using arguments similar to those developed in [4].

Remark 1.2. Let us now explain the meaning of hypotheses **(H1)-(H2)** for almost periodic functions. Let $k_h \in \mathbb{Z}^2$, and let $\phi \in L^\infty([0,\infty)\times\mathbb{T}^2)$ such that

$$\phi(\tau, x_h) = e^{ik_h'\cdot x_h}\sum_{\mu\in M}\hat{\phi}(\mu)e^{i\mu\tau},$$

where M is a countable set. The fact that ϕ as only one horizontal Fourier mode is not crucial, but merely helps focusing on the time spectrum. Then it can be checked easily that for all $\alpha > 0$,

$$\hat{\phi}_\alpha(\lambda, x_h) = \frac{1}{2\pi}e^{ik_h'\cdot x_h}\sum_{\mu\in M}\hat{\phi}(\mu)\frac{2\alpha}{\alpha^2+(\mu-\lambda)^2}.$$

In particular, there exists a constant $C > 0$ such that for all $s > 0$,

$$\begin{aligned}\|\hat{\phi}_\alpha\|_{L^1(\mathbb{R}_\lambda, H^s(\mathbb{T}^2))} &\leq C(1+|k_h|^s)\sum_{\mu\in M}\left|\hat{\phi}(\mu)\right|\int_{\mathbb{R}}\frac{2\alpha}{\alpha^2+(\mu-\lambda)^2}d\lambda\\ &\leq C(1+|k_h|^s)\sum_{\mu\in M}\left|\hat{\phi}(\mu)\right|.\end{aligned}$$

Thus hypothesis **(H1)** is satisfied provided $\left|\hat{\phi}(\mu)\right| < \infty$.

On the other hand, assume that

$$\eta := d(M, \{-1, 1\}) > 0, \tag{1.10}$$

i.e. that there are no frequencies in a neighbourhood of ± 1. Then if $\lambda \in (-1-\eta/2, -1+\eta/2)\cup(1-\eta/2, 1+\eta/2)$, we have

$$|\lambda - \mu| \geq \frac{\eta}{2} \quad \forall \mu \in M,$$

and consequently, setting $V^\pm := (\pm 1 - \eta/2, \pm 1 + \eta/2)$, we have, for all $s > 0$

$$\sup_{\lambda\in V^-\cup V^+}\left\|\hat{\phi}_\alpha(\lambda)\right\|_{H^s(\mathbb{T}^2)} \leq C(s)\frac{1}{\eta}\alpha.$$

Thus hypothesis (1.10) entails **(H2)**. Additionally, hypothesis (1.10) cannot be easily relaxed, as shows the following construction: consider the sequence $\mu_n := 1 - 1/n$, and choose a sequence of positive numbers ϕ_n such that

$$\sum_n \phi_n < \infty.$$

For $\tau \in \mathbb{R}$, set

$$\phi(\tau) := \sum_n \phi_n e^{i\mu_n\tau}.$$

Then for all $\alpha > 0$, for all $k > 0$

$$\hat{\phi}_\alpha(\mu_k) = \sum_n \phi_n \frac{2\alpha}{\alpha^2 + \left(\frac{1}{n}-\frac{1}{k}\right)^2} \geq \frac{2\phi_k}{\alpha}.$$

In particular,

$$\lim_{\alpha\to 0}\hat{\phi}_\alpha(\mu_k) = +\infty$$

for all k, and thus condition **(H2)** is not satisfied.

1.4 Average behaviour at the limit

We have already stressed that the solution w of equation (1.9) is, in general, a random function. Thus one may wonder whether the average behaviour of w at the limit can be characterized. In general, the nonlinearity of equation (1.9) prevents us from deriving an equation, or a system of equations, on the expectation of w, which we denote by $\mathbb{E}[w]$. However, when the torus is non resonant, equation (1.9) decouples, and in this case we are able to exhibit a system of equations satisfied by $\mathbb{E}[w]$.

Let us first recall a few definitions:

Definition 1 (Non-resonant torus). The torus $\mathbb{T}^3 := \mathbb{T}^2 \times [-a, a)$ is said to be **non-resonant** if the following property holds: for all $(k, n) \in \mathbb{Z}^3 \setminus \{0\} \times \mathbb{Z}^3 \setminus \{0\}$,

$$\big(\exists \eta \in \{-1, 1\}^3, \eta_1\lambda_k + \eta_2\lambda_{n-k} - \eta_3\lambda_n = 0\big) \Rightarrow k_3 n_3 = 0. \tag{1.11}$$

We refer to [2] for a discussion of hypothesis (1.11) and its consequences. Let us mention that (1.11) holds for almost all values of $(a, a_1, a_2) \in (0, \infty)^3$. When the torus is non-resonant, the structure of the quadratic form $\bar{Q}$ defined by (1.5) is particularly simple, and the system (1.9) can be decoupled into a two-dimensional Navier-Stokes equation on the vertical average of w, and a linear equation on the z-dependent part (see [3]). The advantage of this decomposition in our case is that the vertical average of $\bar{S}_T(\sigma)$ is deterministic, at least when the group transformation $(\theta_\tau)_{\tau\geq 0}$ acting on E is ergodic (see [20]).

Definition 2 (Ergodic transformation group). Let $(\theta_\tau)_{\tau\in\mathbb{R}}$ be a group of invariant transformations acting on the probability space $(E, \mathcal{A}, m_0)$. The group is said to be ergodic if for all $A \in \mathcal{A}$,

$$(\theta_\tau A \subset A \quad \forall \tau \in \mathbb{R}) \Rightarrow m_0(A) = 0 \text{ or } m_0(A) = 1.$$

We now state the result on the average behaviour at the limit:

Proposition 3. *Assume that the transformation group $(\theta_\tau)_{\tau\in\mathbb{R}}$ is ergodic.*
Let $u_0 \in \mathcal{H} \cap H^{0,1}$, and let $\sigma \in L^\infty_{loc}([0,\infty)_t, L^\infty([0,\infty)_\tau \times E, H^{3/2}(\mathbb{T}^2))$ such that the hypotheses of Theorem 1 are satisfied. Let $w \in L^\infty(E, \mathcal{C}([0,\infty), \mathcal{H} \cap H^{0,1}) \cap L^2_{loc}([0,\infty), H^{1,0}))$ be the unique solution of equation (1.9).

Let $\bar{w} = (\bar{w}_h, 0) \in \mathcal{C}([0,\infty), L^2(\mathbb{T}^2)) \cap L^2_{loc}([0,\infty), H^1(\mathbb{T}^2))$ be the solution of the 2D-Navier-Stokes equation

$$\begin{aligned}
&\partial_t \bar{w}_h + \bar{w}_h \cdot \nabla_h \bar{w}_h - \Delta_h \bar{w}_h + \sqrt{\frac{\nu}{\varepsilon}} \frac{1}{\sqrt{2} a a_1 a_2} \bar{w}_h + \nu\beta \mathbb{E}\left[S_T(\sigma)\right]_h = \nabla_h \bar{p},\\
&\operatorname{div}_h \bar{w}_h = 0,\\
&\bar{w}_{h|t=0}(x_h) = \frac{1}{a_3}\int_0^{a_3} u_{0,h}(x_h, z)\, dz.
\end{aligned}$$

Then the following properties hold:

1. *As $\varepsilon, \nu \to 0$ as in Theorem 1, we have*

$$u^{\varepsilon,\nu} \rightharpoonup \bar{w} \quad \text{in } L^2([0,T] \times \mathbb{T}^2 \times [0,a] \times E).$$

   In particular, the weak limit of $u^{\varepsilon,\nu}$ is a deterministic function.

2. *Assume additionally that the torus $\mathbb{T}^3$ is non resonant. Then*

$$\mathbb{E}[w] = \bar{w} + \tilde{w},$$

   where $\tilde{w}$ solves a linear deterministic equation

$$\begin{aligned}
&\partial_t \tilde{w} + 2\bar{Q}(\bar{w}, \tilde{w}) - \Delta_h \tilde{w} + \sqrt{\frac{\nu}{\varepsilon}} S_B(\tilde{w}) = 0,\\
&\tilde{w}_{|t=0} = u_0 - \bar{w}_{|t=0}.
\end{aligned}$$

1.5 Strategy of proof of Theorem 1

The proof relies on the construction of an approximate solution, obtained as the sum of some interior terms -the largest of which is $\exp(-T/\varepsilon L)w(t)$ -, and some boundary layer terms which restore the horizontal boundary conditions violated by the interior terms. We refer to the works by N. Masmoudi [13, 14], N. Masmoudi and E. Grenier [10], N. Masmoudi and F. Rousset [12], and F. Rousset [18] for an extensive study of boundary layers in rotating fluids, or in incompressible fluids with vanishing vertical viscosity for [13]. We emphasize that *in fine*, all terms will be small in L^2 norm, except $\exp(-T/\varepsilon L)w(t)$.

Following [3] (Chapter 7), let us assume that as $\varepsilon, \nu \to 0$,

$$\begin{aligned} u^{\varepsilon,\nu} &\approx u^{\rm int} + u^{\rm BL}, \\ p^{\varepsilon,\nu} &\approx \frac{1}{\varepsilon} p^{\rm int} + \frac{1}{\varepsilon} p^{\rm BL} + p^{{\rm int},0}, \end{aligned} \tag{1.12}$$

where

$$\begin{aligned} u^{\rm int}(t,x_h,z) &= U\left(t,\frac{t}{\varepsilon},x_h,z\right),\ p^{\rm int}(t,x_h,z) = P\left(t,\frac{t}{\varepsilon},x,y,z\right), \\ u^{\rm BL}(t,x_h,z) &= u_T\left(t,\frac{t}{\varepsilon},x_h,\frac{a-z}{\eta}\right) + u_B\left(t,\frac{t}{\varepsilon},x_h,\frac{z}{\eta}\right), \\ p^{\rm BL}(t,x_h,z) &= p_T\left(t,\frac{t}{\varepsilon},x_h,\frac{a-z}{\eta}\right) + p_B\left(t,\frac{t}{\varepsilon},x_h,\frac{z}{\eta}\right). \end{aligned}$$

Above, η is a small parameter that will be chosen later on. The function $u_T(t,\tau,x_h,\zeta)$ is assumed to vanish as $\zeta \to \infty$ (same for p_T, p_B, u_B).

We then plug the Ansatz (1.12) into equation (1.1), and identify the different powers of ε. In general, there is a coupling between $u^{\rm int}$ and $u^{\rm BL}$: indeed, we have seen that it is natural to expect that

$$U(t,\tau) = \exp(-\tau L)w(t),$$

at first order, and thus $u^{\rm int}$ does not match the horizontal boundary conditions in general. As a consequence, the value of $u^{\rm int}$ at the boundary has to be taken into account when constructing the boundary layer term $u^{\rm BL}$. On the other hand, because of the divergence-free constraint, the third component of $u^{\rm BL}$ does not vanish at the boundary, which means that a small amount of fluid may enter or leave the interior of the domain. This phenomenon is called Ekman suction, and gives rise to a source term (called the Ekman pumping term) in the equation satisfied by $u^{\rm int}$. This leads to some sort of "loop" construction, in which the boundary layer and interior terms are constructed one after the other.

Rather than following this construction step by step, we first explain how a generic boundary layer term is constructed, given arbitrary boundary conditions on the horizontal component of the velocity. We shall see that in general, the vertical component of the boundary layer term does not vanish at the boundary. Thus, in the third section, we explain how a generic interior term $u^{\rm int}$ is constructed, depending on the initial data, and on arbitrary Dirichlet boundary conditions for the vertical component of the velocity. In the fourth section, we detail how the approximate solution is obtained, and the fifth section is devoted to the proof of convergence thanks to energy estimates. At last, we prove Proposition 3 in the sixth section.

2 The boundary layer operator

In this section, we construct a linear boundary layer operator, which maps boundary conditions on the horizontal component of the velocity onto boundary layer terms. The boundary conditions considered here are of three types:

- Stationary Neumann boundary conditions at the surface:

$$\partial_z u^{\rm BL}_{h|z=a} = \beta c_{T,h}\left(t,\frac{t}{\varepsilon},x_h;\omega\right),$$

  where $c_{T,h}(t,\tau,x,y,\omega)$ is a stationary function of τ, $\mathbb{T}^2$-periodic with respect to x_h.

  In this case, we expect $u^{\rm BL}_h$ to be of order $\eta\beta||c_{T,h}||_\infty$ in L^∞.

- Stationary Dirichlet boundary conditions at the bottom:

$$u^{\rm BL}_{h|z=0} = c_{B,h}\left(t, \frac{t}{\varepsilon}, x_h; \omega\right),$$

where $c_{B,h}(t,\tau,x_h)$ is a stationary function of τ, $\mathbb{T}^2$-periodic with respect to x_h. In this case, we expect $u^{\rm BL}_h$ to be of order $||c_{B,h}||_\infty$ in L^∞.

- Almost periodic Dirichlet boundary conditions at the bottom:

$$u^{\rm BL}_{h|z=0} = c_{B,h}\left(t, \frac{t}{\varepsilon}, x_h\right),$$

where the function $c_{B,h}$ is almost periodic, and such that its frequencies are the eigenvalues of the operator L. Hence we consider functions $c_{B,h}$ of the type

$$c_{B,h}(t,\tau,x_h) = \sum_{k\in\mathbb{Z}^3, k\neq 0} \hat{c}_{B,h}(t,k) e^{ik'_h\cdot x_h} e^{-i\lambda_k \tau}, \quad \lambda_k = -\frac{k'_3}{|k'|}. \tag{2.1}$$

Once again, we expect $u^{\rm BL}_h$ to be of order $||c_{B,h}||_\infty$ in L^∞.

The divergence-free condition entails that the third component of $u^{\rm BL}$ is given by the following formulas:

$$\begin{aligned} u_{T,3}(\zeta) &= -\eta \int_\zeta^\infty {\rm div}_h u_{T,h}(\zeta')d\zeta', \\ u_{B,3}(\zeta) &= \eta \int_\zeta^\infty {\rm div}_h u_{T,h}(\zeta')d\zeta'. \end{aligned}$$

Thus $u_{T,3} = \mathcal{O}(\eta^2||c_{T,h}||_{W^{1,\infty}})$, $u_{B,3} = \mathcal{O}(\eta||c_{B,h}||_{W^{1,\infty}})$. At last, in order to be consistent with (1.12), we assume that the pressure inside the boundary layer is given by

$$p \approx \frac{1}{\varepsilon} p^{\rm BL} = \frac{1}{\varepsilon} p_T\left(\frac{a-z}{\eta}\right) + \frac{1}{\varepsilon} p_B\left(\frac{z}{\eta}\right),$$

where $p_T = \mathcal{O}(\eta||c_{T,h}||_\infty)$, $p_B = \mathcal{O}(||c_{B,h}||_\infty)$.

2.1 Stationary Neumann boundary conditions at the top

The construction of boundary layers with stationary boundary conditions is the main novelty of this section. We focus on the boundary layer at the top of the fluid, that is, around $z = a$, and we set

$$\zeta := \frac{a-z}{\eta}.$$

Then the pressure term in the third component of (1.1) is of order $||c_{T,h}||_\infty$, whereas the lowest order term in the left-hand side is of order $\eta^2||c_{T,h}||_{W^{1,\infty}}$. Thus, since η is small, we infer

$$\partial_\zeta p_T = 0,$$

and since p_T vanishes at infinity, we have $p^{\rm BL} = 0$: at first order, the pressure does not vary in the boundary layer. Thus, we now focus on the horizontal component of u_T, which is a solution of

$$\partial_\tau \begin{pmatrix} u_{T,1} \\ u_{T,2} \end{pmatrix} - \frac{\nu\varepsilon}{\eta^2}\partial^2_\zeta \begin{pmatrix} u_{T,1} \\ u_{T,2} \end{pmatrix} + \begin{pmatrix} -u_{T,2} \\ u_{T,1} \end{pmatrix} = 0, \tag{2.2}$$

$$\partial_\zeta u_{T,h|\zeta=0} = -\eta\beta c_{T,h}(\tau,x,y,\omega), \tag{2.3}$$

$$u_{T,h|\zeta=+\infty} = 0. \tag{2.4}$$

We now choose η so that all the terms in (2.2) are of the same order, that is,

$$\eta = \sqrt{\nu\varepsilon}.$$

Moreover, since $c_{T,h}$ is a stationary function of time, it seems natural to look for stationary solutions of (2.2), and thus for fundamental solutions φ_1, φ_2 of (2.2) in the following sense: φ_i $(i = 1, 2)$ is a solution of (2.2) in the sense of distributions and satisfies (2.4), and

$$\partial_\zeta \varphi_{1|\zeta=0} = \delta_0(\tau) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \partial_\zeta \varphi_{2|\zeta=0} = \delta_0(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where δ_0 denotes the Dirac mass at $\tau = 0$. If we can construct φ_1 and φ_2 satisfying the above conditions, then a good candidate for u_T is

$$u_{T,h}(t, \tau, x_h, \zeta; \omega) = -\sqrt{\nu\varepsilon}\beta \sum_{j\in\{1,2\}} \int_0^\infty c_{T,j}(t, \tau - s, x_h; \omega)\varphi_j(s)ds.$$

Hence we now define φ_1, φ_2. Since the fundamental solution of the heat equation is known, let us make the following change of unknow function (see [14]):

$$H_j^\pm = \partial_\zeta \left[e^{\pm i\tau} \left(\varphi_{j,1} \pm i\varphi_{j,2}\varphi_{j,2} \mp i\varphi_{j,1}. \right) \right], \quad j = 1, 2.$$

Then, setting $e_1^\pm := (1, \mp i)$, $e_2^\pm := (\pm i, 1)$, we infer that $H_j^\pm = Ge_j^\pm$, where G satisfies

$$\begin{cases} \partial_\tau G - \partial_\zeta^2 G = 0, \quad \tau > 0, \zeta > 0, \\ G_{|\zeta=0}(\tau) = \delta_0(\tau), \\ G_{|\zeta=+\infty} = 0. \end{cases} \tag{2.5}$$

The boundary condition at $\zeta = 0$ should be understood as follows: for all $\varphi \in \mathcal{C}_b(\mathbb{R})$, for all $\tau > 0$

$$\lim_{\zeta\to 0^+} \left[\int_0^\infty \varphi(\tau - s)G(s, \zeta)ds \right] = \varphi(\tau).$$

It can be checked (see Chapter 4, section 1 in [11]) that

$$G(\tau, \zeta) := \frac{\zeta}{\sqrt{4\pi}\tau^{3/2}} \exp\left(-\frac{\zeta^2}{4\tau} \right) \quad \text{for } \tau > 0,\ \zeta > 0,$$

is a solution of (2.5), which leads to

$$\begin{aligned} \partial_\zeta \varphi_j(\tau, \zeta) &:= \frac{1}{2} \left[e^{-i\tau} H_j^+(\tau, \zeta) + e^{+i\tau} H_j^-(\tau, \zeta) \right] \\ &= \frac{1}{2} G(\tau, \zeta) \left[e^{-i\tau} e_j^+ + e^{+i\tau} e_j^- \right]. \end{aligned}$$

Unfortunately, when we integrate this formula with respect to ζ in order to obtain an explicit expression for $u_{T,h}$, the convolution kernel thus obtained is

$$\varphi_j(\tau, \zeta) = -\frac{1}{\sqrt{4\pi\tau}} \exp\left(-\frac{\zeta^2}{4\tau} \right) \left[e^{-i\tau} e_j^+ + e^{+i\tau} e_j^- \right],$$

and is not integrable near $\tau = +\infty$. Hence, in the spirit of [14], we consider an approximate corrector in the boundary layer: for $\delta > 0$, we set

$$G_\delta(\tau, \zeta) = \frac{\zeta}{\sqrt{4\pi}\tau^{3/2}} \exp\left(-\frac{\zeta^2}{4\tau} - \delta\tau \right).$$

Then the corresponding corrector is given by

$$\begin{aligned} u^\delta_{T,h}(\cdot,\tau,\cdot,\zeta,\omega) &= -\beta\sqrt{\varepsilon\nu}\sum_{j\in\{1,2\}}\int_0^\infty \varphi_j(s,\zeta)\exp(-\delta s)c_{T,j}(\cdot,\tau-s,\cdot;\omega)\,ds \\ &= \frac{\beta\sqrt{\nu\varepsilon}}{\sqrt{4\pi}}\sum_{\pm}\int_0^\infty \frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}\right)(c_{T,h}\pm ic^\perp_{T,h})(\cdot,\tau-s,\cdot,\omega)e^{-\delta s\pm is}ds. \end{aligned} \tag{2.6}$$

The approximate corrector u^δ_T satisfies the exact boundary conditions at $\zeta=0$, and equation (2.2) up to an error term of order δ

$$\partial_\tau u^\delta_{T,h}-\partial^2_\zeta u^\delta_{T,h}+\left(u^\delta_{T,h}\right)^\perp+\delta u^\delta_{T,h}=0.$$

The third component of u^δ_T is then given by

$$u^\delta_{T,3}(\zeta)=-\sqrt{\nu\varepsilon}\int_\zeta^\infty \mathrm{div}_h u^\delta_{T,h},$$

which yields

$$u^\delta_{T,3}(\cdot,\tau,\cdot,\zeta,\omega)=\frac{\nu\varepsilon\beta}{\sqrt{4\pi}}\sum_{\pm}\int_0^\infty \varphi\left(\frac{\zeta}{\sqrt{s}}\right)(\mathrm{div}_h c_{T,h}\mp i\mathrm{rot}_h c_{T,h})(\cdot,\tau-s,\cdot,\omega)e^{-\delta s\pm is}ds,$$

where φ is defined by $\varphi'(\zeta)=\exp\left(-\frac{\zeta^2}{4}\right)$, $\varphi(+\infty)=0$.

In horizontal Fourier variables, we have

$$u^\delta_{T,3}(t,\tau,x_h,\zeta,\omega)=\frac{\nu\varepsilon\beta}{\sqrt{4\pi}}\sum_{k_h\in\mathbb{Z}^2}\sum_{\pm}e^{ik'_h\cdot x_h}\int_0^\infty \varphi\left(\frac{\zeta}{\sqrt{s}}\right)\hat{c}^\pm_{T,h}(t,\tau-s,k_h,\omega)e^{-\delta s\pm is}ds \tag{2.7}$$

where

$$\hat{c}^\pm_{T,h}(k_h)=ik'_h\cdot\hat{c}_{T,h}(k_h)\pm(k'_h)^\perp\cdot\hat{c}_{T,h}(k_h).$$

We define the operator $\mathcal{B}^\delta_T$ by

$$\mathcal{B}^\delta_T[c_{T,h}](t,x_h,z;\omega)=u^\delta_T\left(t,\frac{t}{\varepsilon},x_h,\frac{a-z}{\sqrt{\varepsilon\nu}};\omega\right),$$

where u^δ_T is defined by (2.6)-(2.7).

We now give an estimate on the boundary layer term computed above when assumptions **(H1)-(H2)** are satisfied. The proof is postponed to paragraph 2.4.

Proposition 4. *Assume that $c_{T,h}\in L^\infty([0,\infty)\times\mathbb{R}_\tau\times E\times\mathbb{T}^2)\cap L^\infty(E\times[0,\infty)\times\mathbb{T}^2,\mathcal{C}(\mathbb{R}_\tau))$, and that $c_{T,h}$ satisfies* **(H1)-(H2)**. *Then for all $T>0$, there exists a constant $C_T>0$, such that for all $\delta,\nu,\varepsilon,\beta>0$,*

$$\left|\left|u^\delta_T,\ \zeta\partial_\zeta u^\delta_T\right|\right|_{L^\infty([0,T]\times\mathbb{R}_\tau\times\mathbb{T}^2\times[0,\infty)_\zeta\times E)} \leq C_T\sqrt{\varepsilon\nu}\beta, \tag{2.8}$$

$$\left|\left|u^\delta_T,\ \zeta\partial_\zeta u^\delta_T\right|\right|_{L^\infty([0,T]\times\mathbb{R}_\tau\times E,L^2([0,\infty)_\zeta,H^1(\mathbb{T}^2)))} \leq C_T\sqrt{\varepsilon\nu}\beta. \tag{2.9}$$

2.2 Stationary Dirichlet boundary conditions at the bottom

The construction is the same as for Neumann boundary conditions, and is in fact more simple because we need not integrate with respect to the variable ζ. Thus, with the same notations as above, the boundary layer term at the top is given by

$$u^\delta_{B,h}(t,\tau,x_h,\zeta,\omega)=\frac{1}{2}\sum_{j\in\{1,2\}}\int_0^\infty G_\delta(s,\zeta)\left[e^{-is}e^+_j+e^{+is}e^-_j\right]c_{T,j}(t,\tau-s,x,y;\omega)\,ds, \tag{2.10}$$

and

$$u^{\delta}_{B,3}(t,\tau,x_h,\zeta,\omega) = \frac{\nu\varepsilon}{\sqrt{4\pi}} \sum_{k_h\in\mathbb{Z}^2} \sum_{\pm} e^{ik'_h\cdot x_h} \int_0^{\infty} \frac{1}{\sqrt{s}} \exp\left(-\frac{\zeta^2}{4s}\right) \hat{c}^{\pm}_{T,h}(\cdot,\tau-s,k_h,\omega) e^{-\delta s \pm is} ds. \quad (2.11)$$

The same estimates as in Proposition 4 hold. The corresponding operator is denoted by $\mathcal{B}^{\mathrm{stat}}_B$.

2.3 Almost-periodic Dirichlet boundary conditions at the bottom

In this case, the computation has already been performed by several authors (see for instance [14, 3]); hence we will merely briefly recall the method and the expression of u^0_B. Unlike in [4], no singular behaviour occurs for $k_h \neq 0$ because there is no resonant forcing on the non-homogeneous horizontal modes, meaning that $\lambda_k \neq \pm 1$ when $k_h \neq 0$. For $k_h = 0$, the frequency λ_k is equal to the frequency of rotation of the Earth (i.e. $|\lambda_k| = 1$) and thus we use the so-called "stationary correctors" defined in [4] in that case.

As in the top boundary layer, we have $p_B = 0$, and thus the function $u_{B,h}$ satisfies the evolution equation (2.2), together with the boundary conditions

$$\begin{aligned} u_{B,h}(t,\tau,x_h,\zeta=0) &= c_{B,h}(t,\tau,x_h) \\ &= \sum_{k\in\mathbb{Z}^3,k\neq 0} \hat{c}_{B,h}(t,k) e^{ik'_h\cdot x_h} e^{-i\lambda_k \tau}, \quad (2.12) \\ u^0_{B,h}(t,\tau,x_h,\zeta=+\infty) &= 0. \quad (2.13) \end{aligned}$$

The decomposition (2.12) leads us to search for a corrector u_B satisfying

$$u_{B,h} = \sum_{k\in\mathbb{Z}^3} u_{B,h,k},$$

where each term $u_{B,h,k}$ satisfies (2.13) and (2.2), and

$$u_{B,h,k|\zeta=0}(t,\tau,x_h) = \hat{c}_{B,h}(t,k) e^{-i\lambda_k\tau} e^{ik'_h\cdot x_h}.$$

The periodicity in time of the boundary condition prompts us to choose $u_{B,h,k}$ as a periodic function of τ, with frequency λ_k. Also, it is classical to seek $u_{B,h,k}$ as an exponentially decaying function of ζ; the rate of decay is then dictated by the equation. The precise expression of $u_{B,h,k}$ is the following (see [14]):
First case: $k_h \neq 0$.

In this case, $u_{B,h,k}$ is an exact solution of (2.2), and is equal to

$$u_{B,h,k}(t,\tau,x,y,\zeta) = \sum_{\pm} w^{\pm}_k(t;\omega) e^{-i\lambda_k\tau + ik'_h\cdot x_h - \eta^{\pm}_k \zeta} \quad (2.14)$$

where

$$\begin{aligned} \eta^{\pm}_k &= \sqrt{1\mp\lambda_k}\frac{1\pm i}{\sqrt{2}}, \\ w^{\pm}_k(t;\omega) &= \frac{1}{2}\begin{pmatrix} \hat{c}_{B,1}(t,k) \pm i\hat{c}_{B,2}(t,k) \\ \hat{c}_{B,2}(t,k) \mp i\hat{c}_{B,1}(t,k) \end{pmatrix} = \frac{\hat{c}_{B,1}(t,k)\pm i\hat{c}_{B,2}(t,k)}{2}\begin{pmatrix} 1 \\ \mp i \end{pmatrix}. \end{aligned}$$

The vertical part of the boundary layer is then given by

$$u_{B,3,k}(t,\tau,x,y,\zeta) = \sqrt{\varepsilon\nu}\sum_{\pm}\frac{1}{\eta^{\pm}_k} ik'_h\cdot w^{\pm}_k(t;\omega) e^{-i\lambda_k\tau + ik'_h\cdot x_h - \eta^{\pm}_k\zeta}. \quad (2.15)$$

Second case: $k_h = 0$.
In this case, the construction of the stationary correctors in [4] proves that there are indeed boundary

layers, but which are of order $\sqrt{\nu t}$, and not $\sqrt{\varepsilon\nu}$ in general. Thus the size of the boundary layer depends (slowly) on time.

First, notice that if $k_h = 0$, then $-\lambda_k = \mathrm{sgn}(k_3) = \pm 1$. As in the first case, we decompose $\hat{c}_{B,h}(t,k)$ onto the basis $(1, \pm i)$:

$$\hat{c}_{B,h}(t,k) = \frac{1}{2}\sum_{\pm} \left(\hat{c}_{B,1}(t,k) \mp i\hat{c}_{B,2}(t,k)\right) \begin{pmatrix} 1 \\ \pm i \end{pmatrix}.$$

As a consequence, we have

$$\begin{aligned} \sum_{k_3\in\mathbb{Z}^*} \hat{c}_{B,h}(t,0,k_3)e^{-i\lambda_k\tau} &= \alpha_+(t)e^{i\tau}\begin{pmatrix} 1 \\ i \end{pmatrix} + \alpha_-(t)e^{-i\tau}\begin{pmatrix} 1 \\ -i \end{pmatrix} \\ &+ \gamma_+(t)e^{i\tau}\begin{pmatrix} 1 \\ -i \end{pmatrix} + \gamma_-(t)e^{-i\tau}\begin{pmatrix} 1 \\ i \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \alpha_\pm(t) &= \sum_{k_3,\mathrm{sgn}(k_3)=\pm 1} \left(\hat{c}_{B,1}(t,0,k_3) \mp i\hat{c}_{B,2}(t,0,k_3)\right), \\ \gamma_\pm(t) &= \sum_{k_3,\mathrm{sgn}(k_3)=\pm 1} \left(\hat{c}_{B,1}(t,0,k_3) \pm i\hat{c}_{B,2}(t,0,k_3)\right). \end{aligned}$$

The terms $\gamma_\pm e^{\pm i\tau}(1, \mp i)$ give rise to a classical boundary layer term, namely

$$\sum_{\pm} \gamma_\pm(t)e^{\pm i\tau - \eta^\pm\zeta}\begin{pmatrix} 1 \\ \mp i \end{pmatrix}, \quad \text{with } \eta^\pm = 1 \pm i.$$

For the terms $\alpha_\pm e^{\pm i\tau}(1, \pm i)$, we rather use the following Ansatz (see [4])

$$u^{\mathrm{stat}}(t,x_h,z) = \psi\left(\frac{z}{\sqrt{\nu t}}\right)\sum_{\pm}\alpha_\pm(t)e^{\pm i\frac{t}{\varepsilon}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}. \tag{2.16}$$

In order that u^{stat} is an approximate solution of (the linear part of) equation (1.1), the function ψ must be such that

$$\begin{aligned} -\frac{1}{2}\psi'(X) - \psi''(X) &= 0, \\ \psi_{|X=0} &= 1, \end{aligned}$$

which yields

$$\psi(X) = \frac{1}{\sqrt{\pi}}\int_X^\infty \exp\left(-\frac{u^2}{4}\right) du.$$

With this definition, $u^{\mathrm{stat}}(t)$ vanishes outside a layer of size $\sqrt{\nu t}$ localized near the bottom of the fluid. Hence u^{stat} is an approximate solution of the linear part of equation (1.1), and $u^{\mathrm{stat}}_{|z=a}$ is exponentially small.

We define the operator $\mathcal{B}_B^{\mathrm{per}}$ by

$$\begin{aligned} \mathcal{B}_B^{\mathrm{per}}[c_{B,h}](t,x_h,z) &= u_B\left(t,\frac{t}{\varepsilon},x_h,\frac{z}{\sqrt{\varepsilon\nu}}\right) + u^{\mathrm{stat}}(t,x_h,z) \\ &= \sum_{\substack{k\in\mathbb{Z}^3,\\ k_h\neq 0}} u_{B,h,k}\left(t,\frac{t}{\varepsilon},x_h,\frac{z}{\sqrt{\varepsilon\nu}}\right) + \sum_{\pm}\gamma_\pm(t)e^{\pm i\frac{t}{\varepsilon} - \eta^\pm\frac{z}{\sqrt{\varepsilon\nu}}}\begin{pmatrix} 1 \\ \mp i \end{pmatrix} \\ &+ u^{\mathrm{stat}}(t,x_h,z). \end{aligned}$$

We now give some estimates on the boundary layer terms constructed in this paragraph:

Lemma 2.1. *Let u_B be defined by* (2.14)-(2.15) *and u^{stat} by* (2.16). *Then the following estimates hold, for all $t > 0$*

$$\|u_{B,h}(t),\ \zeta\partial_\zeta u_{B,h}(t)\|_{L^\infty([0,\infty)_\tau,L^2(\mathbb{T}^2\times[0,\infty)_\zeta))} \leq C\left(\sum_{\substack{k\in\mathbb{Z}^3,\\ k_h\neq 0}} |\hat{c}_{B,h}(t,k)|^2\,\frac{|k|}{|k_h|}|k_3|^2\right)^{\frac{1}{2}},$$

$$\|u_{B,h}(t),\ \zeta\partial_\zeta u_{B,h}(t)\|_{L^\infty([0,\infty)_\tau\times\mathbb{T}^2\times[0,\infty)_\zeta)} \leq C\sum_{\substack{k\in\mathbb{Z}^3,\\ k_h\neq 0}} |\hat{c}_{B,h}(t,k)|\,,$$

$$\|u_{B,3}(t),\ \zeta\partial_\zeta u_{B,3}(t)\|_{L^\infty([0,\infty),L^2(\mathbb{T}^2\times[0,\infty)))} \leq C\sqrt{\varepsilon\nu}\left(\sum_{\substack{k\in\mathbb{Z}^3,\\ k_h\neq 0}} |\hat{c}_{B,h}(t,k)|^2\,\frac{|k|^3}{|k_h|}|k_3|^2\right)^{\frac{1}{2}},$$

$$\|u_{B,3}(t),\ \zeta\partial_\zeta u_{B,3}(t)\|_{L^\infty([0,\infty)\times\mathbb{T}^2\times[0,\infty))} \leq C\sqrt{\varepsilon\nu}\sum_{\substack{k\in\mathbb{Z}^3,\\ k_h\neq 0}} |k|\ |\hat{c}_{B,h}(t,k)|\,,$$

and

$$\left\|u^{stat}(t),\ z\partial_z u^{stat}(t)\right\|_{L^2(\mathbb{T}^2\times[0,a])} \leq C(\nu t)^{1/4}\sum_{k_3\in\mathbb{Z}^*} |\hat{c}_B(t,0,k_3)|\,,$$

$$\left\|u^{stat}(t),\ z\partial_z u^{stat}(t)\right\|_{L^\infty(\mathbb{T}^2\times[0,a])} \leq C\sum_{k_3\in\mathbb{Z}^*} |\hat{c}_B(t,0,k_3)|\,.$$

The proof of the above Lemma is left to the reader. Notice that according to the definition of $\eta_k^\pm$, we have

$$C\frac{|k_h|}{|k|} \leq \left|\eta_k^\pm\right| \leq 1 \quad \forall k\in\mathbb{Z}^3.$$

2.4 Estimates on the boundary layer terms

This section is devoted to the proof of Proposition 4.

We focus on the horizontal component of u_T^δ; the vertical one is treated with similar arguments. Recall that $u_{T,h}^\delta$ is given by (2.6); in order to simplify the notation, we denote by c the horizontal boundary condition $c_{T,h}$, and we set $c^\pm := c \pm ic^\perp$.

First, we write

$$\begin{aligned}
u_{T,h}^\delta(\cdot,\tau,\cdot,\zeta,\cdot) &= \frac{\sqrt{\nu\varepsilon}\beta}{\sqrt{4\pi}}\sum_\pm\int_0^\infty \frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}-\delta s\right)c^\pm(\cdot,\tau-s,\cdot)e^{\pm is}\,ds\\
&= \frac{\sqrt{\nu\varepsilon}\beta}{\sqrt{4\pi}}\sum_\pm\int_0^\infty \frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}\right)c_\alpha^\pm(\cdot,\tau-s,\cdot)e^{(-\delta\pm i)s}\,ds \qquad (2.17)\\
&+ \frac{\sqrt{\nu\varepsilon}\beta}{\sqrt{4\pi}}\sum_\pm\int_0^\infty \frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}\right)\left(c^\pm-c_\alpha^\pm\right)(\cdot,\tau-s,\cdot)e^{(-\delta\pm i)s}\,ds. \qquad (2.18)
\end{aligned}$$

The term (2.18) can easily be evaluated thanks to Lemma 1 in the Appendix; notice that since the convergence given in Lemma 1 is not uniform with respect to $\tau\in[0,\infty)$, we cannot derive an estimate in $L^\infty([0,\infty)_\tau)$ at this stage. Hence we keep the variable τ for the time being; there exists a constant

$C>0$ and a number $s>1$ such that for all $\tau\geq 0$, $R>0$,

$$\begin{aligned}
&\left\|\int_0^\infty \frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}-\delta s\right)\left(c^\pm-c_\alpha^\pm\right)(\cdot,\tau-s,\cdot,\omega)e^{\pm is}\,ds\right\|_{L^\infty([0,T]\times E,L^\infty(\mathbb{T}^2))}\\
\leq\ & C||c-c_\alpha||_{L^\infty([0,T]\times E\times[\tau-R,\tau],H^s(\mathbb{T}^2))}\int_0^R\frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}-\delta s\right)\,ds \qquad (2.19)\\
+\ & C\left\|c\right\|_{L^\infty([0,T]\times\mathbb{R}_\tau\times\mathbb{T}^2\times E)}\int_R^\infty\frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}-\delta s\right)\,ds\\
\leq\ & \frac{C}{\delta}||c-c_\alpha||_{L^\infty([0,T]\times E\times[\tau-R,\tau],H^s(\mathbb{T}^2))}\\
+\ & C\left\|c\right\|_{L^\infty([0,T]\times\mathbb{R}_\tau\times\mathbb{T}^2\times E)}\frac{\exp(-\delta R)}{\delta}.
\end{aligned}$$

Above, we have used the Sobolev embedding $H^s(\mathbb{T}^2)\subset L^\infty(\mathbb{T}^2)$ for $s>1$. Choosing $R=\delta^{-2}$, we deduce that

$$\begin{aligned}
&\left\|\int_0^\infty \frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}-\delta s\right)\left(c^\pm-c_\alpha^\pm\right)(\cdot,\tau-s,\cdot,\omega)e^{\pm is}\,ds\right\|_{L^\infty([0,T]\times E,L^\infty(\mathbb{T}^2))}\\
&\qquad\leq \frac{C}{\delta}||c-c_\alpha||_{L^\infty([0,T]\times E\times\left[\tau-\frac{1}{\delta^2},\tau\right],H^s(\mathbb{T}^2))}+C\frac{\exp\left(-\frac{1}{\delta}\right)}{\delta}.
\end{aligned}$$

As for the term (2.17), recalling the definition of c_α, we have

$$\begin{aligned}
&\int_0^\infty\frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}-\delta s\right)c_\alpha^\pm(\cdot,\tau-s,\cdot,\omega)e^{\pm is}\,ds \qquad (2.20)\\
=\ &\int_0^\infty\int_{\mathbb{R}}\frac{1}{\sqrt{s}}e^{-\alpha|\lambda|}\exp\left(-\frac{\zeta^2}{4s}-\delta s\right)\hat{c}_\alpha^\pm(\cdot,\lambda,\cdot,\omega)e^{i\lambda(\tau-s)}e^{\pm is}\,d\lambda\,ds. \qquad (2.21)
\end{aligned}$$

We first evaluate

$$\int_0^\infty\frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}\right)e^{-(\delta+i(\lambda\pm1))s}\,ds.$$

We split the integral into two parts, one going from $s=0$ to $s=1$, and the other from $s=1$ to $s=\infty$. It is obvious that for all $\zeta>0,\delta>0,\lambda\in\mathbb{R}$,

$$\left|\int_0^1\frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}\right)e^{-(\delta+i(\lambda\pm1))s}\,ds\right|\leq\int_0^1\frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}\right)\,ds\leq\frac{1}{2}. \qquad (2.22)$$

Integrating by parts the second integral, we obtain

$$\begin{aligned}
&\int_1^\infty\frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}\right)e^{-(\delta+i(\lambda\pm1))s}\,ds\\
=\ &\frac{1}{\delta+i(\lambda\pm1)}\exp\left(-\frac{\zeta^2}{4}\right)\\
&-\frac{1}{2(\delta+i(\lambda\pm1))}\int_1^\infty\frac{1}{s^{\frac{3}{2}}}\left[1-\frac{\zeta^2}{2s}\right]\exp\left(-\frac{\zeta^2}{4s}\right)e^{-(\delta+i(\lambda\pm1))s}\,ds. \qquad (2.23)
\end{aligned}$$

We are now ready to derive the L^∞ estimate; the function

$$x\mapsto\left(1-\frac{x^2}{2}\right)e^{-\frac{x^2}{4}}$$

is bounded on $\mathbb{R}$. Hence, gathering (2.22) and (2.23), we deduce that there exists a constant C such that for all $\zeta>0,\delta>0,\lambda\in\mathbb{R}$,

$$\left|\int_0^\infty\frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}\right)e^{-(\delta+i(\lambda\pm1))s}\,ds\right|\leq C\left[1+\frac{1}{|\delta+i(\lambda\pm1)|}\right].$$

Inserting this inequality in (2.21), we obtain

$$\begin{aligned}
&\left\| \int_0^\infty \frac{1}{\sqrt{s}} \exp\left(-\frac{\zeta^2}{4s} - \delta s\right) c_\alpha^\pm(\cdot, \tau - s, \cdot, \omega) e^{\mp is}\, ds \right\|_{H^s(\mathbb{T}^2)} \\
\leq\ & C \int_{\mathbb{R}} e^{-\alpha|\lambda|} \left[1 + \frac{1}{|\delta + i(\lambda \mp 1)|}\right] \left\|\hat{c}_\alpha^\pm(\cdot, \lambda, \cdot, \omega)\right\|_{H^s(\mathbb{T}^2)}\, d\lambda \\
\leq\ & C \left[\sup_\alpha ||\hat{\sigma}_{+,\alpha}||_{L^\infty(E, L^1(\mathbb{R}, H^s(\mathbb{T}^2)))} + \int_{V_\pm} \frac{1}{|\delta + i(\lambda \mp 1)|} \left\|\hat{c}_\alpha(\cdot, \lambda, \cdot, \omega)\right\|_{H^s(\mathbb{T}^2)}\, d\lambda\right] \\
& + C \int_{\mathbb{R} \setminus V_\pm} \left\|\hat{c}_\alpha(\cdot, \lambda, \cdot, \omega)\right\|_{H^s(\mathbb{T}^2)}\, d\lambda \\
\leq\ & C \left[\sup_\alpha ||\hat{c}_\alpha||_{L^\infty(E, L^1(\mathbb{R}, H^s(\mathbb{T}^2)))} + \sup_{\lambda \in V_\pm} \left\|\hat{c}_\alpha(\lambda)\right\|_{H^s(\mathbb{T}^2)} \ln(\delta)\right].
\end{aligned}$$

Above, we have used the following facts: there exists a constant $c_1 > 0$ such that

$$|\delta + i(\lambda \mp 1)| \geq |\lambda \mp 1| \geq c_1 \quad \forall \lambda \in \mathbb{R} \setminus V_\pm,$$

and there exists another constant $c_2 > 0$ such that

$$\int_{V_\pm} \frac{1}{|\delta + i(\lambda \mp 1)|} \leq \int_{\pm 1 - c_2}^{\pm 1 + c_2} \frac{1}{\sqrt{\delta^2 + (1 + \lambda)^2}} d\lambda \leq C \ln(\delta).$$

We deduce that for all $\alpha > 0$, for all $\delta > 0$, $\tau \geq 0$,

$$\begin{aligned}
&\left|\left|u_{T,h}^\delta(\tau)\right|\right|_{L^\infty([0,T] \times \mathbb{T}^2 \times [0,\infty)\zeta \times E)} \\
\leq\ & C\sqrt{\varepsilon\nu}\beta \left[1 + \frac{\exp\left(-\frac{1}{\delta}\right)}{\delta}\right] \\
+\ & C\sqrt{\varepsilon\nu}\beta \left[\frac{1}{\delta} ||c - c_\alpha||_{L^\infty([0,T] \times [\tau - \delta^{-1}, \tau] \times E, H^s(\mathbb{T}^2))} + \sup_{\lambda \in V_+ \cup V_-} \left\|\hat{c}_\alpha(\lambda)\right\| \ln(\delta)\right].
\end{aligned}$$

Taking the infimum with respect to α of the right-hand side, with $\delta > 0$ fixed, we deduce that

$$\sup_{\delta > 0} \left|\left|u_{T,h}^\delta\right|\right|_{L^\infty([0,T] \times [0,\infty)\tau \times \mathbb{T}^2 \times [0,\infty)\zeta \times E)} \leq C\sqrt{\varepsilon\nu}\beta.$$

We now turn to the derivation of the L^2 estimate, which is similar to the above computations. The main difference lies in the fact that we need to integrate by parts (2.23) yet another time, which yields

$$\begin{aligned}
&\int_1^\infty \frac{1}{\sqrt{s}} \exp\left(-\frac{\zeta^2}{4s}\right) e^{-(\delta + i(\lambda \pm 1))s}\, ds \\
=\ & \frac{1}{\delta + i(\lambda \pm 1)} \exp\left(-\frac{\zeta^2}{4}\right) - \frac{1}{2(\delta + i(\lambda \pm 1))^2} \left[1 - \frac{\zeta^2}{2}\right] \exp\left(-\frac{\zeta^2}{4}\right) \\
& - \frac{1}{2(\delta + i(\lambda \pm 1))^2} \int_1^\infty \frac{1}{s^{\frac{5}{2}}} \phi\left(\frac{\zeta}{\sqrt{s}}\right) e^{-(\delta + i(\lambda \pm 1))s}\, ds,
\end{aligned}$$

where

$$\phi(x) = -\left(\frac{x^4}{8} - \frac{3x^2}{2} + \frac{3}{2}\right) \exp\left(-\frac{x^2}{4}\right).$$

Consequently, remembering (2.22), we have

$$\begin{aligned}
&\left|\int_0^\infty \frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}\right)e^{-(\delta+i(\lambda\pm1))s}\,ds\right| \\
\leq\ & \int_0^1 \frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}\right)\,ds + \frac{1}{|\delta+i(\lambda\pm1)|}\exp\left(-\frac{\zeta^2}{4}\right) \\
&+\frac{1}{2|\delta+i(\lambda\pm1)|^2}\left|1-\frac{\zeta^2}{2}\right|\exp\left(-\frac{\zeta^2}{4}\right) \\
&+\frac{1}{2|\delta+i(\lambda\pm1)|^2}\int_1^\infty \frac{1}{s^{\frac{5}{2}}}\left|\phi\left(\frac{\zeta}{\sqrt{s}}\right)\right|\,ds.
\end{aligned}$$

Plugging this estimate into (2.21) and using **(H1)-(H2)**, we infer that for all $\zeta>0$, for all $s>0$,

$$\begin{aligned}
&\left\|\int_0^\infty \frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}-\delta s\right)c_\alpha^\pm(\cdot,\tau-s,\cdot,\omega)e^{\pm is}\,ds\right\|_{H^s(\mathbb{T}^2)} \\
\leq\ & C\left[\int_0^1 \frac{1}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}\right)\,ds\right] \\
&+C\exp\left(-\frac{\zeta^2}{4}\right)\left(1+\sup_{\lambda\in V_\pm}\|\hat{c}_\alpha(\lambda)\|_{L^\infty([0,T]\times E,H^s(\mathbb{T}^2)))}\ln(\delta)\right) \\
&+C\left|1-\frac{\zeta^2}{2}\right|\exp\left(-\frac{\zeta^2}{4}\right)\left(1+\sup_{\lambda\in V_\pm}\|\hat{c}_\alpha(\lambda)\|_{L^\infty([0,T]\times E,H^s(\mathbb{T}^2)))}\frac{1}{\delta}\right) \\
&+C\left[\int_1^\infty \frac{1}{s^{\frac{5}{2}}}\left|\phi\left(\frac{\zeta^2}{s}\right)\right|\,ds\right]\left(1+\sup_{\lambda\in V_\pm}\|\hat{c}_\alpha(\lambda)\|_{L^\infty([0,T]\times E,H^s(\mathbb{T}^2)))}\frac{1}{\delta}\right).
\end{aligned}$$

Here, we have used the inequality

$$\int_{V_\mp}\frac{d\lambda}{|\delta+i(\lambda\pm1)|^2}\leq\int_{\mp1-c_2}^{\mp1+c_2}\frac{d\lambda}{\delta^2+(\lambda\pm1)^2}\leq\frac{C}{\delta}.$$

There only remains to prove that each term of the right-hand side has a finite L^2 norm. First, thanks to Jensen's inequality, we have

$$\begin{aligned}
\int_0^\infty\left(\int_0^1\frac{2}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{4s}\right)\,ds\right)^2 d\zeta &\leq \int_0^\infty\int_0^1\frac{2}{\sqrt{s}}\exp\left(-\frac{\zeta^2}{2s}\right)\,ds d\zeta \\
&\leq 2\int_0^1 ds\int_0^\infty e^{-\frac{x^2}{2}}\,dx<\infty.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_0^\infty\left(\int_1^\infty\frac{1}{s^{\frac{5}{2}}}\left|\phi\left(\frac{\zeta}{\sqrt{s}}\right)\right|\,ds\right)^2 d\zeta &\leq C\int_0^\infty\int_1^\infty\frac{1}{s^{\frac{5}{2}}}\left|\phi\left(\frac{\zeta}{\sqrt{s}}\right)\right|^2\,ds d\zeta \\
&\leq C\left(\int_1^\infty\frac{1}{s^2}\right)\left(\int_0^\infty|\phi(x)|^2\,dx\right)<\infty.
\end{aligned}$$

We also have to evaluate the L^2 norm of the integral in (2.19); we have

$$\begin{aligned}
& \int_0^\infty \left[\int_0^\infty \frac{1}{\sqrt{s}} \exp\left(-\frac{\zeta^2}{4s} - \delta s\right) ds\right]^2 d\zeta \\
\underset{\substack{x=\sqrt{\delta}\zeta,\\ u=\delta s}}{\leq} & \frac{1}{\delta^{\frac{3}{2}}} \int_0^\infty \left[\int_0^\infty \frac{1}{\sqrt{u}} \exp\left(-\frac{x^2}{4u} - u\right) du\right]^2 dx \\
\leq & \frac{1}{\delta^{\frac{3}{2}}} \int_0^\infty \int_0^\infty \frac{1}{u} \exp\left(-\frac{x^2}{2u} - u\right) du\, dx \\
\leq & \frac{1}{\delta^{\frac{3}{2}}} \int_0^\infty \int_0^\infty \frac{1}{\sqrt{u}} \exp\left(-\frac{x^2}{2} - u\right) du\, dx \\
\leq & \frac{C}{\delta^{\frac{3}{2}}}.
\end{aligned}$$

Gathering all the terms, we obtain, for all $\alpha, \delta > 0$, for all $\tau > 0$,

$$\begin{aligned}
& \left\| u^\delta_{T,h}(\tau) \right\|^2_{L^\infty([0,T]\times E, L^2([0,\infty)_\zeta, H^s(\mathbb{T}^2)))} \\
\leq & C\beta^2 \varepsilon\nu \frac{||c - c_\alpha||_{L^\infty([0,T]\times[\tau-\delta^{-1},\tau]\times E, H^s(\mathbb{T}^2))}}{\delta^{\frac{3}{2}}} \\
+ & C\beta^2 \varepsilon\nu \left(\frac{\exp\left(-\frac{1}{\delta}\right)}{\delta^{\frac{3}{2}}} + \sup_{\lambda \in V_-} |\hat{\sigma}_{+,\alpha}(\lambda)| \left(\frac{1}{\delta} + \ln(\delta)\right)\right).
\end{aligned}$$

Taking the infimum of the above inequality with respect to α, we infer the L^2 estimate on $u^\delta_{T,h}$. The estimates on $u^\delta_{T,3}$ and $\zeta\partial_\zeta u^\delta_T$ are derived in a similar fashion.

3 The interior operator

This section is devoted to the construction of an approximate solution $u^{\rm int}$ to the evolution equation (1.1) supplemented with the initial condition

$$u^{\rm int}_{|t=0} = u_0 \in \mathcal{H}$$

and with boundary conditions of the type

$$\begin{aligned}
u^{\rm int}_{3|z=0}(t, x_h) &= \sqrt{\varepsilon\nu} c_{B,3}\left(t, \frac{t}{\varepsilon}, x_h\right), \\
u^{\rm int}_{3|z=a}(t, x_h) &= \sqrt{\varepsilon\nu} c_{T,3}\left(t, \frac{t}{\varepsilon}, x_h\right)
\end{aligned} \tag{3.1}$$

where c_B and c_T are periodic with respect to the horizontal variable x_h, and either almost periodic or random and stationary with respect to the fast time variable.

We decompose $u^{\rm int}$ into three terms

$$u^{\rm int}(t) := \bar{u}^{\rm int}\left(t, \frac{t}{\varepsilon}\right) + v^{\rm int}\left(t, \frac{t}{\varepsilon}\right) + \delta u^{\rm int}\left(t, \frac{t}{\varepsilon}\right),$$

where $\bar{u}^{\rm int}$ is the preponderant term, and $v^{\rm int}, \delta u^{\rm int}$ are corrector terms, the roles of which are respectively to ensure that the boundary conditions (3.1) are satisfied, and that $u^{\rm int}$ is an approximate solution of (1.1).

We have already seen in paragraph 1.1 that it is natural to seek $\bar{u}^{\rm int}$ as

$$\bar{u}^{\rm int}(t, \tau) = \mathcal{L}(\tau) w(t),$$

where $\mathcal{L}(\tau) = \exp(-\tau L)$ is the Coriolis semi-group, and

$$w(t) = \sum_{k\in\mathbb{Z}^3} b(t,k)N_k \in \mathcal{H}. \tag{3.2}$$

The construction of w, $v^{\rm int}$ and $\delta u^{\rm int}$ is as follows: first, we define a corrector $v^{\rm int}$ which satisfies the boundary conditions (3.1), but not equation (1.1). Then, we derive an equation on $\bar{u}^{\rm int} + \delta u^{\rm int}$; filtering the oscillating terms in $v^{\rm int}$ yields the equation on w. Then, the term $\delta u^{\rm int}$ is defined so as to take into account the oscillating terms in the equation on $\bar{u}^{\rm int} + \delta u^{\rm int}$.

• **Definition of $v^{\rm int}$.**

We look for a divergence-free function $v^{\rm int}$ satisfying (3.1). Of course, conditions (3.1) do not determine $v^{\rm int}$ unequivocally. A possible choice is

$$v_3^{\rm int}(t,\tau,x) = \frac{\sqrt{\varepsilon\nu}}{a}\left[c_{T,3}\left(t,\tau,x_h\right)z + c_{B,3}\left(t,\tau,x_h\right)(a-z)\right] \tag{3.3}$$

$$v_h^{\rm int}(t,\tau,x) = \frac{\sqrt{\varepsilon\nu}}{a}\sqrt{\varepsilon\nu}\nabla_h\Delta_h^{-1}\left[c_{B,3}\left(t,\tau,x_h\right) - c_{T,3}\left(t,\tau,x_h\right)\right]. \tag{3.4}$$

In fact, if c_B and c_T are both almost periodic functions of the form (2.1), then a more convenient choice can be made, which is the so-called "non-resonant" choice in [14]. In this case, the equation on $\delta u^{\rm int}$ is more simple, since there is no source term due to $v^{\rm int}$. However, we have chosen here not to distinguish between stationary and almost periodic boundary conditions, and thus to work with the expressions (3.3), (3.4).

We give here the statement and proof of a Lemma which will be useful in the construction of $\delta u^{\rm int}$ and w.

Lemma 3.1. *Let $T > 0$ be arbitrary. Assume that the functions $c_{B,3}, c_{T,3}$ are random stationary, and that $\partial_\tau c_{B,3}, \partial_\tau c_{B,3}$ belong to $L^2(\mathbb{T}^2, L^\infty([0,T]\times[0,\infty)\times E))$.*

Let $v \in ([0,T]\times[0,\infty)_\tau, L^2(\mathbb{T}^2\times E))$ such that $\partial_\tau v \in L^\infty_{loc}([0,T]\times[0,\infty)_\tau, L^2(\mathbb{T}^2\times E))$ and

$$\mathrm{div} v = 0, \tag{3.5}$$

$$v_3(t,\tau,x_h,z=0) = \sqrt{\varepsilon\nu}c_{B,3}(t,\tau,x_h), \tag{3.6}$$

$$v_3(t,\tau,x_h,z=1) = \sqrt{\varepsilon\nu}c_{T,3}(t,\tau,x_h),. \tag{3.7}$$

Then as $\theta\to\infty$, the family

$$S_\theta := \frac{1}{\theta}\int_0^\theta \mathcal{L}(-\tau)\mathbb{P}\left[\partial_\tau v + e_3\wedge v\right]\,d\tau$$

converges almost everywhere and in $L^\infty([0,T], L^2(\mathbb{T}^2\times[0,a]\times E))$, and its limit does not depend on the function v. Precisely,

$$\lim_{\theta\to\infty} S_\theta = \frac{\sqrt{\varepsilon\nu}}{\sqrt{aa_1a_2}}\sum_{k\in\mathbb{Z}^3}\frac{|k_h'|}{|k'|^2}\mathcal{E}_{-\lambda_k}\left[\hat{c}_{B,3}(k_h) - (-1)^{k_3}\hat{c}_{T,3}(k_h)\right]N_k, \tag{3.8}$$

where $\hat{c}(k_h) = \frac{1}{\sqrt{a_1a_2}}\int_{\mathbb{T}^2} c(x_h)e^{-ik_h'\cdot x_h}dx_h$.

Remark 3.1. The same result can easily be proved when the functions v, c_B, c_T are assumed to be almost periodic of the form (2.1). (In fact, we recall that the almost periodic case can be embedded in the random stationary one). When (2.1) holds, the limit of S_θ as $\theta\to\infty$ takes the form

$$\frac{\sqrt{\varepsilon\nu}}{\sqrt{aa_1a_2}}\sum_{k\in\mathbb{Z}^3}\frac{|k_h'|}{|k'|^2}\left[\hat{c}_{B,3} - (-1)^{k_3}\hat{c}_{T,3}\right](t,k)N_k.$$

Proof. Let v^1, v^2 be two solutions of (3.5)-(3.7), and let $V = v^1 - v^2$. Notice that $V\in L^\infty([0,T]\times[0,\infty)_\tau; L^2(E,\mathcal{H}))$, and $\partial_\tau V\in L^\infty([[0,T]\times[0,\infty)_\tau; L^2(E\times\mathbb{T}^2\times[0,a]))$. We write

$$\mathcal{L}(-\tau)\mathbb{P}\left[\partial_\tau V + e_3\wedge V\right] = \mathcal{L}(-\tau)\left[\partial_\tau V + LV\right] = \frac{\partial}{\partial\tau}\left[\mathcal{L}(-\tau)V(\tau)\right].$$

Consequently,

$$\frac{1}{\theta}\int_0^\theta \mathcal{L}(-\tau)\left[\partial_\tau V + \mathbb{P}\left(e_3 \wedge V\right)\right]\, d\tau = \frac{\mathcal{L}(-\theta)V(\tau=\theta) - V(\tau=0)}{\theta}.$$

The right-hand side of the above equality vanishes in $L^\infty([0,T]\times E, L^2(\mathbb{T}^2))$ as $\theta \to \infty$. Hence the limit is independent of the choice of v.

In order to complete the proof of the lemma, it is thus sufficient to show that the limit exists for the choice (3.3)-(3.4), and to compute the limit in this case. For all $k = (k_h, k_3) \in \mathbb{Z}^3$, we have

$$\begin{aligned}
\left\langle N_k, \partial_\tau v^{\rm int}\right\rangle &= \frac{\sqrt{\varepsilon\nu}}{a}\int_0^a \cos(k_3' z)\overline{n_h(k)}\cdot\frac{ik_h'}{|k_h'|^2}\left(\partial_\tau \hat{c}_{T,3}(\cdot,k_h) - \partial_\tau \hat{c}_{B,3}(\cdot,k_h)\right)\, dz \\
&+ \frac{\sqrt{\varepsilon\nu}}{a}\int_0^a \overline{n_3(k)}\sin(k_3' z)\left(\partial_\tau \hat{c}_{B,3}(\cdot,k_h)(a-z) + \partial_\tau \hat{c}_{T,3}(\cdot,k_h) z\right)\, dz \\
&= \sqrt{\varepsilon\nu}\,\overline{n_3(k)}\frac{\mathbf{1}_{k_3\neq 0}}{k_3'}\left[\partial_\tau \hat{c}_{B,3}(\cdot,k_h) - (-1)^{k_3}\partial_\tau \hat{c}_{T,3}(\cdot,k_h)\right] \\
&+ \frac{\sqrt{\varepsilon\nu}}{a}\mathbf{1}_{k_3=0}\,\overline{n_h(k)}\cdot\frac{ik_h'}{|k_h'|^2}\left(\partial_\tau \hat{c}_{T,3}(\cdot,k_h) - \partial_\tau \hat{c}_{B,3}(\cdot,k_h)\right).
\end{aligned}$$

Notice that if $k_3 = 0$, then

$$\overline{n_h(k)}\cdot k_h' = 0;$$

consequently, we have

$$\left\langle N_k, \partial_\tau v^{\rm int}\right\rangle = -i\sqrt{\frac{\varepsilon\nu}{aa_1a_2}}\frac{\mathbf{1}_{k_3\neq 0}|k_h'|}{|k'|k_3'}\left[\partial_\tau \hat{c}_{B,3}(t,\tau,k_h;\omega) - (-1)^{k_3}\partial_\tau \hat{c}_{T,3}(t,\tau,k_h;\omega)\right].$$

In a similar way,

$$\begin{aligned}
\left\langle N_k, e_3\wedge v^{\rm int}\right\rangle &= \frac{\sqrt{\varepsilon\nu}}{a}\int_0^a \cos(k_3' z)\overline{n_h(k)}\cdot\frac{i(k_h')^\perp}{|k_h'|^2}\left(\hat{c}_{T,3}(\cdot,k_h) - \hat{c}_{B,3}(\cdot,k_h)\right)\, dz \\
&= \sqrt{\varepsilon\nu}\mathbf{1}_{k_3=0}\overline{n_h(k)}\cdot\frac{i(k_h')^\perp}{|k_h'|^2}\left(\hat{c}_{T,3}(\cdot,k_h) - \hat{c}_{B,3}(\cdot,k_h)\right) \\
&= \sqrt{\frac{\varepsilon\nu}{aa_1a_2}}\mathbf{1}_{k_3=0}\frac{1}{|k_h'|}\left(\hat{c}_{B,3}(\cdot,k_h) - \hat{c}_{T,3}(\cdot,k_h)\right).
\end{aligned}$$

Notice that if $l \in \mathbb{Z}^3$ is such that $l_h \neq k_h$, then

$$\left\langle N_l, \partial_\tau v^{\rm int}\right\rangle = 0, \quad \left\langle N_l, e_3\wedge v^{\rm int}\right\rangle = 0.$$

We deduce from the above calculations that

$$\begin{aligned}
&\mathcal{L}(-\tau)\mathbb{P}(\partial_\tau v^{\rm int} + e_3\wedge v^{\rm int}) \qquad (3.9)\\
= \;& -i\sqrt{\frac{\varepsilon\nu}{aa_1a_2}}\sum_{k\in\mathbb{Z}^3}\frac{\mathbf{1}_{k_3\neq 0}|k_h'|}{k_3'|k'|}e^{i\lambda_k\tau}\left[\partial_\tau \hat{c}_{B,3} - (-1)^{k_3}\partial_\tau \hat{c}_{T,3}\right](t,\tau,k_h;\omega)N_k \\
+ \;& \sqrt{\frac{\varepsilon\nu}{aa_1a_2}}\sum_{k\in\mathbb{Z}^3}\mathbf{1}_{k_3=0}\frac{1}{|k_h'|}\left(\hat{c}_{B,3}(t,\tau,k_h;\omega) - \hat{c}_{T,3}(t,\tau,k_h;\omega)\right)N_k.
\end{aligned}$$

We decompose the sum in the right-hand side into two sums, one bearing on k_h such that $|k_h| > A$,

denoted by $S_{1,A}$, and the other on $|k_h| \leq A$, denoted by $S_{2,A}$, for some $A > 0$ arbitrary. We have

$$\begin{aligned}
&\|S_{1,A}(t,\tau)\|_{L^2}^2 \\
\leq\ & C\varepsilon\nu \left\| \sum_{|k_h|>A} \sum_{k_3\in\mathbb{Z}} \frac{\mathbf{1}_{k_3\neq 0}|k_h'|}{k_3|k'|} e^{i\lambda_k\tau} \left[\partial_\tau \hat{c}_{B,3} - (-1)^{k_3}\partial_\tau \hat{c}_{T,3}\right](t,\tau,k_h;\omega) N_k \right\|_{L^2}^2 \\
+\ & C\varepsilon\nu \left\| \sum_{|k_h|>A} \frac{1}{|k_h|} \left(\hat{c}_{T,3}(t,\tau,k_h;\omega) - \hat{c}_{B,3}(t,\tau,k_h;\omega)\right) N_{k_h,0} \right\|_{L^2} \\
\leq\ & C\varepsilon\nu \sum_{|k_h|>A} \left(|\partial_\tau \hat{c}_{B,3}(t,\tau,k_h;\omega)|^2 + |\partial_\tau \hat{c}_{T,3}(t,\tau,k_h;\omega)|^2\right) \\
&+ \sum_{|k_h|>A} \left(|\hat{c}_{B,3}(t,\tau,k_h;\omega)|^2 + |\hat{c}_{T,3}(t,\tau,k_h;\omega)|^2\right).
\end{aligned}$$

Since $c_B, c_T, \partial_\tau c_B, \partial_\tau c_T$ belong to $L^2(\mathbb{T}^2, L^\infty([0,\infty)\times[0,T]\times E))$, we deduce that the sum $S_{1,A}$ vanishes in $L^\infty([0,T]\times[0,\infty), L^2(\mathbb{T}^2\times[0,a]\times E))$ as $A\to\infty$. Thus we work with A sufficiently large, but fixed, so that $S_{1,A}$ is arbitrarily small in L^2 norm, and we focus on $S_{2,A}$.

For $k\in\mathbb{Z}^3$ fixed, we have, according to Proposition 1,

$$\begin{aligned}
&\frac{1}{\theta}\int_0^\theta e^{i\lambda_k\tau}\left[\partial_\tau \hat{c}_{B,3} - (-1)^{k_3}\partial_\tau \hat{c}_{T,3}\right](t,\tau,k_h;\omega)\,d\tau \\
=\ & -i\lambda_k \frac{1}{\theta}\int_0^\theta e^{i\lambda_k\tau}\left[\hat{c}_{B,3} - (-1)^{k_3}\hat{c}_{T,3}\right](t,\tau,k_h;\omega)\,d\tau \\
+\ & \frac{1}{\theta}\left\{e^{i\lambda_k\theta}\left[\hat{c}_{B,3} - (-1)^{k_3}\hat{c}_{T,3}\right](t,\theta,k_h;\omega) - \left[\hat{c}_{B,3} - (-1)^{k_3}\hat{c}_{T,3}\right](t,0,k_h;\omega)\right\} \\
\xrightarrow[\theta\to\infty]{}\ & -i\lambda_k \mathcal{E}_{-\lambda_k}\left[\hat{c}_{B,3}(t,k_h) - (-1)^{k_3}\hat{c}_{T,3}(t,k_h)\right](\omega)
\end{aligned}$$

in $L^\infty([0,\infty)_t, L^2(E))$. Using Lebesgue's Theorem, we deduce that as $\theta\to\infty$

$$\frac{1}{\theta}\int_0^\theta S_{2,A}(t,\tau)\,d\tau \to \sqrt{\varepsilon\nu}\sum_{|k_h|\leq A}\sum_{k_3\in\mathbb{Z}} \frac{|k_h'|}{|k'|^2}\mathcal{E}_{-\lambda_k}\left[\hat{c}_{B,3}(t,k_h) - (-1)^{k_3}\hat{c}_{T,3}(t,k_h)\right]N_k \tag{3.10}$$

and the convergence holds in $L^\infty([0,T], L^2(\mathbb{T}^2\times[0,a]\times E))$. Moreover, for $c = c_{T,3}$ or $c_{B,3}$, we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^3}\frac{|k_h'|^2}{|k'|^4}\left|\mathcal{E}_{-\lambda_k}[\hat{c}(t,k_h)]\right|^2 &\leq C\sum_{k_3\in\mathbb{Z}^*}\frac{1}{1+|k_3|^2}\|\mathcal{E}_{-\lambda_k}[c(t)]\|_{L^2(\mathbb{T}^2)}^2 \\
&\leq C\|c\|_{L^\infty([0,\infty)\times[0,\infty)\times E, L^2(V_h)}^2.
\end{aligned}$$

Thus the right-hand side of (3.10) converges in $L^2(\mathbb{T}^2\times[0,a]\times E)$ as $A\to\infty$. Eventually, we infer (3.8). □

• Definition of w.

Remember that $u^{\rm int}$ should be an approximate solution of equation (1.1), and that $\delta u^{\rm int}, v^{\rm int}$ are strongly oscillating terms, small in H^1 norm. Consequently, since $\bar{u}^{\rm int}(t,\tau) = \mathcal{L}(\tau)w(t)$, we have

$$\begin{aligned}
&\partial_t u^{\rm int} + u^{\rm int}\cdot\nabla u^{\rm int} + \frac{1}{\varepsilon}e_3\wedge u^{\rm int} - \Delta_h u^{\rm int} - \nu\partial_z^2 u^{\rm int} \\
\approx\ & \mathcal{L}\left(\frac{t}{\varepsilon}\right)\partial_t w + \bar{u}^{\rm int}\cdot\nabla\bar{u}^{\rm int} - \Delta_h\bar{u}^{\rm int} + \frac{1}{\varepsilon}\left[\partial_\tau\delta u^{\rm int} + L\delta u^{\rm int}\right] + \frac{1}{\varepsilon}\left[\partial_\tau v^{\rm int} + e_3\wedge v^{\rm int}\right] \\
=\ & \mathcal{L}\left(\frac{t}{\varepsilon}\right)\left[\partial_t w + Q(\tau,w,w) - \Delta_h w\right] \\
+\ & \frac{1}{\varepsilon}\left[\mathcal{L}(\tau)\,\partial_\tau\left(\mathcal{L}(-\tau)\,\delta u^{\rm int}(t,\tau)\right)\right]_{|\tau=\frac{t}{\varepsilon}} + \Sigma\left(t,\frac{t}{\varepsilon}\right),
\end{aligned}$$

where

$$Q(\tau,w,w)=\mathcal{L}(-\tau)\mathbb{P}\left[\nabla(\mathcal{L}(\tau)w\otimes\mathcal{L}(\tau)w)\right].$$

and Σ is defined by

$$\Sigma(t,\tau):=\frac{1}{\varepsilon}\left[\frac{\partial}{\partial\tau}v^{\rm int}(t,\tau)+e_3\wedge v^{\rm int}(t,\tau)\right]. \tag{3.11}$$

Thus it is natural to choose w and $\delta u^{\rm int}$ such that for all t,τ,

$$\partial_t w+Q(\tau,w,w)-\Delta_h w+\mathcal{L}\left(-\tau\right)\mathbb{P}\Sigma\left(t,\tau\right)+\frac{1}{\varepsilon}\partial_\tau\left[\mathcal{L}\left(-\tau\right)\delta u^{\rm int}(t,\tau)\right]=0. \tag{3.12}$$

The quantity $\mathcal{L}(-\tau)\mathbb{P}\Sigma(t,\tau)$ has already been computed in Lemma 3.1 (see (3.9)). Since w does not depend on τ, the first idea is to average the above equation on a time interval $[0,\theta]$, and to pass to the limit as $\theta\to\infty$ in order to derive an equation for w. We have already proved in Lemma 3.1 that if $c_{B,3},c_{T,3}$ are sufficiently smooth, then

$$\lim_{\theta\to\infty}\frac{1}{\theta}\int_0^\theta\mathcal{L}\left(-\tau\right)\mathbb{P}\Sigma\left(t,\tau\right)\,d\tau = \sqrt{\frac{\nu}{\varepsilon}}\frac{1}{\sqrt{aa_1a_2}}\sum_{k\in\mathbb{Z}^3}\frac{|k_h'|}{|k'|^2}\mathcal{E}_{-\lambda_k}\left[\hat{c}_{B,3}(t,k_h)-(-1)^{k_3}\hat{c}_{T,3}(t,k_h)\right]N_k$$

in $L^\infty_{\rm loc}([0,\infty)_t,L^2(\mathbb{T}^2\times[0,a)\times E))$. Moreover, with the notation (3.2),

$$Q(\tau,w,w)=\sum_{k,l,m\in\mathbb{Z}^3}e^{i(-\lambda_l-\lambda_m+\lambda_k)\tau}b(t,l;\omega)b(t,m;\omega)\left\langle\left(N^l\cdot\nabla\right)N^m,N^k\right\rangle N^k,$$

and it is proved in [3] that if w is sufficiently smooth,

$$\frac{1}{\theta}\int_0^\theta Q(\tau,w,w)\rightharpoonup\bar{Q}(w,w)$$

in the distributional sense, where $\bar{Q}$ is defined by (1.5). Hence, we define w as the unique solution in $L^\infty(E,\mathcal{C}([0,\infty),\mathcal{H}\cap H^{0,1}))\cap L^\infty(E,L^2_{\rm loc}([0,\infty),H^{1,0}))$ of the equation

$$\begin{aligned}&\partial_t w+\bar{Q}(w,w)-\Delta_h w+\sqrt{\frac{\nu}{\varepsilon}}\bar{S}[c_{B,3},c_{T,3}]=0,\\&w_{|t=0}=u_0\in\mathcal{H}\cap H^{0,1},\end{aligned} \tag{3.13}$$

where

$$\bar{S}[c_{B,3},c_{T,3}]:=\frac{1}{\sqrt{aa_1a_2}}\sum_{k\in\mathbb{Z}^3}\frac{|k_h'|}{|k'|^2}\mathcal{E}_{-\lambda_k}\left[\hat{c}_{B,3}(t,k_h)-(-1)^{k_3}\hat{c}_{T,3}(t,k_h)\right]N_k.$$

We refer to Proposition 6.5 p. 145 in [3] and to the comments following Proposition 2 in the Introduction of this paper for existence and uniqueness results about equation (3.13). Notice that if $c_{B,3},c_{T,3}\in L^\infty([0,T)\times[0,\infty)_\tau\times E,H^1(\mathbb{T}^2))$, then $\bar{S}[c_{B,3},c_{T,3}]\in L^\infty([0,T]\times E,H^{0,1})$.

• **Definition of $\delta u^{\rm int}$.**

Once w is defined, there only remains to obtain an equation on $\delta u^{\rm int}$. As stated before, $\delta u^{\rm int}$ is chosen so that equality (3.12) holds for all $\tau\geq0$. According to the above computations, this amounts to taking $\delta u^{\rm int}$ such that

$$\begin{aligned}\frac{\partial}{\partial\tau}\left[\mathcal{L}(-\tau)\delta u^{\rm int}(\tau)\right] &= \varepsilon\bar{Q}(w,w)-\varepsilon Q(\tau,w,w)+\sqrt{\varepsilon\nu}\bar{S}[c_{B,3},c_{T,3}]-\varepsilon\mathcal{L}(-\tau)\mathbb{P}\Sigma(t,\tau),\\ \mathcal{L}(-\tau)\delta u^{\rm int}(\tau) &= \varepsilon\int_0^\tau\left[\bar{Q}(w,w)-Q(s,w,w)\right]\,ds\\ &\quad+\int_0^\tau\left[\sqrt{\varepsilon\nu}\bar{S}[c_{B,3},c_{T,3}]-\varepsilon\mathcal{L}(-s)\mathbb{P}\Sigma(t,s)\right]\,ds\\ \delta u^{\rm int}(\tau) &= \varepsilon\mathcal{L}(\tau)\int_0^\tau\left[\bar{Q}(w,w)-Q(s,w,w)\right]\,ds\\ &\quad+\mathcal{L}(\tau)\int_0^\tau\left[\sqrt{\varepsilon\nu}\bar{S}[c_{B,3},c_{T,3}]-\varepsilon\mathcal{L}(-s)\mathbb{P}\Sigma(t,s)\right]\,ds.\end{aligned} \tag{3.14}$$

Equivalently, $\delta u^{\rm int}$ satifies the equation

$$\partial_\tau \delta u^{\rm int} + L\delta u^{\rm int} = \varepsilon \mathcal{L}(\tau)\left[\bar{Q}(w,w) - \varepsilon Q(\tau,w,w)\right] + \sqrt{\varepsilon\nu}\mathcal{L}(\tau)\bar{S}[c_{B,3},c_{T,3}] - \varepsilon \mathbb{P}\Sigma(t,\tau).$$

We now derive a bound on the coefficients of $\delta u^{\rm int}$:

Lemma 3.2. *Let $T>0$, $N>0$, and let $w \in L^\infty(E,\mathcal{C}([0,T],\mathcal{H}))$ such that*

$$\langle N_k, w(t)\rangle = 0 \quad \forall k, |k|>N,\ \forall t\in[0,T].$$

Let $c_{B,3}, c_{T,3} \in L^\infty([0,T]\times[0,\infty)\times E, L^2(\mathbb{T}^2))$ be such that the assumptions of Lemma 3.1 are satisfied.

Let Σ be given by (3.11), and δu^{int} by (3.14). Then for all $k\in\mathbb{Z}^3$, for all $\eta>0$, there exists a constant $C_{\eta,k}$ such that for all $\tau\geq 0$, for all $\varepsilon,\nu>0$ such that $\nu=\mathcal{O}(\varepsilon)$,

$$\left\|\left\langle N_k, \delta u^{int}(t,\tau)\right\rangle\right\|_{L^\infty([0,T],L^2(E))} \leq (\varepsilon+\sqrt{\varepsilon\nu})(C_{\eta,k}+\eta\tau).$$

Remark 3.2. The above Lemma is stated with a function w having only a finite number of Fourier modes, which is not the case for the solution of (3.13) in general. However, when constructing the approximate solution in paragraph 4.3, we will consider regularizations of the solution w of the envelope equation (1.9), so that this issue is in fact unimportant.

Proof. We begin with the derivation of a bound for the term

$$\begin{aligned}
&\int_0^\tau \left[\bar{Q}(w,w) - Q(s,w,w)\right]\, ds \\
&= \quad - \sum_{\substack{k,l,m \\ \lambda_l+\lambda_m\neq\lambda_k}} \langle N_k, N_l\cdot\nabla N_m\rangle\, b(t,m)b(t,l)\left(\int_0^\tau e^{i(\lambda_k-\lambda_l-\lambda_m)s}ds\right) N_k.
\end{aligned}$$

Notice that the set $(l,m)\in\mathbb{Z}^3\times\mathbb{Z}^3$ such that $b(t,l)b(t,m)\neq 0$ is finite, and included in $B_N\times B_N$. Moreover, if $(l,m)\in B_N\times B_N$ and $\lambda_l+\lambda_m\neq\lambda_k$, then there exists a constant $\alpha_{N,k}>0$ such that

$$|\lambda_l+\lambda_m-\lambda_k|\geq\alpha_{N,k}.$$

As a consequence, we have

$$\left|\left\langle N_k, \int_0^\tau \left[\bar{Q}(w(t),w(t)) - Q(s,w(t),w(t))\right]\, ds\right\rangle\right| \leq \frac{1}{\alpha_{N,k}}\|w\|^2_{L^\infty((0,T)\times\mathbb{T}^2\times[0,a]\times E)}.$$

In a similar way, we now derive a bound on the second term in (3.14). According to Lemma 3.1, we have, for all $k\in\mathbb{Z}^3$,

$$\frac{1}{\tau}\int_0^\tau \langle N_k, \mathcal{L}(-s)\mathbb{P}\Sigma(t,s)\rangle\, ds \to \sqrt{\frac{\nu}{\varepsilon}}\left\langle N_k, \bar{S}[c_{B,3},c_{T,3}]\right\rangle$$

as $\tau\to\infty$, in $L^\infty([0,T],L^2(E))$. Let $\tau_{\eta,k}>0$ such that if $\tau\geq\tau_{\eta,k}$, then

$$\left\|\frac{1}{\tau}\int_0^\tau \langle N_k, \mathcal{L}(-s)\mathbb{P}\Sigma(t,s)\rangle - \left\langle N_k, \sqrt{\frac{\nu}{\varepsilon}}\bar{S}[c_{B,3},c_{T,3}]\right\rangle\right\|_{L^\infty([0,T],L^2(E))} \leq \eta.$$

Now, for $\tau<\tau_{\eta,k}$, we have

$$\begin{aligned}
&\left\|\left\langle N_k, \int_0^\tau \left[\sqrt{\frac{\nu}{\varepsilon}}\bar{S}[c_{B,3},c_{T,3}] - \mathcal{L}(-s)\mathbb{P}\Sigma(t,s)\right] ds\right\rangle\right\|_{L^\infty([0,T],L^2(E))} \\
\leq\ & \tau_{\eta,k}\left\|\left\langle N_k, \sqrt{\frac{\nu}{\varepsilon}}\bar{S}[c_{B,3},c_{T,3}]\right\rangle\right\|_{L^\infty([0,T],L^2(E))} \\
+\ & \sqrt{\tau_{\eta,k}}\int_0^{\tau_{\eta,k}} \|\langle N_k,\Sigma(\cdot,s)\rangle\|_{L^\infty([0,T],L^2(E))}\, ds \\
\leq\ & C_{\eta,k}.
\end{aligned}$$

Gathering all the estimates, we infer the inequality announced in Lemma 3.2. ∎

Definition 3. Let $c_B, c_T \in L^\infty_{\mathrm{loc}}([0,\infty)_t, L^\infty([0,\infty)_\tau \times E, L^2(\mathbb{T}^2)))$ be almost periodic or stationary functions, and let $u_0 \in \mathcal{H} \cap H^{0,1}$. We define the interior operator $\mathcal{U}$ by

$$\mathcal{U}\left[c_B, c_T, w_0\right](t) = \mathcal{L}\left(\frac{t}{\varepsilon}\right) w(t) + v^{\mathrm{int}}\left(t, \frac{t}{\varepsilon}\right) + \delta u^{\mathrm{int}}\left(t, \frac{t}{\varepsilon}\right),$$

where w is the unique solution of equation (3.13) with initial data $w_{|t=0} = u_0$, v^{int} is given by (3.3)-(3.4), and δu^{int} is given by (3.14).

4 Construction of an approximate solution

In this section, we explain how an approximate solution of (1.1) is constructed. To that end, we use the boundary layer and interior operators defined in the previous sections. We first explain how to choose the horizontal boundary conditions for the boundary layer term, together with the vertical boundary condition for the interior term. We then derive an equation for the principal term w, which we call the "envelope equation". At last, we define some additional corrector terms which will be needed in the convergence proof; the latter is postponed to the next section.

4.1 Coupling of the boundary conditions

We set

$$\begin{aligned} u^{\mathrm{BL}} &= \mathcal{B}^{\mathrm{per}}_B(c_{B,h}) + \mathcal{B}^{\delta}_T(c_{T,h}), \\ u^{\mathrm{int}} &= \mathcal{U}(u_0, c_{B,3}, c_{T,3}) \end{aligned}$$

where the boundary conditions c_B, c_T are yet to be defined.

In order to match the boundary conditions (1.2), we must take u^{BL} and u^{int} such that

$$\begin{aligned} \left(u^{\mathrm{BL}}_h + u^{\mathrm{int}}_h\right)_{|z=0} &= o(c), \\ \partial_z \left(u^{\mathrm{BL}}_h + u^{\mathrm{int}}_h\right)_{|z=1} &= o(c), \\ \left(u^{\mathrm{BL}}_3 + u^{\mathrm{int}}_3\right)_{|z=0} &= o(\sqrt{\varepsilon\nu}c), \\ \left(u^{\mathrm{BL}}_3 + u^{\mathrm{int}}_3\right)_{|z=1} &= o(\sqrt{\varepsilon\nu}c), \end{aligned}$$

denoting by c the order of magnitude of c_T, c_B, in a sense to be made clear later on.

We now examine each of the boundary conditions independently. Since $\mathcal{L}(\tau)w$ is the principal term in u^{int}, we will neglect the horizontal components of v^{int} and δu^{int} at $z = 0$ and $z = a$.

- *Horizontal condition at $z = 0$.* The Dirichlet boundary condition leads to

$$c_{B,h}(t, \tau, x_h; \omega) = -\sum_{k \in \mathbb{Z}^3} e^{-i\lambda_k \tau} e^{ik'_h \cdot x_h} b(t, k; \omega) \begin{pmatrix} n_1(k) \\ n_2(k) \end{pmatrix}. \tag{4.1}$$

  Thus $c_{B,h}$ is almost periodic in the fast time variable τ, and we set

$$u_B(t, \tau, x_h, \zeta; \omega) = \sum_{\substack{k \in \mathbb{Z}^3, \\ k_h \neq 0}} \sum_{\pm} e^{ik'_h \cdot x_h} e^{-i\lambda_k \tau} e^{-\eta^{\pm}_k \zeta} \begin{pmatrix} w^{\pm}_k(t; \omega) \\ \sqrt{\varepsilon\nu} \frac{ik'_h \cdot w^{\pm}_k}{\eta^{\pm}_k} \end{pmatrix} \tag{4.2}$$

  where $w^{\pm}_k$ is defined by

$$w^{\pm}_k(t; \omega) = -\frac{1}{2} b(t, k; \omega) \begin{pmatrix} n_1(k) \pm i n_2(k) \\ n_2(k) \mp i n_1(k) \end{pmatrix}$$

  and

$$u^{\mathrm{stat}}(t, x) = \psi\left(\frac{z}{\sqrt{\nu t}}\right) \sum_{\pm} \alpha^{\pm}(t; \omega) e^{\pm i \frac{t}{\varepsilon}} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} \tag{4.3}$$

where

$$\alpha^{\pm}(t;\omega) = \mp \frac{1}{\sqrt{aa_1a_2}} \sum_{\substack{k_3\in\mathbb{Z}^*,\\ \mathrm{sgn}(k_3)=\pm 1}} b(t,0,k_3;\omega).$$

Notice that since $\hat{c}_{B,h}(t,k) = -b(t,k)n_h(k)$, we have $\gamma_{\pm}(t) = 0$, with the notation of Section 2.

- *Vertical condition at $z = 0$.* Since $w \in \mathcal{H}$, we have $w_{3|z=0} = 0$. Thus we take $c_{B,3}$ such that

$$c_{B,3} := -u_{B,3|\zeta=0}(\varepsilon\nu)^{-1/2},$$

and thus

$$c_{B,3}(t,\tau,x_h;\omega) = -\sum_{\substack{k\in\mathbb{Z}^3,\\ k_h\neq 0}} \sum_{\pm} \frac{ik'_h\cdot w_k^{\pm}}{\eta_k^{\pm}} e^{ik'_h\cdot x_h} e^{-i\lambda_k\tau}.$$

Hence $c_{B,3}$ is also almost periodic in the fast time variable τ.

- *Horizontal condition at $z = a$.* Since $\partial_z u^{\mathrm{int}}_{h|z=a} = 0$, we merely take

$$c_{T,h}(t,\tau,x_h;\omega) = \sigma(t,\tau,x_h;\omega). \tag{4.4}$$

Hence $c_{T,h}$ is random and stationary in the fast time variable, and thus yields a boundary layer term equal to

$$u^{\delta}_{T,h}(t,\tau,x_h,\zeta;\omega) = \frac{\beta\sqrt{\nu\varepsilon}}{\sqrt{4\pi}} \sum_{\pm} \int_0^{\infty} \frac{1}{\sqrt{s}} \exp\left(-\frac{\zeta^2}{4s} - \delta s\right) (\sigma \pm i\sigma^{\perp})(t,\tau-s,x_h,\omega) e^{\pm is}\, ds \tag{4.5}$$

and

$$u^{\delta}_{T,3}(t,\tau,x,\omega) = \frac{\nu\varepsilon\beta}{\sqrt{4\pi}} \sum_{\pm} \int_0^{\infty} \varphi\left(\frac{\zeta}{\sqrt{s}}\right) \left[\mathrm{div}_h\sigma \mp i\mathrm{rot}_h\sigma\right](t,\tau-s,x_h;\omega) e^{-\delta s\pm is} ds. \tag{4.6}$$

- *Vertical condition at $z = a$.* The calculation is similar to that at $z = 0$. We infer

$$\begin{aligned} c_{T,3} &= -u_{T,3|\zeta=0}(\varepsilon\nu)^{-1/2} \\ c_{T,3}(t,\tau,x_h;\omega) &= \frac{\beta\sqrt{\nu\varepsilon}}{2} \sum_{\pm} \int_0^{\infty} \left[\mathrm{div}_h\sigma \mp i\mathrm{rot}_h\sigma\right](t,\tau-s,x_h;\omega) e^{-\delta s\pm is} ds. \end{aligned} \tag{4.7}$$

The relations (4.1), (4.4) allow us to write c_B in terms of the solution w of the envelope equation (3.13). Conversely, equation (3.13) depends on $c_{B,3}$, and thus on the coefficients $b(t,k)$ of w. In other words, there is a coupling between the boundary condition at the bottom for u^{BL}, and the equation satisfied by w. Since w is the only non-vanishing term in L^2 norm, we choose (as is usually done in the rotating fluids literature) to write an explicit equation for w, and to express u^{BL} in terms of w.

4.2 The envelope equation

The goal of this paragraph is to compute the term $\bar{S}[c_{B,3}, c_{T,3}]$ occurring in equation (3.13) when $c_{B,3}, c_{T,3}$ are given by (4.4) and (4.7) respectively. Since $c_{B,3}$ is almost periodic, easy calculations lead to

$$\begin{aligned} \mathcal{E}_{-\lambda_k}(\hat{c}_{B,3}(t,k_h)) &= -\mathbf{1}_{k_h\neq 0} \sum_{\pm} \frac{ik'_h\cdot w_k^{\pm}}{\eta_k^{\pm}} \\ &= \frac{1}{2\sqrt{a_1a_2a}} \mathbf{1}_{k_h\neq 0} b(t,k;\omega)|k'_h| \sum_{\pm} \frac{1\pm\lambda_k}{\sqrt{1\mp\lambda_k}} \frac{1\pm i}{\sqrt{2}}. \end{aligned}$$

There remains to compute the coefficients $\mathcal{E}_{-\lambda_k}(\hat{c}_{T,3}(t,k_h))$; since the boundary condition $c_{T,3}$ depends on the small parameter δ, the corresponding Ekman pumping term will depend on δ as well. However, thanks Proposition 1, the non-resonance hypotheses **(H1)-(H2)** allow us to pass to the limit in the coefficients $\mathcal{E}_{-\lambda_k}(\hat{c}_{T,3}(t,k_h))$ as δ vanishes, and thus to derive a limit equation for the function w.

• First, by definition of $\mathcal{E}_\lambda$, we have, for all $k_h \in \mathbb{Z}^2$, for all $\lambda \in \mathbb{R}$,

$$\begin{aligned}
&\mathcal{E}_\lambda\left[\hat{c}_{T,3}(t,k_h)\right](\omega)\\
=\;& \frac{\sqrt{\varepsilon\nu}\beta}{2}\sum_{\pm}\lim_{\theta\to\infty}\frac{1}{\theta}\int_0^\theta\int_0^\infty \hat{\sigma}^{\pm}(t,\tau-s,k_h;\omega)e^{-\delta s-i\lambda\tau\pm is}\,ds\,d\tau\\
=\;& \frac{\sqrt{\varepsilon\nu}\beta}{2}\sum_{\pm}\lim_{\theta\to\infty}\int_0^\infty\left(\frac{1}{\theta}\int_0^\theta \hat{\sigma}^{\pm}(t,\tau,k_h;\theta_{-s}\omega)e^{-i\lambda\tau}\,d\tau\right)e^{-\delta s\pm is}\,ds,
\end{aligned}$$

where

$$\hat{\sigma}^{\pm}(k_h) = ik_h'\cdot\hat{\sigma}(k_h)\mp k_h'\cdot(\hat{\sigma}(k_h))^{\perp}.$$

Thanks to Lebesgue's dominated convergence Theorem and Proposition 1, we infer, for all $\delta>0$,

$$\begin{aligned}
\mathcal{E}_\lambda\left[\hat{c}_{T,3}(t,k_h)\right](\omega) &= \frac{\sqrt{\varepsilon\nu}\beta}{2}\sum_{\pm}\int_0^\infty \mathcal{E}_\lambda\left[\hat{\sigma}^{\pm}(t,k_h)\right](\theta_{-s}\omega)e^{-\delta s\pm is}\,ds\\
&= \frac{\sqrt{\varepsilon\nu}\beta}{2}\sum_{\pm}\int_0^\infty \mathcal{E}_\lambda\left[\hat{\sigma}^{\pm}(t,k_h)\right](\omega)e^{-\delta s\pm is+i\lambda s}\,ds\\
&= \frac{\sqrt{\varepsilon\nu}\beta}{2}\sum_{\pm}\mathcal{E}_\lambda\left[\hat{\sigma}^{\pm}(t,k_h)\right](\omega)\frac{-1}{-\delta+i(\lambda\pm1)}.
\end{aligned}$$

Since $\sigma\in L^\infty_{\rm loc}([0,\infty)_t,L^\infty([0,\infty)_\tau\times E,H^{3/2}(\mathbb{T}^2))$, we know (see Proposition 6.5 p. 145 in [3]) that for all $\delta>0$, for all $u_0\in\mathcal{H}\cap H^{0,1}$, there exists a unique solution $w^\delta\in L^\infty(E,\mathcal{C}([0,\infty),\mathcal{H}\cap H^{0,1}))\cap L^\infty(E,L^2_{\rm loc}([0,\infty),H^{1,0}))$ of the equation

$$\begin{aligned}
&\partial_t w^\delta+\bar{Q}(w^\delta,w^\delta)+\sqrt{\frac{\nu}{\varepsilon}}S_B(w^\delta)+\nu\beta S_T^\delta(\sigma)=0\\
&w^\delta(t=0)=w_0;
\end{aligned}\tag{4.8}$$

the operators S_B and S_T^δ are defined by

$$\begin{aligned}
S_B(u)&=\sum_{k\in\mathbb{Z}^3}\langle N_k,u\rangle\,A_kN_k,\\
S_T^\delta(\sigma)&=\frac{1}{2\sqrt{aa_1a_2}}\sum_{k\in\mathbb{Z}^3}\sum_{\pm}\frac{(-1)^{k_3}|k_h'|}{|k'|^2}\frac{\mathcal{E}_{-\lambda_k}\left[\hat{\sigma}^{\pm}(k_h)\right]}{-\delta+i(-\lambda_k\pm1)},
\end{aligned}\tag{4.9}$$

where

$$A_k:=\frac{|k_h'|^2}{2\sqrt{2}a_1a_2a|k'|^2}\sum_{\pm}\frac{1\pm\lambda_k}{\sqrt{1\mp\lambda_k}}\,(1\pm i).$$

Notice $\Re(A_k)\geq0$; hence the Ekman pumping due to the Dirichlet condition at $z=0$ induces a damping term in the envelope equation.

• The idea is then to pass to the limit in $S_T^\delta(\sigma)$ as $\delta\to0$ when σ satisfies **(H1)-(H2)**, using (1.8). Let us admit for the time being that the last property of Proposition 1 holds, i.e.

$$\exists\eta>0,\ \forall\lambda\in[-1-\eta,-1+\eta]\cup[1-\eta,1+\eta],\ \mathcal{E}_\lambda(\sigma)=0.\tag{4.10}$$

Property (4.10) entails that the second sum in the right-hand side of (4.9) bears only on the triplets (k_1,k_2,k_3) such that

$$|\lambda_k-1|\geq\eta,\ |\lambda_k+1|\geq\eta,$$

which entails

$$|k_3| \leq C(\eta)|k_h|.$$

Consequently, since $\sigma \in L^\infty([0,\infty)^2 \times E, H^1(\mathbb{T}^2))$, we deduce that $S_T^\delta(\sigma)$ converges as $\delta \to 0$ in $L^\infty([0,\infty) \times E; L^2(\mathbb{T}^2 \times [0,a)))$ towards

$$S_T(\sigma) := \frac{1}{2\sqrt{aa_1a_2}} \sum_{\substack{k\in\mathbb{Z}^3,\\ k_h\neq 0}} \frac{(-1)^{k_3}}{|k_h'|} \left(\lambda_k k_h' - i(k_h')^\perp\right) \cdot \mathcal{E}_{-\lambda_k}\left[\hat{\sigma}(k_h)\right] N^k. \tag{4.11}$$

Thus for all $T > 0$, the source term $S_T^\delta(\sigma)$ remains bounded in $L^\infty((0,T) \times E, H^{0,1})$ as $\delta \to 0$; thus w^δ is bounded, uniformly in δ, in $L^\infty(E, \mathcal{C}([0,T], \mathcal{H} \cap H^{0,1}) \cap L^2([0,T], H^{1,0}))$. Moreover, let w be the unique solution in $L^\infty(E, \mathcal{C}([0,\infty), \mathcal{H} \cap H^{0,1})) \cap L^\infty(E, L^2_{\rm loc}([0,\infty), H^{1,0}))$ of

$$\begin{aligned} &\partial_t w + \bar{Q}(w,w) - \Delta_h w + \sqrt{\frac{\nu}{\varepsilon}} S_B(w) + \nu\beta S_T(\sigma) = 0, \\ &w_{|t=0} = u_0. \end{aligned} \tag{4.12}$$

A standard energy estimate leads to the following error bound, for all $T > 0$,

$$\begin{aligned} ||w - w_\delta||_{L^\infty([0,T]\times E, L^2)} + ||\nabla_h(w - w_\delta)||_{L^\infty(E, L^2([0,T]\times\mathbb{T}^2\times[0,a]))} \\ \leq C\nu\beta ||S_T(\sigma) - S_T^\delta(\sigma)||_{L^\infty(E, L^2([0,T]\times\mathbb{T}^2\times[0,a]))}. \end{aligned} \tag{4.13}$$

Thus, when constructing the approximate solution in the next paragraph, we will use the function w^δ, but we will keep in mind that w^δ converges towards w as δ vanishes.

• Let us now turn to the proof of property (4.10) (which is the same as (1.8)). We choose $\eta_0 > 0$ such that

$$[-1-\eta_0, -1+\eta_0] \subset V_-, \quad [1-\eta_0, 1+\eta_0] \subset V_+.$$

For $\lambda \in \mathbb{R}$ arbitrary, and for $\theta > 0$, we have

$$\begin{aligned} & \left\| \frac{1}{\theta} \int_0^\theta \sigma(\tau,\omega) e^{-i\lambda\tau}\, d\tau \right\|_{L^\infty([0,T], H^s(\mathbb{T}^2))} \\ = & \left\| \frac{1}{\theta} \int_0^\theta (\sigma - \sigma_\alpha + \sigma_\alpha)(\tau,\omega) e^{-i\lambda\tau}\, d\tau \right\|_{L^\infty([0,T], H^s(\mathbb{T}^2))} \\ \leq & ||\sigma - \sigma_\alpha||_{L^\infty((0,\theta)\times[0,T]\times E, H^s(\mathbb{T}^2))} \\ & + \frac{1}{\theta} \left\| \int_0^\theta \int_{\mathbb{R}} e^{-\alpha|\mu| + i\mu\tau - i\lambda\tau} \hat{\sigma}_\alpha(\mu)\, d\mu\, d\tau \right\|_{L^\infty([0,T], H^s(\mathbb{T}^2))} \\ \leq & ||\sigma - \sigma_\alpha||_{L^\infty((0,\theta)\times[0,T]\times E, H^s(\mathbb{T}^2))} \\ & + \left\| \int_{\mathbb{R}} e^{-\alpha|\mu|} \frac{e^{i(\mu-\lambda)\theta} - 1}{i(\mu-\lambda)\theta} \hat{\sigma}_\alpha(\mu)\, d\mu \right\|_{H^s(\mathbb{T}^2)} \\ \leq & ||\sigma - \sigma_\alpha||_{L^\infty((0,\theta)\times[0,T]\times E, H^s(\mathbb{T}^2))} \\ & + \left(\sup_{\mu\in V_-\cup V_+} \|\hat{\sigma}_\alpha(\mu)\|_{L^\infty([0,T]\times E, H^s(\mathbb{T}^2))} \right) (|V_+| + |V_-|) \\ & + \int_{\mathbb{R}\setminus(V_-\cup V_+)} e^{-\alpha|\mu|} \left| \frac{e^{i(\mu-\lambda)\theta} - 1}{i(\mu-\lambda)\theta} \right| \|\hat{\sigma}_\alpha(\mu)\|_{L^\infty([0,T]\times E, H^s(\mathbb{T}^2))}\, d\mu\, d\tau. \end{aligned}$$

Let us now evaluate the last integral when λ is close to ± 1, say for instance

$$|\lambda - 1| \leq \frac{\eta_0}{2}.$$

Then if $\mu \in \mathbb{R} \setminus (V_- \cup V_+)$, we have $|\mu - 1| \geq \eta_0$, and thus

$$|\mu - \lambda| \geq \frac{\eta_0}{2}.$$

In particular,

$$\left| \frac{e^{i(\mu-\lambda)\theta} - 1}{i(\mu-\lambda)\theta} \right| \leq \frac{2}{|\mu-\lambda|\theta} \leq \frac{C}{\theta}.$$

Hence, for all $\theta > 0$, for λ such that $|\lambda \pm 1| \leq \eta_0/2$, the following inequality holds for all $\alpha > 0$

$$\begin{aligned} & \left\| \frac{1}{\theta} \int_0^\theta \sigma(\tau,\omega) e^{-i\lambda\tau} \, d\tau \right\|_{L^\infty([0,T],L^2(E,H^s(\mathbb{T}^2)))} \\ \leq \quad & ||\sigma - \sigma_\alpha||_{L^\infty([0,\theta]\times[0,T]\times E,H^s(\mathbb{T}^2))} + \frac{C}{\theta} \\ + \quad & \sup_{\mu\in V_-\cup V_+} \|\hat{\sigma}_\alpha(\mu)\|_{L^\infty([0,T]\times E,H^s(\mathbb{T}^2))} \left(|V_+| + |V_-|\right). \end{aligned}$$

In the above inequality, we first take θ large enough, so that the left-hand side is close to $\|\bar{\sigma}(\lambda)\|$, and C/θ is small. Then we let α go to zero, with θ fixed; we deduce that

$$\bar{\sigma}(\cdot,\lambda) = 0 \quad \forall \lambda \text{ such that } d(\lambda, \pm 1) \leq \frac{\eta_0}{2}.$$

4.3 The approximate solution

The approximate solution is obtained as the sum of some interior terms and some boundary layer terms; although we have to construct several correctors in order to obtain a good approximation of the function u^γ, we emphasize that all terms vanish in L^2 norm, except the solution w^δ of the approximated envelope equation (4.8). In this paragraph, we build the correctors step by step, using the general constructions of the previous sections. At each step, we will give some bounds on the corresponding term.

• *First step. The interior term at the main order.*

We have seen that the interior term at main order is given as the solution of some envelope equation, and that when all parameters $\varepsilon, \nu, \beta, \delta$ vanish, the envelope equation becomes (4.12). However, we are not able to construct the boundary layer terms at the top for $\delta = 0$, and thus we must keep an approximated solution of the envelope equation, namely w^δ. Moreover, when constructing the corrector terms $u^{\mathrm{BL}}, \delta u^{\mathrm{int}}, v^{\mathrm{int}}$, we will need some high regularity estimates in space and time on w^δ, which are in general not available for w^δ or w. Thus we introduce a regularization of w^δ with respect to the time variable, and we truncate the large frequencies in w^δ. Let $\chi \in \mathcal{D}(\mathbb{R})$, with

$$\chi(t) = 0 \quad \forall t \in [0,\infty), \quad \chi(t) = 0 \quad \forall t \in (-\infty, -1],$$
$$\chi(t) \geq 0 \quad \forall t \in \mathbb{R}, \ \int_{\mathbb{R}} \chi = 1.$$

For $n \in \mathbb{N}^*$, set $\chi_n := n^{-1}\chi(\cdot/n)$, and define, for $n, N > 0$,

$$w^\delta_{n,N} := \mathbb{P}_N \left[w^\delta *_t \chi_n \right] = (\mathbb{P}_N w^\delta) *_t \chi_n,$$

where $\mathbb{P}_N$ stands for the projection onto the vector space generated by N_k for $|k| \leq N$. The convolution in time is well-defined thanks to the assumptions on the support of χ. We have clearly

$$\lim_{n,N\to\infty} \sup_{\delta>0} \|w^\delta - w^\delta_{n,N}\|_{L^\infty([0,T]\times E,L^2)} = 0,$$
$$\lim_{n,N\to\infty} \sup_{\delta>0} \|w^\delta - w^\delta_{n,N}\|_{L^\infty(E,L^2([0,T],H^{1,0}))} = 0.$$

We recall that $\bar{Q}$ is bilinear continuous from

$$L^\infty([0,T],H^{0,1}) \times L^2([0,T],H^{1,0}) \quad \text{into} \quad L^2([0,T],H^{-1,0}).$$

(see Proposition 6.6 in [3] for a proof of this non trivial fact). Precisely, for $a, b \in H^1 \cap \mathcal{H}$, it can be proved, using the methods of [3], that there exists a constant $C > 0$ such that

$$\begin{aligned} \|\bar{Q}(a,b)\|_{H^{-1,0}} &\leq C\|a\|_{L^2}^{1/2}\|a\|_{H^{1,0}}^{1/2}\|b\|_{L^2}^{1/2}\|b\|_{H^{1,0}}^{1/2} \\ &\quad + C\|\partial_3 a\|_{L^2}\|b\|_{L^2}^{1/2}\|b\|_{H^{1,0}}^{1/2} + C\|\partial_3 b\|_{L^2}\|a\|_{L^2}^{1/2}\|a\|_{H^{1,0}}^{1/2}. \end{aligned} \tag{4.14}$$

It is easily deduced from the above inequality that $w_{n,N}^\delta$ is an approximate solution of (3.13), with an error term $r_{n,N}^\delta$ which vanishes in $L^2([0,T], H^{-1,0})$ as $n, N \to \infty$. Indeed, we have

$$\begin{aligned} r_{n,N}^\delta &= \bar{Q}(w_{n,N}^\delta, w_{n,N}^\delta) - \mathbb{P}_N\bar{Q}(w^\delta, w^\delta) * \chi_n + \nu\beta\mathbb{P}_N S_T^\delta[\sigma - \sigma * \chi_n] \\ &= \left[(\mathbb{P} - \mathbb{P}_N)\bar{Q}(w^\delta, w^\delta)\right] * \chi_n + \left[\bar{Q}(\mathbb{P}_N w^\delta, \mathbb{P}_N w^\delta) - \bar{Q}(w^\delta, w^\delta)\right] * \chi_n \\ &\quad + \left[\bar{Q}(w_{n,N}^\delta, w_{n,N}^\delta) - \bar{Q}(\mathbb{P}_N w^\delta, \mathbb{P}_N w^\delta) * \chi_n\right] + \nu\beta\mathbb{P}_N S_T^\delta[\sigma - \sigma * \chi_n]. \end{aligned}$$

The convergence towards zero of the last three terms follows from the continuity of $\bar{Q}$ in $H^{-1,0}$ and the regularity of σ. We thus focus on the first one, which we write

$$\begin{aligned} &\bar{Q}(w_{n,N}^\delta(t), w_{n,N}^\delta(t)) - \bar{Q}(\mathbb{P}_N w^\delta, \mathbb{P}_N w^\delta) * \chi_n \\ = &\int_{\mathbb{R}} \bar{Q}(w_{n,N}^\delta(t), \mathbb{P}_N w^\delta(u))\chi_n(t-u)\,du - \int_{\mathbb{R}} \bar{Q}(\mathbb{P}_N w^\delta(u), \mathbb{P}_N w^\delta(u))\chi_n(t-u)\,du \\ = &\int_{\mathbb{R}} \bar{Q}(w_{n,N}^\delta(t) - \mathbb{P}_N w^\delta(u), \mathbb{P}_N w^\delta(u))\chi_n(t-u)\,du, \end{aligned}$$

and thus, using inequality (4.14) together with the $L^\infty([0,T], H^{0,1})$ bound on w^δ, we infer

$$\begin{aligned} &\left\|\bar{Q}(w_{n,N}^\delta(t), w_{n,N}^\delta(t)) - \bar{Q}(w^\delta, w^\delta) * \chi_n(t)\right\|_{H^{-1,0}} \\ \leq\ & C\int_{\mathbb{R}} \left\|w_{n,N}^\delta(t) - \mathbb{P}_N w^\delta(u)\right\|_{H^{1,0}}^{1/2} \left\|\mathbb{P}_N w^\delta(u)\right\|_{H^{1,0}}^{1/2} \chi_n(t-u)\,du \\ +\ & C\int_{\mathbb{R}} \left\|w_{n,N}^\delta(t) - \mathbb{P}_N w^\delta(u)\right\|_{H^{0,1}} \left\|\mathbb{P}_N w^\delta(u)\right\|_{H^{1,0}} \chi_n(t-u)\,du \\ +\ & C\int_{\mathbb{R}} \left\|w_{n,N}^\delta(t) - \mathbb{P}_N w^\delta(u)\right\|_{H^{1,0}} \left\|\mathbb{P}_N w^\delta(u)\right\|_{H^{0,1}} \chi_n(t-u)\,du. \end{aligned}$$

Eventually, we get

$$\begin{aligned} &\left\|\bar{Q}(w_{n,N}^\delta(t), w_{n,N}^\delta(t)) - \bar{Q}(w^\delta, w^\delta) * \chi_n(t)\right\|_{L^\infty(E, L^2([0,T], H^{-1,0}))} \\ \leq\ & C \sup_{|h| \leq \frac{1}{n}} \left\|w^\delta - \tau_h w^\delta\right\|_{L^\infty(E, L^2([0,T], H^{1,0}))} + C \sup_{|h| \leq \frac{1}{n}} \left\|w^\delta - \tau_h w^\delta\right\|_{L^\infty([0,T]\times E, H^{0,1})}, \end{aligned}$$

where $\tau_h w : (t,x) \mapsto w(t+h, x)$. The right-hand side of the above inequality vanishes as $n \to \infty$, uniformly in δ.

Hence we work with $w_{n,N}^\delta$ instead of w from now on; for all $k, s > 0$, there exists a constant $C_{n,N}(k,s)$ such that

$$\|\partial_t^k w_{n,N}^\delta\|_{L^\infty(E, W^{k,\infty}([0,T], H^s(\mathbb{T}^2\times[0,a])))} \leq C_{n,N}(k,s).$$

In the sequel, we denote by $C_{n,N}$ all constants depending on n and N (and possibly T), but not on δ.

• *Second step. The boundary layer terms at the first order.*

The boundary condition $c_{B,h}$ is given by (4.1), where $b(t,k)$ is replaced by

$$b_{n,N}^\delta(t,k) := \left\langle N_k, w_{n,N}^\delta(t)\right\rangle.$$

Thus the boundary layer term at the bottom, u_B, is defined by formula (4.2), and the stationary corrector u^{stat} by (4.3). Similarly, the boundary layer term at the bottom, u_T, is given by formulas (4.5)-(4.6). According to paragraph 2.4 and to the previous step, the boundary layer term

$$u^{\text{BL}}(t, x_h, z) := u_B\left(t, \frac{t}{\varepsilon}, x_h, \frac{z}{\sqrt{\varepsilon\nu}}\right) + u_T\left(t, \frac{t}{\varepsilon}, x_h, \frac{a-z}{\sqrt{\varepsilon\nu}}\right) + u^{\text{stat}}(t,x)$$

satisfies the following estimates

$$\begin{aligned}
&\left\|u^{\rm BL},\ z\partial_z u^{\rm BL},\ (z-a)\partial_z u^{\rm BL}\right\|_{L^\infty([0,T]\times\mathbb{T}^2\times[0,a]\times E)} \leq C_{n,N} + C\sqrt{\varepsilon\nu}\beta, \qquad (4.15)\\
&\left\|u^{\rm BL}\right\|_{L^\infty([0,T]\times E, H^{1,0})} \leq C_{n,N}\left[(\varepsilon\nu)^{1/4}+\nu^{1/4}\right] + C(\varepsilon\nu)^{3/4}\beta,\\
&\left\|z\partial_z u^{\rm BL},\ (z-a)\partial_z u^{\rm BL}\right\|_{L^\infty([0,T]\times E, L^2(\mathbb{T}^2\times[0,a]))} \leq C_{n,N}\nu^{1/4} + C(\varepsilon\nu)^{3/4}\beta.
\end{aligned}$$

Moreover, $u^{\rm BL}$ is an approximate solution of the linear part of equation (1.1) (we will treat the quadratic term at the very end, when the whole approximate solution has been constructed), with an error term equal to

$$\begin{aligned}
&[(\partial_t-\Delta_h)u_B]\left(t,\frac{t}{\varepsilon},x_h,\frac{z}{\sqrt{\varepsilon\nu}}\right) + \varphi\left(\frac{z}{\sqrt{\nu t}}\right)\sum_{\pm}\partial_t\alpha^{\pm}(t)e^{\pm i\frac{t}{\varepsilon}}\\
+\quad &\left[\left(\partial_t-\Delta_h+\frac{\delta}{\varepsilon}\right)u_T\right]\left(t,\frac{t}{\varepsilon},x_h,\frac{a-z}{\sqrt{\varepsilon\nu}}\right),
\end{aligned}$$

where ∂_t is the derivation operator with respect to the macroscopic time variable. Thanks to the first step, we have

$$\begin{aligned}
&\left[\int_0^T\int_{\mathbb{T}^2\times[0,a]}\int_E\left|\partial_t u_B\left(t,\frac{t}{\varepsilon},x_h,\frac{z}{\sqrt{\varepsilon\nu}}\right)\right|^2 dm_0(\omega)\,dz\,dx_h\,dt\right]^{1/2}\\
&\qquad\qquad \leq C_{n,N}\left[(\varepsilon\nu)^{1/4}+\nu^{1/4}\right] + C(\varepsilon\nu)^{3/4}\beta,
\end{aligned}$$

(the same bound holds with u_B replaced by u_T), whereas the terms $\Delta_h u_B, \Delta_h u_T$ are bounded in $L^\infty(E,L^2([0,T],H^{-1,0}))$ by

$$C_{n,N}\left[(\varepsilon\nu)^{1/4}+\nu^{1/4}\right] + C(\varepsilon\nu)^{3/4}\beta.$$

Moreover,

$$\frac{\delta}{\varepsilon}\left\|u_T\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right\|_{L^\infty([0,T]\times E,L^2)} \leq C\frac{\delta}{\varepsilon}(\varepsilon\nu)^{3/4}\beta \leq C\frac{\delta}{\sqrt{\varepsilon}}.$$

At last, the error term due to $u^{\rm stat}$ satisfies

$$\left\|\varphi\left(\frac{z}{\sqrt{\nu t}}\right)\sum_{\pm}\partial_t\alpha^{\pm}(t)e^{\pm i\frac{t}{\varepsilon}}\right\|_{L^\infty(E,L^2(\mathbb{T}^2\times[0,a]))} \leq C_{n,N}\nu^{1/4}.$$

• *Third step. The interior corrector terms v^{int} and δu^{int}.*

We now define the correctors $v^{\rm int}$ and $\delta u^{\rm int}$ as in (3.3)-(3.4) and (3.14) respectively, where the boundary conditions $c_{B,3}$ and $c_{T,3}$ are given by (4.4), (4.7), and $w = w^\delta_{n,N}$ in (3.14). Recall that we have assumed $\nu = \mathcal{O}(\varepsilon)$ and $\sqrt{\nu\varepsilon}\beta = \mathcal{O}(1)$, so that the boundary conditions $c_{B,3}$ and $c_{T,3}$ are of order one in L^∞. More precisely, using the fact that $w^\delta_{n,N}$ has a finite number of Fourier modes on the one hand, and **(H1)-(H2)** on the other, we deduce that

$$\|v^{\rm int}\|_{L^\infty([0,T]\times[0,\infty)\times\mathbb{T}^2\times[0,a])} \leq C\left(\sqrt{\varepsilon\nu}\|w^\delta_{n,N}\|_{L^\infty([0,T,V_0^3)} + \nu\varepsilon\beta\right) \leq C_{n,N}\sqrt{\nu\varepsilon};$$

moreover, the boundary conditions $c_{B,3}$, $c_{T,3}$ satisfy the assumptions of Lemma 3.1 thanks to the hypotheses on σ. Thus, according to Lemma 3.2,

$$\forall\eta>0,\ \forall k\in\mathbb{Z}^3,\ \exists C_{\eta,k}>0,\ \left\|\left\langle N_k,\delta u^{\rm int}\left(t,\frac{t}{\varepsilon}\right)\right\rangle\right\|_{L^\infty([0,T],L^2(E))} \leq \eta + C_{\eta,k}\varepsilon.$$

Thus we set, for $K>0$ arbitrary,

$$\delta u^{\rm int}_K := \mathbb{P}_K\delta u^{\rm int} = \sum_{|k|\leq K}\left\langle N_k,\delta u^{\rm int}\right\rangle N_k.$$

According to the above convergence result, for all $K \in \mathbb{N}$, we have

$$\left\| \delta u_K^{\rm int}\left(t, \frac{t}{\varepsilon}\right)\right\|_{L^\infty([0,T],L^2(E,W^{1,\infty}(\mathbb{T}^2\times[0,a])))} \to 0 \quad \text{as } \varepsilon, \nu \to 0.$$

Moreover, there exists a constant $C_{n,N}$ such that

$$\left\| \delta u_K^{\rm int}\left(t, \frac{t}{\varepsilon}\right)\right\|_{L^\infty([0,T]\times E,W^{1,\infty}(\mathbb{T}^2\times[0,a]))} \leq C_{n,N}.$$

By replacing $\delta u^{\rm int}$ by $\delta u_K^{\rm int}$, we have introduced an error term in (3.12) which is bounded in $L^\infty([0,\infty)\times E, L^2([0,T]\times\mathbb{T}^2\times[0,a])$ by

$$\begin{aligned}
&\left\|(\mathbb{P}-\mathbb{P}_K)\left[\bar{Q}(w_{n,N}^\delta, w_{n,N}^\delta)\right]\right\|_{L^\infty(E,L^2([0,T]\times\mathbb{T}^2\times[0,a]))}\\
&+\left\|(\mathbb{P}-\mathbb{P}_K)\left[Q(s, w_{n,N}^\delta, w_{n,N}^\delta)\right]\right\|_{L^\infty([0,\infty)\times E,L^2([0,T]\times\mathbb{T}^2\times[0,a]))}\\
&+\sqrt{\frac{\nu}{\varepsilon}}\left\|(\mathbb{P}-\mathbb{P}_K)\left[\bar{S}[c_{B,3}, c_{T,3}]\right]\right\|_{L^\infty(E,L^2([0,T]\times\mathbb{T}^2\times[0,a]))}\\
&+\|(\mathbb{P}-\mathbb{P}_K)\Sigma\|_{L^\infty([0,\infty)\times E,L^2([0,T]\times\mathbb{T}^2\times[0,a]))}.
\end{aligned}$$

If $\nu = \mathcal{O}(\varepsilon)$, and $\sqrt{\nu\varepsilon}\beta = \mathcal{O}(1)$, all terms vanish as $K\to\infty$ uniformly in ε, ν, δ. Thus, we choose $K > 0$ sufficiently large (but fixed) so that the error term in the equation is $o(1)$, and we work with $\delta u_K^{\rm int}$ from now on. Notice that K depends on n and N in general.

Let

$$u^{\rm int}(t) := \mathcal{L}\left(\frac{t}{\varepsilon}\right) w_{n,N}^\delta(t) + v^{\rm int}\left(t, \frac{t}{\varepsilon}\right) + \delta u_K^{\rm int}\left(t, \frac{t}{\varepsilon}\right); \tag{4.16}$$

we have defined $v^{\rm int}$ and $\delta u^{\rm int}$ so that $u^{\rm int}$ is an approximate solution of equation (1.1), with an error term which we now evaluate in $L^2([0,T]\times\mathbb{T}^2\times[0,a]\times E) + L^2([0,T]\times E, H^{-1,0})$. Apart from the one mentioned above, which is due to the truncation of the large spatial frequencies in $\delta u^{\rm int}$, the error term is equal to

$$\begin{aligned}
&-\Delta_h v^{\rm int}\left(t, \frac{t}{\varepsilon}\right) + \partial_t(\delta u_K^{\rm int} + v^{\rm int})\left(t, \frac{t}{\varepsilon}\right) - \Delta_h \delta u_K^{\rm int}\left(t, \frac{t}{\varepsilon}\right) - \nu\partial_z^2 \delta u_K^{\rm int}\left(t, \frac{t}{\varepsilon}\right)\\
&+\left[u^{\rm int}\cdot\nabla\right](\delta u_K^{\rm int} + v^{\rm int})\left(t, \frac{t}{\varepsilon}\right) + \left[(\delta u_K^{\rm int} + v^{\rm int})\left(t, \frac{t}{\varepsilon}\right)\cdot\nabla\right]\mathcal{L}\left(\frac{t}{\varepsilon}\right) w_{n,N}^\delta(t).
\end{aligned}$$

The term $-\Delta_h v^{\rm int}(t, t/\varepsilon)$ is bounded in $L^2([0,T]\times E, H^{-1,0})$ by

$$\|c_{B,3}\|_{L^\infty([0,T]\times[0,\infty)_\tau\times E,H^1(\mathbb{T}^2))} + \|c_{T,3}\|_{L^\infty([0,T]\times[0,\infty)_\tau\times E,H^1(\mathbb{T}^2))} = o(1).$$

All the remaining error terms are bounded in $L^2([0,T]\times\mathbb{T}^2\times[0,a]\times E) := L^2(Q)$ by

$$\begin{aligned}
&\|\partial_t \delta u_K^{\rm int}\|_{L^\infty([0,T]_t\times\left[0,\frac{T}{\varepsilon}\right]_\tau, L^2(E\times\mathbb{T}^2\times[0,a]))} + \|\partial_t v^{\rm int}\|_{L^\infty([0,\infty)_\tau, L^2(Q))}\\
+\quad &\|\delta u_K^{\rm int}\|_{L^\infty([0,T]_t\times\left[0,\frac{T}{\varepsilon}\right]_\tau, L^2(E,H^2))}\\
+\quad &\|u^{\rm int}\|_{L^\infty}\|\delta u_K^{\rm int} + v^{\rm int}\|_{L^\infty([0,T]_t\times\left[0,\frac{T}{\varepsilon}\right]_\tau, L^2(E,H^1))}\\
+\quad &\|u^{\rm int}\|_{L^\infty(E,L^2([0,T],H^1))}\|\delta u_K^{\rm int} + v^{\rm int}\|_{L^2(E,L^\infty([0,T]_t\times\left[0,\frac{T}{\varepsilon}\right]_\tau\times\mathbb{T}^2\times[0,a]))}\\
=\quad &o(1).
\end{aligned}$$

Above, we have used the fact that $w_{n,N}^\delta$, and whence $v^{\rm int}$, $\delta u_K^{\rm int}$, are smooth with respect to the time variable t; thus the $o(1)$ means that for all n, N, K, the limit of the above expression as ε vanishes is zero, uniformly in δ.

To sum things up, we have

$$\partial_t u^{\rm int} + u^{\rm int}\cdot\nabla u^{\rm int} - \Delta_h u^{\rm int} - \nu\partial_z^2 u^{\rm int} + \nabla p = w_1^{\rm rem} + w_2^{\rm rem},$$

where $w_1^{\rm rem} = o(1)$ in $L^2([0,T]\times E\times\mathbb{T}^2\times[0,a])$ and $w_2^{\rm rem} = o(1)$ in $L^2([0,T]\times E, H^{-1,0})$. Moreover,

$$u^{\rm int}_{|t=0} = u_0 + o(1) \quad \text{in } L^2(\mathbb{T}^2\times[0,a]),$$

and there exists a constant $C_{n,N}$ such that

$$\|u^{\rm int}\|_{L^\infty([0,T]\times\mathbb{T}^2\times[0,a]\times E)} + \|u^{\rm int}\|_{L^\infty([0,T]\times E, H^1(\mathbb{T}^2\times[0,a]))} \leq C_{n,N}.$$

• *Fourth step. The boundary layer term at the second order.*

At this stage, we have proved that $u^{\rm int}$ (resp. $u^{\rm BL}$) is an approximate solution of the evolution equation (1.1) (resp. of its linear part); moreover, the boundary layer term $u^{\rm BL}$ and the corrector $v^{\rm int}$ have been built so that the boundary conditions are satisfied at the leading order. Precisely, we have

$$\begin{aligned}
u^{\rm BL}_{h|z=0}(t) + u^{\rm int}_{h|z=0}(t) &= v^{\rm int}_{h|z=0}(t,t/\varepsilon) + \delta u^{\rm int}_{K,h|z=0}(t,t/\varepsilon) + u_{T,h|\zeta=\frac{a}{\sqrt{\varepsilon\nu}}}(t,t/\varepsilon),\\
\partial_z\left(u^{\rm BL}_{h|z=a}(t) + u^{\rm int}_{h|z=a}(t)\right) &= \beta\sigma(t,t/\varepsilon) + \frac{1}{\sqrt{\varepsilon\nu}}\partial_\zeta u_{B,h|\zeta=\frac{a}{\sqrt{\varepsilon\nu}}}(t,t/\varepsilon) + \partial_z u^{\rm stat}_{h|z=a}(t),\\
u^{\rm BL}_{3|z=0}(t) + u^{\rm int}_{3|z=0}(t) &= u_{T,3|\zeta=\frac{a}{\sqrt{\varepsilon\nu}}}(t,t/\varepsilon),\\
u^{\rm BL}_{3|z=a}(t) + u^{\rm int}_{3|z=a}(t) &= u_{B,3|\zeta=\frac{a}{\sqrt{\varepsilon\nu}}}(t,t/\varepsilon).
\end{aligned}$$

The terms $u_{T|\zeta=\frac{a}{\sqrt{\varepsilon\nu}}}$, $u_{B|\zeta=\frac{a}{\sqrt{\varepsilon\nu}}}$ and $u^{\rm stat}_{|z=a}$ are exponentially small, thus satisfy the assumptions of Lemma 2 in the Appendix; they will be taken care of at the very last step. But in general, setting $\tilde{c}_{B,h} := v^{\rm int}_{h|z=0} + \delta u^{\rm int}_{K,h|z=0}$, the quantity $\varepsilon^{-1}\tilde{c}_{B,h}$ does not vanish. Thus, we define another boundary layer term in order to restore the Dirichlet boundary condition at $z=0$. We now have to make precise which parts are almost periodic or random stationary in $\tilde{c}_{B,h}(t,\tau)$. We have

$$v^{\rm int}_{h|z=0} = v^{\rm int}_h = \sqrt{\varepsilon\nu}\nabla_h\Delta_h^{-1}(c_{T,3}) - \sqrt{\varepsilon\nu}\nabla_h\Delta_h^{-1}(c_{B,3}).$$

The first term in the right-hand side is clearly random and stationary, whereas the second one is almost periodic. Concerning the term $\delta u^{\rm int}_K$, the situation is not so clear. Using (3.14), we write

$$\delta u^{\rm int}_K(t,\tau) = \sum_{|k|\leq K} e^{-i\lambda_k\tau}\delta b_k(t,\tau)N_k,$$

where

$$\begin{aligned}
\delta b_k(t,\tau) &:= \varepsilon\left\langle N_k, \int_0^\tau \left(\bar{Q}(w^\delta_{n,N}, w^\delta_{n,N}) - Q(s, w^\delta_{n,N}, w^\delta_{n,N})\right)\, ds\right\rangle\\
&+ \left\langle N_k, \int_0^\tau \sqrt{\varepsilon\nu}\bar{S}[c_{B,3}, c_{T,3}] - \varepsilon\mathcal{L}(-s)\mathbb{P}\Sigma(t,s)\right\rangle.
\end{aligned}$$

According to Lemma 3.2,

$$\sup_{t\in[0,T]}\left\|\delta b_k\left(t,\frac{t}{\varepsilon}\right)\right\|_{L^2(E)} = o(1),$$

and

$$\sup_{t\in[0,T]}\left\|\frac{\partial}{\partial t}\delta b_k\left(t,\frac{t}{\varepsilon}\right)\right\|_{L^\infty(E)} = \mathcal{O}(1).$$

Thus we forget the fact that δb_k depends on the microscopic time variable τ, and we merely treat $\delta u^{\rm int}_K$ as an almost periodic function. Hence we use the construction in paragraph 2.3, and we set

$$\begin{aligned}
\delta u^{\rm BL} &:= \mathcal{B}^{\rm per}_D\left(-\delta u^{\rm int}_{K,h|z=0} + \sqrt{\varepsilon\nu}\nabla_h\Delta_h^{-1}(c_{B,3})\right)\\
&+ \mathcal{B}^{\rm stat}_N\left(\sqrt{\varepsilon\nu}\nabla_h\Delta_h^{-1}(c_{T,3})\right).
\end{aligned}$$

As before, it is easily proved that $\delta u^{\rm BL}$ is an approximate solution of the evolution equation (1.1), with an error term which is $o(1)$ in $L^2([0,T],H^{-1,0})$.

• *Fifth step. The "stopping" corrector.*

Let us now examine the remaining boundary conditions.

▷ Horizontal component at $z=0$: this term is the simplest of all. We have

$$\delta_{B,h}(t) := \left(u_h^{\rm int}(t) + u_h^{\rm BL}(t) + \delta u_h^{\rm BL}(t)\right)_{|\zeta=\frac{a}{\sqrt{\varepsilon\nu}}} = u_{T,h|\zeta=\frac{a}{\sqrt{\varepsilon\nu}}}(t,t/\varepsilon),$$

and thus, using the same arguments as in Proposition 4, we prove that there exists a constant C such that

$$\|\delta_{B,h}(t)\|_{H^3(\mathbb{T}^2)} \leq C\exp\left(-\frac{a}{\sqrt{\varepsilon\nu}}\right)$$
$$\|\partial_t\delta_{B,h}(t)\|_{H^3(\mathbb{T}^2)} \leq \frac{C}{\varepsilon}\exp\left(-\frac{a}{\sqrt{\varepsilon\nu}}\right).$$

Since $\varepsilon^{-k}\exp\left(-a/\sqrt{\varepsilon\nu}\right) = o(1)$ for all $k\in\mathbb{N}^*$, $\delta_{B,h}$ satisfies the conditions of Lemma 2 in the Appendix.

▷ Vertical component at $z=0$: we compute

$$\delta_{B,3}(t) := \left(u_3^{\rm int}(t) + u_3^{\rm BL}(t) + \delta u_3^{\rm BL}(t)\right)_{|z=0} = u_{T,3|\zeta=\frac{a}{\sqrt{\varepsilon\nu}}}(t,t/\varepsilon) + \delta u_{3|z=0}^{\rm BL}(t).$$

It is easily proved that $u_{T,3|\zeta=a/\sqrt{\varepsilon\nu}}(t,t/\varepsilon)$ satisfies the hypotheses of Lemma 2, provided σ is sufficiently smooth. Concerning $\delta u_3^{\rm BL}$, we have, according to the assumptions on σ,

$$\|\delta u_{3|z=0}^{\rm BL}\|_{L^\infty([0,T],L^2(E,H^3(\mathbb{T}^2))} \leq o(\sqrt{\varepsilon\nu}) + C(\nu\varepsilon)^{3/2}\beta,$$
$$\|\partial_t\delta u_{3|z=0}^{\rm BL}\|_{L^\infty([0,T],L^2(E,H^3(\mathbb{T}^2)))} = o(1).$$

Thus $\delta_{B,3}$ satisfies the conditions of Lemma 2. Notice that the regularity conditions of Lemma 2 account for the H^4 regularity assumption on σ (see **(H1)-(H2)**).

▷ Horizontal component at $z=a$:

$$\begin{aligned}\delta_{T,h}(t) &= \partial_z\left(u_h^{\rm int}(t) + u_h^{\rm BL}(t) + \delta u_h^{\rm BL}(t)\right)_{|z=a} - \frac{1}{\beta}\sigma \\ &= \frac{1}{\sqrt{\varepsilon\nu}}\partial_\zeta u_{B,h|\zeta=\frac{a}{\sqrt{\varepsilon\nu}}}(t,t/\varepsilon) + \partial_z u_{h|z=a}^{\rm stat}(t) + \partial_z\delta u_{h|z=a}^{\rm BL}(t).\end{aligned}$$

For all $s>0$, we have

$$\left\|\partial_z u_{h|z=a}^{\rm stat}\right\|_{L^\infty([0,T]\times E,H^s(\mathbb{T}^2))} \leq C_{n,N}\frac{1}{\sqrt{\nu T}}\exp\left(-\frac{a^2}{4\nu T}\right) = o(\varepsilon),$$
$$\left\|\partial_t\partial_z u_{h|z=a}^{\rm stat}(t)\right\|_{L^\infty([0,T]\times E,H^s(\mathbb{T}^2))} \leq C_{n,N}\frac{1}{\nu^{3/2}}\exp\left(-\frac{a^2}{4\nu T}\right) = o(\varepsilon).$$

(Remember that $\nu=\mathcal{O}(\varepsilon)$.) Thus all terms of the right-hand side are exponentially small as ε vanishes, and satisfy the conditions of Lemma 2.

▷ Vertical component at $z=a$: let

$$\begin{aligned}\delta_{T,3}(t) &:= \left(u_3^{\rm int}(t) + u_3^{\rm BL}(t) + \delta u_3^{\rm BL}(t)\right)_{|z=a} \\ &= u_{B,3|\zeta=\frac{a}{\sqrt{\varepsilon\nu}}}(t,t/\varepsilon) + \delta u_{3|\zeta=\frac{a}{\sqrt{\varepsilon\nu}}}^{\rm BL}(t).\end{aligned}$$

Once again, $\delta_{T,3}$ is exponentially small in all H^s norms, and thus matches the conditions of Lemma 2.

We thus define $u^{\rm stop}$, given by Lemma 2, so that

$$u_{h|z=0}^{\rm stop} = -\delta_{B,h},\quad \partial_z u_{h|z=a}^{\rm stop} = -\delta_{T,h}$$
$$u_{3|z=0}^{\rm stop} = -\delta_{B,3},\quad u_{3|z=a}^{\rm stop} = -\delta_{T,3},$$

and such that $u^{\rm stop}$ is an approximate solution of the linear part of equation (1.1), with an error term which is $o(1)$ in L^2.

We now define

$$\begin{aligned} u^{\rm app} &:= u^{\rm int} + u^{\rm BL} + \delta u^{\rm BL} + u^{\rm stop} && (4.17)\\ &= u^{\rm int} + u^{\rm rem}. && (4.18)\end{aligned}$$

By construction, the remainder $u^{\rm rem}$ is $o(1)$ in $L^\infty([0,T], L^2(E\times\mathbb{T}^2\times[0,a])$ and $u^{\rm app}$ satisfies conditions (1.2). The goal of the next section is to prove that $u^{\rm app}$ is an approximate solution of (1.1), and to conclude that $u^{\varepsilon,\nu} - u^{\rm app}$ vanishes thanks to an energy estimate.

5 Energy estimates - proof of convergence

In the previous section, we have constructed a function $u^{\rm app}$, given by (4.17), where $u^{\rm int}$ - given by (4.16) - is an approximate solution of equation (1.1), and $u^{\rm BL}, \delta u^{\rm BL}, u^{\rm stop}$, defined respectively in the second, fourth and fifth step of paragraph 4.3, are approximate solutions of the linear part of equation (1.1) and are all $o(1)$ in L^2.

The organization of this section is as follows: first, we prove that $u^{\rm app}$ is indeed an approximate solution of (1.1). Then we evaluate the difference between $u^{\varepsilon,\nu}$ and $u^{\rm app}$ thanks to an energy estimate. At last, we prove Theorem 1 by conveniently choosing the parameters n, N, δ occurring in $u^{\rm app}$.

• Let us first prove that the function $u^{\rm app}$ is an approximate solution of equation (1.1). The core of the proof lies in the following Lemma:

Lemma 5.1 (Non linear estimate on the remainder term). *For all n, N, as $\varepsilon, \nu, \beta \to 0$ with $\nu = \mathcal{O}(\varepsilon)$ and $\beta\sqrt{\varepsilon\nu} = \mathcal{O}(1)$, we have*

$$\sup_{\delta>0} \left\| u^{int}\cdot\nabla u^{rem} + u^{rem}\cdot\nabla u^{int} + u^{rem}\cdot\nabla u^{rem}\right\|_{L^2([0,T]\times\mathbb{T}^2\times[0,a]\times E)} \to 0.$$

Proof. First, we have

$$\begin{aligned} & \left\|(u^{\rm rem}\cdot\nabla)\, u^{\rm int}\right\|_{L^2([0,T]\times\mathbb{T}^2\times[0,a]\times E)} \\ \leq\ & \|u^{\rm rem}\|_{L^2([0,T]\times\mathbb{T}^2\times[0,a]\times E)} \left\|u^{\rm int}\right\|_{L^\infty([0,T]\times E, W^{1,\infty})} \\ \leq\ & C_{n,N}\left(\|u^{\rm BL}\|_{L^2} + \|\delta u^{\rm BL}\|_{L^2} + \|u^{\rm stop}\|_{L^2}\right). \end{aligned}$$

The right-hand side vanishes thanks to the estimates of the previous section.

The other terms are slightly more complicated. We write

$$\begin{aligned} u^{\rm int}\cdot\nabla u^{\rm rem} + u^{\rm rem}\cdot\nabla u^{\rm rem} &= u^{\rm app}\cdot\nabla u^{\rm rem} \\ &= u^{\rm app}\cdot\nabla u^{\rm stop} + u^{\rm app}\cdot\nabla\left(u^{\rm BL} + \delta u^{\rm BL}\right). \end{aligned}$$

The first term in the right-hand side is bounded in $L^2([0,T]\times E\times\mathbb{T}^2\times[0,a])$ by

$$\|u^{\rm app}\|_{L^\infty}\|u^{\rm stop}\|_{L^2([0,T]\times E, H^1)} \leq C_{n,N}\varepsilon.$$

We thus focus on the second term, which we further split into

$$u_h^{\rm app}\cdot\nabla_h\left(u^{\rm BL} + \delta u^{\rm BL}\right) + u_3^{\rm app}\partial_z\left(u^{\rm BL} + \delta u^{\rm BL}\right).$$

We have

$$\begin{aligned} & \left\|u_h^{\rm app}\cdot\nabla_h\left(u^{\rm BL} + \delta u^{\rm BL}\right)\right\|_{L^2([0,T]\times E\times\mathbb{T}^2\times[0,a])} \\ \leq\ & \|u^{\rm app}\|_{L^\infty([0,T]\times E\times\mathbb{T}^2\times[0,a])}\|u^{\rm BL} + \delta u^{\rm BL}\|_{L^2([0,T]\times E, H^{1,0})} \\ \leq\ & C_{n,N}\left(\nu^{1/4} + (\varepsilon\nu)^{3/4}\beta\right). \end{aligned}$$

We split the other term as follows

$$\begin{aligned}\left\|u_3^{\rm app}\partial_z\left(u^{\rm BL}+\delta u^{\rm BL}\right)\right\|^2_{L^2(\mathbb{T}^2\times[0,a]}&=\int_{\mathbb{T}^2}\int_0^{a/2}\left|u_3^{\rm app}\partial_z\left(u_B+\delta u^{\rm BL}\right)\right|^2\\&+\int_{\mathbb{T}^2}\int_0^{a/2}\left|u_3^{\rm app}\partial_z u_T\right|^2\\&+\int_{\mathbb{T}^2}\int_{a/2}^{a}\left|u_3^{\rm app}\partial_z\left(u_B+\delta u^{\rm BL}\right)\right|^2\\&+\int_{\mathbb{T}^2}\int_{a/2}^{a}\left|u_3^{\rm app}\partial_z u_T\right|^2.\end{aligned}$$

For $z\geq a/2$, $t>0$, we have

$$\left|\partial_z\left(u_B+\delta u^{\rm BL}\right)(t)\right|^2\leq C_{n,N}\left[(\varepsilon\nu)^{-1}\exp\left(-\frac{ca}{\sqrt{\varepsilon\nu}}\right)+\frac{1}{\nu t}\exp\left(-\frac{ca}{\sqrt{\nu t}}\right)\right]$$

and thus

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^2}\int_{a/2}^{a}\left|u_3^{\rm app}\partial_z\left(u_B+\delta u^{\rm BL}\right)\right|^2\\\leq\;&C_{n,N}\left[(\varepsilon\nu)^{-1}\exp\left(-\frac{ca}{\sqrt{\varepsilon\nu}}\right)+\exp\left(-\frac{ca}{\sqrt{\nu T}}\right)\right].\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^2}\int_0^{a/2}\left|u_3^{\rm app}\partial_z u_T\right|^2\\\leq\;&C_{n,N}\beta^2\exp\left(-\frac{ca}{\sqrt{\varepsilon\nu}}\right)\leq C(\varepsilon\nu)^{-1}\exp\left(-\frac{ca}{\sqrt{\varepsilon\nu}}\right).\end{aligned}$$

We now evaluate the two remaining terms. The idea is the following: since $u_3^{\rm app}$ vanishes at the boundary, we have

$$\begin{aligned}u_3^{\rm app}(z)&\approx Cz\quad\text{for } z=o(1),\\\text{and } u_3^{\rm app}(z)&\approx C(z-a)\quad\text{for } z-a=o(1),\end{aligned}$$

and $z\partial_z u_B$, $(z-a)\partial_z u_T$ are evaluated in (4.15). Moreover, we can split $u^{\rm app}$ into

$$\begin{aligned}u^{\rm app}(t)&=\left[\mathcal{L}\left(\frac{t}{\varepsilon}\right)w^{\delta}_{n,N}(t)+\delta u^{\rm int}_K\left(t,\frac{t}{\varepsilon}\right)\right]+\left[v^{\rm int}\left(t,\frac{t}{\varepsilon}\right)+u^{\rm BL}(t)\right]\\&+\left[\delta u^{\rm BL}(t)+u^{\rm stop}(t)\right].\end{aligned}$$

By definition of $v^{\rm int}$ and $u^{\rm stop}$, the vertical component of each of the three terms in brackets vanishes at $z=0$ and $z=a$; additionally, the first term is bounded in $L^\infty([0,T]\times E,W^{1,\infty})$ by a constant $C_{n,N}$, while the (vertical components of the) second and third ones are respectively of order

$$C_{n,N}\left(\sqrt{\varepsilon\nu}+(\varepsilon\nu)^{3/4}\right)\quad\text{and}\quad C_{n,N}(\varepsilon\nu)^{5/4}+o(\varepsilon)$$

in $L^\infty([0,T]\times E,H^{1,0})$. Once again, the term $o(\varepsilon)$ must be understood as

$$\forall n,N,\ \lim_{\varepsilon,\nu\to0}\sup_{\delta>0}\varepsilon^{-1}\|u^{\rm stop}\|=0.$$

As a consequence, we have

$$\begin{aligned}&\int_{\mathbb{T}^2}\int_0^{a/2}\left|u_3^{\rm app}(t)\partial_z\left(u_B+\delta u^{\rm BL}\right)(t)\right|^2\\\leq\;&\left\|z^{-1}\left[\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)w^{\delta}_{n,N}\right)_3(t)+\delta u^{\rm int}_{K,3}\left(t,\frac{t}{\varepsilon}\right)\right]\right\|^2_{L^\infty}\left\|z\partial_z\left(u_B+\delta u^{\rm BL}\right)(t)\right\|^2_{L^2}\\+\;&\left\|z^{-1}\left[v_3^{\rm int}\left(t,\frac{t}{\varepsilon}\right)+u_3^{\rm BL}(t)+\delta u_3^{\rm BL}(t)+u_3^{\rm stop}(t)\right]\right\|^2_{L^2}\left\|z\partial_z\left(u_B+\delta u^{\rm BL}\right)(t)\right\|^2_{L^\infty}.\end{aligned}$$

Using Hardy's inequality together with the divergence-free property, we infer that

$$\begin{aligned}
&\int_{\mathbb{T}^2}\int_0^{a/2}\left|u_3^{\rm app}(t)\partial_z\left(u_B+\delta u^{\rm BL}\right)(t)\right|^2\\
\leq\ & C_{n,N}\left\|\partial_z\bar{u}_3^{\rm int}(t)+\delta u_{K,3}^{\rm int}\left(t,\frac{t}{\varepsilon}\right)\right\|_{L^\infty}^2\nu^{1/2}\\
+\ & C_{n,N}\left\|\partial_z\left[v_3^{\rm int}\left(t,\frac{t}{\varepsilon}\right)+u_3^{\rm BL}(t)+\delta u_3^{\rm BL}(t)+u_3^{\rm stop}(t)\right]\right\|_{L^2}^2\\
\leq\ & C_{n,N}\left\|\mathcal{L}\left(\frac{t}{\varepsilon}\right)w_{n,N}^\delta(t)+\delta u_K^{\rm int}\left(t,\frac{t}{\varepsilon}\right)\right\|_{W^{1,\infty}}^2\nu^{1/2}\\
+\ & C_{n,N}\left\|v_h^{\rm int}\left(t,\frac{t}{\varepsilon}\right)+u_h^{\rm BL}(t)+\delta u_h^{\rm BL}(t)+u_h^{\rm stop}(t)\right\|_{H^{1,0}}^2\\
\leq\ & o(1).
\end{aligned}$$

The term

$$\int_{\mathbb{T}^2}\int_{a/2}^a|u_3^{\rm app}(t)\partial_z u_T(t)|^2$$

is treated in a similar way. Gathering all the terms, we deduce the convergence result stated in Lemma 5.1. □

In the rest of this section, we denote by $w_1^{\rm rem}$ any term which satisfies

$$\forall n,N,\quad \lim_{\varepsilon\to0}\sup_{\delta>0}\|w_1^{\rm rem}\|_{L^2([0,T]\times E\times\mathbb{T}^2\times[0,a])}=0, \tag{5.1}$$

and by $w_2^{\rm rem}$ any term which satisfies

$$\forall n,N,\quad \lim_{\varepsilon\to0}\sup_{\delta>0}\|w_2^{\rm rem}\|_{L^2([0,T]\times E,H^{-1,0})}=0. \tag{5.2}$$

According to paragraph 4.3 and to Lemma 5.1, $u^{\rm app}$ satisfies an equation of the type

$$\begin{aligned}
&\partial_t u^{\rm app}+u^{\rm app}\cdot\nabla u^{\rm app}+\frac{1}{\varepsilon}e_3\wedge u^{\rm app}-\Delta_h u^{\rm app}-\nu\partial_z^2u^{\rm app}\\
&\qquad=\nabla p+w_1^{\rm rem}+w_2^{\rm rem}+r_{n,N}^\delta+\mathcal{O}\left(\frac{\delta}{\sqrt{\varepsilon}}\right)_{L^2},
\end{aligned} \tag{5.3}$$

We recall that the remainder $r^{n,N}$ satisfies

$$\lim_{n,N\to\infty}\sup_{\varepsilon,\nu,\delta}\|r_{n,N}^\delta\|_{L^2([0,T]\times E,H^{-1,0})}=0.$$

Equation (5.3) is supplemented with the boundary conditions (1.2) and the initial condition

$$u_{|t=0}^{\rm app}=w_0+\delta w_0^1+\delta w_0^2,$$

where δw_0^1 and δw_0^2 are such that

$$\begin{aligned}
\lim_{n,N\to\infty}\sup_{\delta,\varepsilon,\nu}\|\delta w_0^1\|_{L^\infty(E,L^2(\mathbb{T}^2\times[0,a]))}&=0,\\
\forall n,N,\quad \lim_{\varepsilon,\nu\to0}\sup_{\delta>0}\|\delta w_0^2\|_{L^\infty(E,L^2(\mathbb{T}^2\times[0,a]))}&=0.
\end{aligned}$$

In order to avoid too heavy notation, we will simply write

$$u_{|t=0}^{\rm app}=w_0+o(1).$$

• We now evaluate the difference between $u^{\varepsilon,\nu}$ and $u^{\rm app}$ thanks to an energy estimate. The function $u^{\varepsilon,\nu}-u^{\rm app}$ is a solution of

$$\begin{aligned}
&\partial_t(u^{\varepsilon,\nu}-u^{\rm app})+\frac{1}{\varepsilon}e_3\wedge(u^{\varepsilon,\nu}-u^{\rm app})-\Delta_h(u^{\varepsilon,\nu}-u^{\rm app})-\nu\partial_z^2(u^{\varepsilon,\nu}-u^{\rm app})\\
=\;&\nabla p'+w_1^{\rm rem}+w_2^{\rm rem}-r^\delta_{n,N}+\mathcal{O}\left(\frac{\delta}{\sqrt{\varepsilon}}\right)_{L^2}\\
-\;&(u^{\varepsilon,\nu}\cdot\nabla)(u^{\varepsilon,\nu}-u^{\rm app})-\left[(u^{\varepsilon,\nu}-u^{\rm app})\cdot\nabla\right]u^{\rm app}.
\end{aligned}$$

Taking the scalar product the above equation by $u^{\varepsilon,\nu}-u^{\rm app}$ and using the Cauchy-Schwarz inequality, we deduce that for all $t>0$, for almost every $\omega\in E$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^{\varepsilon,\nu}(t,\omega)-u^{\rm app}(t,\omega)\|_{L^2}^2+\frac{1}{2}\|u^{\varepsilon,\nu}(t,\omega)-u^{\rm app}(t,\omega)\|_{H^{1,0}}^2\\
\leq\;&\int_{\mathbb{T}^2\times[0,a]}\left|\left[\left((u^{\varepsilon,\nu}(t,\omega)-u^{\rm app}(t,\omega))\cdot\nabla\right)u^{\rm app}(t,\omega)\right]\cdot(u^{\varepsilon,\nu}(t,\omega)-u^{\rm app}(t,\omega))\right|\\
+\;&\|w_1^{\rm rem}(t,\omega)\|^2_{L^2(\mathbb{T}^2\times[0,a])}+\|w_2^{\rm rem}(t,\omega)\|^2_{H^{-1,0}}\\
+\;&\left\|r^\delta_{n,N}\right\|^2_{H^{-1,0}}+C\frac{\delta^2}{\varepsilon}+C\|u^{\varepsilon,\nu}(t,\omega)-u^{\rm app}(t,\omega)\|_{L^2}^2.
\end{aligned}$$

In the above inequality, we have dropped the term $\nu\|\partial_z(u^{\varepsilon,\nu}-u^{\rm app})\|_{L^2}^2$ in the right-hand side. We now evaluate the term

$$\int_{\mathbb{T}^2\times[0,a]}\left|\left((u^{\varepsilon,\nu}-u^{\rm app})\cdot\nabla\right)u^{\rm app}\cdot(u^{\varepsilon,\nu}-u^{\rm app})\right|.$$

First, let us write

$$u^{\rm app}=\left[u^{\rm int}+u^{\rm stop}\right]+\left[u^{\rm BL}+\delta u^{\rm BL}\right].$$

The function $u^{\rm int}+u^{\rm stop}$ is bounded in $L^\infty([0,T]\times E,W^{1,\infty}(\mathbb{T}^2\times[0,a])$ by a constant $C_{n,N}$; similarly, $\nabla_h(u^{\rm BL}+\delta u^{\rm BL})$ is bounded in $L^\infty([0,T]\times E\times\mathbb{T}^2\times[0,a])$. As a consequence, we have

$$\begin{aligned}
&\int_{\mathbb{T}^2\times[0,a]}\left|(u^{\varepsilon,\nu}-u^{\rm app})\cdot\nabla\left[u^{\rm int}+u^{\rm stop}\right]\cdot(u^{\varepsilon,\nu}-u^{\rm app})\right|\\
+\;&\int_{\mathbb{T}^2\times[0,a]}\left|(u_h^{\varepsilon,\nu}-u_h^{\rm app})\cdot\nabla_h\left[u^{\rm BL}+\delta u^{\rm BL}\right]\cdot(u^{\varepsilon,\nu}-u^{\rm app})\right|\\
\leq\;&C_{n,N}\|u^{\varepsilon,\nu}-u^{\rm app}\|^2_{L^2([T^2\times[0,a])}.
\end{aligned}$$

There remains to derive a bound for the term

$$\int_{\mathbb{T}^2\times[0,a]}\left|(u_3^{\varepsilon,\nu}-u_3^{\rm app})\partial_z\left[u^{\rm BL}+\delta u^{\rm BL}\right]\cdot(u^{\varepsilon,\nu}-u^{\rm app})\right|;$$

the calculations are quite similar to those of Lemma 5.1. We first split the integral on $[0,a]$ into two integrals, one bearing on $[0,a/2]$ and the other on $[a/2,a]$. The term u_T (resp. $u_B+\delta u^{\rm BL}$) is exponentially small on $[0,a/2]$ (resp. on $[a/2,a]$), and thus we neglect it in the final estimate. Moreover, we have for instance

$$\begin{aligned}
&\int_0^{a/2}\int_{\mathbb{T}^2}\left|(u_3^{\varepsilon,\nu}-u_3^{\rm app})\partial_z\left[u_B+\delta u^{\rm BL}\right]\cdot(u^{\varepsilon,\nu}-u^{\rm app})\right|\\
\leq\;&\left\|\frac{1}{z}(u_3^{\varepsilon,\nu}-u_3^{\rm app})\right\|_{L^2}\left\|z\partial_z\left[u_B+\delta u^{\rm BL}\right]\right\|_{L^\infty}\|u^{\varepsilon,\nu}-u^{\rm app}\|_{L^2}\\
\leq\;&C\|\partial_z(u_3^{\varepsilon,\nu}-u_3^{\rm app})\|_{L^2(\mathbb{T}^2\times[0,a])}\|u^{\varepsilon,\nu}-u^{\rm app}\|_{L^2(\mathbb{T}^2\times[0,a])}\\
\leq\;&C\|u^{\varepsilon,\nu}-u^{\rm app}\|_{H^{1,0}}\|u^{\varepsilon,\nu}-u^{\rm app}\|_{L^2(\mathbb{T}^2\times[0,a])}.
\end{aligned}$$

Eventually, we infer that

$$\begin{aligned}
&\int_{\mathbb{T}^2\times[0,a]}\left|(u_3^{\varepsilon,\nu}-u_3^{\rm app})\partial_z\left[u^{\rm BL}+\delta u^{\rm BL}\right]\cdot(u^{\varepsilon,\nu}-u^{\rm app})\right|\\
\leq\;&C\|u^{\varepsilon,\nu}-u^{\rm app}\|^2_{L^2(\mathbb{T}^2\times[0,a])}+C\|u^{\varepsilon,\nu}-u^{\rm app}\|_{H^{1,0}}\|u^{\varepsilon,\nu}-u^{\rm app}\|_{L^2(\mathbb{T}^2\times[0,a])}.
\end{aligned}$$

Gathering all the above estimates and integrating on E, we deduce that

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\|u^{\varepsilon,\nu}-u^{\rm app}\right\|^2_{L^2(E\times\mathbb{T}^2\times[0,a])}+\left\|u^{\varepsilon,\nu}-u^{\rm app}\right\|^2_{L^2(E,H^{1,0})}\\
\leq\;& C\left\|u^{\varepsilon,\nu}-u^{\rm app}\right\|^2_{L^2(E\times\mathbb{T}^2\times[0,a])}\\
+\;& \left\|w_1^{\rm rem}\right\|^2_{L^2(E\times\mathbb{T}^2\times[0,a])}+\left\|w_2^{\rm rem}\right\|^2_{L^2(E,H^{-1,0})}+\left\|r^\delta_{n,N}\right\|^2_{L^2(E,H^{-1,0})}+\frac{C\delta^2}{\varepsilon}.
\end{aligned}$$

Using Gronwall's Lemma, we infer that for all $t\in[0,T]$,

$$\begin{aligned}
&\left\|(u^{\varepsilon,\nu}-u^{\rm app})(t)\right\|^2_{L^2(E\times\mathbb{T}^2\times[0,a])}+\int_0^t\left\|u^{\varepsilon,\nu}-u^{\rm app}\right\|^2_{L^2(E,H^{1,0})} \qquad (5.4)\\
\leq\;& C\left[\left\|w_1^{\rm rem}\right\|^2_{L^2([0,T]\times E\times\mathbb{T}^2\times[0,a])}+\left\|w_2^{\rm rem}\right\|^2_{L^2([0,T]\times E,H^{-1,0})}\right]\\
+\;& C\left[\left\|r^\delta_{n,N}\right\|^2_{L^2([0,T]\times E,H^{-1,0})}\frac{\delta^2}{\varepsilon}\right].
\end{aligned}$$

• We are now ready to prove Theorem 1. Let us write

$$\begin{aligned}
u^{\varepsilon,\nu}(t)-\mathcal{L}\left(\frac{t}{\varepsilon}\right)w(t) \;=\;& \left[u^{\varepsilon,\nu}-u^{\rm app}\right](t)+\left[u^{\rm app}(t)-\mathcal{L}\left(\frac{t}{\varepsilon}\right)w^\delta_{n,N}(t)\right]\\
&+\left[\mathcal{L}\left(\frac{t}{\varepsilon}\right)[w^\delta_{n,N}-w^\delta](t)\right]+\mathcal{L}\left(\frac{t}{\varepsilon}\right)[w^\delta-w](t),
\end{aligned}$$

where

- ▷ the term $u^{\varepsilon,\nu}-u^{\rm app}$ satisfies the energy estimate (5.4);
- ▷ the term $u^{\rm app}(t)-\mathcal{L}\left(\frac{t}{\varepsilon}\right)w^\delta_{n,N}(t)$ is equal to $u^{\rm rem}+v^{\rm int}+\delta u^{\rm int}_K$, and thus vanishes in $L^\infty([0,T],L^2(E,H^{1,0}))$ as $\varepsilon,\nu\to0$, uniformly in $\delta>0$, and for all n,N,K;
- ▷ the term $w^\delta_{n,N}-w^\delta$ vanishes as $n,N\to\infty$ uniformly in δ,ε,ν according to the first step in paragraph 4.3;
- ▷ the term $w^\delta-w$ vanishes as $\delta\to0$, uniformly in ε,ν, according to (4.13).

Let $\eta>0$ be arbitrary. We first take n_0,N_0 large enough so that for all $\delta>0$, $\varepsilon,\nu,\beta>0$,

$$\begin{aligned}
\left\|r^\delta_{n_0,N_0}\right\|^2_{L^\infty([0,T]\times E,H^{-1,0})}&\leq\eta,\\
\left\|w^\delta_{n,N}-w^\delta\right\|^2_{L^\infty([0,T]\times E,L^2)},\left\|w^\delta_{n,N}-w^\delta\right\|^2_{L^\infty(E,L^2([0,T],H^{1,0}))}&\leq\eta.
\end{aligned}$$

Remembering properties (5.1)-(5.2), we deduce that there exists $\varepsilon_0,\nu_0>0$ such that for all δ, for all $\varepsilon<\varepsilon_0,\nu<\nu_0$ with $\nu\leq C\varepsilon$ and $\beta\sqrt{\varepsilon\nu}\leq C$,

$$\begin{aligned}
\left\|w_1^{\rm rem}\right\|^2_{L^2([0,T]\times E\times\mathbb{T}^2\times[0,a])}&\leq\eta,\\
\left\|w_2^{\rm rem}\right\|^2_{L^2([0,T]\times E,H^{-1,0})}&\leq\eta,\\
\left\|u^{\rm app}(t)-\mathcal{L}\left(\frac{t}{\varepsilon}\right)w^\delta_{n_0,N_0}(t)\right\|^2_{L^\infty([0,T],L^2(E,H^{1,0}))}&\leq\eta.
\end{aligned}$$

At this stage, we have, for all $\delta>0$, for all ε,ν,β such that $0<\varepsilon<\varepsilon_0$ and $\nu=\mathcal{O}(\varepsilon)$, $\sqrt{\nu\varepsilon}\beta=\mathcal{O}(1)$,

$$\begin{aligned}
&\left\|u^{\varepsilon,\nu}(t)-\mathcal{L}\left(\frac{t}{\varepsilon}\right)w(t)\right\|^2_{L^2(E\times\mathbb{T}^2\times[0,a])}+\int_0^t\left\|u^{\varepsilon,\nu}(s)-\mathcal{L}\left(\frac{s}{\varepsilon}\right)w(s)\right\|^2_{L^2(E,H^{1,0})}ds\leq\\
&\qquad C\eta+C\|w^\delta-w\|^2_{L^\infty([0,T],L^2(E\times\mathbb{T}^2\times[0,a]))}+C\|w^\delta-w\|^2_{L^2([0,T]\times E,H^{1,0})}+\frac{C\delta^2}{\varepsilon}.
\end{aligned}$$

We now let $\delta \to 0$ in the right-hand side, and we obtain

$$\left\| u^{\varepsilon,\nu}(t) - \mathcal{L}\left(\frac{t}{\varepsilon}\right) w(t) \right\|^2_{L^2(E\times T^2\times[0,a])} + \int_0^t \left\| u^{\varepsilon,\nu}(s) - \mathcal{L}\left(\frac{s}{\varepsilon}\right) w(s) \right\|^2_{L^2(E,H^{1,0})} ds \leq C\eta$$

for ε, ν small enough. The convergence result is thus proved.

6 Mean behaviour at the limit

This section is devoted to the proof of Proposition 1. Let us recall what the issue is: in general, the source term S_T in (1.9) is a random function, and thus so is w. Hence, our goal is to derive an equation, or a system of equations, on $\mathbb{E}[w]$. We emphasize that such a derivation is not always possible, because of the nonlinear term $\bar{Q}(w,w)$. However, we shall prove that the vertical average of w_h, denoted by $\bar{w}_h$, is always a deterministic function. Moreover, if the torus is nonresonant (see (1.11)), then $w - \bar{w}$ solves a linear equation, and thus in this particular case we can derive an equation for $\mathbb{E}[w - \bar{w}]$.

Our first result is the following:

Lemma 6.1. *Assume that the group transformation $(\theta_\tau)_{\tau\in\mathbb{R}}$ is ergodic. Let $u_0 \in \mathcal{H} \cap H^1$, and let w be the solution of* (1.9)*. Set*

$$\bar{w}_h = \frac{1}{a}\int_0^a w_h.$$

Then $\bar{w}$ is the unique solution in $\mathcal{C}([0,\infty), L^2(\mathbb{T}^2)) \cap L^2_{loc}([0,\infty), H^1(\mathbb{T}^2))$ of the two-dimensional Navier-Stokes equation

$$\begin{cases} \partial_t \bar{w}_h + \bar{w}_h \cdot \nabla \bar{w}_h - \Delta_h \bar{w}_h + \dfrac{\sqrt{\nu}}{aa_1a_2\sqrt{2\varepsilon}} \bar{w}_h + \nu\beta \mathbb{E}\left[S_T(\sigma)\right]_h = 0, \\ \bar{w}_{h|t=0} = \frac{1}{a}\int_0^a w_{0,h}. \end{cases} \tag{6.1}$$

In particular, $\bar{w}_h$ is a deterministic function.

Proof. Let us recall that if

$$\phi = \sum_{k\in\mathbb{Z}^3} \hat{\phi}(k) N_k \in \mathcal{H},$$

then

$$P_h(\phi) := \frac{1}{a}\int_0^a \phi_h = \sum_{k_h\in\mathbb{Z}^2} \hat{\phi}(k_h,0) n_h(k_h,0).$$

Thus we have to project equation (1.9) onto the horizontal modes, which correspond to $k_3 = 0$. It is easily checked that

$$P_h\left(S_B(w)\right) = S_{B,h}(\bar{w}_h) = \frac{1}{\sqrt{2}aa_1a_2}\bar{w}_h,$$

and we recall (see [14] and Proposition 6.2 in [3]) that there exists a function $\bar{p} \in L^2(\mathbb{T}^2)$ such that for all $w \in H^1 \cap \mathcal{H}$

$$P_h(\bar{Q}(w,w)) = (\bar{w}_h \cdot \nabla_h)\bar{w}_h + \nabla_h \bar{p}.$$

Thus we only have to prove that

$$P_h(S_T(\sigma)) = \mathbb{E}\left[S_{T,h}(\sigma)\right],$$

almost surely in E. We use the following fact, of which we postpone the proof: if $\lambda \in \mathbb{R}$, $k_h \in \mathbb{Z}^2$, then

$$\mathbb{E}[\mathcal{E}_\lambda[\sigma]] = \begin{cases} \mathbb{E}[\sigma] & \text{if } \lambda = 0, \\ 0 & \text{else.} \end{cases} \tag{6.2}$$

Moreover, if $\lambda = 0$, then

$$\mathcal{E}_\lambda[\sigma] = \mathbb{E}[\sigma] \quad \text{almost surely.} \tag{6.3}$$

Note also that $\lambda_k = 0$ if and only if $k_3 = 0$. Remembering (4.11), we deduce that

$$\begin{aligned}\mathbb{E}[S_{T,h}(\sigma)] &= -\frac{i}{aa_1a_2}\sum_{k_h\in\mathbb{Z}^2}\frac{1}{|k_h'|^2}(k_h')^\perp\cdot\mathbb{E}[\hat\sigma(k_h)]\begin{pmatrix}ik_2'\\-ik_1'\end{pmatrix}\\ &= -\frac{i}{aa_1a_2}\sum_{k_h\in\mathbb{Z}^2}\frac{1}{|k_h'|^2}(k_h')^\perp\cdot\mathcal{E}_0[\hat\sigma(k_h)]\begin{pmatrix}ik_2'\\-ik_1'\end{pmatrix}\\ &= P_h[S_T(\sigma)].\end{aligned}$$

Thus the lemma is proved, pending the derivation of (6.2) and (6.3). Concerning (6.2), the invariance of the probability measure m_0 with respect to θ_τ entails that

$$\mathbb{E}\left[\mathcal{E}_\lambda[\sigma]\right] = \mathbb{E}[\sigma]\lim_{\theta\to\infty}\frac{1}{\theta}\int_0^\theta e^{-i\lambda\tau}\,d\tau,$$

and (6.2) follows easily. Equality (6.3) is a consequence of Birkhoff's ergodic theorem (see [20]). □

The first point in Proposition 3 follows easily from the above Lemma (together with Theorem 1), by simply noticing that the sequence

$$\exp\left(-\frac{t}{\varepsilon}L\right)w(t) = \sum_k e^{-i\lambda_k\frac{t}{\varepsilon}}b(t,k;\omega)N_k$$

weakly converges in $L^2([0,T]\times\mathbb{T}^2\times[0,a]\times E)$ towards

$$\sum_{\substack{k\in\mathbb{Z}^3,\\ \lambda_k=0}} b(t,k;\omega)N_k = \bar w(t).$$

Remark 6.1. Notice that

$$\mathrm{rot}_h P_h[S_T(\sigma)] = -\mathbb{E}\left[\mathrm{rot}_h\sigma\right].$$

Hence we recover the result of [5]: the vorticity $\phi := \mathrm{rot}_h\bar w_h$ is a solution of

$$\partial_t\phi + \bar w_h\cdot\nabla_h\phi - \Delta_h\phi + \frac{\sqrt{\nu}}{aa_1a_2\sqrt{2\varepsilon}}\phi = \nu\beta\mathbb{E}\left[\mathrm{rot}_h\sigma\right].$$

From now on, we assume that the torus is nonresonant (see (1.11)). Consequently, with $\bar w = (\bar w_h, 0)$, we have

$$\bar Q(w-\bar w, w-\bar w) = 0.$$

Moreover, using (6.2)-(6.3), it is easily checked that

$$\mathbb{E}\left[S_{T,3}(\sigma)\right] = 0.$$

Setting $u = w - \bar w$, we deduce that u solves a linear equation, namely

$$\partial_t u + 2\bar Q(u,\bar w) - \Delta_h u + \sqrt{\frac{\nu}{\varepsilon}}S_B(u) + \nu\beta S_T(\sigma) - \nu\beta\mathbb{E}[S_T(\sigma)] = 0.$$

Since $\bar w$ is deterministic, we have

$$\mathbb{E}\left[\bar Q(u,\bar w)\right] = \bar Q(\mathbb{E}[u],\bar w).$$

Hence we can further decompose u into $\tilde w + \tilde u$, where $\tilde w$ is deterministic and does not depend on σ, and $\tilde u$ is random with zero average. The precise result is stated in the following lemma, from which Proposition 3 follows immediately:

Lemma 6.2. *Assume that the hypotheses of Proposition 3 hold. Then*

$$w = \bar w + \tilde w + \tilde u$$

where:

- *the function $\bar{w}$ is deterministic and satisfies* (6.1)*;*
- *the function $\tilde{w}$ is deterministic and satisfies*

$$\begin{cases} \partial_t \tilde{w} + 2\bar{Q}(\bar{w}, \tilde{w}) - \Delta_h \tilde{w} + \sqrt{\frac{\nu}{\varepsilon}} S_B(\tilde{w}) = 0, \\ \tilde{w}_{|t=0} = u_0 - \bar{w}_{|t=0}; \end{cases}$$

- *the function $\tilde{u}$ is random, with zero average, and satisfies*

$$\begin{cases} \partial_t \tilde{u} + 2\bar{Q}(\bar{w}, \tilde{u}) - \Delta_h \tilde{u} + \sqrt{\frac{\nu}{\varepsilon}} S_B(\tilde{u}) + \nu\beta S_T(\sigma) - \nu\beta \mathbb{E}[S_T(\sigma)] = 0, \\ \tilde{u}_{|t=0} = 0. \end{cases}$$

Appendix A: convergence of the family σ_α

Lemma 1. *Let $T > 0$. Assume that $\sigma \in L^\infty([0,T] \times E, \mathcal{C}(\mathbb{R})) \cap L^\infty([0,T] \times \mathbb{R}_\tau \times E)$. Then for all $T' > 0$,*

$$\sigma_\alpha - \sigma \to 0 \quad \text{in } L^\infty((0,T) \times (0,T') \times E) \quad \text{as } \alpha \to 0.$$

Proof. By definition of σ_α, we have

$$\begin{aligned} \sigma_\alpha(t,\tau,\omega) &= \frac{1}{2\pi} \int_{\mathbb{R}\times\mathbb{R}} \exp(-\alpha|\lambda| - \alpha|s|) e^{i\lambda(\tau - s)} \sigma(t,s,\omega)\, ds\, d\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-\alpha|s|) \frac{2\alpha}{\alpha^2 + (\tau - s)^2} \sigma(t,s,\omega)\, ds \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \exp(-\alpha|\tau + \alpha s|) \frac{1}{1+s^2} \sigma(t, \tau + \alpha s, \omega)\, ds. \end{aligned}$$

Consequently,

$$\begin{aligned} \sigma(t,\tau,\omega) - \sigma_\alpha(t,\tau,\omega) &= \frac{1}{\pi} \int_{\mathbb{R}} \exp(-\alpha|\tau + \alpha s|) \frac{1}{1+s^2} \left[\sigma(t,\tau,\omega) - \sigma(t,\tau+\alpha s,\omega)\right] ds \\ &\quad + \frac{1}{\pi} \sigma(t,\tau,\omega) \int_{\mathbb{R}} \left[1 - \exp(-\alpha|\tau + \alpha s|)\right] \frac{1}{1+s^2}\, ds. \end{aligned}$$

The convergence result of Lemma 1 follows easily. □

Appendix B: proof of Proposition 1

Let $\lambda \in \mathbb{R}$ be arbitrary, and let $\phi \in L^1(E)$.

Consider the probability space

$$E_\lambda := E \times [0, 2\pi), \quad P_\lambda := P \otimes \frac{d\mu}{2\pi},$$

where μ is the standard Lebesgue measure on $[0, 2\pi]$. Let us define the following group of transformations, acting on (E_λ, P_λ)

$$\mathcal{T}_\tau^\lambda(\omega, \varphi) := (\theta_\tau \omega, \varphi - \lambda\tau \bmod 2\pi), \quad \tau \in \mathbb{R}.$$

Then it is easily checked that $\mathcal{T}_\tau^\lambda$ is measure-preserving for all $\tau \in \mathbb{R}$. And if $T > 0$, we have, for all $\varphi \in [0, 2\pi]$,

$$\begin{aligned} \int_0^T \Phi(\theta_\tau \omega) e^{-i\lambda\tau}\, d\tau &= e^{-i\varphi} \int_0^T \Phi(\theta_\tau \omega) e^{i\varphi - i\lambda\tau}\, d\tau \\ &= e^{-i\varphi} \int_0^T \Psi\left(\mathcal{T}_\tau^\lambda(\omega, \varphi)\right)\, d\tau, \end{aligned}$$

where the function $\Psi \in L^1(E_\lambda)$ is defined by

$$\Psi(\omega, \varphi) := \Phi(\omega) e^{i\varphi}.$$

Hence, according to Birkhoff's ergodic theorem (see [20]), there exists a function $\Psi^\lambda \in L^1(E_\lambda)$, invariant by the group of transformations $\left(\mathcal{T}_\tau^\lambda\right)_{\tau\in\mathbb{R}}$, such that

$$\frac{1}{T}\int_0^T \Phi(\theta_\tau\omega) e^{-i\lambda\tau}\, d\tau \to e^{i\varphi}\Psi^\lambda(\omega,\varphi),$$

P_λ - almost surely in E_λ and in $L^1(E_\lambda)$. Moreover, the function

$$(\omega, \Phi) \mapsto e^{i\varphi}\Psi^\lambda(\omega,\varphi)$$

clearly does not depend on φ. Hence, we set

$$\Phi^\lambda(\omega) := e^{i\varphi}\Psi^\lambda(\omega,\varphi) \quad \forall (\omega,\varphi) \in E_\lambda,$$

and we have proved that

$$\frac{1}{T}\int_0^T \Phi(\theta_\tau\omega) e^{-i\lambda\tau}\, d\tau \to \Phi^\lambda(\omega)$$

almost surely in ω and in $L^1(E)$.

Now, since Ψ^λ is invariant by the group $\left(\mathcal{T}_\tau^\lambda\right)_{\tau\in\mathbb{R}}$ and Φ^λ does not depend on φ, we have, almost surely in ω,

$$\begin{aligned}
\Phi^\lambda(\theta_\tau\omega) &= e^{i\varphi}\Psi^\lambda(\theta_\tau\omega,\varphi) \\
&= e^{i\varphi - i\lambda\tau}\Psi^\lambda(\theta_\tau\omega, \varphi - i\lambda\tau \bmod 2\pi) \\
&= e^{i\varphi - i\lambda\tau}\Psi^\lambda\left(\mathcal{T}_\tau^\lambda(\omega,\varphi)\right) \\
&= e^{i\varphi - i\lambda\tau}\Psi^\lambda(\omega,\varphi) \\
&= e^{-i\lambda\tau}\Phi^\lambda(\omega).
\end{aligned}$$

This completes the proof of Proposition 1.

Appendix C: the stopping Lemma

Lemma 2 (Stopping condition). *Let $T¿0$, and let $\delta_0, \delta_1 \in L^\infty([0,T], H^2(\mathbb{T}^2))$ be two families such that*

$$\int (\delta_{1,3} - \delta_{0,3}) dx_h = 0$$

and such that as $\varepsilon \to 0$,

$$\frac{1}{\varepsilon}\|\delta_i\|_{L^\infty([0,T],H^1(\mathbb{T}^2))} \to 0, \quad \|\delta_i\|_{L^\infty([0,T],H^3(\mathbb{T}^2))} \to 0 \text{ and } \|\partial_t\delta_i\|_{L^\infty([0,T],H^1(\mathbb{T}^2))} \to 0.$$

Then there exists a family $w \in L^\infty([0,T], L^2(\mathbb{T}^2 \times [0,a]))$ with $\nabla \cdot w = 0$ such that

$$w_{|z=0} = \delta_0, \quad w_{3|z=1} = \delta_{1,3} \text{ and } \partial_z w_{h|z=1} = \delta_{1,h}$$

and such that as $\varepsilon \to 0$,

$$\|w\|_{L^\infty([0,T],L^2)} \to 0, \ \left\|\partial_t w + \frac{1}{\varepsilon}Lw - \nu\partial_{zz}w - \Delta_h w\right\|_{L^\infty([0,T],L^2)} \to 0.$$

For a proof of the above Lemma, see [4].

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References

[1] Grégoire Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal. (1992), no. 23, 1482–1518.

[2] A. Babin, A. Mahalov, and B. Nicolaenko, *Integrability and regularity of 3D Euler and equations for uniformly rotating fluids*, Comput. Math. Appl. **31** (1996), no. 9, 35–42.

[3] J.-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier, *Mathematical geophysics*, Oxford Lecture Series in Mathematics and its Applications, vol. 32, The Clarendon Press Oxford University Press, Oxford, 2006, An introduction to rotating fluids and the Navier-Stokes equations.

[4] A.-L. Dalibard and L. Saint-Raymond, *Mathematical study of resonant wind-driven oceanic motions*, preprint, 2008.

[5] B. Desjardins and E. Grenier, *On the homogeneous model of wind-driven ocean circulation*, SIAM J. Appl. Math. **60** (2000), no. 1, 43–60 (electronic).

[6] I. Gallagher and L. Saint-Raymond, *On the influence of the earth's rotation on geophysical flows*, Handbook of Mathematical Fluid Dynamics (Susan Friedlander and Denis Serre, eds.), vol. 4, Elsevier, 2007.

[7] David Gérard-Varet, *Highly rotating fluids in rough domains*, J. Math. Pures Appl. (9) **82** (2003), no. 11, 1453–1498.

[8] A.E. Gill, *Atmosphere-Ocean dynamics*, International Geaophysics series, vol. 4, 1982.

[9] E. Grenier, *Oscillatory perturbations of the Navier-Stokes equations*, J. Math. Pures Appl. (9) **76** (1997), no. 6, 477–498.

[10] E. Grenier and N. Masmoudi, *Ekman layers of rotating fluids, the case of well prepared initial data*, Comm. Partial Differential Equations **22** (1997), no. 5-6, 953–975.

[11] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic type*, American Mathematical Society, 1968.

[12] N. Masmoudi and F. Rousset, *Stability of oscillating boundary layers in rotating fluids*, preprint, 2007.

[13] Nader Masmoudi, *The Euler limit of the Navier-Stokes equations, and rotating fluids with boundary*, Arch. Rational Mech. Anal. **142** (1998), no. 4, 375–394.

[14] ———, *Ekman layers of rotating fluids: the case of general initial data*, Comm. Pure Appl. Math. **53** (2000), no. 4, 432–483.

[15] G. Papanicolaou and S. R. S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, Rigorous results in Statistical Mechanics and Quantum Field Theory (J. Fritz, J. L. Lebaritz, and D. Szasz, eds.), Proc. Colloq. Random Fields, vol. 10, Coll. Math. Soc. Janos Bolyai, 1979, pp. 835–873.

[16] J. Pedlosky, *Geophysical fluid dynamics*, Springer, 1979.

[17] ———, *Ocean Circulation theory*, Springer, 1996.

[18] F. Rousset, *Stability of large Ekman boundary layers in rotating fluids*, Arch. Ration. Mech. Anal. **172** (2004), no. 2, 213–245.

[19] Steven Schochet, *Fast singular limits of hyperbolic PDEs*, J. Differential Equations **114** (1994), no. 2, 476–512.

[20] Ya. G. Sinai, *Introduction to ergodic theory*, Princeton University Press, Princeton, N.J., 1976, Translated by V. Scheffer, Mathematical Notes, 18.