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Normal modes of layered elastic media and application to diffuse fields

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Abstract

The spectral decomposition of the elastic wave operator in a layered isotropic half-space is derived by means of standard functional analytic methods. Particular attention is paid to the coupled P - SV waves. The problem is formulated directly in terms of displacements which leads to a 2×2 Sturm-Liouville system. The resolvent kernel (Green function) is expressed in terms of simple plane-wave solutions. Application of Stone's formula leads naturally to eigenfunction expansions in terms of generalized eigenvectors with oscillatory behavior at infinity. The generalized normal mode expansion is employed to define a diffuse field as a white noise process in modal space. By means of a Wigner transform, we calculate vertical to horizontal kinetic energy ratios in layered media, as a function of depth and frequency. Several illustrative examples are considered including energy ratios near a free surface, in the presence of a soft layer. Numerical comparisons between the generalized eigenfunction summation and a classical locked-mode approximation demonstrate the validity of the approach. The impact of the local velocity structure on the energy partitioning of a diffuse field is illustrated.

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I. INTRODUCTION

Since its introduction in elastodynamics by Weaver¹, the diffuse field concept and the principle of equipartition of elastic waves have been successfully applied to both field² and laboratory experiments³. It is an important ingredient of the spectacular reconstruction of Green's function from thermal noise⁴ or seismic coda waves⁵. The principle of equipartition can also be used to predict the partition of energy in diffuse fields^{1,2,6}. Yet, the practical implementation of the equipartition principle in a configuration as simple as a half-space poses some serious technical difficulties⁶. The simplest mathematical formulation of equipartition relies on the the existence of a complete normal mode basis. In the past, simple waveguides or a homogeneous half-space have been considered^{6,7}. In this work, we construct a complete set of normal modes of the elastic wave equation from the general spectral theory of self-adjoint operators in Hilbert space^{8,9}. A diffuse field is then represented as a white noise distributed over the complete set of normal modes independent of the nature of the spectrum (discrete/continuous), or medium (open/closed). The paper is organized as follows: in section II, the spectral problem is introduced and general properties of the elastodynamic operator are summarized. In section III, the key mathematical results are presented and illustrated on a simple problem. Sections IV-V present the central results of the paper. The spectral theory of a layered elastic half-space is derived in detail and applied to energy partitioning of diffuse fields. Other applications of the theory are briefly discussed in conclusion.

II. PROBLEM STATEMENT

The elastodynamic equation governing seismic wave propagation can be written as

$$\partial_t^2 |u\rangle = -\mathbf{L} |u\rangle, \quad (1)$$

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or more explicitly in the the position representation:

$$\partial_t^2 u_i(\mathbf{x}) = \frac{1}{\rho(\mathbf{x})} \partial_j C_{ijkl} \partial_k u_l(\mathbf{x}). \quad (2)$$

In equation (1), the minus sign has been introduced in order to deal with a positive definite operator.

In an isotropic medium where properties depend only on the depth coordinate z , the elastodynamic equation decouples into independent scalar (SH) and vectorial (P - SV) equations. Our goal is to obtain a complete set of normal modes for the latter problem. Following Ref.10, we reduce the three-dimensional problem to a coupled system of second order differential equations. Translational invariance suggests to look for solutions of the form $\mathbf{u}(z)e^{ikx}$. Additionally, cylindrical symmetry enables one to work in a single plane of propagation and ignore the third space dimension. Taking all the symmetries into account yields an eigenvalue problem for a second-order differential operator \mathbf{L}^{10}

$$\begin{cases} \langle z|\mathbf{L}|u\rangle = -\frac{1}{\rho(z)} \partial_z \langle z|\tau|u\rangle + kB^t \partial_z \langle z|u\rangle + k^2 C \langle z|u\rangle = \lambda \langle z|u\rangle \\ \langle z|\tau|u\rangle = A \partial_z \langle z|u\rangle + kB \langle z|u\rangle \end{cases}. \quad (3)$$

The operator τ provides the tractions generated by the displacement field $|u\rangle$ and ρ denotes the density. The matrices A , B and C are defined as

$$\begin{aligned} A &= \rho(z) \begin{pmatrix} \alpha^2(z) & 0 \\ 0 & \beta^2(z) \end{pmatrix}, & B &= \rho(z) \begin{pmatrix} 0 & 2\beta^2(z) - \alpha^2(z) \\ \beta^2(z) & 0 \end{pmatrix}, \\ C &= \begin{pmatrix} \beta^2(z) & 0 \\ 0 & \alpha^2(z) \end{pmatrix}, \end{aligned} \quad (4)$$

where α and β denote the longitudinal and shear wavespeed, respectively. The vector $|u\rangle$ has components (iu_z, u_x) , where u_x and u_z denote the original displacement field. This change of variables can be represented by a unitary transformation U and turns the original problem into an equivalent, easier problem. Indeed, the multiplication of the vertical component by i makes \mathbf{L} a real operator which simplifies the calculations in the layered case. Once the eigenvectors of the transformed operator have been obtained, those pertaining to the original

operator are recovered by application of the inverse operation U^\dagger . As shown previously^{11,12}, the elastodynamic operator is positive definite and self-adjoint in the Hilbert space with scalar product

$$\langle u|v\rangle = \int dz \rho(z) u_i(z)^* v_i(z). \quad (5)$$

The precise domain of definition of \mathbf{L} is given in Ref.11 but loosely speaking, the operator acts on functions of finite elastic deformation energy. In equation (3) and (5) we have introduced the Dirac bra-ket notation for its compactness and convenience. Since it will be used throughout the paper, we have summarized the main properties of the Dirac formalism in appendix A.

III. STONE FORMULA AND A SIMPLE APPLICATION

A. Mathematical results

The Stone formula¹³ is a powerful functional analytic result for self-adjoint operators in a Hilbert space. This result is particularly useful for the mathematical formulation of scattering theory^{8,9,14}. In this paper, we will formally apply this formula to the elastic wave equation to obtain generalized eigenfunction expansions. The eigenvectors $|e\rangle$ are solutions of the equation $\mathbf{L}|e\rangle = \lambda|e\rangle$ but do not belong to the domain of the operator, i.e., they are not solutions with finite energy. These eigenvectors are required to construct the complete modal solutions of the elastic wave equation¹⁵. Our work generalizes previous results obtained in the case of a homogeneous half-space^{11,16}. Before stating the Stone formula, we introduce the resolvent $\mathbf{G}(\lambda)$ of the operator \mathbf{L} :

$$\mathbf{G} = (\mathbf{L} - \lambda\mathbf{I})^{-1}. \quad (6)$$

Because \mathbf{L} is self-adjoint, its spectrum is a subset of the real axis and therefore the resolvent is defined for $\text{Im}\lambda \neq 0$. In position and polarisation space, the resolvent is represented by the Green matrix

$$\langle z|(\mathbf{L} - \lambda\mathbf{I})^{-1}|z'\rangle = G_{ij}(\lambda, z, z'), \quad (7)$$

which possesses the Hermitian symmetry

$$G_{ij}(\lambda, z, z') = G_{ji}(\lambda^*, z', z)^*. \quad (8)$$

We first recall the fundamental spectral theorem and the Stone formula in an abstract setting. To every self-adjoint operator \mathbf{L} one can associate a projection operator valued function (measure) \mathbf{P}_λ having the following properties¹³

$$\mathbf{f}(\mathbf{L}) = \int_{-\infty}^{+\infty} f(\lambda) d\mathbf{P}_\lambda, \quad (9)$$

The spectral theorem guarantees that the spectral family \mathbf{P}_λ is orthogonal, diagonalizes the operator \mathbf{L} and provides the completeness relation. A practical method to construct the spectral projector \mathbf{P}_λ is provided by the Stone formula, which relates functions of a self-adjoint operator to the discontinuity of its resolvent across the real axis¹³:

$$f(\mathbf{L}) = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{+\infty} [\mathbf{G}(\lambda + i\epsilon) - \mathbf{G}(\lambda - i\epsilon)] f(\lambda) d\lambda. \quad (10)$$

For a positive operator, the integral can be taken from 0 to $+\infty$ only because the spectrum is a subset of the positive real axis. As a particular case of equation (10), one can formally obtain the completeness relation:

$$\mathbf{I} = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{+\infty} [\mathbf{G}(\lambda + i\epsilon) - \mathbf{G}(\lambda - i\epsilon)] d\lambda. \quad (11)$$

For an operator with a mixed spectrum, with both discrete and continuous parts, equation (11) will take the following form:

$$\mathbf{I} = \int d\lambda \sum_m |e_m(\lambda)\rangle \langle e_m(\lambda)| + \sum_n |e_n\rangle \langle e_n|, \quad (12)$$

where $|e_m(\lambda)\rangle$ denotes generalized eigenfunctions of the continuum -we assume that no singular continuous spectrum exists- that are normalized in the following sense:

$$\langle e_m(\lambda) | e_{m'}(\lambda') \rangle = \delta_{mm'} \delta(\lambda - \lambda'). \quad (13)$$

In equation (12), we have introduced a discrete index in the continuous part of the spectrum, which corresponds to the possibility that the space of generalized eigenfunction be multidimensional.

B. Generalized eigenvectors in homogeneous full-space

We consider the system (3) with constant parameters $\alpha_\infty, \beta_\infty$ and ρ_∞ . First we construct the resolvent $\mathbf{L} - \lambda\mathbf{I}$ outside of the real axis. In order to uniquely define the square root of a complex number z we adopt the following convention: $\sqrt{z} = |z|^{1/2}e^{i\theta/2}$, with $\theta = \arg(z) \in [0, 2\pi)$. This definition guarantees that $\text{Im}\sqrt{z} > 0$ when z lies outside the positive real axis. The 2×2 system of coupled second-order equation (3) has four linearly independent solutions $|u_{+\infty}^P(\lambda)\rangle, |u_{-\infty}^P(\lambda)\rangle, |u_{+\infty}^S(\lambda)\rangle, |u_{-\infty}^S(\lambda)\rangle$, whose properties are summarized below:

$$\begin{aligned} \langle z|u_{\pm\infty}^P(\lambda)\rangle &= \frac{1}{\epsilon_\alpha(\lambda)} \begin{pmatrix} \pm iq_\alpha(\lambda) \\ k \end{pmatrix} e^{\pm iq_\alpha(\lambda)z}, \\ \langle z|u_{\pm\infty}^S(\lambda)\rangle &= \frac{1}{\epsilon_\beta(\lambda)} \begin{pmatrix} k \\ \mp iq_\beta(\lambda) \end{pmatrix} e^{\pm iq_\beta(\lambda)z}. \end{aligned} \quad (14)$$

In equation (14), the prefactors $\epsilon_{\alpha,\beta}$ ensure that the plane waves are energy normalized¹⁰:

$$\epsilon_{\alpha,\beta} = \frac{1}{\sqrt{2\rho_\infty\sqrt{\lambda}q_{\alpha,\beta}}}, \quad (15)$$

and the vertical wavenumbers $q_{\alpha,\beta}$ are defined as

$$q_\alpha = \sqrt{\frac{\lambda}{\alpha_\infty^2} - k^2}, \quad q_\beta = \sqrt{\frac{\lambda}{\beta_\infty^2} - k^2}. \quad (16)$$

The following symmetry relations will prove useful:

$$\begin{aligned} \langle z|u_{+\infty}^P(\lambda^*)\rangle &= -\langle z|u_{+\infty}^P(\lambda)\rangle^*, \\ \langle z|u_{-\infty}^P(\lambda^*)\rangle &= -\langle z|u_{-\infty}^P(\lambda)\rangle^*, \end{aligned} \quad (17)$$

with completely analogous relations involving $|u_{+\infty}^S(\lambda)\rangle, |u_{-\infty}^S(\lambda)\rangle$. When $\text{Im}\lambda = 0$, $\lambda > \alpha_\infty^2 k^2$, assuming a time dependence of the form $e^{-i\sqrt{\lambda}t}$, $(u_{+\infty}^P, u_{-\infty}^P) - (u_{+\infty}^S, u_{-\infty}^S)$ behave like (outgoing, incoming) plane P - S waves at $+\infty$, respectively. Using this set of solutions, we can construct the Green matrix for $\text{Im}\lambda \neq 0$, which obeys the following equation:

$$-\rho_\infty^{-1}\partial_z\tau[G]_{ij}(z, z') + K_{ik}(z)G_{kj}(\lambda, z, z') - \lambda G_{ij}(z, z') = \rho_\infty^{-1}\delta_{ij}\delta(z - z'), \quad (18)$$

where we have introduced the differential operator $K(z) = k^2C + kB^t\partial_z$. The Green matrix should possess the following properties:

1. For $z > z'$, each column of G must be a linear combination of solutions with finite energy at $+\infty$ ($|u_{+\infty}^P\rangle, |u_{+\infty}^S\rangle$).
2. For $z < z'$, each column of G must be a linear combination of solutions with finite energy at $-\infty$ ($|u_{-\infty}^P\rangle, |u_{-\infty}^S\rangle$).
3. G obeys the symmetry relation (8)
4. G is continuous on the diagonal $z = z'$.
5. The traction matrix of G has a jump discontinuity on the diagonal: $\tau[G]_{ij}|_{z'^-}^{z'^+} = -\delta_{ij}$

Properties (1)-(5) are straightforward generalizations to systems of 2^{nd} order differential equations of the standard Sturm-Liouville theory¹⁶. Properties (1)-(3) indicate that $\mathbf{G}(\lambda)$ should have the following form

$$\mathbf{G}(\lambda) = \begin{cases} K_1(\lambda) |u_{-\infty}^P(\lambda)\rangle \langle u_{+\infty}^P(\lambda)^*| + K_2(\lambda) |u_{-\infty}^S(\lambda)\rangle \langle u_{+\infty}^S(\lambda)^*| & z < z' \\ K_1(\lambda) |u_{+\infty}^P(\lambda)\rangle \langle u_{-\infty}^P(\lambda)^*| + K_2(\lambda) |u_{+\infty}^S(\lambda)\rangle \langle u_{-\infty}^S(\lambda)^*| & z > z' \end{cases}. \quad (19)$$

In equation (19) we have separated the Green matrix into a P and an S wave part. Application of the continuity condition of the Green matrix on the diagonal (4) yields $K_1(\lambda) = K_2(\lambda)$. Next we apply the condition (5) for the discontinuity of the traction on the diagonal to obtain:

$$K_1(\lambda) = K_2(\lambda) = \frac{i}{\sqrt{\lambda}}. \quad (20)$$

The next step is to study the discontinuity of the resolvent across the real axis. We will denote by $|u_{+\infty}^{P+}\rangle$ and $|u_{+\infty}^{P-}\rangle$ the limiting values of the vectors $|u_{+\infty}^P(\lambda)\rangle$ as λ approaches the real axis from above and below, respectively, with analogous definitions for $|u_{-\infty}^{P+}\rangle$ and $|u_{-\infty}^{S+}\rangle, \dots$. The jump of the resolvent $\mathbf{G}^+ - \mathbf{G}^-$ across the real axis depends on the relations among these functions. For $z < z'$, one obtains

$$\mathbf{G}^+ - \mathbf{G}^- = \frac{i}{\sqrt{\lambda^+}} (|u_{-\infty}^{P+}\rangle \langle (u_{+\infty}^{P+})^*| + |u_{-\infty}^{P-}\rangle \langle (u_{+\infty}^{P-})^*| + |u_{-\infty}^{S+}\rangle \langle (u_{+\infty}^{S+})^*| + |u_{-\infty}^{S-}\rangle \langle (u_{+\infty}^{S-})^*|). \quad (21)$$

Three cases have to be distinguished:

(1) $\lambda > \alpha_\infty^2 k^2$. In this case $q_{\alpha,\beta}$ and $\epsilon_{\alpha,\beta}$ are real and all the waves are propagating. Applying the following symmetry relations,

$$\begin{aligned} |(u_{+\infty}^{P+})^* \rangle &= |u_{-\infty}^{P+} \rangle & |u_{+\infty}^{P-} \rangle &= - |(u_{+\infty}^{P+})^* \rangle \\ |(u_{+\infty}^{S+})^* \rangle &= |u_{-\infty}^{S+} \rangle & |u_{+\infty}^{S-} \rangle &= - |(u_{+\infty}^{S+})^* \rangle, \end{aligned} \quad (22)$$

one can rewrite the resolvent discontinuity (21) as a sum of projection operators on a generalized eigenfunction basis:

$$\mathbf{G}^+ - \mathbf{G}^- = \frac{i}{\sqrt{\lambda^+}} (|u_{-\infty}^{P+} \rangle \langle u_{-\infty}^{P+}| + |u_{+\infty}^{P+} \rangle \langle u_{+\infty}^{P+}| + |u_{-\infty}^{S+} \rangle \langle u_{-\infty}^{S+}| + |u_{+\infty}^{S+} \rangle \langle u_{+\infty}^{S+}|). \quad (23)$$

From equation (23), one concludes that there are four linearly independent generalized eigenfunctions with eigenvalue $\lambda > \alpha_\infty^2 k^2$. For instance, the first term on the left-hand side of equation (23) is an orthogonal projector on the space of upgoing P waves.

(2) $\alpha_\infty^2 k^2 > \lambda > \beta_\infty^2 k^2$. q_α is pure imaginary and q_β is real which corresponds to evanescent P waves and propagating S waves. Upon application of the following symmetry relations:

$$\begin{aligned} |(u_{-\infty}^{P-}) \rangle &= -i |u_{-\infty}^{P+} \rangle & |u_{+\infty}^{P-} \rangle &= -i |u_{+\infty}^{P+} \rangle \\ |(u_{+\infty}^{S+})^* \rangle &= |u_{-\infty}^{S+} \rangle & |u_{+\infty}^{S-} \rangle &= - |(u_{+\infty}^{S+})^* \rangle, \end{aligned} \quad (24)$$

equation (21) simplifies to

$$\mathbf{G}^+ - \mathbf{G}^- = \frac{i}{\sqrt{\lambda^+}} (|u_{-\infty}^{S+} \rangle \langle u_{-\infty}^{S+}| + |u_{+\infty}^{S+} \rangle \langle u_{+\infty}^{S+}|). \quad (25)$$

There are two linearly independent S eigenfunctions.

(3) $\lambda < \beta_\infty^2 k^2$. There are no singularities and the resolvent is continuous:

$$\mathbf{G}^+ - \mathbf{G}^- = \mathbf{0}. \quad (26)$$

Because of the symmetry of the Green's function (8), the same results are obtained for $z > z'$. The completeness relation now follows from the Stone formula (11). The continuous parameter λ is analogous to a squared eigenfrequency. Introducing $\lambda = \omega^2$, and generalized

eigenvectors $|e_{+\infty}^P(\omega)\rangle, \dots$ the resolution of the identity writes as a sum of integrals over frequencies:

$$\mathbf{I} = \int_{\alpha_{\infty}k}^{+\infty} d\omega (|e_{+\infty}^P(\omega)\rangle \langle e_{+\infty}^P(\omega)| + |e_{-\infty}^P(\omega)\rangle \langle e_{-\infty}^P(\omega)|) + \int_{\beta_{\infty}k}^{+\infty} d\omega (|e_{+\infty}^S(\omega)\rangle \langle e_{+\infty}^S(\omega)| + |e_{-\infty}^S(\omega)\rangle \langle e_{-\infty}^S(\omega)|). \quad (27)$$

The eigenvectors $|e_{+\infty}^P(\omega)\rangle, \dots$ can be deduced from the basis vectors $|u_{+\infty}^P(\lambda)\rangle$ and form an orthonormal set. Let us for instance consider the eigenvector $|e_{+\infty}^P(\omega)\rangle$:

$$\langle z|e_{+\infty}^P(\omega)\rangle = \frac{1}{\sqrt{2\pi\rho_{\infty}\omega q_{\alpha}}} \begin{pmatrix} -iq_{\alpha} \\ k \end{pmatrix} e^{+iq_{\alpha}z}, \quad \langle e_{+\infty}^P(\omega)|e_{+\infty}^P(\omega')\rangle = \delta(\omega - \omega'), \quad (28)$$

where $q_{\alpha} = \sqrt{\frac{\omega^2}{\alpha_{\infty}^2} - k^2}$. Multiplying the wavefunctions $\langle z|e_{+\infty}^P(\omega)\rangle$ by the phase term $\frac{1}{\sqrt{2\pi}}e^{ikx}$ we can form compound eigenvectors $|\psi^{P,S}\rangle$ that can be used to expand 2-D in-plane vector fields. The expansion takes a particularly simple form if instead of using a frequency integral, the new variables $p_x = \sqrt{\frac{\omega^2}{\alpha_{\infty}^2} - k^2}$ and $p_z = \sqrt{\frac{\omega^2}{\beta_{\infty}^2} - k^2}$ are introduced in the P and S integrals of equation (27), respectively. In order to obtain eigenfunctions of the elastodynamic operator in its standard form, we apply the unitary transformation U^{\dagger} to obtain a fairly simple completeness relation:

$$\mathbf{I} = \iint_{\mathbb{R}^2} dp_x dp_z (|\psi^P(p_x, p_z)\rangle \langle \psi^P(p_x, p_z)| + |\psi^S(p_x, p_z)\rangle \langle \psi^S(p_x, p_z)|), \quad (29)$$

where $|\psi^P(p_x, p_z)\rangle$ and $|\psi^S(p_x, p_z)\rangle$ are simple plane P and SV waves:

$$\begin{aligned} \langle r|\psi^P(p_x, p_z)\rangle &= \hat{p} \frac{e^{ip_x x + ip_z z}}{2\pi\sqrt{\rho_{\infty}}} \\ \langle r|\psi^S(p_x, p_z)\rangle &= \hat{p}^{\perp} \frac{e^{ip_x x + ip_z z}}{2\pi\sqrt{\rho_{\infty}}}, \end{aligned} \quad (30)$$

where \hat{p} and \hat{p}^{\perp} denote unit vectors parallel and perpendicular to the wavevector \mathbf{p} , respectively:

$$\hat{p} = \frac{1}{\sqrt{p_x^2 + p_z^2}} \begin{pmatrix} p_x \\ p_z \end{pmatrix}, \quad \hat{p}^{\perp} = \frac{1}{\sqrt{p_x^2 + p_z^2}} \begin{pmatrix} -p_z \\ p_x \end{pmatrix}. \quad (31)$$

$|\psi^P(p_x, p_z)\rangle$ and $|\psi^S(p_x, p_z)\rangle$ are eigenvectors of the elastic wave equation with eigenvalues $\alpha_\infty^2(p_x^2 + p_y^2)$ and $\beta_\infty^2(p_x^2 + p_y^2)$, respectively. They form a complete set of normal modes that decompose 2-D vector fields into longitudinal and transverse parts.

IV. GENERALIZED EIGENFUNCTIONS IN STRATIFIED MEDIA

A. Construction of the resolvent

We begin with the construction of the resolvent, which parallels the development of the previous section. To make comparison easier we denote by ρ_∞ , α_∞ and β_∞ the density and seismic wave speeds in the half-space located at depth $z > 0$. The medium is assumed to be composed of a stack of n layers and is bounded by a free surface at $z = -z_n$, $z_n > 0$. We wish to find the eigenfunction expansion associated with the eigenvalue problem $\mathbf{L}|u\rangle = \lambda|u\rangle$ introduced in equation (3) and supplemented with the free surface traction condition $\langle z|\tau|u\rangle = 0$ at $z = 0$ and the continuity of stresses and displacements at each interface. To do so, we introduce four linearly independent vector solutions of the elastic wave equation (3) denoted by $|u_{+\infty}^P(\lambda)\rangle$, $|u_{-\infty}^P(\lambda)\rangle$, $|u_{+\infty}^S(\lambda)\rangle$, $|u_{-\infty}^S(\lambda)\rangle$. In the half-space $z > 0$, their analytical form is given by equation (14). Their analytical dependence in the stack of layer is obtained by integrating the equations of motion (3) from depth $z = 0^+$ to $z = -z_n^+$ by applying continuity of stresses and displacements at each interface. Note that, in general, these solutions do *not* verify the stress free condition at $z = 0$. Following Ref.10, we introduce two linearly independent solutions $|u_0^P(\lambda)\rangle$, $|u_0^S(\lambda)\rangle$ of the wave equation that verify the stress-free condition at $z = 0$

$$\begin{aligned} |u_0^P(\lambda)\rangle &= |u_{-\infty}^P(\lambda)\rangle + r^{pp}(\lambda) |u_{+\infty}^P(\lambda)\rangle + r^{ps}(\lambda) |u_{+\infty}^S(\lambda)\rangle \\ |u_0^S(\lambda)\rangle &= |u_{-\infty}^S(\lambda)\rangle + r^{sp}(\lambda) |u_{+\infty}^P(\lambda)\rangle + r^{ss}(\lambda) |u_{+\infty}^S(\lambda)\rangle \end{aligned}, \quad (32)$$

where r^{pp} , r^{ps} , r^{sp} and r^{ss} are the reflection coefficients of the stack of layer including the free surface. They are uniquely specified by the boundary conditions and the form of the solutions in the half-space. For $\text{Im}\lambda = 0$, $\text{Re}\lambda > \alpha_\infty^2 k^2$, the solutions (32) behave like propagating P and S waves incident from $+\infty$ together with their reflections. They are

solutions of the wave equation with infinite energy that verify the free surface condition. For $\text{Im}\lambda \neq 0$, we remark that because the matrix elements of \mathbf{L} are real, the following relation holds: $\mathbf{L} |u_0^P(\lambda)^*\rangle = \lambda^* |u_0^P(\lambda)\rangle = \mathbf{L} |u_0^P(\lambda^*)\rangle$. Using equation (17), which is still valid in the stratified case, the following symmetry relations are established:

$$|u_0^{P,S}(\lambda^*)\rangle = -|u_0^{P,S}(\lambda)\rangle, \quad (33)$$

$$r^{ab}(\lambda^*) = r^{ab}(\lambda)^*, \quad (34)$$

where $a, b = p, s$. In addition, using the method of propagation invariants developed in Ref.17, we can prove the following reciprocity relation:

$$r^{ps}(\lambda) = r^{sp}(\lambda). \quad (35)$$

To construct the Green matrix we again make use of the properties (1)-(3) above. Condition (1) is replaced by the requirement that the columns of G be linear combinations of the vectors $|u_0^P\rangle, |u_0^S\rangle$ for $z < z'$. This suggests to try the following form for $\mathbf{G}(\lambda)$:

$$\mathbf{G}(\lambda) = \begin{cases} K_1 |u_0^P(\lambda)\rangle \langle u_{+\infty}^P(\lambda)^*| + K_2 |u_0^S(\lambda)\rangle \langle u_{+\infty}^S(\lambda)^*| & z < z' \\ K_1 |u_{+\infty}^P(\lambda)\rangle \langle u_0^P(\lambda)^*| + K_2 |u_{+\infty}^S(\lambda)\rangle \langle u_0^S(\lambda)^*| & z > z' \end{cases}, \quad (36)$$

where use has been made of equation (34). In essence, equation (36) is completely similar to equation (19): the Green function is separated into a “ P ” and an “ S ” wave part, although the vectors $|u_0^P\rangle, |u_0^S\rangle$ are neither purely longitudinal nor purely transverse, due to the mode conversions at the medium boundaries. We must now determine the values of K_1 and K_2 in order to fulfill conditions (4)-(5). When the source is located in the half-space $z > 0$, we can make use of the analytical form of the vectors $|u_0^{P,S}\rangle$ and $|u_{+\infty}^{P,S}\rangle$. Following the same steps as in the homogeneous case and employing the reciprocity relation (35) one obtains:

$$K_1(\lambda) = K_2(\lambda) = \frac{i}{\sqrt{\lambda}}. \quad (37)$$

Let us now show that this expression is valid throughout the stack of layers. The key point is to verify conditions (4)-(5). Let us first introduce the following 4-dimensional displacement-stress vectors :

$$w_{\infty}^{P,S}(\lambda, z) = \begin{pmatrix} u_{+\infty}^{P,S}(\lambda, z) \\ \tau[u_{+\infty}^{P,S}(\lambda, z)] \end{pmatrix}, \quad (38)$$

and

$$w_0^{P,S}(\lambda, z) = \begin{pmatrix} u_0^{P,S}(\lambda, z) \\ \tau[u_0^{P,S}(\lambda, z)] \end{pmatrix}. \quad (39)$$

The conditions (4)-(5) will be satisfied provided the following relation holds at arbitrary depth z in the medium:

$$\begin{aligned} \frac{i}{\sqrt{\lambda}} (w_{\infty}^P(\lambda, z)w_0^P(\lambda, z)^t - w_0^P(\lambda, z)w_{\infty}^P(\lambda, z)^t \\ + w_{\infty}^S(\lambda, z)w_0^S(\lambda, z)^t - w_0^S(\lambda, z)w_{\infty}^S(\lambda, z)^t) = N, \end{aligned} \quad (40)$$

where we have introduced the notations :

$$N = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (41)$$

Relation (40) is verified in the half-space $z > 0$ as discussed above. The displacement-stress vector satisfies a first-order linear system $dw(\lambda, z)/dz = A(\lambda, z)w(\lambda, z)$, equivalent to the second order system (3) , which can be integrated with the aid of a propagator matrix:

$$w(\lambda, z) = P(\lambda, z, z')w(\lambda, z'). \quad (42)$$

Since the propagator matrix has the symplectic symmetry¹⁰:

$$P(\lambda, z, z')NP(\lambda, z, z')^t = N, \quad (43)$$

relation (40) can be propagated at any depth in the stratified half-space by left and right multiplication of equation (40) by P and its transpose, respectively. Note that in the case of a stratified medium, the symmetry relation (43) can be verified by direct but tedious calculations.

B. Singularities of the resolvent

The next step is to reexamine the discontinuity of the resolvent (36) across the real axis. This task is significantly facilitated by the unitary and symmetry properties of the reflection coefficients. As shown in Ref.17, the energy normalization of the eigenvectors is of fundamental importance in this respect. Other normalizations are of course possible but they lead to much more awkward calculations. We will denote by $|u(\lambda^+)\rangle$ - $(|u(\lambda^-)\rangle)$ the limiting value of the ket $|u(\lambda)\rangle$ as λ approaches the real axis from above-(below). The interrelations among the kets $|u_{+\infty}^{P,S}\rangle$, $|u_{-\infty}^{P,S}\rangle$ across the real axis determine the form of the resolvent discontinuity. These relations, in turn, depend essentially on the analytical forms of these kets, i.e., on whether the P and S waves are propagating or evanescent in the half-space $z > 0$. As in the homogenous case, this leads to a separation of the real axis into three distinct domains, as shown in Figure 1.

(1) For propagating P and S waves: $\lambda > \alpha^2 k^2$, the following relations apply:

$$\begin{aligned} |u_{\pm\infty}^P(\lambda^+)\rangle &= |u_{\mp\infty}^P(\lambda^+)^*\rangle & |u_{\pm\infty}^S(\lambda^+)\rangle &= |u_{\mp\infty}^S(\lambda^+)^*\rangle \\ |u_{\pm\infty}^P(\lambda^-)\rangle &= -|u_{\pm\infty}^P(\lambda^+)^*\rangle & |u_{\pm\infty}^S(\lambda^-)\rangle &= -|u_{\pm\infty}^S(\lambda^+)^*\rangle \end{aligned} \quad (44)$$

These relations, together with the analytical properties of the reflection coefficients, enable us to write the kets $|u_0^{P,S}(\lambda^-)\rangle$ in terms of the kets $|u_{\pm\infty}^{P,S}(\lambda^+)\rangle$ only:

$$\begin{aligned} |u_0^P(\lambda^-)\rangle &= -|u_{+\infty}^P(\lambda^+)\rangle - \overline{r^{pp}} |u_{-\infty}^P(\lambda^+)\rangle - \overline{r^{ps}} |u_{-\infty}^S(\lambda^+)\rangle, \\ |u_0^S(\lambda^-)\rangle &= -|u_{+\infty}^S(\lambda^+)\rangle - \overline{r^{ss}} |u_{-\infty}^S(\lambda^+)\rangle - \overline{r^{sp}} |u_{-\infty}^P(\lambda^+)\rangle, \end{aligned} \quad (45)$$

where the reflection coefficients are defined by their limiting value as λ approaches the real axis from above. For notational convenience, we have denoted the complex conjugation with an overbar. When both P and S waves are propagating in the half-space, the matrix of reflection coefficients is unitary¹⁷:

$$R = \begin{pmatrix} r^{pp} & r^{ps} \\ r^{sp} & r^{ss} \end{pmatrix}, \quad RR^\dagger = E. \quad (46)$$

For $z < z'$, the discontinuity of the resolvent writes:

$$\mathbf{G}^+ - \mathbf{G}^- = \frac{i}{\sqrt{\lambda^+}} (|u_0^P(\lambda^+)\rangle \langle u_{-\infty}^P(\lambda^+)| + |u_0^S(\lambda^+)\rangle \langle u_{-\infty}^S(\lambda^+)| - |u_0^P(\lambda^-)\rangle \langle u_{+\infty}^P(\lambda^+)| - |u_0^S(\lambda^-)\rangle \langle u_{+\infty}^S(\lambda^+)|). \quad (47)$$

Using equations (45) and (46) and after some algebra, one obtains:

$$\mathbf{G}^+ - \mathbf{G}^- = \frac{i}{\sqrt{\lambda^+}} (|u_0^P(\lambda^+)\rangle \langle u_0^P(\lambda^+)| + |u_0^S(\lambda^+)\rangle \langle u_0^S(\lambda^+)|). \quad (48)$$

The same result applies for $z > z'$ and we have therefore put the discontinuity of the resolvent in suitable dyadic form. It appears that for $\lambda > \alpha^2 k^2$ there are two linearly independent eigenvectors. They are energy normalized incident P and S waves incident from $+\infty$ together with their reflections.

(2) For $\alpha^2 k^2 > \lambda > \beta^2 k^2$, there are propagating S and evanescent P waves in the half-space $z > 0$ and the following relations apply:

$$|u_{\pm\infty}^P(\lambda^+)\rangle = -i |u_{\pm\infty}^P(\lambda^+)^*\rangle \quad \text{and} \quad |u_{\pm\infty}^S(\lambda^+)\rangle = |u_{\mp\infty}^S(\lambda^+)^*\rangle, \quad (49)$$

which yields:

$$\begin{aligned} |u_0^P(\lambda^-)\rangle &= -i |u_{-\infty}^P(\lambda^+)\rangle - i \overline{r^{pp}} |u_{+\infty}^P(\lambda^+)\rangle - \overline{r^{ps}} |u_{-\infty}^S(\lambda^+)\rangle \\ |u_0^S(\lambda^-)\rangle &= - |u_{+\infty}^S(\lambda^+)\rangle - \overline{r^{ss}} |u_{-\infty}^S(\lambda^+)\rangle - i \overline{r^{sp}} |u_{+\infty}^P(\lambda^+)\rangle \end{aligned} \quad (50)$$

For propagating S and evanescent P waves in the half-space, the unitary relations take a more complicated form¹⁷: $\overline{r^{ss}} r^{sp} = i \overline{r^{sp}}$. With the aid of equation (49), this implies $\overline{r^{ss}} |u_0^S(\lambda^+)\rangle = - |u_0^S(\lambda^-)\rangle$. For $z < z'$, using the unitary relations, the resolvent discontinuity can be put into the following form:

$$\begin{aligned} \mathbf{G}^+ - \mathbf{G}^- &= - [i |u_0^P(\lambda^+)\rangle + |u_0^P(\lambda^-)\rangle] \langle u_{+\infty}^P(\lambda^+)| \\ &\quad + |u_0^S(\lambda^+)\rangle [\langle u_{-\infty}^S(\lambda^+)| + \overline{r^{ss}} \langle u_{+\infty}^S(\lambda^+)|] \end{aligned} \quad (51)$$

Using the symmetry relation $r^{pp} - \overline{r^{pp}} = i \overline{r^{sp}}$, the following relation is established:

$$i |u_0^P(\lambda^+)\rangle + |u_0^P(\lambda^-)\rangle = -\overline{r^{sp}} |u_0^S(\lambda^+)\rangle, \quad (52)$$

which in turn implies:

$$\mathbf{G}^+ - \mathbf{G}^- = \frac{i}{\sqrt{\lambda^+}} |u_0^S(\lambda^+)\rangle \langle u_0^S(\lambda^+)|. \quad (53)$$

This last equation has the desired form and shows that in the range of λ considered there is only one generalized eigenfunction. In the half-space, it consists of the sum of incident, reflected propagating S wave and a reflected evanescent P wave. Note that for some λ in the range $]\beta_\infty^2 k^2, \alpha_\infty^2 k^2[$, it may happen that the vector $|u_{+\infty}^P\rangle$ satisfies the zero traction condition at the free surface. Such a situation can be considered as accidental, since a slight modification of the thickness or velocity in one the layers is likely to make the mode disappear (Y. Colin de Verdière, personal communication) . Such possible complications will be neglected.

(3) For $\beta^2 k^2 > \lambda$, both P and S waves are evanescent in the half-space $z > 0$. In this case the reflection matrix R is simultaneously symmetric and hermitian, so that all its coefficients are real. Further, using the relations:

$$|u_{\pm\infty}^P(\lambda^-)\rangle = -i |u_{\pm\infty}^P(\lambda^+)\rangle \quad \text{and} \quad |u_{\pm\infty}^S(\lambda^-)\rangle = -i |u_{\pm\infty}^S(\lambda^+)\rangle, \quad (54)$$

one establishes:

$$|u_0^{P,S}(\lambda^-)\rangle = -i |u_0^{P,S}(\lambda^+)\rangle, \quad (55)$$

which seems to implies that the resolvent discontinuity vanishes identically. This is not true as there can exist values of λ for which the reflection coefficients have poles. For these particular points of the spectrum, the kets $|u_0^{P,S}(\lambda^+)\rangle$ can be written solely as linear combinations of the kets $|u_{+\infty}^{P,S}(\lambda^+)\rangle$ ¹⁰. This means that there exist eigenvectors with finite energy that satisfy the traction free condition, which are the well-known Rayleigh surface waves. To each pole, we can associate a one dimensional eigenspace spanned by a normalized Rayleigh wave eigenvector. For a given value of the horizontal wavenumber k , there can be n distinct solutions denoted by $\lambda_0 < \lambda_1 < \dots < \lambda_n$. The index serves to label the different surface waves mode branches. The index 0 refers to the fundamental Rayleigh mode.

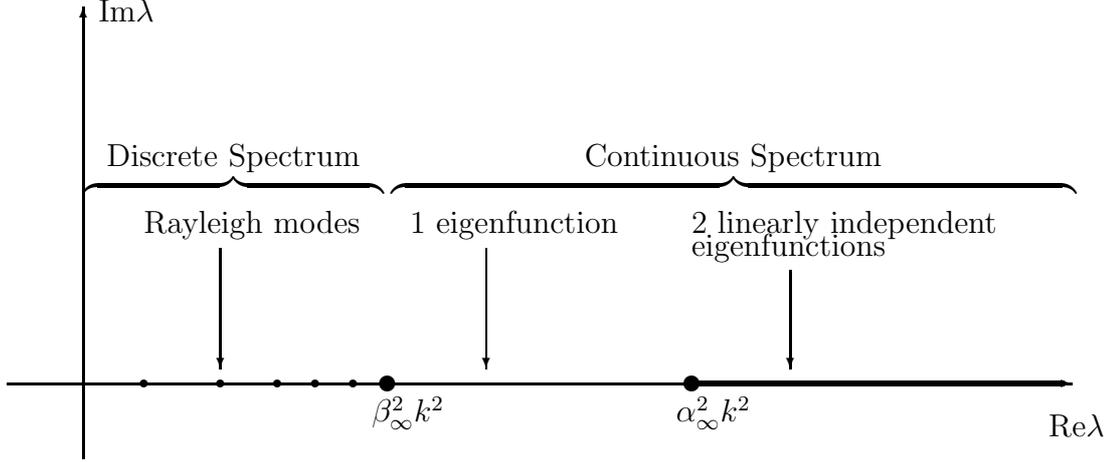


FIG. 1. Spectral properties of the elastodynamic operator in a layered half-space. λ is the eigenparameter.

C. Eigenfunction expansions

We have now identified all the singularities of the resolvent on the real axis and are in position to write down the completeness relation for 2-D in-plane problems using formula (10). In the frequency domain, introducing the new variable $\omega^2 = \lambda^+$, one obtains:

$$\begin{aligned} \mathbf{I} = & \int_{-\infty}^{+\infty} dk \sum_n |e^R(\omega_n)\rangle \langle e^R(\omega_n)| + \int_{-\infty}^{+\infty} dk \int_{|\beta_\infty k|}^{|\alpha_\infty k|} d\omega |e^S(\omega)\rangle \langle e^S(\omega)| \\ & + \int_{-\infty}^{+\infty} dk \int_{|\alpha_\infty k|}^{+\infty} d\omega (|e^S(\omega)\rangle \langle e^S(\omega)| + |e^P(\omega)\rangle \langle e^P(\omega)|), \end{aligned} \quad (56)$$

where the compound generalized eigenvectors $|e^{S,P,R}\rangle$ are defined as:

$$\langle r|e^{S,P}(\omega)\rangle = \sqrt{2} \frac{e^{ikx}}{\sqrt{2\pi}} \langle z|u_0^{S,P}(\omega^2)\rangle \quad \text{and} \quad \langle r|e^R(\omega)\rangle = \frac{e^{ikx}}{\sqrt{2\pi}} \langle z|u^R(\omega_n)\rangle. \quad (57)$$

In the first of equations (57), the $\sqrt{2}$ factor is the result of introducing of the new variable ω , whereas the last equation defines properly surface waves with unit-energy normalized z -eigenfunction. Although this is implicit in equation (57), the reader should keep in mind that all eigenvectors are functions of the horizontal wavenumber k . This applies to the eigenfrequencies ω_n as well as the number of branches. As in the case of the homogeneous

space, it is possible to replace the frequency integral by a vertical wavenumber integral in equation (56) to obtain:

$$\begin{aligned} \mathbf{I} = & \int_{-\infty}^{+\infty} dp_x \sum_n |\psi^R(\omega_n)\rangle \langle \psi^R(\omega_n)| + \int_{-\infty}^{+\infty} dp_x \int_{-\infty}^0 dp_z |\psi^S(p_x, p_z)\rangle \langle \psi^S(p_x, p_z)| \\ & + \int_{-\infty}^{+\infty} dp_x \int_{-\infty}^0 dp_z |\psi^P(p_x, p_z)\rangle \langle \psi^P(p_x, p_z)|. \end{aligned} \quad (58)$$

The eigenvectors $|\psi^{P,S,R}\rangle$ are obtained by application of the transformation U^\dagger to the kets $|e^{S,P,R}\rangle$. The eigenvectors $|\psi^{P,S}\rangle$ correspond to properly normalized incoming P and S waves exactly as given in equations (30)-(31) together with their reflections from the stack of layers. This is a general result in scattering theory: the eigenvectors of the medium+scatterer can be obtained by calculating the complete response of the scatterer to the unperturbed eigenvectors^{8,18}. This is basically what the double integrals of equation (58) say. Note that by calculating the scattering of plane waves, one does not obtain directly the surface waves. They show up as poles of the reflection coefficient located on the real axis and, in that sense, can be compared to the bound states in scattering theory.

V. APPLICATION TO DIFFUSE FIELDS

A diffuse field is defined as a random field where all the modes are equally represented. More precisely, according to Ref.1, it is a narrow-band signal which is a sum of normal modes of the system excited randomly at equal energy:

$$\begin{aligned} u(\mathbf{x}, t) = & \sum_{n,R,L} \iint_{\omega_n \in B} \frac{dp_x dp_y}{2\pi} A_n^{R,L}(p_x, p_y) \psi_n^{R,L}(z) e^{i(p_x x + p_y y)} e^{-i\omega_n^{R,L}(p_x, p_y)t} \\ & + \int_{p_z < 0} dp_z \sum_{P,S} \iint_{\omega \in B} \frac{dp_x dp_y}{2\pi} A^{P,S}(p_x, p_y, p_z) \psi_{p_z}^{P,S}(z) e^{i(p_x x + p_y y)} e^{-i\omega^{P,S}(p_x, p_y, p_z)t}. \end{aligned} \quad (59)$$

In equation (59), B denotes the frequency band of the signal and u refers to the complex analytic signal whose real part is the measured displacement vector. In a diffuse field, the

amplitudes $A_n^{R,L}$ and $A^{P,S}$ are assumed to be white-noise random processes:

$$\begin{aligned} \langle A_n^{R,L} A^{P,S} \rangle &= 0, \\ \langle A_n^{R,L}(p_x, p_y) A_m^{R,L}(p'_x, p'_y)^* \rangle &= \sigma^2 \delta^{R,L} \delta_{nm} \delta(p_x - p'_x) \delta(p_y - p'_y), \\ \langle A^{P,S}(p_x, p_y, p_z) A^{P,S}(p'_x, p'_y, p'_z)^* \rangle &= \sigma^2 \delta^{P,S} \delta(p_z - p'_z) \delta(p_x - p'_x) \delta(p_y - p'_y). \end{aligned} \quad (60)$$

In the case of a semi-infinite medium, the sum over all modes involves a continuous index that labels the eigenfunctions of the continuous spectrum and a discrete index that refers to the surface wave mode branch.

Formulas (59)-(60) are now applied to the calculation of the vertical to horizontal kinetic energy ratio of a diffuse field in a stratified half-space. The vertical (or horizontal) kinetic energy density at central frequency ω_0 is obtained by means of the Wigner distribution of the complex analytic wavefield:

$$E_z(t, \tau, \mathbf{x}) = \frac{1}{2} \rho(z) \langle \partial_t u_z(t + \tau/2, \mathbf{x}) \partial_t u_z(t - \tau/2, \mathbf{x})^* \rangle. \quad (61)$$

In equation (61), the brackets denote an ensemble average. Inserting the spectral decomposition (59), applying the equipartition principle (60), and taking a Fourier transform with respect to the time variable τ , the local vertical kinetic energy density of a diffuse field at circular frequency ω_0 is obtained:

$$\begin{aligned} E_z(\omega_0, z) &= \frac{\rho(z) \omega_0^2 \sigma^2}{8\pi^2} \sum_{n,R,L} \iint_{\mathbb{R}^2} dp_x dp_y |\psi_{n,z}^{R,L}(z)|^2 \delta(\omega_0 - \omega_n^{R,L}(p_x, p_y)) \\ &+ \frac{\rho(z) \omega_0^2 \sigma^2}{8\pi^2} \int_{p_z < 0} dp_z \sum_{P,S} \iint_{\mathbb{R}^2} dp_x dp_y |\psi_{p_z,z}^{P,S}(z)|^2 \delta(\omega_0 - \omega^{P,S}(p_x, p_y, p_z)). \end{aligned} \quad (62)$$

In equation (62), the delta functions represent the density of states of the surface and body waves, which are closely related to the imaginary part of the Green's function¹⁹. The link between the Wigner distribution of the wavefield and the Green's function is the theoretical basis of the Green's function reconstruction in diffuse fields^{18,20,21}. Introducing cylindrical and spherical coordinates in the double and triple integrals of equation (62), respectively,

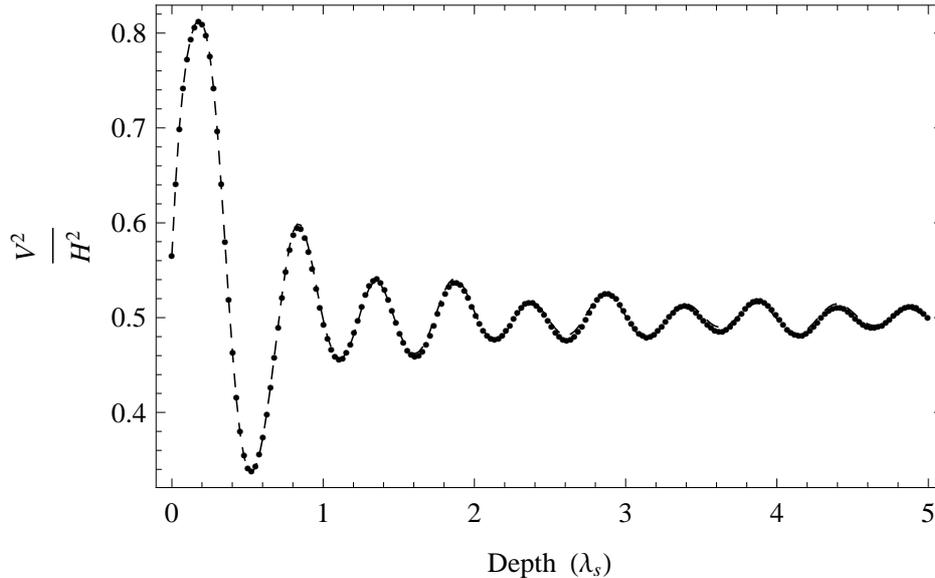


FIG. 2. Depth dependence of the vertical to horizontal kinetic energy ratio near the free surface of a Poisson half-space. Dots: locked-mode approximation. Dashed line: generalized eigenfunctions summation. The depth unit is the shear wavelength λ_s

one finds:

$$\begin{aligned}
 E_z(\omega_0, z) = & \sum_{n,R,L} \frac{\rho(z)\omega_0^3\sigma^2}{4\pi c_n^{R,L} u_n^{R,L}} |\psi_{n,z}^{R,L}(z)|^2 \\
 & + \sum_{P,S} \frac{\rho(z)\omega_0^4\sigma^2}{4\pi v_{p,s}^3} \int_{\pi/2}^{\pi} d\theta \sin\theta |\psi_{p_z,z}^{P,S}(z)|^2 \Big|_{p_z = \frac{\omega_0}{v_{p,s}} \cos\theta}.
 \end{aligned} \tag{63}$$

In equation (63), we have introduced the following notations: c_n and u_n are the phase and group velocities of the n^{th} surface wave mode, respectively; v_p -(v_s) stands for α_∞ -(β_∞). The calculation of the eigenfunctions and the remaining sum and integral have to be performed numerically.

In Figure (2), we consider the vicinity of a free surface which has been previously investigated by several authors^{2,6,7}. The calculations were performed both with formula (63) and with a locked-mode approximation where the medium is bounded at great depth by a rigid boundary where the displacements vanish exactly. In the latter case, the eigenvalue problem in the depth coordinate only has a discrete spectrum, which is standard. For more

details on the locked-mode technique and geophysical applications, the reader is referred to Ref.22. The outcome of the two calculations for a 3-D medium are superposed in Figure 2. In the locked-mode technique, the lower boundary is at a depth of 16 shear wavelengths. A total of 32 Love and 50 Rayleigh modes were found. The agreement is very satisfactory and confirms the validity of our approach. Our work shows that the generalized eigenfunctions of the continuum can be treated like standard normal modes, albeit with a continuous index. The results presented in Figure (2) were first obtained in Refs.2,7, using the eigenfunctions of a thick plate, the so-called Lamb modes. Our calculations illustrate the fact that the elastodynamic operator is in the limit point case at $+\infty$ according to the classification of Ref.23. This simply means that independent of the self-adjoint boundary condition imposed at the lower boundary -Neumann, Dirichlet or mixed-, the eigenfunctions converge to a common limit as the depth of that boundary tends to ∞ . A few wavelengths away from the free surface, the kinetic energy ratio oscillates around 0.5, which is the expected value in a homogeneous half-space.

We now investigate the case of a soft layer overlying a homogeneous space. In the layer, the P and S wave velocity is one third smaller than in the half-space and the density is reduced by a factor 2. The vertical to horizontal kinetic energy ratio is plotted as a function of depth in Figure 3. The unit depth is the shear wavelength inside the layer and the interface between the layer and the half-space lies at about 0.2 unit depth. In the locked-mode approximation, two situations were considered where the rigid boundary is located at 5.25 and 42 wavelengths below the bottom of the layer. In the former case, 7 Love and 11 Rayleigh modes were found, while in the latter case we identified a total 56 Love and 88 Rayleigh modes. In the generalized eigenfunction expansion method, only the fundamental Love and Rayleigh modes are present. Thus, in the locked-mode method, the higher Love and Rayleigh modes serve as an approximation for the propagating P and S waves incident from below the layer. As illustrated in Figure 3, the agreement between the two methods is extremely good and as expected, improves as the depth of the rigid boundary increases. In the vicinity of the layer, the vertical to horizontal kinetic energy ratio shows rapid variations

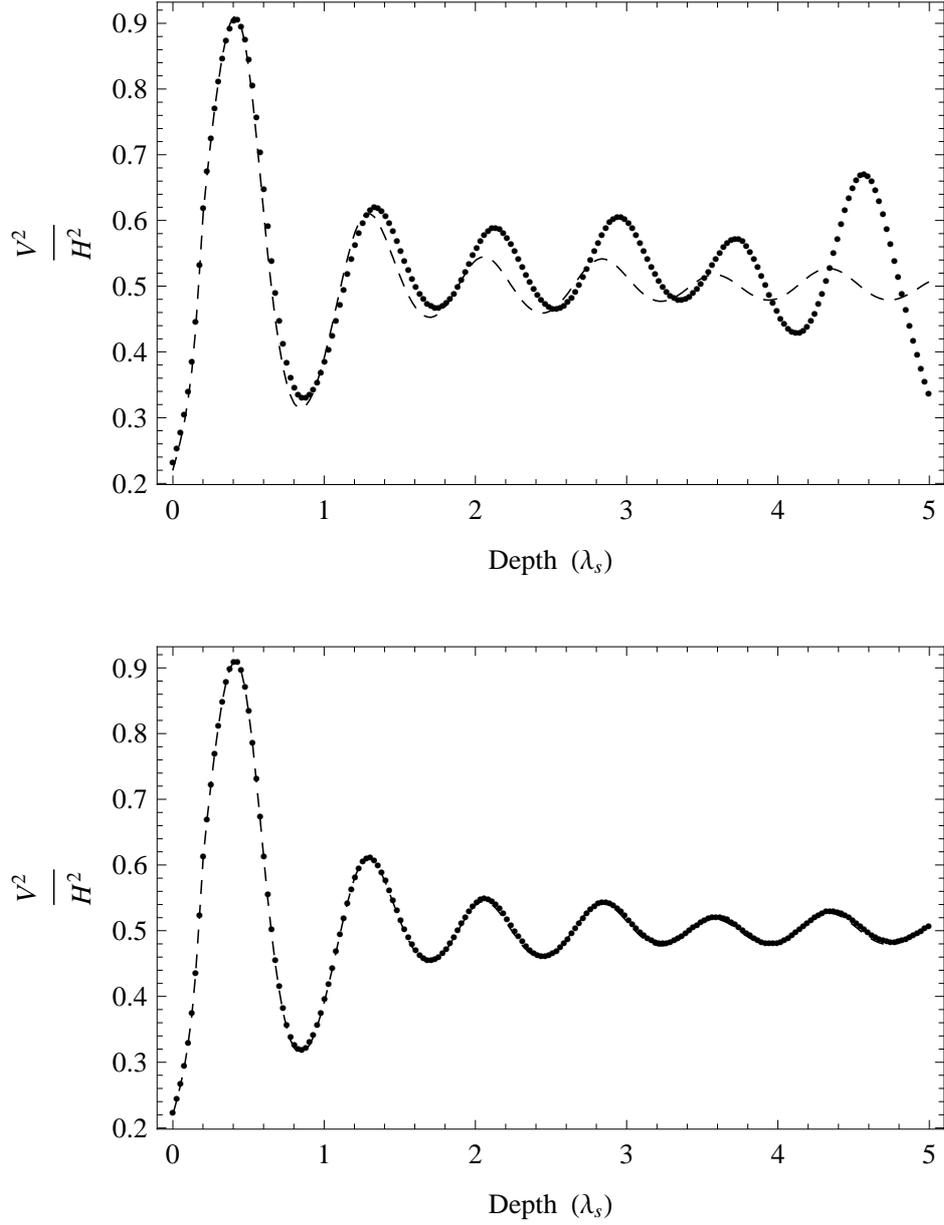


FIG. 3. Depth dependence of the vertical to horizontal kinetic energy ratio in the presence of a soft layer. Dots: locked-mode approximation. Dashed line: generalized eigenfunctions summation. The depth unit is the shear wavelength λ_s . In the locked mode approximation, the lower boundary is located at a depth of $5.25\lambda_s$ (Top) and $42\lambda_s$ (Bottom), respectively.

and, at greater depth, converges towards the ratio of a homogeneous space.

In Figure 4, we investigate the frequency dependence of the V^2/H^2 kinetic energy ratio at the surface of a solid with a soft layer at the surface, where the shear velocity and density are reduced by a factor 2 with respect to the underlying Poisson half-space. The ratio of longitudinal to shear wavespeeds in the layer is taken to be 2.5. To facilitate the interpretation of the V^2/H^2 calculations, we show in the Top panel the normalized density of states of body, Rayleigh and Love waves as a function of frequency. The frequency unit is the fundamental resonance frequency of the layer for vertically propagating shear waves, $f_0 = \beta/4h$, where β is the shear wavespeed and h is the layer thickness. In Figure 4, we have also indicated high- and low-frequency asymptotics of the V^2/H^2 energy ratio. At high frequency, we expect the waves to be largely insensitive to the properties of the underlying half-space. We therefore calculate an approximate high-frequency V^2/H^2 ratio by replacing the true model with a simple half-space where the P to S velocity ratio equals 2.5. The expected value is 0.513, in good agreement with the full calculations. At low-frequency, we can neglect the presence of the shallow layer to recover the classical 0.56 ratio at the surface of a Poisson solid. The V^2/H^2 ratio is slightly smaller at high frequencies mainly because of the increased amount of horizontally polarized shear waves. To the contrary, the V^2/H^2 energy ratio of the Rayleigh wave tends to increase with the P to S velocity ratio. Close to the resonance frequency f_0 of the layer, the vertical to horizontal kinetic energy ratio drops dramatically. Careful analysis reveals that this is caused by the increasing contribution of the fundamental Love mode to the density of states at the surface. Around twice the resonance frequency, the V^2/H^2 ratio presents a marked overshoot which is due to the appearance of Rayleigh waves higher modes (i.e. body waves trapped in the low velocity layer) which are preferentially polarized on the vertical axis. At high frequencies, the energy density is largely dominated by the Rayleigh and Love waves trapped in the low-velocity layer. The fluctuations of the local density of states slowly decrease with increasing frequency.

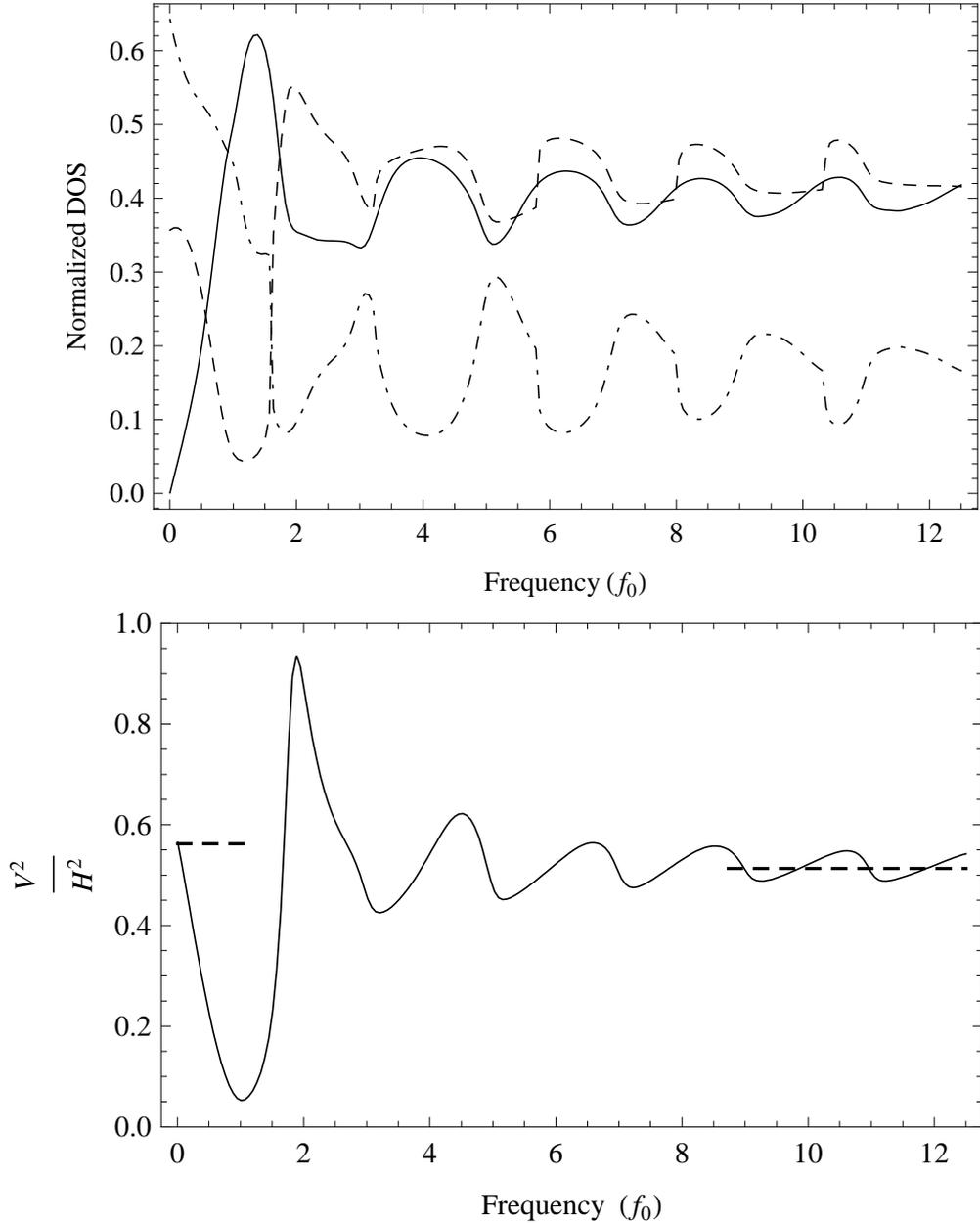


FIG. 4. Properties of a diffuse field at the surface of a half-space with a superficial low-velocity, low-density layer. Top: Normalized density of states of Rayleigh (dashed line), Love (solid line) and bulk waves (dash-dotted line) as a function of normalized frequency. The resonance frequency f_0 of vertically propagating S waves in the layer is the unit frequency. Bottom: Frequency dependence of the vertical to horizontal kinetic energy ratio. Dashed lines: low- and high-frequency asymptotics. Solid line: generalized eigenfunctions summation.

VI. CONCLUSION

In this work, we have shown that the definition of a diffuse field as a white noise in modal space can be applied equally well to closed¹ and open systems²⁴. In the framework of the spectral theory presented in this paper, discrete, continuous or mixed spectra can be treated on the same footing. Other applications of the generalized normal mode expansion could be envisaged, such as the calculation of synthetic seismograms for geophysical applications¹⁵. The theory could also be used to give more rigorous foundations to empirical civil engineering techniques. In particular, the large drop of the vertical to horizontal kinetic energy ratio in diffuse fields close to the resonance frequency of a low-velocity layer sounds reminiscent of the so-called Nakamura's method used for site effect evaluation with ambient noise²⁵. Although extremely popular, the limitations of the technique are still to be understood. The diffuse field concept offers a potentially useful tool for this purpose.

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I would like to express my gratitude to Pr. Y. Colin de Verdière for his lectures on mathematical topics closely related to the present article. This paper has also been influenced by discussions with Dr. H. Nakahara, who suggested to examine the wave content of diffuse fields.

APPENDIX A: SUMMARY OF DIRAC FORMALISM

In this appendix, we summarize the bra-ket notations used in this paper. A vector or ket of an abstract vector space will be denoted by: $|e\rangle$. The representation of the vector in position space is written as: $\langle x|e\rangle = e(x)$. To each ket of our space of function, we associate a bra denoted by $\langle e| = |e\rangle^\dagger$. In position space, a bra has representation $\langle e|x\rangle = e(x)^*$, where $*$ denotes complex conjugation. Mathematically speaking, a bra would be more appropriately defined as a linear functional acting on a space of test functions. However, we will not insist on these technicalities. We shall also make use of the conjugated versions of kets such that:

$\langle x|e^*\rangle = e(x)^*$. The scalar product between two vectors $|e\rangle$ and $|f\rangle$ is denoted by

$$\langle e|f\rangle = \int dx \rho(x) e_i(x)^* f_i(x), \quad (\text{A1})$$

where a summation over the repeated index i is implied. The completeness relation or resolution of the identity for a set of eigenvectors $|e_p\rangle$ is written as:

$$\sum_p |e_p\rangle \langle e_p| = \mathbf{I}, \quad (\text{A2})$$

where \mathbf{I} denotes the identity in the abstract vector space, and p denotes a label running over the whole set of eigenfunctions. Equation (A2) introduces the outer product between a bra and a ket. The outer product of two (properly normalized) eigenvectors is an orthogonal projector on the subspace generated by $|e\rangle$. In the position representation, equation (A2) reads

$$\sum_p \langle x|e_p\rangle \langle e_p|x'\rangle = \delta_{ij} \frac{\delta(x-x')}{\rho(x')}. \quad (\text{A3})$$

The appearance of the weighting function ρ in the denominator is consistent with the definition of the scalar product (A1). The matrix elements in position and polarization space of a general outer product between a ket $|e\rangle$ and a bra $\langle f|$ are given by

$$\langle x|e\rangle \langle f|x'\rangle = e_i(x) f_j(x')^*. \quad (\text{A4})$$

The matrix elements of an abstract operator \mathbf{L} in position space are given by the kernel of an integral operator. If \mathbf{L} represents a differential operator, it will be assumed to be diagonal in position space.

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