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TOWARDS NEW SCHEMES: A LIE-GROUP APPROACH OF THE CBKDV AND ITS DERIVED EQUATIONS

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ABSTRACT. The aim of this paper is to propose methods that enable us to build new numerical schemes, which preserve the Lie symmetries of the original differential equations. To this purpose, the compound Burgers-Korteweg-de Vries (*CBKDV*) equation is considered. The particular case of the Burgers equation is taken as a numerical example, and the resulting semi-invariant scheme is exposed.

1. Introduction. Finite difference equations used to approximate the solutions of a differential equation generally do not respect the symmetries of the original equation, and can lead to inaccurate numerical results. Usually, specific equations are considered, for which the authors build a scheme preserving the symmetries of the original differential equation. Yet, it is more interesting to directly consider a class of differential equations, in order to obtain more general results.

Using the work of Yanenko [4] and Shokin [3], who applied the Lie group theory to finite difference equations by means of the differential approximation, in conjunction with the approach of Ames et al. [8], we generalize results developed in [1], [2], and expose the invariance condition for a differential approximation of the *CBKDV* equation. A numerical example, in the particular case of the Burgers equation, is developed.

2. Notion of symmetry.

2.1. Definitions.

Definition 2.1. A r -parameter Lie group is a r -dimensional smooth manifold G_r , which has the group properties, such that the group operation of multiplication and inversion are smooth maps.

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Especially, a Lie group G_r is defined as a group of continuous transformations which act on an open subset of the Euclidean space \mathbb{R}^k of variables, which change under the action of G_r . We presently concentrate on a local group, the transformations of which are close to the identity transformation.

Definition 2.2. A r -parameter Lie group G_r is a group of point transformations, which acts on $X \times U$, the space of the independent variables and the dependent ones:

$$G_r = \{x_i^* = \phi_i(x, u, a); u_j^* = \varphi_j(x, u, a), i = 1, \dots, m; j = 1, \dots, n; a = (a_1, \dots, a_r)\} \quad (1)$$

where $x \in X \subset \mathbb{R}^m$ and $u \in U \subset \mathbb{R}^n$.

G_r locally satisfies the group axioms: existence of an identity element, associativity, inversibility, closure under the binary composition operation. The transformation corresponding to a zero parameter is the identity transformation. Expand the transformations by means of a Taylor series at the zero value of the parameter a :

$$\begin{aligned} x_i^* &= x_i + a_\alpha \left. \frac{\partial \phi_i}{\partial a_\alpha} \right|_{a=0} + \mathcal{O}(a_\alpha^2), \quad \alpha = 1, \dots, r \\ u_j^* &= u_j + a_\alpha \left. \frac{\partial \varphi_j}{\partial a_\alpha} \right|_{a=0} + \mathcal{O}(a_\alpha^2), \quad \alpha = 1, \dots, r \end{aligned} \quad (2)$$

The derivatives of ϕ_i and φ_j with respect to the parameter a_α are smooth functions, called *infinitesimals of the group G_r* . Denote by ξ_i^α and η_j^α the infinitesimals of G_r . The point transformation group G_r can be represented by means of the operator \mathbf{L}_α :

$$\mathbf{L}_\alpha = \xi_i^\alpha(x, u) \frac{\partial}{\partial x_i} + \eta_j^\alpha(x, u) \frac{\partial}{\partial u_j}, \quad i = 1, \dots, m; j = 1, \dots, n; \alpha = 1, \dots, r \quad (3)$$

The operators \mathbf{L}_α , $\alpha = 1, \dots, r$ are called the *infinitesimal operators* of G_r . $\{\mathbf{L}_\alpha, \alpha = 1, \dots, r\}$ represents the set of tangent vectors to the manifold G_r , when the zero value is assigned to the parameter a . The set is a basis of the Lie-algebra of the infinitesimal operators of G_r , the dimension of which is the same as the one of the Lie group G_r .

The knowledge of the \mathbf{L}_α enables us to determine the point transformations of the group G_r by solving the equations:

$$\frac{\partial x_i^*}{\partial a_\alpha} = \xi_i^\alpha(x^*, u^*), \quad \frac{\partial u_j^*}{\partial a_\alpha} = \eta_j^\alpha(x^*, u^*), \quad i = 1, \dots, m; j = 1, \dots, n; \alpha = 1, \dots, r \quad (4)$$

in conjunction with the initial conditions:

$$x_i^*|_{a=0} = x_i; \quad u_j^*|_{a=0} = u_j \quad (5)$$

Example 2.3. The Galilean transformation group

The Galilean transformations correspond to time dependent translations of a reference frame :

$$G = \{\mathcal{T} : (x, t, u) \mapsto (x^*, t^*, u^*) = (x + \epsilon t, t, u + \epsilon)\}$$

where ϵ is the translation constant velocity.

\mathcal{T} and its inverse function are continuous. The infinitesimals functions of the group are $(\xi_x^\epsilon, \xi_t^\epsilon, \eta_u^\epsilon) = (t, 0, 1)$

2.2. Symmetry properties of differential equations. The notion of symmetry is a tool for generating new solutions of differential equations. Let us review the main aspects of the application of the Lie group theory to differential equations. Consider a system of l^{th} -order differential equations:

$$\mathcal{F}^\lambda(x, u, u^{(k_1)}, u^{(k_1, k_2)}, \dots, u^{(k_1 \dots k_l)}) = 0, \quad \lambda = 1, \dots, q \quad (6)$$

Denote by $u^{(k_1 \dots k_p)}$ the vector, the components of which are partial derivatives of order p , namely, $u_j^{(k_1 \dots k_p)} = \frac{\partial^p u_j}{\partial x_{k_1} \dots \partial x_{k_p}}$ $j = 1, \dots, n$ and $k_1, \dots, k_p \in \{1, \dots, m\}$.

Denote by $x = (x_1, \dots, x_m)$ the independent variables, $u = (u_1, \dots, u_n)$ the dependent variables, and $(x_{k_1} \dots x_{k_p})$ a set of elements of the independent variables.

Equation (6) is a subset of $X \times U^{(l)}$, a prolongation of the space $X \times U$ to the space of the partial derivatives of u with respect to x up to order l . $X \times U^{(l)}$, which is a smooth manifold, is called the l -th order jet space of $X \times U$. In order to take into account the derivative terms involved in the differential equation, the action of G_r on $X \times U$ needs to be prolonged to the space of the derivatives of the dependent variables.

Denote by $\tilde{G}_r^{(l)}$ a r -parameter Lie group of point transformations acting on an open subset $M^{(l)}$ of the l -th order jet space $X \times U^{(l)}$ of the independent variables x , dependent variables u and the partial derivatives of u with respect to x .

The l^{th} -prolongation operator of G_r is:

$$\tilde{\mathbf{L}}_\alpha^{(l)} = \xi_i^\alpha(x, u) \frac{\partial}{\partial x_i} + \eta_j^\alpha(x, u) \frac{\partial}{\partial u_j} + \sigma_j^{\alpha, (k_1)} \frac{\partial}{\partial u_j^{(k_1)}} + \dots + \sigma_j^{\alpha, (k_1 \dots k_l)} \frac{\partial}{\partial u_j^{(k_1 \dots k_l)}}, \quad (7)$$

$i = 1, \dots, m; j = 1, \dots, n; \alpha = 1, \dots, r.$

The infinitesimal functions ξ_i^α , η_j^α , $\sigma_j^{\alpha, (k_1)}$ and $\sigma_j^{\alpha, (k_1 \dots k_o)}$ are given by:

$$\begin{aligned} \xi_i^\alpha &= \frac{\partial \phi_i}{\partial a_\alpha} \Big|_{a=0}, \quad \eta_j^\alpha = \frac{\partial \varphi_j}{\partial a_\alpha} \Big|_{a=0}, \quad \sigma_j^{\alpha, (k_1)} = \frac{\mathcal{D} \eta_j^\alpha}{\mathcal{D} x_{k_1}} - \sum_{i=1}^m \frac{\partial u_j}{\partial x_i} \frac{\mathcal{D} \xi_i^\alpha}{\mathcal{D} x_{k_1}} \\ \sigma_j^{\alpha, (k_1 \dots k_o)} &= \frac{\mathcal{D} \sigma_j^{\alpha, (k_1 \dots k_{o-1})}}{\mathcal{D} x_{k_o}} - \sum_{i=1}^m \frac{\partial^o u_j}{\partial x_i \partial x_{k_1} \dots \partial x_{k_{o-1}}} \frac{\mathcal{D} \xi_i^\alpha}{\mathcal{D} x_{k_o}}, \quad o = 2, \dots, l \end{aligned} \quad (8)$$

where: $\frac{\mathcal{D}}{\mathcal{D} x_k} = \frac{\partial}{\partial x_k} + \sum_{j=1}^n \frac{\partial u_j}{\partial x_k} \frac{\partial}{\partial u_j}$

Theorem 2.4. *The system of l^{th} -order differential equations is invariant under the group $\tilde{G}_r^{(l)}$ if and only if it satisfies the following infinitesimal invariance criterion:*

$$\tilde{\mathbf{L}}_\alpha^{(l)} \mathcal{F}^\lambda \Big|_{\mathcal{F}^\lambda=0} = 0, \quad \alpha = 1, \dots, r; \quad \lambda = 1, \dots, q \quad (9)$$

3. Lie group of a differential approximation. The symmetry group analysis of the differential approximation uses the techniques of the Lie group theory applied to differential equations. The differential approximations involve step size variable which change under the group action.

The technique of symmetry analysis is not directly applied to the finite difference schemes. It is based on the differential approximation, which describes approximately the numerical solution behaviour at a reference point of the mesh. Thus the concept of differential approximation is a local object, which can not systematically detect a mesh change. Despite the local behaviour of the symmetry group analysis,

the invariant method based on the differential approximation has enabled one to establish interesting properties of symmetry group of finite difference schemes. The finite difference scheme, which approximates the differential system (6), can be written as:

$$\Lambda^\lambda(x, u, h, Tu) = 0, \quad \lambda = 1, \dots, q \quad (10)$$

where $h = (h_1, h_2, \dots, h_m)$ denotes the space step vector, and $T = (T_1, T_2, \dots, T_m)$ the shift-operator along the axis of the independent variables, defined by:

$$T_i[u](x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_m) = u(x_1, x_2, \dots, x_{i-1}, x_i + h_i, x_{i+1}, \dots, x_m). \quad (11)$$

Definition 3.1. The differential system:

$$\begin{aligned} \mathcal{P}^\lambda(x, u, u^{(k_1)}, \dots, u^{(k_1 \dots k_{l'})}) &= \mathcal{F}^\lambda(x, u, u^{(k_1)}, \dots, u^{(k_1 \dots k_{l'})}) \\ &+ \sum_{\beta=1}^s \sum_{i=1}^m (h_i)^{l_\beta} \mathcal{R}_i^\lambda(x, u, u^{(k_1)}, \dots, u^{(k_1 \dots k_{l'} \lambda_{i,i})}), \\ \lambda &= 1, \dots, q; \quad l' = \max_{(\lambda,i)} l'_{\lambda,i} \end{aligned} \quad (12)$$

is called the s^{th} -order differential approximation of the finite difference scheme (10). In the specific case $s = 1$, the above system is called the first differential approximation.

The differential system (3) is obtained from the algebraic system (10) by applying Taylor series expansion to the components of Tu about the point $x = (x_1, \dots, x_m)$ and truncating the expansion to a given finite order.

Denote by G'_r a group of transformations acting on an open subset M' of $X \times U \times H$ the space of the independent variables, the dependent variables and the step size variables :

$$G'_r = \{x_i^* = \phi_i(x, u, a); u_j^* = \varphi_j(x, u, a); h_i^* = \psi_i(x, u, h, a), \quad i = 1, \dots, m; \quad j = 1, \dots, n\} \quad (13)$$

by \mathbf{L}_α' the basis infinitesimal operator of G'_r :

$$\mathbf{L}_\alpha' = \xi_i^\alpha(x, u) \frac{\partial}{\partial x_i} + \eta_j^\alpha(x, u) \frac{\partial}{\partial u_j} + \zeta_i^\alpha(x, u, h) \frac{\partial}{\partial h_i}, \quad \alpha = 1, \dots, r \quad (14)$$

where $\zeta_i^\alpha = \left. \frac{\partial \psi_i}{\partial a_\alpha} \right|_{a=0}$, $\alpha = 1, \dots, r$

and by $\tilde{G}_r^{(l')}$ a group of transformation acting on an open subset $M^{(l')}$ of the space of the independent variables, the dependent variables and the step size variables and the partial derivatives involved in the differential system . The l'^{th} -prolongation operator of G'_r , $\tilde{\mathbf{L}}_\alpha^{(l')}$ can be written as:

$$\tilde{\mathbf{L}}_\alpha^{(l')} = \mathbf{L}_\alpha' + \sum_{j=1}^n \sum_{p=1}^{l'} \sigma_j^{\alpha, (k_1 \dots k_p)} \frac{\partial}{\partial u_j^{(k_1 \dots k_p)}} \quad (15)$$

Theorem 3.2. The differential approximation (3) is invariant under the group $\tilde{G}_r^{(l')}$ if and only if

$$\tilde{\mathbf{L}}_\alpha^{(l')} \mathcal{P}^\lambda((x, u, u^{(k_1)}, \dots, u^{(k_1 \dots k_{l'})})) \Big|_{\mathcal{P}^\lambda=0} = 0, \quad \alpha = 1, \dots, r; \quad \lambda = 1, \dots, q \quad (16)$$

$$\text{or } \left[\tilde{\mathbf{L}}_\alpha^{(l)} \mathcal{F}^\lambda + \tilde{\mathbf{L}}_\alpha^{(l')} \left(\sum_{\beta=1}^s \sum_{i=1}^m (h_i)^{l_\beta} \mathcal{R}_i^\lambda \right) \right] \Big|_{\mathcal{P}^\lambda=0} = 0, \quad \alpha = 1, \dots, r; \quad \lambda = 1, \dots, q \quad (17)$$

4. Determination of the infinitesimal functions. The calculation of Lie groups of differential equations with pencil and paper is tedious and may induce errors. The size of related equations increases with the number of the symmetry variables, and the order of the differential equations. A large amount of packages have been created using software programs with symbolic manipulations, such as **Mathematica**,

MACSYMA, Maple, REDUCE, AXIOM, MuPAD. Schwarz [19] wrote algorithms for REDUCE and AXIOM computer algebra systems, Vu and Carminati [16] worked on DESOLVE, a Maple program, Herod [17] and Baumann [18] developed Mathematica programs.

The authors have implemented a Mathematica package [1], for the determination of the Lie group of the differential approximation of one dimensional model equations. Theorems 2.4 and 3.2, respectively give the algorithmic procedure for the determination of Lie group of any differential system and differential approximation.

The theorem infinitesimal invariance criteria involved the independent variables x , the dependent ones u , products of the partial derivatives of u with respect to x , the unknown infinitesimal functions ξ_i^α , and η_j^α , $i = 1, \dots, m$; $j = 1, \dots, n$ and their partial derivatives with respect to x and u .

The partial derivatives of the infinitesimal functions with respect to x and u come from the coefficients of prolonged Lie algebra vector field (8).

Equations (9) and (17) are simplified by means of the conditions (6) and (3). This simplification manipulation eliminates some derivatives of u . These equations are then solved algebraically with respect to the partial derivatives of the dependent variables, handled as independent variables. Denote by w the vector, the components of which are these variables. Since the whole equation holds for all the w components, each coefficient in front of the products of the w components has to be zero. This leads to a linear overdetermined system of partial differential equation, with respect to the infinitesimal functions, called the *determining equations* of the Lie group of the related differential system.

The solution of the overdetermined system can be found either by using elementary methods of the theory of linear partial differential equations or by using a polynomial form for the infinitesimals. The last technique provides a linear system of algebraic equations with respect to the polynomial coefficients. But it can not find infinitesimals with transcendental functions.

The resolution of the determining equations yields explicitly the expression of ξ_i^α , η_j^α , $\alpha = 1, \dots, r$, $i = 1, \dots, m$, $j = 1, \dots, n$. Then relations (4) and (5) provide the calculation of group transformations from the infinitesimal expression.

5. Symmetries of the CBKDV equation. The invariance condition of theorem 2.4, which enables us to obtain the expression of the infinitesimal operators, can be written as:

$$\tilde{\mathbf{L}}_\alpha^{(2)} \mathcal{F} \Big|_{u_t + \alpha u u_x + \beta u^2 u_x + \mu u_{xx} - s u_{xxx} = 0} = 0, \quad \alpha = 1, \dots, r; \quad (18)$$

Equation (18) provides an overdetermined system of linear partial differential equations for the infinitesimal functions.

Solving these equations, by means of a symbolic calculus tool (Mathematica), provides the expression of the infinitesimal functions of a two-parameter group:

$$\xi_1^\alpha = a_0, \quad \xi_2^\alpha = b_0, \quad \eta^\alpha = 0$$

where a_0, b_0 are constants.

The two-dimensional Lie algebra of the group G is generated by the following operators:

$$\mathbf{L}_1 = \frac{\partial}{\partial x}, \quad \mathbf{L}_2 = \frac{\partial}{\partial t}$$

which respectively correspond to:

- the space translation : $(x, t, u, \nu) \mapsto (x + x_0, t, u, \nu)$;
- the time translation : $(x, t, u, \nu) \mapsto (x, t + t_0, u, \nu)$;

x_0, t_0 are constants.

It is important to note that, in the case of particular cases of the CBKDV equation, when, for instance, some of the coefficients are equal to zero, more symmetries can be obtained.

6. The specific case of the Burgers equation. Yanenko [4] and Shokin [3] symmetry analysis is applied to finite difference schemes for solving the Burgers equation. Thus, the symmetries of the Burgers equation, which are finally broken by the finite difference discretization, are determined. The techniques exposed in [3] and [4] enable one to construct differential approximations, which preserve the symmetries of the original differential system. We call the related finite difference scheme a *semi-invariant* scheme, in so far as the invariance condition is weaker than the one of the other invariance methods, defined as direct invariance methods. Indeed, the approach in [3] and [4] does not deal with the invariance of the algebraic equations, which govern the mesh evolution.

The differential equation has provided important characteristics for numerical schemes, in the study of numerical stability, dissipation and dispersive property. In [3], [4] and [8], the differential approximation has been revealed as a practical and receivable tool for symmetry analysis of finite difference scheme.

A comparison is made between the numerical solutions of the Burgers equation for some standard schemes and the semi-invariant one.

6.1. Symmetries of the Burgers equation. The Burgers equation, which is a particular case of the CBKDV equation, with $\alpha = 1$, $\beta = 0$, $s = 0$, $\mu = -\nu$, can be written as:

$$\mathcal{F}(x, t, u, \nu, u_x, u_t, u_{xx}) = u_t + u u_x - \nu u_{xx} = 0 \quad (19)$$

where $\nu \geq 0$ is the dynamic viscosity.

Let us denote by G a group of continuous transformations of the Burgers equation acting on an open subset M of the space the independent variables (x, t) , the dependent variable u , and the viscosity ν . The viscosity is taken as a symmetry variable in order to enable us to take into account variations of the Reynolds number.

The six-dimensional Lie algebra of the group G is generated by the following operators:

$$\begin{aligned} \mathbf{L}_1 &= \frac{\partial}{\partial x}, \quad \mathbf{L}_2 = \frac{\partial}{\partial t}, \quad \mathbf{L}_3 = x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} \\ \mathbf{L}_4 &= xt \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} + (-ut + x) \frac{\partial}{\partial u}, \quad \mathbf{L}_5 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad \mathbf{L}_6 = -t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} + \nu \frac{\partial}{\partial \nu} \end{aligned} \quad (20)$$

which respectively correspond to:

- the space translation : $(x, t, u, \nu) \mapsto (x + \epsilon_1, t, u, \nu)$;
- the time translation : $(x, t, u, \nu) \mapsto (x, t + \epsilon_2, u, \nu)$;
- the dilatation : $(x, t, u, \nu) \mapsto (\epsilon_3 x, \epsilon_3^2 t, \epsilon_3^{-1} u, \nu)$;
- the projective transformation : $(x, t, u, \nu) \mapsto \left(\frac{x}{1 - \epsilon_4 t}, \frac{t}{1 - \epsilon_4 t}, x \epsilon_4 + u(1 - \epsilon_4 t), \nu \right)$;
- the Galilean transformation : $(x, t, u, \nu) \mapsto (x + \epsilon_5 t, t, u + \epsilon_5, \nu)$;
- the dilatation : $(x, t, u, \nu) \mapsto (x, \epsilon_6^{-1} t, \epsilon_6 u, \epsilon_6 \nu)$.

$(\epsilon_i)_{i=1, \dots, 6}$ are constants.

As expected, the Burgers equation, as a particular form of the CBKDV equation, admits more symmetries, yielding a richer Lie algebra.

6.2. Symmetries of first differential approximations. Denote by h the mesh size, τ the time step, N_x the number of mesh points, N_t the number of time steps, and u_i^n , $i \in \{0, \dots, N_t\}$, $n \in \{0, \dots, N_x\}$ the discrete approximation of $u(ih, n\tau)$. In order to shorten the size of the finite difference scheme expressions, we use the following notations introduced by Hildebrand in [20]:

$$\begin{aligned}\delta(u_i^n) &= \frac{u_{i+\frac{1}{2}}^n - u_{i-\frac{1}{2}}^n}{h}, & \mu(u_i^n) &= \frac{u_{i+\frac{1}{2}}^n + u_{i-\frac{1}{2}}^n}{2} \\ \delta^+(u_i^n) &= \frac{u_{i+1}^n - u_i^n}{h}, & \delta^-(u_i^n) &= \frac{u_i^n - u_{i-1}^n}{h}, & E^\alpha u_i^n &= u_{i+\alpha}^n\end{aligned}$$

The Burgers equation can be discretized by means of:

- the FTCS (forward-time and centered-space) scheme:

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{\mu\delta}{h} \left(\frac{u^2}{2}\right)_i - \nu \frac{\delta^2}{h^2} u_i^n = 0$$

- the Lax-Wendroff scheme:

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{\mu\delta}{h} \left(\frac{u^2}{2}\right)_i - \nu \frac{\delta^2}{h^2} u_i^n + A_i^n = 0$$

where:

$$\begin{aligned}A_i^n &= - \frac{\tau}{2h^2} \left[E^{\frac{1}{2}} u_i^n \delta^+ \left(\frac{u^2}{2}\right)_i - E^{-\frac{1}{2}} u_i^n \delta^- \left(\frac{u^2}{2}\right)_i \right] - \frac{\nu^2 \tau}{2} \left[\frac{\delta^4}{h^4} u_i^n \right] \\ &+ \frac{\nu\tau}{2h^3} \left[E^{\frac{1}{2}} u_i^n \delta^2 (E^{\frac{1}{2}} u_i^n) - E^{-\frac{1}{2}} u_i^n \delta^2 (E^{-\frac{1}{2}} u_i^n) \right] + \frac{\nu\tau}{2} \left[\frac{\mu\delta^3}{h^3} \left(\frac{u^2}{2}\right)_i \right]\end{aligned}$$

- the Crank-Nicolson scheme:

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{\mu\delta}{h} \left[\left(\frac{u^2}{2}\right)_i^{n+1} + \left(\frac{u^2}{2}\right)_i^n \right] - \nu \frac{\delta^2}{h^2} [u_i^{n+1} + u_i^n] = 0$$

The linear stability properties and the related orders of approximation are:

- the FTCS scheme: $S \leq \frac{1}{2}$, $CFL \leq 1$, $CFL^2 \leq 2S$; $\mathcal{O}(\tau, h^2)$
- the Lax-Wendroff scheme: $S^* \leq \frac{1}{2}$, $CFL \leq 1$; $\mathcal{O}(\tau^2, h^2)$
- the Crank-Nicolson scheme: unconditional stability; $\mathcal{O}(\tau^2, h^2)$

where $CFL = \frac{a\tau}{h}$, $S = \frac{\nu\tau}{h^2}$ and $S^* = \left(\nu + \frac{ahCFL}{2}\right) \frac{\tau}{h^2}$.

Consider u_i^n as a function of the time step τ , and of the mesh size h , expand it at a given order by means of its Taylor series, and neglect the $\mathcal{O}(\tau^\alpha)$ and $\mathcal{O}(h^\beta)$ terms, where α and β depend on the order of the schemes. This yields the differential representation of the finite difference equation.

The following differential representations are obtained:

- for the FTCS scheme:

$$u_t + \frac{1}{2}(u^2)_x - \nu u_{xx} + \frac{\tau}{2}g_2 + \frac{h^2}{12}(u^2)_{xxx} - \frac{\nu h^2}{12}u_{xxxx} = 0$$

- for the Lax-Wendroff scheme:

$$u_t + \frac{1}{2}(u^2)_x - \nu u_{xx} + \frac{\tau^2}{6}g_3 + \frac{h^2}{12}(u^2)_{xxx} - \frac{\nu h^2}{12}u_{xxxx} = 0$$

- for the Crank-Nicolson scheme:

$$u_t + \frac{1}{2}(u^2)_x - \nu u_{xx} + \tau^2 \left(\frac{g_3}{6} + \frac{1}{4}(g_1^2 + u g_2)_x - \frac{\nu}{4}(g_2)_{xx} \right) + \frac{h^2}{12}(u^2)_{xxx} - \frac{\nu h^2}{12}u_{xxxx} = 0$$

where $g_1 = -\left(\frac{u^2}{2}\right)_x + \nu u_{xx}$, $g_2 = (-g_1 u)_x + \nu(g_1)_{xx}$, $g_3 = (-g_2 u - g_1^2)_x + \nu(g_2)_{xx}$

Denote by G' the group of transformations of a first differential approximation acting on an open subset M' of the space of the independent variables (x, t) and the dependent variable u , the step size variables (h, τ) and the viscosity ν .

The l'^{th} -prolongation of G' can be written as:

$$\tilde{\mathbf{L}}_{\alpha}^{(l')} = \xi_1^{\alpha} \frac{\partial}{\partial x} + \xi_2^{\alpha} \frac{\partial}{\partial t} + \eta^{\alpha} \frac{\partial}{\partial u} + \sum_{p=1}^{l'} \sigma_j^{\alpha, (k_1 \dots k_p)} \frac{\partial}{\partial u_j^{(k_1 \dots k_p)}} + \zeta_1^{\alpha} \frac{\partial}{\partial h} + \zeta_2^{\alpha} \frac{\partial}{\partial \tau} + \theta^{\alpha} \frac{\partial}{\partial \nu} \quad (21)$$

where l' has been defined in **definition 3.1**.

Theorem 3.2 enables us to obtain the necessary and sufficient condition of invariance of the first differential approximation \mathcal{P} :

$$\tilde{\mathbf{L}}_{\alpha}^{(l')} \mathcal{P} \Big|_{\mathcal{P}=0} = 0 \quad (22)$$

Theorem 3.2 is applied to the differential representations of the above schemes. The resolution of the determining equations of each first differential approximation yields the 4-parameter group (see [1]):

$$\begin{aligned} \xi_1^{\alpha} &= a + b x, & \xi_2^{\alpha} &= c + (2b - d) t, & \eta^{\alpha} &= (-b + d) u \\ \zeta_1^{\alpha} &= b h, & \zeta_2^{\alpha} &= (2b - d) \tau, & \theta^{\alpha} &= e \nu \end{aligned} \quad (23)$$

The 4-dimensional Lie algebra of G' is generated by:

$$\begin{aligned} \mathbf{L}_1 &= \frac{\partial}{\partial x}, & \mathbf{L}_2 &= \frac{\partial}{\partial t}, & \mathbf{L}'_3 &= x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} + h \frac{\partial}{\partial h} + 2\tau \frac{\partial}{\partial \tau} \\ \mathbf{L}'_4 &= -t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} - \tau \frac{\partial}{\partial \tau} + \nu \frac{\partial}{\partial \nu} \end{aligned} \quad (24)$$

These operators are respectively related to:

- the space translation : $(x, t, u, h, \tau, \nu) \mapsto (x + \epsilon_1, t, u, h, \tau, \nu)$;
- the time translation : $(x, t, u, h, \tau, \nu) \mapsto (x, t + \epsilon_2, u, h, \tau, \nu)$;
- the dilatation : $(x, t, u, h, \tau, \nu) \mapsto (\epsilon_3 x, \epsilon_3^2 t, \epsilon_3^{-1} u, \epsilon_3 h, \epsilon_3^2 \tau, \nu)$;
- the dilatation : $(x, t, u, h, \tau, \nu) \mapsto (x, \epsilon_4^{-1} t, \epsilon_4 u, h, \epsilon_4^{-1} \tau, \epsilon_4 \nu)$;

where $(\epsilon_i)_{i=1, \dots, 4}$ are constants.

The above finite difference equations are preserved by the space translation, the time translation and both dilatations.

Approximating the Burgers equation by the above finite difference equations results in the loss of the projective and Galilean transformations.

6.3. A semi-invariant scheme. Two analogous methods provide a direct symmetry analysis of finite difference schemes and can lead to the definition of adapted evolutionary meshes, whose geometrical structure is preserved by the entire group. The first direct method has been introduced by Dorodnitsyn [11] and is based on Lie algebra techniques, using the infinitesimal operators. The second method has been introduced by Olver [9] and is based on the theory of the Cartan moving frame. The method proposed by Yanenko [4] and Shokin [3] consists of a symmetry study of the differential approximation. Although the last method is not fully exact, the numerical results in [3] and [8] has proved its effectiveness.

The scheme proposed below is associated to an uniform orthogonal mesh.

We propose to approximate the Burgers equation by the finite difference scheme:

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{1}{h} \left(\mu \delta - \frac{\mu \delta^3}{6} \right) \left(\frac{u^2}{2} \right)_i^n - \nu \frac{1}{h^2} \left(\delta^2 - \frac{\delta^4}{12} \right) (u_i^n) - h \left(\Omega_{i+\frac{1}{2}}^n \delta^+ - \Omega_{i-\frac{1}{2}}^n \delta^- \right) u_i^n = 0 \quad (25)$$

where $\Omega_i^n = \Omega(x_i, t_n, u_i^n)$ is defined next so that the related differential representation is preserved by the symmetries of the Burgers equation. The scheme has

second-order accuracy in space and first-order accuracy in time. The derivatives $(u^2)_x$ and u_{xx} are approximated by fourth order accuracy difference expressions:

$$\left(\frac{\mu\delta}{h} - \frac{\mu\delta^3}{6h}\right)(u_i^n) = \left(u_x - \frac{h^4}{30}u_{5x}\right)_i^n + \mathcal{O}(h^6), \quad \left(\frac{\delta^2}{h^2} - \frac{\delta^4}{12h^2}\right)(u_i^n) = \left(u_{xx} - \frac{h^4}{90}u_{6x}\right)_i^n + \mathcal{O}(h^6) \quad (26)$$

The truncation error of the difference scheme (25) can be written as:

$$\epsilon = \frac{\tau}{2}u_{tt} - h^2\left(\Omega u_x\right)_x + \mathcal{O}(\tau^2) + \mathcal{O}(h^4)$$

u_{tt} is replaced by an expression involving partial derivatives with respect to x , by using the Burgers equation. Replacing the obtained expression in the truncation error leads to:

$$\epsilon = \left(Cu_x\right)_x - \frac{\nu\tau}{2}\left(uu_{xx}\right)_x - \frac{\nu\tau}{2}\left(\frac{u^2}{2}\right)_{xxx} + \frac{\nu^2\tau}{2}u_{xxxx} + \mathcal{O}(\tau^2) + \mathcal{O}(h^4)$$

where $C = \frac{\tau}{2}u^2 - h^2\Omega$.

It is convenient for the calculation of C that the truncation error is reduced to:

$$\epsilon = \left(Cu_x\right)_x + \mathcal{O}(\tau^2) + \mathcal{O}(h^4)$$

The related finite difference scheme is the following first order accuracy in time and second order accuracy in space:

$$\begin{aligned} \frac{u_i^{n+1} - u_i^n}{\tau} + \frac{1}{h}\left(\mu\delta - \frac{\mu\delta^3}{6}\right)\left(\frac{u^2}{2}\right)_i^n - \nu\frac{1}{h^2}\left(\delta^2 - \frac{\delta^4}{12}\right)(u_i^n) - h\left(\Omega_{i+\frac{1}{2}}^n\delta^+ - \Omega_{i-\frac{1}{2}}^n\delta^-\right)u_i^n \\ + \frac{\nu\tau}{2}\left(u_{i+\frac{1}{2}}^n\frac{\mu\delta^2}{h^2}(u_{i+\frac{1}{2}}^n) - u_{i-\frac{1}{2}}^n\frac{\mu\delta^2}{h^2}(u_{i-\frac{1}{2}}^n)\right) - \frac{\nu^2\tau}{2}\frac{\delta^4}{h^4}u_i^n + \frac{\nu\tau}{2}\frac{\mu\delta^3}{h^3}\left(\frac{u^2}{2}\right)_i^n = 0 \end{aligned} \quad (27)$$

and the differential approximation can be written as:

$$\mathcal{P}(x, t, u, \nu, u_x, u_t, u_{xx}) = u_t + u u_x - \nu u_{xx} + (Cu_x)_x = 0 \quad (28)$$

The von Neumann stability analysis of scheme (27) under a linearized form provides the following necessary conditions for S , CFL and $\Omega_\tau = \Omega\tau$:

$$CFL^2 - 2S - 2\Omega_\tau \leq 0, \quad 0 \leq (4S)/3 - 2S^2 + \Omega_\tau \leq 1/2 \quad (29)$$

If Ω is sufficiently close to zero, these conditions become then sufficient for the linear formulation.

6.4. Numerical application. The artificial viscosity term $(Cu_x)_x$ is build so as to preserve the symmetries of the Burgers equation. C is function of (x, t, u, τ, h) .

The Burgers equation is solved numerically for the semi-invariant scheme, and the classical schemes mentioned above. The numerical solutions are determined in the frame (F1), with an uniform and orthogonal mesh, in the frame (F2) and (F3), which are respectively obtained with the Galilean transformation $(x, t, u, \nu) \mapsto (x + 0.25 t, t, u + 0.25, \nu)$ and a higher translation velocity $(x, t, u, \nu) \mapsto (x + 0.5 t, t, u + 0.5, \nu)$. In (F2) and (F3) the orthogonality of the mesh is not preserved. The parameters, which control the numerical stability, i.e. the CFL number and the cell Reynolds number, are preserved by the Galilean transformation. Any change of the frame does not modify their value. These parameters satisfy the linear stability conditions, which are sufficient for all the numerical applications realized below.

Figures 1 display the time evolution, of the L^2 norm of the error, for the above numerical schemes, in the different frames. A shock wave type solution is initially introduced. The CFL and the cell Reynolds number are respectively equal to 4 and 0.5. Non-invariant schemes are sensitive to the change of frame. Thus, the numerical error of the FTCS scheme oscillates after the application of a translation with a velocity of 0.25. It increases sharply after a translation with a velocity of 0.5.

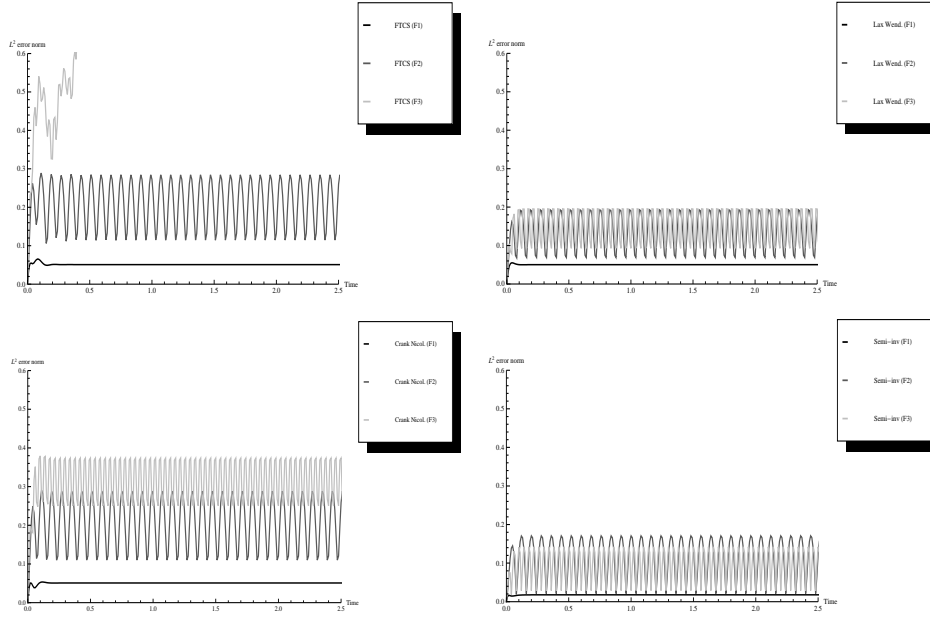


FIGURE 1. Evolution in time of the L^2 norm of the error in (F1), (F2), (F3) when $Re_h = 4.0$, $CFL = 0.5$

The linear numerical stability is satisfied, though the discretization parameters of the CFL and the cell Reynolds numbers are situated on the boundary between the region where the solutions rapidly blow up and the one where the solutions remain bounded for certain time steps. The introduction of an additional error due to the loss of the Galilean transformation leads to the numerical solution blow-up.

We notice the dependence of numerical error, for the semi-invariant scheme, upon the frame. The change of the frame causes the apparition of slight oscillations.

Figures 2 display the time evolution, of the L^2 norm of the error, for the above numerical schemes, in the same frames. This time, a wavy solution is initially introduced. The CFL number and the cell Reynolds number are respectively equal to 0.2 and 0.06. We clearly notice the dependence of numerical error, for the four numerical scheme, upon the frame.

7. Conclusion. Starting from the general *CBKDV* equation, we have set methods that enable us to build new numerical schemes. The application to the Burgers equation can be seen as a restricted case of the *CBKDV* equation.

The invariance method based on the differential approximation does not enable us to build schemes, which are rigorously invariant under the action of the group of the original equations. Like in the case of classical finite difference methods for aerodynamics, the method based on the differential approximation breaks the symmetries, which act on the independent variables, and transform the geometrical properties of the mesh. This symmetry action is observed for the Galilean transformation and has been applied for the numerical resolution of the Burgers equation. Under the action of the Galilean transformation, a non-invariant discretization leads to the loss of the mesh orthogonality.

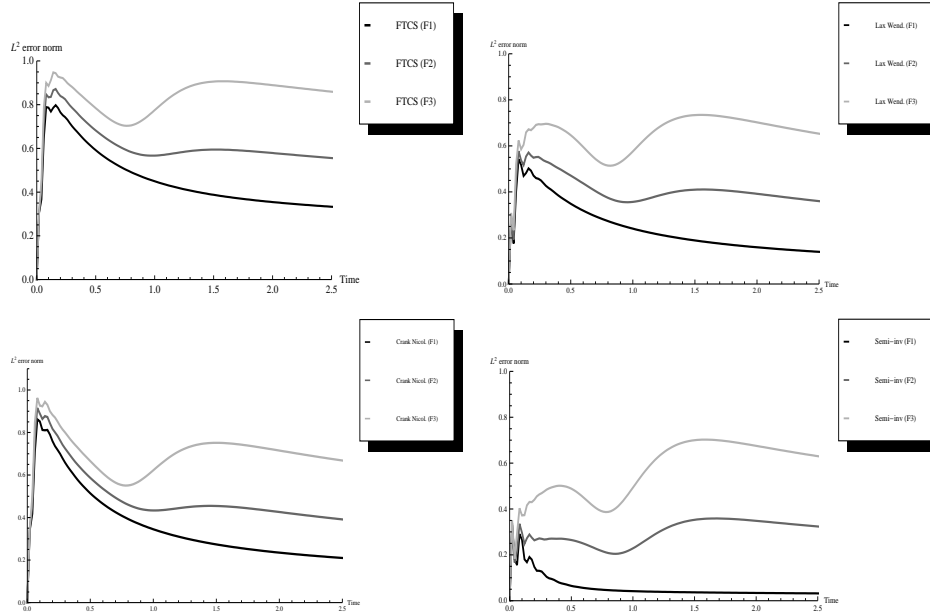


FIGURE 2. Evolution in time of the L^2 norm of the error in (F1), (F2), (F3) when $Re_h =$, $CFL =$

If, on the one hand, the method based on the differential approximation would provide the exact invariance if the higher order terms of the numerical error were considered, it would yield, on the other hand, a scheme too complicated to be implemented. In order to preserve rigorously the symmetries, we need to use direct methods of symmetry analysis of finite difference schemes, which take into account the transformation of all the finite difference variables and preserve the geometrical properties of the mesh.

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