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# Examples of Fast and Slow Convergence of 2D Asynchronous Cellular Systems

## Towards a Classification of the Totalistic Rules Under Asynchronous Updating

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**Abstract.** This article studies the convergence properties of cellular automata under fully asynchronous updating, i.e., when a single cell is selected at random at each time unit. We tackle this question for the two-dimensional totalistic cellular automata. As a first step for studying this class, we focus on a few examples that are, in our view, representative of the diversity of the behaviours found in dimension two. The richness of the evolutions we exhibit underlines that the updating scheme plays a central role in the evolution of a cellular automaton.

## 1 Introduction

In a recent series of studies, cellular automata (CA) have been examined in the light of their robustness to asynchronous updating: the question is to determine to which extent the behaviour of a cellular automaton is due to the method used to update the cells. In particular, what happens if this method is probabilistic, for example when each cell has a probability  $\alpha$  to be fired at each time step (the  $\alpha$ -asynchronous dynamics), or when a single cell is updated at random at each time step (the fully asynchronous dynamics)?

For the sake of conciseness, we refer to [3, 1] for a review of works related to asynchronism and for a discussion on updating schemes. One of the first analytical results on asynchronous CA led to classify a small set of one-dimensional rules (the double-quiescent Elementary CA) according to their convergence properties [3]. The present paper is intended as a first step to broaden the scope of our research to the two-dimensional case. We follow the pioneering work by D. Regnault *et al.* which concerned the Minority rule [2]. We select some totalistic two-dimensional rules that are, in our view, representative of the different classes of behaviour in two dimensional CA under fully asynchronous updating. Among these rules, we distinguish three examples for which we apply techniques already used in one dimension. Furthermore, we exhibit two rules for which a finer geometrical analysis is needed. This provides us with a new type of asymptotic convergence time, specific to the two-dimensional case.

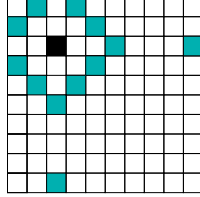
## 2 Definitions

**Topology of the environment.** Let  $A$  be the two-dimensional square grid  $\{1, \dots, L\} \times \{1, \dots, L\}$ , with toric boundary conditions (*i.e.*, we identify  $\{1, \dots, L\}$  with  $\mathbb{Z}/L\mathbb{Z}$ ). We denote by  $n = L^2$  the total number of cells.

For a cell  $c$  and an integer  $k$ , we define the *sphere*  $\partial B(c, k)$  as:

$$\partial B(c, k) = \{c' \in A \mid d(c, c') = k\},$$

where  $d$  is the usual *max* distance on the torus. Below is a representation of the sphere  $\partial B(c, 3)$ , with  $c$  in black and  $L = 10$ :



To each cell  $c$  we associate a state  $\sigma_c$  in  $\{0, 1\}$ , and we call  $\sigma = \{\sigma_c\}_{c \in A}$  a *configuration*. To express the locality of the interactions, we introduce the function  $N : A \rightarrow A^5$  that associates to each cell  $c \in A$ , its *von Neumann* neighbourhood:

$$N(c) = \{c, c + \mathbf{n}, c - \mathbf{n}, c + \mathbf{e}, c - \mathbf{e}\},$$

where  $\mathbf{n}, \mathbf{e}$  denote the vectors  $(0, 1)$  and  $(1, 0)$ .

**Totalistic 2D Cellular Automata.** A *local transition rule* is a function  $\phi : \{0, 1\}^5 \rightarrow \{0, 1\}$ . In this paper, we consider only the *totalistic functions*, *i.e.*, functions  $\phi$  such that:

$$\phi(q_1, \dots, q_5) = f(q_1 + \dots + q_5),$$

where  $f : \{0, \dots, 5\} \rightarrow \{0, 1\}$ . There are 64 such totalistic rules ; we denote the totalistic rule  $\phi$  associated to a function  $f$  by the code  $\mathbb{T}i$  where  $i = f(0) \cdot 2^0 + f(1) \cdot 2^1 + \dots + f(5) \cdot 2^5$ .

We restrict our study to the *fully asynchronous dynamics*: only one cell is updated at each time step. To define this type of asynchronism, we associate to each local rule  $\phi$  a *global rule*  $\Phi : \{0, 1\}^A \times A \rightarrow \{0, 1\}^A$  where  $\Phi(\sigma, \tilde{c})$  is the configuration obtained by updating cell  $\tilde{c}$  in configuration  $\sigma$ , according to the local rule  $\phi$ . More formally, if we write  $\sigma' = \Phi(\sigma, c)$ :

$$\sigma'_c = \begin{cases} f(\sigma_c + \sigma_{c+\mathbf{n}} + \sigma_{c-\mathbf{n}} + \sigma_{c+\mathbf{e}} + \sigma_{c-\mathbf{e}}) & \text{for } c = \tilde{c}; \\ \sigma_c & \text{otherwise.} \end{cases}$$

Let us denote by  $U_t$  the cell updated at time  $t$ . A local rule  $\phi$  and a sequence  $(U_t)_{t \in \mathbb{N}}$  defines the sequence of configurations  $(\sigma^t)_{t \in \mathbb{N}}$ :

$$\begin{aligned} \sigma^0 &\in \{0, 1\}^A, \\ \sigma^{t+1} &= \Phi(\sigma^t, U_t), \text{ for } t \in \mathbb{N}. \end{aligned}$$

As we are interested in random updates of the system, we assume that  $(U_t)_{t \in \mathbb{N}}$  is a sequence of independent random variables, uniformly sampled from  $\Lambda$ . This random sequence makes  $(\sigma^t)$  a stochastic dynamical system in  $\{0, 1\}^\Lambda$ . Our objective is to determine the convergence properties of this system.

For a local rule  $\phi$ , we denote by  $\mathfrak{F}_L^\phi$  the set of the fixed points of the global rule  $\Phi$  associated to  $\phi$ . It consists of the configurations that remain unchanged, whatever the cell selected for updating:

$$\mathfrak{F}_L^\phi = \{\sigma \in \{0, 1\}^\Lambda \mid \text{for any cell } c \in \Lambda, \Phi(\sigma, c) = \sigma\}.$$

Note that for a given  $\phi$ , the sets of fixed points under synchronous and asynchronous updating are identical.

**Definition 1** For a given function  $\phi$  and an initial configuration  $\sigma^0$ , let  $T_\phi(\sigma^0)$  be the time of convergence of the sequence  $(\sigma^t)_{t \in \mathbb{N}}$ , that is, the random variable:

$$T_\phi(\sigma^0) = \min\{t \in \mathbb{N} \mid \sigma^t \in \mathfrak{F}_L^\phi\},$$

with  $\min \emptyset = +\infty$ . The Worst Expected Convergence Time (WECT) of rule  $\phi$  is given by:

$$\text{WECT}_\phi(n) = \max_{\sigma^0} \mathbb{E}[T_\phi(\sigma^0)].$$

Intuitively, if we think of cellular automata as models of physical or artificial systems, studying the WECT provides us with an estimation of the maximum time needed to go back to equilibrium when a perturbation is applied.

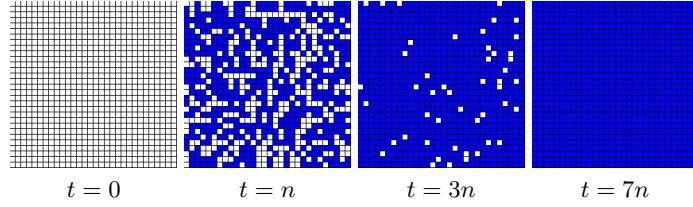
Fatès *et al.* showed that the asymptotic behaviour of  $\text{WECT}_f(n)$  provides a relevant classification of the Elementary Cellular Automata with two quiescent states [3]. More precisely, they have shown that these rules may be classified into 5 families, according to whether  $\text{WECT}_f(n)$  is  $\Theta(n \log n)$ ,  $\Theta(n^2)$ ,  $\Theta(n^3)$ ,  $\Theta(n2^n)$  or infinite<sup>3</sup>.

### 3 Convergence Times

The examples of 2D totalistic rules that we exhibit show that we recover some of the classes mentioned above, as well as a new one:  $\Theta(n^{3/2})$ . The five different behaviours found in 2D are summarised in table below:

Name	$\phi$	$\text{WECT}_\phi(n)$	Observed behaviour
Coupon collector	T63	$\Theta(n \log n)$	Fast convergence to $1^A$
Epidemic	T62	$\approx n^{3/2}$	Fast convergence to $1^A$
Majority	T56	$\Theta(n^2)$	Fast convergence to a local equilibrium
Erratic	T10	$> \lambda^n$	Slow convergence (metastability)
Parity Counter	T21	infinite	Noise-like evolution

<sup>3</sup> We write  $f_n = \Theta(g_n)$  when there exist positive numbers  $C^-, C^+$  such that, for  $n$  large enough,  $C^- g_n \leq f_n \leq C^+ g_n$ .



**Fig. 1.** Example of a simulation for the *coupon collector* rule  $\mathbb{T}63$ . Cells in state 1 (resp. 0) are coloured in black (resp. white), a convention which is kept throughout the paper. Simulations were obtained with the *FiatLux* CA simulator [4].

### 3.1 A *Coupon Collector* automaton

Our examination begins with the rule  $\mathbb{T}63$ :

$s$	0	1	2	3	4	5
$f(s)$	1	1	1	1	1	1

An update sets the cell into state 1 whatever the initial state. Under the synchronous dynamics  $\mathbb{T}63$  reaches the fixed point  $1^A$  in one step. In the asynchronous case, this fixed point is attained once each cell has been updated at least once (see Fig. 1). This kind of process often arises in the analysis of algorithms ; it is usually called a *Coupon collector* process.

#### Theorem 1

$$\text{WECT}_{63}(n) = \Theta(n \log n).$$

**Sketch of proof.** Remark that when  $k$  cells among the  $n$  cells are still in state 0, it takes in average  $n/k$  time steps to update one of them. Thus, the slowest convergence time holds for  $\sigma^0 = 0^A$ . It is equal to:

$$\frac{n}{n} + \frac{n}{n-1} + \dots + \frac{n}{1} = \Theta(n \log n).$$

### 3.2 The *Majority* rule

We now turn to the *Majority* rule  $\mathbb{T}56$ :

$s$	0	1	2	3	4	5
$f(s)$	0	0	0	1	1	1

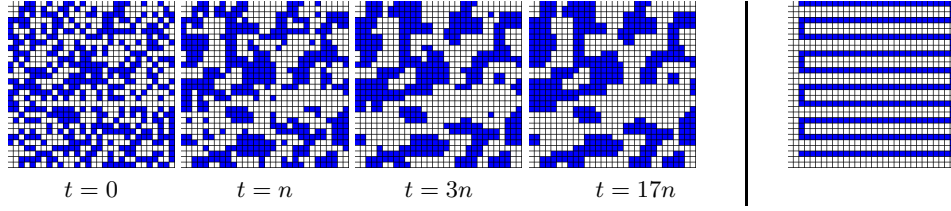
An update sets cell  $c$  to the state that is most present in  $N(c)$ . The global effect of the rule is to converge quickly to equilibrium (see Fig. 2).

#### Theorem 2

$$\text{WECT}_{56}(n) = \Theta(n^2).$$

**Sketch of proof.** We begin by introducing some notations. The *similarity function*  $\mathcal{I}$  counts the number of neighbouring cells in the same state:





$$\mathcal{I}(\sigma) = \text{card}\{(c, c') \in \Lambda \mid d(c, c') = 1 \text{ and } \sigma_c = \sigma_{c'}\}.$$



**Fig. 2.** Left : Example of a simulation for rule T56, when  $\sigma^0$  is a random uniform configuration. Right : A snake-like configuration gives a lower bound on the WECT.

For a configuration  $\sigma$ , we define  $M_{q,r}(\sigma)$  as the numbers of cells in state  $q \in Q$ , with  $r$  neighbours in state 1.

**Upper bound.** We obtain an upper bound on the convergence time by showing that the sequence  $\mathcal{I}(\sigma^t)$  is increasing. To see why this holds, let us list all the cases where a cell may change its state:

Pattern	$M_{0,4}$	$M_{0,3}$	$M_{1,0}$	$M_{1,1}$
				
$\Delta\mathcal{I}$	+8	+4	+8	+4

With the notation  $\Delta\mathcal{I}_t = \mathcal{I}(\sigma^{t+1}) - \mathcal{I}(\sigma^t)$ , we have that:

$$\mathbb{E}[\Delta\mathcal{I}_t \mid \sigma^t] = \frac{1}{L^2} (8M_{0,4} + 4M_{0,3} + 8M_{1,0} + 4M_{1,1})(\sigma^t).$$

Remark that if  $\sigma^t$  is not a fixed point, then this quantity is greater than  $4/L^2$ . Using arguments on Lyapunov functions, we obtain that, for any initial configuration  $\sigma^0$ :

$$\mathbb{E}[T_{56}(\sigma^0)] \leq \frac{L^2}{4} \mathcal{I}(\sigma^0).$$

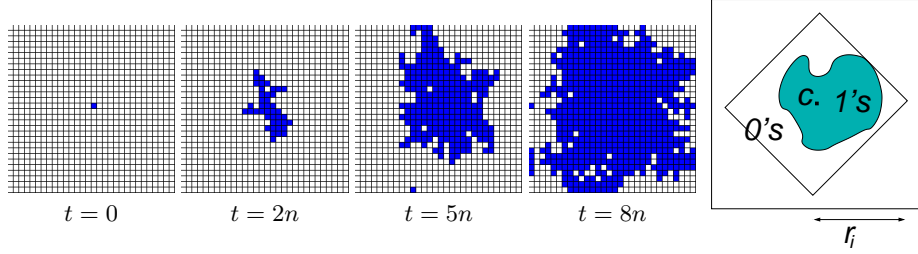
As  $\mathcal{I}$  is bounded by  $4L^2$ , it follows that:

$$\mathbb{E}[T_{56}(\sigma^0)] \leq L^4 = n^2.$$

**Lower bound.** To obtain a lower bound on the convergence time, we consider the *snake-like* initial configuration drawn in Fig.2. Intuitively, this initial configuration is chosen in order to ensure that the only way of reaching equilibrium is to “shrink” the snake by updating its two extremities. The snake is made of about  $n/3$  cells in state 1 ; as the probability to update one of the two extremities is  $2/n$ , we obtain:

$$\mathbb{E}[T_{56}(\sigma^0)] \geq C^- n^2.$$

As the lower bound and the upper bound scale in  $n^2$ , the theorem is proved. Remark that the evolution of the *Majority* rule is different in 1D and 2D since the WECT of the *Majority* rule in 1D is of order  $n \log n$  [3].



**Fig. 3.** Left : Example of a simulation for rule T62. Right : Schematic view of a configuration where a region of 1's reaches the sphere  $\partial(c, r_i)$ .

### 3.3 The epidemic automaton

Let us consider the epidemic rule T62:

$s$	0	1	2	3	4	5
$f(s)$	0	1	1	1	1	1

A cell in state 0 (healthy) turns to state 1 (infected) if one of its neighbours is in state 1 ; it then remains in this state.

Up to now, the classes of convergence that we have met were already known as they were identified for 1D cellular automata [3]. This rule has a new type of convergence, specific to the two-dimensional case.

**Theorem 3** *There exist two constants  $C^-, C^+$  such that, for  $n$  large enough,*

$$\frac{C^-}{\log n} n^{3/2} \leq \text{WECT}_{62}(n) \leq C^+ (\log n) n^{3/2}.$$

**Remark.** We conjecture that one may tighten these bounds and show that  $\text{WECT}_{62}(n)$  is actually of order  $n^{3/2}$ .

**Sketch of proof.** The first step is to prove that the largest convergence time is obtained for the configurations which contain only one cell in state 1. For such an initial configuration, the region of 1's spreads all over the grid (see Fig. 3). The second step consists in estimating the time needed for this region to cover the whole grid.

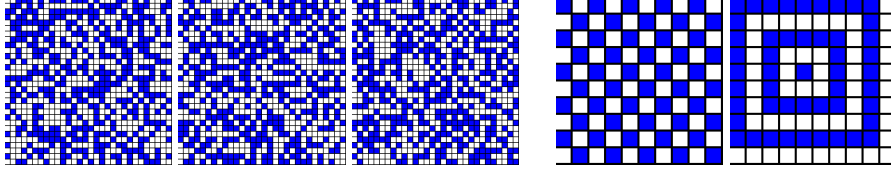
To prove the first step, let us introduce a partial order  $\prec$  on  $\{0, 1\}^A$  by:

$$\sigma \prec \eta \text{ if and only if } \forall c \in A, \sigma_c \leq \eta_c.$$

It is then enough to note that  $\Phi$  conserves the order  $\prec$  in the sense that:

$$\forall U_t \in A, \sigma \prec \eta \Rightarrow \Phi(\sigma, U_t) \prec \Phi(\eta, U_t).$$

We therefore obtain:  $\sigma^t = \mathbf{1}^A \Rightarrow \eta^t = \mathbf{1}^A$  ; which proves the first step. To obtain the bounds around  $n^{3/2}$ , we analyse how the region of 1's spreads over  $A$  as the system evolves. The key point is to cut  $A$  in concentric spheres  $\partial B(c, r_i)$  for some suitably chosen integers  $(r_i)$  and to estimate the time needed for the region of 1's to reach every  $\partial B(c, r_i)$  (see Fig. 3 - Right). The choice of the sequence  $r_i$  is different for the two bounds of the theorem.



**Fig. 4.** Left: Three typical noise-like configurations observed in the evolution of the *erratic* rule T10. Right: Two fixed points of this rule.

### 3.4 An *erratic* automaton

When exploring systematically the 64 totalistic rules, we observed that some of them evolved in a metastable regime: the observation of the configurations attained gives the impression that the system evolves “erratically” and will never converge. However, a detailed analysis show that these rules do possess fixed points and that some of them may be attained with a convenient sequence of updates.

Among these rules, we focus on rule T10:

$s$	0	1	2	3	4	5
$f(s)$	0	1	0	1	0	0

We restrict our analysis to the even values of  $L$ .

**Theorem 4** *There exists a constant  $\lambda > 1$  such that, for  $n$  large enough,*

$$\lambda^n \leq \text{WECT}_{10}(n) < +\infty.$$

**Sketch of proof.** The proof we suggest consists of three parts. The first part is geometric in the sense where we need to describe the set of fixed points and to evaluate its size as a function of  $L$ . Then, we have to prove that, starting from any initial configuration, the system reaches a fixed point if we apply an appropriate sequence of updates. The third part is somehow paradoxical since we need to *increase* the randomness of the problem by perturbing the dynamics in order to use tools from Markov Chain Theory.

More precisely, for  $0 < \varepsilon < 1$ , let  $\phi^\varepsilon$  be the probabilistic local rule defined

by the following table:

$s$	0	1	2	3	4	5
$f(s)$	0	1	0	1	$\varepsilon$	0

The effect of  $\phi^\varepsilon$  is identical to  $\phi$ , unless the updated cell  $c$  has four neighbours in state 1 ; in this case,  $c$  takes the state 1 with probability  $\varepsilon$ . The benefit is that the introduction of this  $\varepsilon$  makes the sequence  $(\sigma^t)$  a *reversible* Markov Chain (see *e.g.*[5]). This allows us to calculate the convergence properties of  $\phi^\varepsilon$ . The technical point is to show that the bounds obtained for  $\varepsilon \rightarrow 0$  display a form of continuity in  $\varepsilon$  which allows us to find similar bounds for the original model ( $\varepsilon = 0$ ).



### 3.5 The Parity Counter

We finish our exploration with rule  $\mathbb{T}42$ :

$s$	0	1	2	3	4	5
$f(s)$	0	1	0	1	0	1

It is sometimes referred to as the *Parity Counter*: an update on cell  $c$  turns it into 1 if and only if the number of 1's in  $N(c)$  is odd.

**Theorem 5** *If  $\sigma^0$  is not a fixed point, then  $T_{42}(\sigma^0) = +\infty$ .  
Consequently,*

$$\text{WECT}_{42}(n) = +\infty.$$

**Sketch of proof.** To show that  $T_{42}(\sigma^0)$  is infinite when  $\sigma^0$  is not a fixed point, we need to prove that, unlike the *erratic* automaton, there is no sequence of updates leading from  $\sigma^0$  to a fixed point. The argument is that all the transitions are reversible: remark that updating a cell  $c$  changes both its state and the “parity” of  $N(c)$ , and therefore  $c$  remains unstable under the application of the rule (although other cells in  $N(c)$  may become stable).

### 3.6 Concluding remark

According to our simulations, the spectrum of convergence times that we studied here covers most of the totalistic rules. We ask whether there are other convergence types, for example in  $\theta(n^3)$  or in  $\theta(n^4)$ .

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