



HAL
open science

Modular functionals and perturbations of Nakano spaces

Itai Ben Yaacov

► **To cite this version:**

| Itai Ben Yaacov. Modular functionals and perturbations of Nakano spaces. 2008. hal-00259298v1

HAL Id: hal-00259298

<https://hal.science/hal-00259298v1>

Preprint submitted on 27 Feb 2008 (v1), last revised 23 Feb 2009 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

MODULAR FUNCTIONALS AND PERTURBATIONS OF NAKANO SPACES

ITAÏ BEN YAACOV

ABSTRACT. We settle several questions regarding the model theory of Nakano spaces left open by the PhD thesis of Poitevin [Poi06].

We start by studying isometric Banach lattice embeddings of Nakano spaces, showing that in dimension two and above such embeddings have a particularly simple and rigid form.

We use this to show that in the Banach lattice language the modular functional is definable and that complete theories of atomless Nakano spaces are model complete. We also show that up to arbitrarily small perturbations of the exponent Nakano spaces are \aleph_0 -categorical and \aleph_0 -stable. In particular they are stable.

INTRODUCTION

Nakano spaces are a generalisation of L_p function spaces in which the exponent p is allowed to vary as a measurable function of the underlying measure space. The PhD thesis of Pedro Poitevin [Poi06] studies Nakano spaces as Banach lattices from a model theoretic standpoint. More specifically, he viewed Nakano spaces as continuous metric structures (in the sense of continuous logic, see [BU]) in the language of Banach lattices, possibly augmented by a predicate symbol Θ for the modular functional, showed that natural classes of such structures are elementary in the sense of continuous first order logic, and studied properties of their theories.

In the present paper we propose to answer a few questions left open by Poitevin.

- First, Poitevin studies Nakano spaces in two natural languages: that of Banach lattices, and the same augmented with an additional predicate symbol for the modular functional. It is natural to ask whether these languages are truly distinct, i.e., whether adding the modular functional adds new structure.
- Even if the naming of the modular functional does not add structure, it does give quantifier elimination in atomless Nakano spaces. While it is clear that

Date: 28 February 2008.

2000 Mathematics Subject Classification. 03C98,46E30.

Research initiated during the workshop “Model theory of metric structures”, American Institute of Mathematics Research Conference Centre, 18 to 22 September 2006.

Research supported by ANR chaire d’excellence junior (projet THEMOMET) and by the European Commission Marie Curie Research Network ModNet.

without it quantifier elimination is impossible, it is natural to ask whether model completeness is.

- Poitevin showed that the theory of atomless Nakano space where the exponent function is *bounded away from one* is stable. What about the general case?
- Similarly, if the exponent is constant, i.e., if we are dealing with classical atomless L_p spaces, it is known (see [BBH]) that the theory of these spaces is \aleph_0 -categorical and \aleph_0 -stable. On the other hand, it is quite easy to verify complete theories of atomless Nakano spaces are non \aleph_0 -categorical and non \aleph_0 -stable once the essential range of the exponent is infinite. It is therefore natural to ask whether, up to small perturbations of the exponent, a complete theory of atomless Nakano spaces is \aleph_0 -categorical and \aleph_0 -stable. A positive answer would mean that the theory of atomless Nakano spaces is stable settling the previous item as well.

In this paper we answer all of these questions positively (where a negative answer to the first question is considered positive). It is organised as follows:

Section 1 consist purely of functional analysis, and requires no familiarity with model theory. After a few general definitions we study mappings between vector lattices of measurable functions and then more specifically between Nakano spaces. Our main result is:

Theorem. *Let $\theta: L_{p(\cdot)}(X, \mathfrak{B}, \mu) \hookrightarrow L_{q(\cdot)}(Y, \mathfrak{C}, \nu)$ be a Banach lattice isometric embedding of Nakano spaces of dimension at least two. Then up to a measure density change on Y and identification between subsets of X and of Y (and thus between measurable functions on X and on Y) θ is merely an extension by zeros from X to $Y \supseteq X$. In particular $p = q|_X$ and $\mu = \nu|_{\mathfrak{B}}$.*

It follows that such embeddings respect the modular functional and extend the essential range of the exponent function.

In Section 2 we expose the model theoretic setting for the paper. In particular, we quote the main results of Poitevin's PhD thesis [Poi06].

In Section 3 we prove our main model theoretic results:

Theorem. *The modular functional is definable in every Nakano Banach lattice (i.e., naming it in the language does not add structure). Moreover, it is uniformly definable in the class of Nakano spaces of dimension at least two, and in fact both sup-definable and inf-definable there.*

Theorem. *The theory of atomless Nakano spaces with a fixed essential range for the exponent function is model complete in the Banach lattice language.*

In Section 4 we study perturbations of the exponent function, showing that small perturbations thereof yield small perturbations of the structures. Up to such perturbations the theory of atomless Nakano spaces is \aleph_0 -stable, and every completion thereof is \aleph_0 -categorical. In particular all Nakano space are stable.

Appendix A consist of the adaptation of a few classical model theoretic results used in this paper to continuous logic.

Finally, Appendix B contains some approximation results for the modular functional which were used in earlier versions of this paper to be superseded later by Theorem 1.10, but which nonetheless might be useful.

1. SOME FUNCTIONAL ANALYSIS

1.1. Nakano spaces. Let (X, \mathfrak{B}, μ) be an arbitrary measure space, and let $L_0(X, \mathfrak{B}, \mu)$ be the space of all measurable functions $f: X \rightarrow \mathbb{R}$ up to equality a.e. (Since we wish to consider function spaces as Banach lattices it will be easier to consider the case of real-valued functions.)

Let $p: X \rightarrow [1, \infty)$ be an essentially bounded measurable function. We define the *modular functional* $\Theta_{p(\cdot)}: L_0(X, \mathfrak{B}, \mu) \rightarrow [0, \infty]$ by:

$$\Theta_{p(\cdot)}(f) = \int |f(x)|^{p(x)} d\mu.$$

We define the corresponding *Nakano space* as:

$$L_{p(\cdot)}(X, \mathfrak{B}, \mu) = \{f \in L_0(X, \mathfrak{B}, \mu) : \Theta_{p(\cdot)}(f) < \infty\}.$$

If $f \in L_{p(\cdot)}(X, \mathfrak{B}, \mu)$ then there exists a unique number $c \geq 0$ such that $\Theta_{p(\cdot)}(f/c) = 1$, and we define $\|f\| = \|f\|_{p(\cdot)} = c$. This is a norm, making $L_{p(\cdot)}(X, \mathfrak{B}, \mu)$ a Banach space. With the point-wise minimum and maximum operations it is a Banach lattice.

Remark. In the literature $\Theta_{p(\cdot)}$ is usually merely referred to as the *modular*. Being particularly sensitive regarding parts of speech we will nonetheless refer to it throughout as the modular *functional*.

1.2. Strictly localisable spaces. In this paper we will consider the class of Nakano spaces from a model-theoretic point of view. This means we will have to admit arbitrarily large Nakano spaces (e.g., κ -saturated for arbitrarily big κ) and therefore arbitrarily large measure spaces. In particular, we cannot restrict our attention to σ -finite measure spaces. In order to avoid pathologies which may arise with arbitrary measure spaces we will require a weaker assumption:

Recall from [Fre03]:

Definition 1.1. A measure space (X, \mathfrak{B}, μ) is *strictly localisable* if it can be expressed as a disjoint union of measure spaces of finite measure, i.e., if X admits a partition as $\bigcup_{i \in I} X_i$ such that:

- (i) For all $i \in I$: $X_i \in \mathfrak{B}$ and $\mu(X_i) < \infty$.
- (ii) For all $A \subseteq X$: $A \in \mathfrak{B}$ if and only if $A \cap X_i \in \mathfrak{B}$ for all $i \in I$, in which case $\mu(A) = \sum \mu(A \cap X_i)$.

In this case the family $\{X_i : i \in I\}$ *witnesses* that (X, \mathfrak{B}, μ) is strictly localisable.

For example every σ -finite measure space is strictly localisable. On the other hand, if (X, \mathfrak{B}, μ) is an arbitrary measure space we can find a maximal family $\mathcal{X} = \{X_i : i \in I\} \subseteq \mathfrak{B}$ of almost disjoint sets, $0 < \mu(X_i) < \infty$, and define $(X', \mathfrak{B}', \mu') = \coprod_{i \in I} (X_i, \mathfrak{B}|_{X_i}, \mu|_{X_i})$, with an obvious mapping $\theta: L_0(X, \mathfrak{B}, \mu) \rightarrow L_0(X', \mathfrak{B}', \mu')$. This does not lose any information that interests us: in particular, θ restricts to an isometric isomorphism of Nakano spaces $\theta: L_{p(\cdot)}(X, \mathfrak{B}, \mu) \rightarrow L_{\theta p(\cdot)}(X', \mathfrak{B}', \mu')$

We will therefore allow ourselves:

Convention 1.2. In this paper every measure space and every inclusion of measure spaces is assumed to be strictly localisable.

Let us state a few very easy facts concerning strictly localisable measure spaces. All of them are probably somewhere in the literature.

The following is immediate:

Fact 1.3. *Let $\mathcal{X} = \{X_i : i \in I\}$ witness that (X, \mathfrak{B}, μ) is strictly localisable. If $\mathcal{X}' = \{X'_j : j \in J\} \subseteq \mathfrak{B}$ is another partition of X refining \mathcal{X} , splitting each X_i into at most countably many subsets, then \mathcal{X}' is a witness as well.*

For example, we may use strict localisability as a replacement for the σ -finiteness assumption in the Radon-Nikodým theorem. (Weirdly enough we did not manage to find this result in [Fre03, Chapter 23].)

Fact 1.4. *Let (X, \mathfrak{B}, μ) be a strictly localisable measure space and let ν be another measure on (X, \mathfrak{B}) . Then $\nu \ll \mu$ if and only if there is a measurable function $\zeta: X \rightarrow (0, \infty)$ such that for every positive measurable function $f: \int f d\nu = \int f \zeta d\mu$.*

Moreover, in this case (X, \mathfrak{B}, ν) is strictly localisable as well, and the function ζ is unique up to equality a.e., denoted $\frac{d\nu}{d\mu}$.

Proof. Let $\mathcal{X} = \{X_i : i \in I\}$ witness that (X, \mathfrak{B}, μ) is strictly localisable and apply the classical Radon-Nikodým theorem on each X_i . We have for all $A \in \mathfrak{B}$: $\nu(A) = \int_A \zeta d\mu = \sum_{i \in I} \int_{A \cap X_i} \zeta d\mu$. In particular, ν is σ -finite on each X_i . Thus by Fact 1.3 we may further refine \mathcal{X} so that $\nu(X_i)$ is always finite. Then \mathcal{X} witnesses that (X, \mathfrak{B}, ν) is strictly localisable as well. ■_{1.4}

If μ and ν are two equivalent measures on (X, \mathfrak{B}) (i.e., $\mu \ll \nu$ and $\nu \ll \mu$) then each is obtained from the other by a mere *density change*. The corresponding Nakano spaces are naturally isomorphic.

Fact 1.5. *Let μ and ν be two equivalent measures on (X, \mathfrak{B}) such that (X, \mathfrak{B}, μ) is strictly localisable (and therefore (X, \mathfrak{B}, ν) as well), and let $p: X \rightarrow [1, r]$ be measurable. Let $(N, \Theta) = L_{p(\cdot)}(X, \mathfrak{B}, \mu)$ and $(N', \Theta') = L_{p(\cdot)}(X, \mathfrak{B}, \nu)$ be the corresponding Nakano spaces with their modular functionals. For $f \in N$ define $D_{\mu, \nu} f = (d\mu/d\nu)^{1/p} f$. Then $D_{\mu, \nu} f \in N'$ and $D_{\mu, \nu}: (N, \Theta) \simeq (N', \Theta')$ is an (isometric) isomorphism.*

Proof. One calculates:

$$\begin{aligned}\Theta'(D_{\mu,\nu}f) &= \int ((d\mu/d\nu)^{1/p}|f|)^p d\nu \\ &= \int |f|^p(d\mu/d\nu)d\nu \\ &= \int |f|^p d\mu = \Theta(f).\end{aligned}\quad \blacksquare_{1.5}$$

It follows that $f \in N \implies D_{\mu,\nu}f \in N'$, and $D_{\mu,\nu}$ clearly respects the linear and lattice structures, as well as Θ . It follows that it respects the norm as well. It has an inverse $D_{\nu,\mu}$.

1.3. Mappings between function space lattices. Let (X, \mathfrak{B}, μ) be a measure space, $L \subseteq L_0(X, \mathfrak{B}, \mu)$ be a vector sub-lattice which contains all characteristic functions of finite measure sets. Assume furthermore that L is *Dedekind σ -complete*, namely that if $\mathcal{A} \subseteq L$ is a bounded countable family then the point-wise supremum $\bigvee \mathcal{A} \in L$. For example, L could be $L_{p(\cdot)}(X, \mathfrak{B}, \mu)$.

Let (Y, \mathfrak{C}, ν) be another measure space. Let $\theta: L \rightarrow L_0(Y, \mathfrak{C}, \nu)$ be a vector lattice homomorphism respecting countable suprema which in addition sends characteristic functions to characteristic functions. We will allow ourselves the following abuse of notation: if $A \in \mathfrak{B}$ has finite measure and $\theta(\chi_A) = \chi_B$, $B \in \mathfrak{C}$ then we will write $\theta(A) = B$ (even though this is only defined up to null measure). In particular, instead of writing $\theta(\chi_A)$ we will write $\chi_{\theta A}$.

Lemma 1.6. *Under the assumptions above, θ extends to a unique vector lattice homomorphism $\hat{\theta}: L_0(X, \mathfrak{B}, \mu) \rightarrow L_0(Y, \mathfrak{C}, \nu)$ which respects countable suprema. Moreover, for every Borel function $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ which fixes zero (i.e., which sends $0 \in \mathbb{R}^n$ to $0 \in \mathbb{R}$) and every tuple $\bar{f} \in L_0(X, \mathfrak{B}, \mu)$ we have $\hat{\theta}(\varphi(\bar{f})) = \varphi(\theta(\bar{f}))$.*

Proof. Let us write L_0 for $L_0(X, \mathfrak{B}, \mu)$, and let L_0^+ be its positive cone.

Let us first consider the case where $\mu(X), \nu(Y) < \infty$. In this case L contains all positive bounded functions. For $f \in L_0^+$ we have to define $\hat{\theta}(f) = \hat{\theta}(\bigvee_{k \in \mathbb{N}} f \wedge k) = \bigvee_{k \in \mathbb{N}} \theta(f \wedge k)$, and then extend to arbitrary $f \in L_0$ by $\hat{\theta}(f) = \hat{\theta}(f^+) - \hat{\theta}(f^-)$. We now need to make sure this verifies the requirements.

First of all we need to make sure that if $f \in L_0^+$ then $\hat{\theta}(f) = \bigvee_{k \in \mathbb{N}} \theta(f \wedge k)$ exists, i.e., that it is finite a.e. Indeed, since f is assumed to be finite a.e. the sequence of sets $A_k = \{x: f(x) \geq k\}$ decreases to zero, whereby $\theta(A_k)$ decrease to zero as well. we have $f \wedge (k+m) \leq k+m\chi_{A_k}$ whereby $\theta(f \wedge (k+m)) \leq k+m\chi_{\theta(A_k)}$, so $\theta(f \wedge (k+m))(y) \leq k$ outside $\theta(A_k)$. Thus $\hat{\theta}(f) \leq k$ outside $\theta(A_k)$, and we can conclude that $\hat{\theta}(f) \in L_0(Y, \mathfrak{C}, \nu)$. Since θ respects countable suprema $\hat{\theta}$ extends it. Once we prove linearity it will follow that $\hat{\theta}$ extends θ on all of L_0 .

Let us verify some properties of $\hat{\theta}$ on L_0^+ . If $\bigvee_{m \in \mathbb{N}} f_m$ exists then $\hat{\theta}(\bigvee_{m \in \mathbb{N}} f_m) = \bigvee_{m \in \mathbb{N}, k \in \mathbb{N}} f_m \wedge k = \bigvee_{m \in \mathbb{N}} \hat{\theta}(f_m)$. Similarly $\hat{\theta}(f) \wedge \hat{\theta}(g) = \bigvee_{k, \ell \in \mathbb{N}} [\theta(f \wedge k) \wedge \theta(g \wedge \ell)] = \bigvee_{k, \ell \in \mathbb{N}} [\theta(f \wedge g \wedge k \wedge \ell)] = \hat{\theta}(f \wedge g)$. If $f = \sum_{m \in \mathbb{N}} f_m$ where $f_m \in L_0^+$, $f_m \wedge f_{m'} = 0$ for $m \neq m'$ then $\theta(f_m) \wedge \theta(f_{m'}) = 0$ as well and $\hat{\theta}(f) = \hat{\theta}(\bigvee_m f_m) = \bigvee_m \hat{\theta}(f_m) = \sum_m \hat{\theta}(f_m)$.

We wish to show that if $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ is Borel fixing zero and $\bar{f} \in L_0^n$ then $\hat{\theta}(\varphi(\bar{f})) = \varphi(\hat{\theta}(\bar{f}))$.

We claim that if $A \subseteq (0, \infty)^n$ is a Borel set and $\bar{f} \in (L_0^+)^n$ then $\theta\{\bar{f}(x) \in A\} = \{\hat{\theta}\bar{f}(y) \in A\}$. Indeed, for a single f we have $\hat{\theta}(f) = \hat{\theta}(\bigvee_{t \in \mathbb{Q}^+} t\chi_{\{f > t\}}) = \bigvee_{t \in \mathbb{Q}^+} t\chi_{\theta\{f > t\}}$, whereby $\{\hat{\theta}(f) > t\} = \theta\{f > t\}$. Our claim follows for the case $A = (t_0, \infty) \times \dots \times (t_{n-1}, \infty)$ and we may use the properties of θ to climb up the Borel hierarchy.

Assume first that $\bar{f}(x) \in (0, \infty)^n \cup \{0\}$ for all x and that $\varphi \geq 0$. Letting $A_t = \{x \in (0, \infty)^n: \varphi(x) > t\}$:

$$\theta\{\varphi(\bar{f}(x)) > t\} = \theta\{\bar{f}(x) \in A_t\} = \{\hat{\theta}\bar{f}(y) \in A_t\} = \{\varphi(\hat{\theta}\bar{f}(y)) > t\},$$

whereby $\hat{\theta}(\varphi(\bar{f})) = \varphi(\hat{\theta}(\bar{f}))$. For general \bar{f} , let $S = \{1, 0, -1\}^n \setminus \{0\}$, and for $s \in S$ let $A_s = \{x \in X: \text{sgn}(f) = s\}$. On each A_s we may drop those f_i s which are constantly zero and replace those which are negative with their absolute value, making the necessary modifications to φ , obtaining by the previous argument

$$\hat{\theta}(\varphi(\chi_{A_s}\bar{f})) = \varphi(\hat{\theta}(\chi_{A_s}\bar{f})),$$

whereby:

$$\hat{\theta}(\varphi(\bar{f})) = \sum_{s \in S} \hat{\theta}(\varphi(\chi_{A_s}\bar{f})) = \sum_{s \in S} \varphi(\hat{\theta}(\chi_{A_s}\bar{f})) = \varphi(\hat{\theta}(\bar{f}))$$

Finally, for general φ we can split it to the positive and negative part and then put them back together. It follows in particular that $\hat{\theta}$ is a linear lattice homomorphism.

Now let us consider the case where X is an arbitrary measure space. Let $\{X_i: i \in I\} \subseteq \mathfrak{B}$ be a maximal family of almost disjoint sets of finite non zero measure such that in addition $\theta(\chi_{X_i}) \neq 0$. Since $\nu(Y)$ is assumed finite such a family must be at most countable, so we can write it as $\{X_k: k < \omega\}$. Let $X' = \bigcup X_k$. Then for every $f \in L$ we have $\theta(f) = \theta(f\chi_{X'}) = \sum_k \theta(f\chi_{X_k})$ (verify first for $f \geq 0$ and then extend by linearity), so we may restrict to each X_k , reducing to the case already considered, then checking that $\hat{\theta}(f) = \sum_k \hat{\theta}(f\chi_{X_k})$ works.

Finally, if (Y, \mathfrak{C}, ν) is merely strictly localisable then let this be witnessed by $\{Y_i: i \in I\}$. Then we can first extend $\theta_i = \chi_{Y_i}\theta: L \rightarrow (Y_i, \mathfrak{C}|_{Y_i}, \nu|_{Y_i})$ to $\hat{\theta}_i$ and then obtain $\hat{\theta}$ by gluing. ■_{1.6}

Lemma 1.7. *Continue with previous assumptions, and add that if $\mu(A) < \infty$ then $\nu(\theta A) = \mu(A)$. Then for every function $f \in L_1(X, \mathfrak{B}, \mu)$: $\int f d\mu = \int \hat{\theta}f d\nu$.*

Proof. This holds by assumption for characteristic functions of finite measure sets, from which we deduce it for simple positive functions, positive functions and finally general functions. $\blacksquare_{1.7}$

1.4. Embeddings of Nakano Banach lattices. We will now prove the main functional analysis results of this paper.

Lemma 1.8. *Let $N = L_{p(\cdot)}(X, \mathfrak{B}, \mu)$ and $N' = L_{q(\cdot)}(Y, \mathfrak{C}, \nu)$ be two Nakano spaces, and let $\theta: N \rightarrow N'$ be an isometric embedding of Banach lattices which sends characteristic functions to characteristic functions. Assume furthermore that $\dim N \geq 2$. Then:*

- (i) $\hat{\theta}(p) = q\chi_{\hat{\theta}X}$.
- (ii) For all finite measure $A: \nu(\theta A) = \mu(A)$.

Proof. First of all the hypotheses of Lemma 1.6 are verified with $N = L$, so $\hat{\theta}$ exists. Let $Y_0 = \hat{\theta}X \in \mathfrak{C}$ be the support of the range of θ .

$$\begin{aligned} C_1 &= \{y \in Y_0: \hat{\theta}p(y) < q(y)\}, \\ C_2 &= \{y \in Y_0: \hat{\theta}p(y) > q(y)\}, \\ C &= C_1 \cup C_2 = \{y \in Y_0: \hat{\theta}p(y) \neq q(y)\} \end{aligned}$$

Then $C_1, C_2, C \in \mathfrak{C}$ and we need to show that $\nu(C) = 0$. Let $A, B \in \mathfrak{B}$ be such that $0 < \mu(A), \mu(B) < \infty$. For $t \in [0, 1]$, let

$$\begin{aligned} f_t &= \chi_A \left(\frac{t}{\mu(A)} \right)^{\frac{1}{p(x)}} + \chi_B \left(\frac{1-t}{\mu(B)} \right)^{\frac{1}{p(x)}}, \\ g_t &= \theta(f_t) = \chi_{\theta A} \left(\frac{t}{\mu(A)} \right)^{\frac{1}{\hat{\theta}p(y)}} + \chi_{\theta B} \left(\frac{1-t}{\mu(B)} \right)^{\frac{1}{\hat{\theta}p(y)}}, \end{aligned}$$

Then $\Theta(f_t) = 1 \implies \|f_t\| = 1 \implies \|g_t\| = 1 \implies \Theta'(g_t) = 1$. In other words:

$$\Theta'(g_t) = \int_{\theta A} \left(\frac{t}{\mu(A)} \right)^{\frac{q}{\hat{\theta}p}} d\nu + \int_{\theta B} \left(\frac{1-t}{\mu(B)} \right)^{\frac{q}{\hat{\theta}p}} d\nu = 1$$

Substituting $t = 0$ and $t = 1$ we see that in particular $\nu(A)$ and $\nu(B)$ are both positive and finite. We may therefore differentiate under the integral sign for $t \in (0, 1)$, obtaining:

$$\begin{aligned} 0 = \frac{d}{dt} \Theta'(g_t) &= \int_{\theta A \cap C} \frac{q}{\mu(A)\hat{\theta}p} \left(\frac{t}{\mu(A)} \right)^{\frac{q}{\hat{\theta}p}-1} d\nu + \int_{\theta A \setminus C} \frac{q}{\mu(A)\hat{\theta}p} d\nu \\ &\quad - \int_{\theta B \cap C} \frac{q}{\mu(B)\hat{\theta}p} \left(\frac{1-t}{\mu(B)} \right)^{\frac{q}{\hat{\theta}p}-1} d\nu - \int_{\theta B \setminus C} \frac{q}{\mu(B)\hat{\theta}p} d\nu \end{aligned}$$

If $\nu(\theta A \cap C_2) > 0$ then $\lim_{t \rightarrow 0} \frac{d}{dt} \Theta'(g_t) = +\infty \neq 0$ which is impossible, so $\nu(\theta A \cap C_2) = 0$, and considering $t \rightarrow 1$ we see that $\nu(\theta B \cap C_2) = 0$ as well. We may therefore substitute $t = 0$ and $t = 1$ and obtain:

$$\begin{aligned} 0 &= \int_{\theta A \setminus C} \frac{q}{\mu(A) \hat{\theta} p} d\nu \\ &\quad - \int_{\theta B \cap C} \frac{q}{\mu(B) \hat{\theta} p} \left(\frac{1}{\mu(B)} \right)^{\frac{q}{\hat{\theta} p} - 1} d\nu - \int_{\theta B \setminus C} \frac{q}{\mu(B) \hat{\theta} p} d\nu \\ &= \int_{\theta A \cap C} \frac{q}{\mu(A) \hat{\theta} p} \left(\frac{1}{\mu(A)} \right)^{\frac{q}{\hat{\theta} p} - 1} d\nu + \int_{\theta A \setminus C} \frac{q}{\mu(A) \hat{\theta} p} d\nu \\ &\quad - \int_{\theta B \setminus C} \frac{q}{\mu(B) \hat{\theta} p} d\nu, \end{aligned}$$

whereby

$$\int_{\theta A \cap C} \frac{q}{\mu(A) \hat{\theta} p} \left(\frac{1}{\mu(A)} \right)^{\frac{q}{\hat{\theta} p} - 1} d\nu = - \int_{\theta B \cap C} \frac{q}{\mu(B) \hat{\theta} p} \left(\frac{1}{\mu(B)} \right)^{\frac{q}{\hat{\theta} p} - 1} d\nu$$

This is only possible if both are zero, i.e., if $\nu(\theta A \cap C) = \nu(\theta B \cap C) = 0$.

We have shown that $\nu(\theta A \cap C) = \nu(\theta B \cap C) = 0$ for every $A, B \in \mathfrak{B}$ disjoint of finite non zero measure. If N had dimension ≤ 1 this would be vacuous, but as we assume that it has dimension ≥ 2 we have in fact $\nu(\theta A \cap C) = 0$ for all $A \in \mathfrak{B}$ such that $\mu(A) < \infty$. It follows that $\nu(C) = \nu(Y_0 \cap C) = 0$, i.e., that $\hat{\theta} p = q \chi_{Y_0}$.

Now let $A \in \mathfrak{B}$ be of finite non zero measure, $h = \mu(A)^{-1/p(x)}$. Then $\Theta(h) = 1 \implies 1 = \Theta'(\theta(h)) = \nu(\theta A) / \mu(A)$. ■_{1.8}

Remark. A special case of this result was independently obtained at the same time by Poitevin and Raynaud [PR, Lemma 6.1].

The technical assumption that θ sends characteristic functions to such (i.e., acts on measurable sets) is easy to obtain via a density change:

Lemma 1.9. *Let $N = L_{p(\cdot)}(X, \mathfrak{B}, \mu)$ and $N' = L_{q(\cdot)}(Y, \mathfrak{C}, \nu)$ be two Nakano spaces, and let $\theta: N \rightarrow N'$ be an isometric embedding of Banach lattices. Then there is a measure λ on (Y, \mathfrak{C}) , equivalent to ν , such that $D_{\nu, \lambda} \circ \theta: N \rightarrow N'' = L_{q(\cdot)}(Y, \mathfrak{C}, \lambda)$ sends characteristic functions to characteristic functions, where $D_{\nu, \lambda}: N' \rightarrow N''$ is the density change isomorphism from Fact 1.5.*

Proof. Let $\{X_i: i \in I\} \subseteq \mathfrak{B}$ and $\{Y_j: j \in J\} \subseteq \mathfrak{C}$ witness that X and Y are strictly localisable. Possibly replacing them with refinements as in Fact 1.3 we may assume that $I \subseteq J$ and that for $i \in I$ the set Y_i is the support of $\theta(\chi_{X_i})$. Define $\zeta: Y \rightarrow \mathbb{R}^+$ by

$$\zeta = \sum_{i \in I} \theta(\chi_{X_i})^q + \sum_{j \in J \setminus I} \chi_{Y_j}.$$

This function is measurable and non zero a.e., allowing us to define another measure λ by $d\lambda = \zeta d\nu$. Then ν and λ are equivalent measures, and $D_{\nu,\lambda} \circ \theta(\chi_{X_i}) = \chi_{Y_i}$. Since this is an embedding of Banach lattices it follows that it sends every characteristic function to a characteristic function. $\blacksquare_{1.9}$

Putting everything together we obtain:

Theorem 1.10. *Let $N = L_{p(\cdot)}(X, \mathfrak{B}, \mu)$ and $N' = L_{q(\cdot)}(Y, \mathfrak{C}, \nu)$ be two Nakano spaces, $\dim N \geq 2$, and let $\theta: N \rightarrow N'$ be an isometric embedding of Banach lattices. Then up to a measure density change on Y :*

- (i) θ sends characteristic functions to such.
- (ii) $\hat{\theta}(p) = q\chi_{\hat{\theta}X}$.
- (iii) For all finite measure A : $\nu(\theta A) = \mu(A)$.

Proof. Immediate from Lemma 1.9 and Lemma 1.8. $\blacksquare_{1.10}$

Corollary 1.11. *Let $(N, \Theta) = L_{p(\cdot)}(X, \mathfrak{B}, \mu)$, $(N', \Theta') = L_{q(\cdot)}(Y, \mathfrak{C}, \nu)$ be two Nakano spaces, $\dim N \geq 2$, and let $\theta: N \rightarrow N'$ be an embedding of Banach lattices. Then θ respects the modular functional: $\Theta = \Theta' \circ \theta$.*

Proof. According to Fact 1.5 a density change on Y does not alter Θ' . Thus we may assume that θ is as in the conclusion of Theorem 1.10. By Lemma 1.7 we then obtain for all $f \in N$:

$$\Theta' \circ \theta(f) = \int |\theta(f)|^q d\nu = \int |\theta(f)|^{\hat{\theta}p} d\nu = \int \hat{\theta}(|f|^p) d\nu = \int |f|^p d\mu = \Theta(f).$$

$\blacksquare_{1.11}$

Corollary 1.12. *Let $(N, \Theta) = L_{p(\cdot)}(X, \mathfrak{B}, \mu)$ and $(N', \Theta') = L_{q(\cdot)}(Y, \mathfrak{C}, \nu)$ be two Nakano spaces, $\dim N \geq 2$, and let $\theta: N \rightarrow N'$ be an embedding of Banach lattices. Then $\text{ess rng } p \subseteq \text{ess rng } q$. If the band generated by $\theta(N)$ in N' is all of N' (so in particular, if θ is an isomorphism) then $\text{ess rng } p = \text{ess rng } q$.*

Proof. The density change does not modify p and thus neither its range, so again we may assume that θ is as in the conclusion of Theorem 1.10. It is also not difficult to see that $\text{ess rng } p = \text{ess rng } \hat{\theta}p \setminus \{0\} \subseteq \text{ess rng } q$. If the band generated by $\theta(N)$ in N' is all of N' then $\hat{\theta}X = Y$ and $q = \hat{\theta}p$. $\blacksquare_{1.12}$

In the case where θ is an isomorphism this has already been proved by Poitevin [Poi06, Proposition 3.4.4].

2. MODEL THEORY OF NAKANO SPACES

2.1. The model theoretic setting. We will assume familiarity with the general setting of continuous first order logic, as exposed in [BU] or [BBHU]. Since continuous logic only allows bounded metric structures we cannot treat Banach spaces directly. The

two standard solutions for this are either to consider a Banach space as a multi-sorted structure, with a sort for $\bar{B}(0, n)$ (the closed ball of radius n) for each n , or to restrict our consideration to the first of these sorts, i.e., the closed unit ball (a third solution is to consider it as an unbounded metric structure, see [Benb]). While Poitevin chose to use the former we consider the latter to be preferable, so a few words regarding the difference in approaches is in order.

The unit ball of a Banach space is, first of all, a complete convex space, i.e., a complete metric space equipped with a convex combination operation from an ambient Banach space. Such structures were characterised by Machado [Mac73] in a language containing all convex combinations, and this characterisation can be expressed in continuous logic. There are advantages to a minimalistic language, though, so we prefer to work in a language consisting of a single function symbol $\frac{x+y}{2}$. Convex combinations with coefficients of the form $\frac{k}{2^n}$ can be obtained as more complex terms in this language, and arbitrary convex combinations with real coefficients are obtained as limits (as our structures are by definition complete), so this language is quite sufficient. While it follows from Machado's work that an axiomatisation of unit balls of Banach spaces exists in this language, it seems preferable to put an explicit axiomatisation of this kind on record along with a complete (outline of a) proof.

Let T_{cvx} consist of the following axioms:

$$\begin{aligned}
 \text{(ID)} \quad & \forall x \left[\frac{x+x}{2} = x \right], & \text{i.e.,} \quad & \sup_x \left[d \left(\frac{x+x}{2}, x \right) \right] = 0, \\
 \text{(PRM)} \quad & \forall xyz \left[\frac{1}{2} \left(\frac{x+y}{2} + \frac{z+t}{2} \right) = \frac{1}{2} \left(\frac{z+x}{2} + \frac{t+y}{2} \right) \right], & & \text{etc.} \\
 \text{(HOM)} \quad & \forall xyz \left[d \left(\frac{x+z}{2}, \frac{y+z}{2} \right) = \frac{d(x,y)}{2} \right].
 \end{aligned}$$

We will usually be interested in subsets of Banach spaces which are not only convex, but also contain zero and are symmetric around it (i.e., $-x$ exists for all x). The unit ball is such a space, but is not the only interesting one (another one is the unit ball of a von Neumann algebra with a normalised finite trace τ : it is a *proper* subset of the unit ball of the Hilbert spaces with inner product $\langle x, y \rangle = \tau(x^*y)$). The natural language for such *symmetric* convex spaces is

$$\mathcal{L}_{Bs} = \{0, -, \frac{x+y}{2}, \|\cdot\|\}$$

We will use $\frac{x-y}{2}$ as shorthand for $\frac{x+(-y)}{2}$. Since we wish to admit the unit ball of a Banach space as a structure in this language we will interpret the distinguished distance symbol as half the usual distance $d(x, y) = \|\frac{x-y}{2}\|$, noticing the latter is an atomic formula. We define T_{sc} (for symmetric convex) as T_{cvx} along with:

$$\begin{aligned}
 \text{(SYM)} \quad & \forall x \left[\frac{x-x}{2} = 0 \right] \\
 \text{(NORM)} \quad & \forall x \left[d(x, 0) = \frac{1}{2}\|x\| \right]
 \end{aligned}$$

Finally, we define T_{Bs} , the theory of (unit balls of) Banach spaces as T_{sc} along with

$$(FULL) \quad \forall x \exists y \left[\|x\| \geq \frac{1}{2} \text{ or } \frac{y}{2} = x \right] \quad \text{i.e.,} \quad \sup_x \inf_y \left[\left(\frac{1}{2} \div \|x\| \right) \wedge d\left(\frac{y+0}{2}, x\right) \right] = 0$$

Theorem 2.1. (i) *The models of T_{cvx} are precisely complete convex subsets of diameter ≤ 1 of Banach spaces.*

(ii) *The models of T_{sc} are precisely complete convex subsets of unit balls of Banach spaces which are symmetric around zero.*

(iii) *The models of T_{Bs} are precisely closed unit balls of Banach spaces.*

Proof. For each of the assertions it is clear that all the said structures are models, so we prove the converse. We will start by examining the case of T_{cvx} , reducing it to that of T_{sc} .

From the axioms we can deduce commutativity and a variant of the triangle inequality:

$$(COMM) \quad \frac{x+y}{2} = \frac{1}{2} \left(\frac{x+x}{2} + \frac{y+y}{2} \right) = \frac{1}{2} \left(\frac{y+x}{2} + \frac{y+x}{2} \right) = \frac{y+x}{2}$$

$$(TRI2) \quad d\left(\frac{x+y}{2}, \frac{z+w}{2}\right) \leq d\left(\frac{x+y}{2}, \frac{z+y}{2}\right) + d\left(\frac{z+y}{2}, \frac{z+w}{2}\right) = \frac{d(x,z)+d(y,w)}{2}$$

Now let $C \models T_{cvx}$. Let $C - C$ be the set of all formal differences $x - y$ for $x, y \in C$, and define $d_0(x - x', y - y') = d\left(\frac{x+y'}{2}, \frac{y+x'}{2}\right)$. This is a pseudo-metric. Indeed, symmetry and reflexivity are clear, and for transitivity one checks:

$$\begin{aligned} d\left(\frac{x+z'}{2}, \frac{z+x'}{2}\right) &= 2d\left(\frac{1}{2} \left(\frac{x+z'}{2} + \frac{y+y'}{2} \right), \frac{1}{2} \left(\frac{z+x'}{2} + \frac{y+y'}{2} \right)\right) \\ &= 2d\left(\frac{1}{2} \left(\frac{x+y'}{2} + \frac{y+z'}{2} \right), \frac{1}{2} \left(\frac{z+y'}{2} + \frac{y+x'}{2} \right)\right) \\ &\stackrel{(TRI2)}{\leq} d\left(\frac{x+y'}{2}, \frac{y+x'}{2}\right) + d\left(\frac{y+z'}{2}, \frac{z+y'}{2}\right) \end{aligned}$$

Thus $d_0(x - y, z - t) = 0$ defines an equivalence \sim relation on $C - C$, and d_0 induces a metric on $C_- = (C - C)/\sim = \{[x - y] : x, y \in C\}$. It is straightforward to verify that $\frac{[x-y]+[z-t]}{2} = \left[\frac{x+z}{2} - \frac{y+t}{2}\right]$, $0 = [x-x]$ and $-[x-y] = [y-x]$ are well defined and render C_- a model of T_{sc} . Finally, if $x_0 \in C$ is any fixed element then $x \mapsto [x - x_0]$ is an embedding of C in C_- which respects convex combination and shrinks distances by a factor of 2. It follows that if we prove that C_- embeds in a Banach space, so does C . We thus reduced the first assertion to the second.

We now work modulo T_{sc} . First, observe that $d(x, y) = 2d\left(\frac{x-y}{2}, \frac{y-y}{2}\right) = 2d\left(\frac{x-y}{2}, 0\right) = \left\|\frac{x-y}{2}\right\|$. Thus the relation between the distance and the norm is as expected.

A similar reasoning shows that $\frac{x+y}{2} = 0$ implies $d(y, -x) = 2d\left(\frac{x+y}{2}, \frac{x-x}{2}\right) = 0$, so $y = -x$. It follows that $-(-x) = x$ and that $-\frac{x+y}{2} = \frac{-x-y}{2}$ (since $\frac{1}{2} \left(\frac{x+y}{2} + \frac{-x-y}{2} \right) = \frac{1}{2} \left(\frac{x-x}{2} + \frac{y-y}{2} \right) = \frac{0+0}{2} = 0$).

Fix a model $S \models T_{sc}$. For $x \in S$, let us define $\frac{1}{2}x = \frac{x+0}{2}$, and by induction we can further define $2^{-n}x$ for all n . If there is y such that $x = \frac{1}{2}y$ then y is unique (indeed, if z were another such element then $0 = d(x, x) = \frac{1}{2}d(y, z)$ so $y = z$), and we may

unambiguously write $y = 2x$. If $2\frac{x+y}{2}$ exists we write it as $x + y$. It follows from the definition that $x + 0 = x$ and $x + (-x) = 0$. By definition we have $\frac{1}{2}(x + y) = \frac{x+y}{2}$ (provided that $x + y$ exists), and applying the permutation axiom we get $\frac{1}{2}x + \frac{1}{2}y = \frac{x+y}{2}$, from which it follows that $\frac{1}{2}(-x) = -\frac{1}{2}x$ and $\frac{1}{2}(x + y) = \frac{1}{2}x + \frac{1}{2}y$ (provided $x + y$ exists).

From the commutativity of $\frac{x+y}{2}$ it follows that $x + y = y + x$, by which we mean that one exists if and only if the other does, in which case they are equal. Similarly, by the permutation axioms, if $x+y$ and $y+z$ exist then $(x+y)+z = x+(y+z)$. This means we can write something like $\sum_{i < k} x_i$ unambiguously, without having to specify either parentheses or order, as long as we know that for every subset $w \subseteq k$ the partial sum $\sum_{i \in w} x_i$ exists in some order and with some organisation of the parentheses. In particular, this means that $\sum_{i < m} k_i 2^{-n_i} x_i$ always makes sense for $n_i \in \mathbb{N}$, $k_i \in \mathbb{Z}$ satisfying $\sum 2^{-n_i} |k_i| \leq 1$, and that sums and differences of such expressions behave as expected (in particular: $2^{-n-1}x + 2^{-n-1}x = 2^{-n}x$). It follows that $k2^{-n}(\ell 2^m x) = (k\ell)2^{-n-m}x$.

It follows directly from the axioms that $\|\frac{1}{2}x\| = \frac{1}{2}d(\frac{1}{2}x, 0) = \frac{1}{2} \cdot \frac{1}{2}d(x, 0) = \frac{1}{2}\|x\|$. We obtain $\|x\| = 2d(0, x) = 2\|\frac{0-x}{2}\| = \|-x\|$, and if $x+y$ exists then $\|x+y\| = 2d(x+y, 0) \leq 2d(x+y, y) + 2d(y, 0) = 2\|\frac{(x+y)-y}{2}\| + 2\|\frac{y-0}{2}\| = \|x\| + \|y\|$. By induction on n one proves first that $\|2^{-n}x\| = 2^{-n}\|x\|$, and then that for all $0 \leq k \leq 2^n$: $\|k2^{-n}x\| = k2^{-n}\|x\|$. It follows that $\|\sum_{i < m} k_i 2^{-n} x_i\| \leq 2^{-n} \sum |k_i|$.

Thus for every $\alpha \in [-1, 1]$ we can define αx as a limit of $k_n 2^{-n} x$. We obtain that $\sum \alpha_i x_i$ always makes sense if $\sum |\alpha_i| \leq 1$, $\alpha(\beta x) = (\alpha\beta)x$, $(\alpha + \beta)x = \alpha x + \beta x$ (provided that $|\alpha + \beta| \leq 1$), $\alpha(x+y) = \alpha x + \alpha y$ (provided that $x+y$ exists), and $\|\alpha x\| = |\alpha|\|x\|$. We also have $d(\alpha x, \alpha y) = \|\frac{\alpha x - \alpha y}{2}\| = |\alpha| \|\frac{x-y}{2}\| = |\alpha|d(x, y)$, so in particular $\alpha x = \alpha y \implies x = y$ for $|\alpha| \neq 0$.

We can now define $B_0 = \mathbb{R}^{>0} \times S$, and define $(\alpha, x) \sim (\beta, y)$ if $\frac{\alpha}{\alpha+\beta}x = \frac{\beta}{\alpha+\beta}y$. It is straightforward to verify using results from the previous paragraph that \sim is an equivalence relation, and that the following operations are well defined on $B = B_0/\sim$ and render it a normed vector space over \mathbb{R} :

$$\beta[\alpha, x] = \begin{cases} [\alpha\beta, x] & \beta > 0 \\ [-\alpha\beta, -x] & \beta < 0 \\ [1, 0] & \beta = 0 \end{cases}$$

$$[\alpha, x] + [\beta, y] = \left[\alpha + \beta, \frac{\alpha}{\alpha + \beta}x + \frac{\beta}{\alpha + \beta}y \right]$$

$$\|[\alpha, x]\| = \alpha\|x\|.$$

Our structure S embeds in the unit ball of B via $x \mapsto [1, x]$.

The last assertion now follows immediately. ■_{2.1}

When dealing with models of T_{Bs} we will allow ourselves to omit the halving operation when no ambiguity may arise. Thus, for example, we will write $x + y + z = t + w$ instead of $\frac{1}{2} \left(\frac{x}{2} + \frac{y+z}{2} \right) = \frac{1}{2} \frac{t+w}{2}$, and so on.

We can now extend this to expansion of Banach spaces. For example, the natural languages for (unit balls of) Banach lattices is:

$$\mathcal{L}_{Bl} = \mathcal{L}_{Bs} \cup \left\{ \frac{x \wedge y}{2}, \frac{x \vee y}{2} \right\}.$$

The class of Banach lattices is again elementary, its theory T_{Bl} obtained by adding to T_{Bs} the usual axioms for Banach lattices. As above we will omit the halving operation in the notation, allowing ourselves to write $(x \wedge y) \vee x = x$ instead of $\frac{1}{2} \left(\frac{x \wedge y}{2} \vee \frac{1}{2} x \right) = \frac{1}{2} \frac{1}{2} x$, and so on.

Since we are dealing specifically with Nakano spaces, we may consider them in the language $\mathcal{L}_{Bl}^\Theta = \mathcal{L}_{Bl} \cup \{\Theta\}$ where Θ will interpret the modular functional. However, there is a small caveat here: the modular functional $\Theta_{p(\cdot)}$ is indeed uniformly continuous on the unit ball of $L_{p(\cdot)}(X, \mathfrak{B}, \mu)$, but its precise uniform continuity modulus depends on the essential bound of the exponent function p .

Convention 2.2. We fix here, once and for all, a *uniform* bound $1 \leq r < \infty$ on p . Thus all Nakano spaces considered henceforth will be of the form $L_{p(\cdot)}(X, \mathfrak{B}, \mu)$ where $p: X \rightarrow [1, r]$.

Let $K \subseteq [1, r]$ be compact. We will consider the following classes of structures:

$$\mathcal{N}_K^\Theta = \{ \mathcal{L}_{Bl}^\Theta\text{-structures isomorphic to some } (L_{p(\cdot)}(X, \mathfrak{B}, \mu), \Theta_{p(\cdot)}) \text{ with ess rng } p = K \},$$

$$\mathcal{N}_K = \{ N \upharpoonright_{\mathcal{L}_{Bl}} : N \in \mathcal{N}_K^\Theta \}$$

$$= \{ \mathcal{L}_{Bl}\text{-structures isomorphic to some } L_{p(\cdot)}(X, \mathfrak{B}, \mu) \text{ with ess rng } p = K \},$$

$$\mathcal{N}_{\subseteq K}^\Theta = \bigcup \{ \mathcal{N}_{K'}^\Theta : \emptyset \neq K' \subseteq K \text{ compact} \},$$

$$= \{ \mathcal{L}_{Bl}^\Theta\text{-structures isomorphic to some } (L_{p(\cdot)}(X, \mathfrak{B}, \mu), \Theta_{p(\cdot)}) \text{ with ess rng } p \subseteq K \},$$

$$\mathcal{N}_{\subseteq K} = \bigcup \{ \mathcal{N}_{K'} : \emptyset \neq K' \subseteq K \text{ compact} \} = \{ N \upharpoonright_{\mathcal{L}_{Bl}} : N \in \mathcal{N}_{\subseteq K}^\Theta \}$$

$$= \{ \mathcal{L}_{Bl}\text{-structures isomorphic to some } L_{p(\cdot)}(X, \mathfrak{B}, \mu) \text{ with ess rng } p \subseteq K \}.$$

(Of course, strictly speaking, these are the classes of the unit balls rather than of entire spaces.)

Given the uniform bound we fixed before, the largest classes we will consider are $\mathcal{N}_{\subseteq [1, r]}$ and $\mathcal{N}_{\subseteq [1, r]}^\Theta$, respectively.

Fact 2.3. *Each of the classes \mathcal{N}_K^Θ , \mathcal{N}_K , $\mathcal{N}_{\subseteq K}^\Theta$ and $\mathcal{N}_{\subseteq K}$ is elementary.*

Proof. This is just [Poi06, Proposition 3.8.2]. While the case of $\mathcal{N}_{\subseteq K}^\Theta$ is not mentioned there explicitly all the ingredients are there (in particular, as each class of the form \mathcal{N}_K^Θ is closed under ultraroots, so are classes of the form $\mathcal{N}_{\subseteq K}^\Theta$). ■_{2.3}

We may impose additional requirement, such as the dimension being greater than 1, or the lattice (equivalently, the underlying measure space) being atomless. These are first order conditions as well. For the first one we would like to say that there are functions x and y such that $\|x\| = \|y\| = 1$ and $|x| \wedge |y| = 0$, i.e.:

$$\inf_{x,y} \neg\|x\| \vee \neg\|y\| \vee \||x| \wedge |y|\| = 0.$$

Unfortunately, the absolute value $|x|$ is not a term in our language. On the other hand, $\frac{1}{2}|x| = \frac{x \vee -x}{2}$ is, and we may follow the convention of omitting the halving operation. Alternatively, one could argue that the mapping $x \mapsto |x|$ is *definable*, as $y = |x| \iff d(\frac{|x|}{2}, \frac{y}{2}) = 0$, so the axiom above does make sense directly (see [Bena] for a general discussion of definable functions in continuous logic). Similarly, atomlessness is expressible by:

$$\sup_x \inf_y \||y\| - \||x| - y|\| \vee \|y \wedge (|x| - y)\| = 0.$$

The classes of Nakano spaces of dimension at least 2 will be denoted $2\mathcal{N}_K$, $2\mathcal{N}_K^\ominus$, etc. The classes of atomless Nakano spaces will be denoted \mathcal{AN}_K , \mathcal{AN}_K^\ominus , etc.

Fact 2.4. *Assume $L_{p(\cdot)}(X, \mathfrak{B}, \mu) \in 2\mathcal{N}_K$ ($\in 2\mathcal{N}_{\subseteq K}$). Then $\text{ess rng } p = K$ ($\subseteq K$).*

Proof. [Poi06, Proposition 3.4.4]. ■_{2.4}

Remark. There are a few minor gaps in the proof of [Poi06, Proposition 3.4.4], which are corrected in a subsequent article by Poitevin. Since Fact 2.4 also follows from our more general result Corollary 1.12 these gaps should not trouble our reader.

Fact 2.5. *The theory $\text{Th}(\mathcal{AN}_K^\ominus)$ eliminates quantifiers. It follows that it is complete, as is $\text{Th}(\mathcal{AN}_K)$.*

Proof. [Poi06, Theorem 3.9.4]. ■_{2.5}

Fact 2.6. *Let $K \subseteq (1, \infty)$ be compact (so $\min K > 1$). Then the theory $\text{Th}(\mathcal{AN}_K)$ is stable.*

Proof. [Poi06, Theorem 3.10.9]. ■_{2.6}

In fact, we are cheating here a little, as Poitevin proved his results in a different language, or rather in “the same language” but in a different structure. He follow the approach described in the paragraphs following [BU, Example 4.5], viewing a Banach space N as multi-sorted structure consisting of a sort $N_m = \bar{B}(0, m)$ for each $0 < m < \omega$. The corresponding language for Banach lattices, to which we will refer here as $\mathcal{L}_{Bl, \omega}$, consists of the obvious embedding mappings between sorts, plus multiplication by (say, rational) scalars and the binary operations $+$, \wedge and \vee going from sorts or pairs of sorts to an appropriate target sort (e.g., $+$: $N_m \times N_k \rightarrow N_{m+k}$, or $\frac{1}{2}x$: $N_2 \rightarrow N_1$). The predicate symbols norm and distance can have values greater than one, but they are still bounded on each sort and thus still fit in the framework of continuous logic. Similarly, one can

define $\mathcal{L}_{Bl,\omega}^\Theta$ as $\mathcal{L}_{Bl,\omega}$ along with a predicate symbol Θ on each sort, and again in every Nakano space Θ is uniformly continuous and bounded on each sort.

In order to obtain the axiomatisability result we claimed from Poitevin's it will suffice to show that a Nakano space (viewed as an $\mathcal{L}_{Bl,\omega}$ - or $\mathcal{L}_{Bl,\omega}^\Theta$ -structure) is bi-interpretable with its unit ball (viewed as an \mathcal{L}_{Bl} - or \mathcal{L}_{Bl}^Θ -structure), and that this interpretation is uniform. For the quantifier elimination result we will have to show in addition that this interpretation is quantifier-free.

It is not difficult to see that $\mathcal{L}_{Bl,\omega}$ puts on the unit ball sort the same quantifier-free structure as \mathcal{L}_{Bl} , so "our" structures are trivially quantifier-free interpretable in Poitevin's, and the same works when naming Θ . Conversely, given a Banach lattice $N = (N_m : 0 < m < \omega)$, one can recover the Banach lattice structure on the sort N_m from that on N_1 by dilation. Thus, when considering Nakano spaces as Banach lattices, the two structures are bi-interpretable (the interpretation being uniform and quantifier-free) and there is no problem.

The converse is somewhat more complicated when the modular functional Θ is named. Indeed, every Banach lattice term τ is a $\mathcal{L}_{Bl,\omega}$ -term (possibly with a very high target sort), and $\Theta(\tau)$ is an atomic $\mathcal{L}_{Bl,\omega}^\Theta$ -formula. On the other hand, when allowing calculations only within the confines of the unit ball, all we know is that there is some $s > 0$ such that for every $0 < t \leq s$, $t\tau$ can be arbitrarily well approximated by \mathcal{L}_{Bl} -terms, so $\Theta(t\tau)$ is a limit of atomic \mathcal{L}_{Bl}^Θ -formulae (i.e., is a quantifier-free \mathcal{L}_{Bl}^Θ -definable predicate). We need to show that from this information one can recover $\Theta(\tau)$. For this we essentially repeat the proof of [Poi06, Lemma 3.4.1], showing that the limited information at our disposal suffices for its conclusion.

We recall a version Stone-Weierstrass density Theorem:

Fact 2.7. *Let X be a compact Hausdorff space and let $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{R})$ be a sub-algebra which separates points and vanishes nowhere. Then \mathcal{A} is dense in $\mathcal{C}(X, \mathbb{R})$.*

Lemma 2.8. *Let $0 < s \leq 1$, and let \mathcal{A}_s to be the family of all functions $f: [1, r] \rightarrow \mathbb{R}$ of the form $f(x) = \sum_{i < n} a_i t_i^x$, where $a_i \in \mathbb{R}$ and $0 < t_i \leq s$. Then \mathcal{A}_s is a dense sub-algebra of $\mathcal{C}([1, r], \mathbb{R})$.*

Proof. If $0 < t, t' \leq s$ then $t^x t'^x = (tt')^x$ and $0 < tt' \leq s^2 \leq s$. Thus \mathcal{A}_s is indeed a sub-algebra. Clearly it separates points and vanishes nowhere (indeed, the sub-algebra generated by $(s/2)^x$ will suffice). ■_{2.8}

Lemma 2.9. *For a Borel probability measure μ on $[1, r]$, define $\theta_\mu: [0, 1] \rightarrow \mathbb{R}$ by*

$$\theta_\mu(t) = \int_{[1,r]} t^x d\mu(x).$$

Then μ is uniquely determined by the restriction of θ_μ to $[0, s]$ for any $0 < s \leq 1$.

Proof. Let $f \in \mathcal{A}_s$, say $f(x) = \sum_{i < n} a_i t_i^x$. Then we can recover $\int_{[1,r]} f(x) d\mu(x)$ as $\sum_{i < n} a_i \theta_\mu(t_i)$. Since \mathcal{A}_s is dense in $\mathcal{C}([1,r], \mathbb{R})$ and μ has finite mass, we can recover $\int_{[1,r]} f(x) d\mu(x)$ for all $f \in \mathcal{C}([1,r], \mathbb{R})$. This determines μ uniquely. $\blacksquare_{2.9}$

Lemma 2.10. *Let $(N, \Theta) = (L_{p(\cdot)}(X, \mathfrak{B}, \mu), \Theta_{p(\cdot)})$, let \bar{g} be a tuple in the unit ball of N , and let $\tau(\bar{x})$ be a Banach lattice term. Then the quantifier-free \mathcal{L}_{Bl}^Θ -type of \bar{g} determines $\Theta(\tau(\bar{g}))$.*

Proof. Let $f = \tau(\bar{g})$. Since $\Theta(f) = \Theta(|f|)$ and $|\tau|$ is also a term, we may assume that $f \geq 0$. Up to a density change, we may further assume that $f = \|f\| \chi_A$ where $\mu(A) = 1$.

Restrict everything to A , and let π_A be the image of $\mu \upharpoonright_A$ under $p \upharpoonright_A$. Then π_A is a probability measure on $[1,r]$, and for all t : $\Theta(tf) = \theta_{\pi_A}(t\|f\|)$. As we said above, there is some $s > 0$ such that the quantifier-free type of \bar{g} gives us $\Theta(tf)$ for all $0 < t \leq s$, i.e., $\theta_{\pi_A}(t)$ for all $0 < t \leq s/\|f\|$. By the results above this determines π_A , and thus determines $\theta_{\pi_A}(t)$ for all t . In particular, it determines $\theta_{\pi_A}(\|f\|) = \Theta(f)$. $\blacksquare_{2.10}$

So let $\bar{f} = f_{<n}$ be a tuple in a Nakano space N , say that $f_i \in N_{m_i}$, and let $g_i = f_i/m_i$. Then $\bar{g} \in N_1$, and the quantifier-free $\mathcal{L}_{Bl,\omega}^\Theta$ -type of \bar{f} can be recovered from the quantifier-free \mathcal{L}_{Bl}^Θ -type of \bar{g} . We have thus shown:

Fact 2.11. *Let $N_\omega = (N_n : 0 < n < \omega)$ be the $\mathcal{L}_{Bl,\omega}^\Theta$ -structure (or $\mathcal{L}_{Bl,\omega}$ -structure) associated to a Nakano Banach lattice $N = L_{p(\cdot)}(X, \mathfrak{B}, \mu)$. Then each of the sorts N_n can be interpreted by dilation in the unit ball sort N_1 . This interpretation is moreover uniform (i.e., does not depend on N) and quantifier-free.*

It follows that the $\mathcal{L}_{Bl,\omega}^\Theta$ -structure (respectively, $\mathcal{L}_{Bl,\omega}$ -structure) and \mathcal{L}_{Bl}^Θ -structure (respectively, \mathcal{L}_{Bl} -structure) associated to N are uniformly quantifier-free bi-interpretable.

Therefore, Poitevin's axiomatisability and quantifier-elimination results do transfer to our formalism. Conversely, this means that once we show that the modular functional of a Nakano space is \mathcal{L}_{Bl} -definable in the unit ball (e.g., as in Theorem 3.1), this result transfers from the unit ball to the whole space.

3. DEFINABILITY OF THE MODULAR FUNCTIONAL

This section contains the main model theoretic results of this paper. We start with the definability result.

Theorem 3.1. *The modular functional Θ is uniformly \mathcal{L}_{Bl} -definable in $2\mathcal{N}_{\subseteq[1,r]}^\Theta$. Moreover, it is both uniformly inf-definable and sup-definable and can be used to axiomatise $2\mathcal{N}_{\subseteq[1,r]}^\Theta$ modulo the axioms for $2\mathcal{N}_{\subseteq[1,r]}^\Theta$.*

More precisely:

- (i) *There exists a \mathcal{L}_{Bl} -definable predicate $\varphi_\Theta(x)$ such that $(N, \Theta) \models \Theta(x) = \varphi_\Theta(x)$ for all $(N, \Theta) \in 2\mathcal{N}_{\subseteq[1,r]}^\Theta$.*

- (ii) *There quantifier-free \mathcal{L}_{Bl} -formulae $\psi_n(x, \bar{y}_n)$ and $\chi_n(x, \bar{z}_n)$ such that in all Nakano spaces of dimension at least two:*

$$\Theta(x) = \varphi_\Theta(x) = \lim_{n \rightarrow \infty} \inf_{\bar{y}_n} \psi_n(x, \bar{y}_n) = \lim_{n \rightarrow \infty} \sup_{\bar{z}_n} \chi_n(x, \bar{z}_n),$$

each of the limits converging uniformly and at a uniform rate.

- (iii) *The theory $\text{Th}(2\mathcal{N}_{\subseteq[1,r]}^\Theta)$ is equivalent to $\text{Th}(2\mathcal{N}_{\subseteq[1,r]}) \cup \{\Theta(x) = \varphi_\Theta(x)\}$.*

Proof. By Corollary 1.11 every $N \in 2\mathcal{N}_{\subseteq[1,r]}$ admits at most one expansion to $(N, \Theta) \in 2\mathcal{N}_{\subseteq[1,r]}^\Theta$. As these are elementary classes, one can apply Theorem A.1 (Beth's theorem for continuous logic) in order to obtain φ_Θ .

Using Corollary 1.11 again we see that φ_Θ is constant in $2\mathcal{N}_{\subseteq[1,r]}$ (see Definition A.2). By Theorem A.4 it is both inf-definable and sup-definable there.

The last item is immediate. ■_{3.1}

Corollary 3.2. *For a fixed compact $K \subseteq [1, r]$, the modular functional is uniformly \mathcal{L}_{Bl} -definable in \mathcal{N}_K^Θ .*

In particular the modular functional is \mathcal{L}_{Bl} -definable in every Nakano Banach lattice.

Proof. If $K = \{p_0\}$ is a single point, we have $\Theta(f) = \|f\|^{p_0}$. Otherwise $\mathcal{N}_K^\Theta = 2\mathcal{N}_K^\Theta \subseteq 2\mathcal{N}_{\subseteq[1,r]}^\Theta$ and we can apply Theorem 3.1. ■_{3.2}

We have shown that naming the modular functional does not add structure. Still, in the case of an atomless Nakano space naming Θ does give something, namely quantifier elimination. It is clear that without Θ quantifier elimination would be impossible: the complete \mathcal{L}_{Bl}^Θ -type of a function contains, among other information, the essential range of p on its support, and there is no way of recovering this information from the quantifier-free \mathcal{L}_{Bl} -type of a single positive function, as it is determined by its norm alone.

A next-best would be to obtain model completeness. Indeed, all the work for obtaining it is already done.

Theorem 3.3. *For every compact $K \subseteq [1, r]$ the (theory of the) class \mathcal{AN}_K is model complete.*

Proof. Follows from Corollary 1.11 and the quantifier elimination in \mathcal{AN}_K^Θ . ■_{3.3}

Our previous work also yields:

Proposition 3.4. *The (theories of the) classes \mathcal{AN}_K^Θ , \mathcal{AN}_K , $\mathcal{AN}_{\subseteq K}^\Theta$ and $\mathcal{AN}_{\subseteq K}$ are all inductive.*

Proof. By Theorem 1.10, any increasing chain of inclusions $N_0 \subseteq N_1 \subseteq \dots \subseteq N_i \subseteq \dots$ of models of any of these theories can be presented as an increasing sequence of measure algebras (\mathfrak{B}_i, μ_i) along with extensions of the exponent functions p_i . Letting $(\mathfrak{B}, \mu) = \bigcup (\mathfrak{B}_i, \mu_i)$ and $p = \bigcup p_i$ we see that $\bigcup N_i = L_{p(\cdot)}(\mathfrak{B}, \mu)$ and $\text{ess rng}(p) = \overline{\bigcup \text{ess rng}(p_i)}$. ■_{3.4}

4. PERTURBATIONS OF THE EXPONENT

Intuitively, a small change to the exponent function p should not change the structure of a Nakano space by too much. We formalise this intuitive idea, showing that small perturbations of the exponent form indeed a perturbation system in the sense of [Benb]. We show that up to such perturbations, every complete theory of Nakano spaces is \aleph_0 -categorical and \aleph_0 -stable. In case p is constant (i.e., K is a singleton), we already know (see, e.g., [BBH]) that the theory is \aleph_0 -stable and \aleph_0 -categorical without perturbation. Indeed, no perturbation of p is possible in this case, so it is a special case of what we prove below.

4.1. Preliminary computations. We seek to find bounds for $1 + \gamma^s$ in terms of $(1 + \gamma)^s$, and for $1 - \gamma^s$ in terms of $(1 - \gamma)^s$, where $\gamma \in [0, 1]$ and $s \in [1/r, r]$. The function $\frac{1 + \gamma^s}{(1 + \gamma)^s}$ is well behaved, i.e., continuous as a function of two variables, and will not cause trouble. The function $\frac{1 - \gamma^s}{(1 - \gamma)^s}$ is badly behaved near $\gamma = 1$, so we will only use it for $\gamma \in [0, \frac{1}{2}]$. For $\gamma \in [\frac{1}{2}, 1]$ we will have to consider another function, namely $\varphi(\gamma, s) = \frac{\ln(1 - \gamma^s)}{\ln(1 - \gamma)}$, which is, for the time being, only defined for $\gamma \in (0, 1)$ and $s > 0$. We calculate its limit as $\gamma \rightarrow 1$ for a fixed $s > 0$ making several uses of l'Hôpital's rule (marked with *):

$$\begin{aligned} \lim_{\gamma \rightarrow 1} \frac{\ln(1 - \gamma^s)}{\ln(1 - \gamma)} &=^* \lim_{\gamma \rightarrow 1} \frac{-s\gamma^{s-1}(1 - \gamma^s)^{-1}}{-(1 - \gamma)^{-1}} = \lim_{\gamma \rightarrow 1} \frac{s\gamma^{s-1}(1 - \gamma)}{1 - \gamma^s} \\ &=^* \lim_{\gamma \rightarrow 1} \frac{s(s-1)\gamma^{s-2} - s^2\gamma^{s-1}}{-s\gamma^{s-1}} = \frac{-s}{-s} = 1. \end{aligned}$$

It is therefore natural to extend φ by $\varphi(1, s) = 1$. This function is continuous in each variable for $s > 0$ and $\gamma \in (0, 1]$, and we wish to show that it is continuous as a function of two variables. In fact, all we need is to show it is continuous on $[\frac{1}{2}, 1] \times [1, r]$.

Assume $\gamma \in (0, 1)$, $s \in [1, r]$. A straightforward verification leads to:

$$\frac{\ln(1 - \gamma^r)}{\ln(1 - \gamma)} \leq \frac{\ln(1 - \gamma^s)}{\ln(1 - \gamma)} \leq 1,$$

whereby:

$$\left| 1 - \frac{\ln(1 - \gamma^s)}{\ln(1 - \gamma)} \right| \leq \left| 1 - \frac{\ln(1 - \gamma^r)}{\ln(1 - \gamma)} \right|.$$

Thus $\lim_{\gamma \rightarrow 1} \frac{\ln(1 - \gamma^s)}{\ln(1 - \gamma)} = 1$ *uniformly* for $s \in [1, r]$, and $\varphi(\gamma, s)$ is indeed continuous on $[\frac{1}{2}, 1] \times [1, r]$.

We now define for $1 \leq s \leq r$:

$$\begin{aligned} A_s &= \inf \left\{ \frac{\ln(1 - \gamma^s)}{s \ln(1 - \gamma)} : \gamma \in [\frac{1}{2}, 1) \right\} \leq \frac{1}{s}, \\ B_s^- &= \sup \left\{ \frac{1 - \gamma^t}{(1 - \gamma)^t} : \gamma \in [0, \frac{1}{2}], t \in [1/s, s] \right\}, \\ B_s^+ &= \sup \left\{ \frac{1 + \gamma^t}{(1 + \gamma)^t} : \gamma \in [0, 1], t \in [1/s, s] \right\}, \\ B_s &= \max\{B_s^-, B_s^+\}. \end{aligned}$$

By continuity of $\varphi(\gamma, s)$, and as $\varphi(\gamma, 1) = 1$ for all γ : $\lim_{s \rightarrow 1} A_s = \lim_{s \rightarrow 1} \frac{1}{s} = 1$. Similarly $\lim_{s \rightarrow 1} B_s = 1$.

In particular we have for $s \in [1, r]$ and $\gamma \in [\frac{1}{2}, 1)$: $A_s \leq \frac{\ln(1 - \gamma^s)}{s \ln(1 - \gamma)}$ whereby $s A_s \ln(1 - \gamma) \geq \ln(1 - \gamma^s)$ and thus $(1 - \gamma)^{s A_s} \geq 1 - \gamma^s$.

Lemma 4.1. *Let $\alpha, \beta \in [-1, 1]$ and $1/s \leq t \leq s$. Then*

$$|\operatorname{sgn}(\alpha)|\alpha|^t - \operatorname{sgn}(\beta)|\beta|^t| \leq \max \left\{ |\alpha - \beta|^{A_s t}, B_s |\alpha - \beta|^t \right\}.$$

Proof. We may assume that $|\alpha| \geq |\beta|$ by symmetry. We may further assume that $\alpha, \beta \neq 0$. Assume first that $\operatorname{sgn}(\alpha\beta) = -1$. Then:

$$\begin{aligned} |\operatorname{sgn}(\alpha)|\alpha|^t - \operatorname{sgn}(\beta)|\beta|^t| &= |\alpha|^t(1 + |\beta/\alpha|^t) \\ &\leq |\alpha|^t B_s (1 + |\beta/\alpha|^t) \\ &= B_s |\alpha - \beta|^t. \end{aligned}$$

A similar argument shows that when $\operatorname{sgn}(\alpha\beta) = 1$ and $|\beta/\alpha| \leq 1/2$:

$$|\operatorname{sgn}(\alpha)|\alpha|^t - \operatorname{sgn}(\beta)|\beta|^t| \leq B_s |\alpha - \beta|^t.$$

Finally, assume $\operatorname{sgn}(\alpha\beta) = 1$ and $|\beta/\alpha| \geq 1/2$. We use the fact that $|\alpha| \leq 1$ and $A_s \leq 1/s < 1$ imply that $|\alpha| \leq |\alpha|^{A_s}$:

$$\begin{aligned} |\operatorname{sgn}(\alpha)|\alpha|^t - \operatorname{sgn}(\beta)|\beta|^t| &= |\alpha|^t(1 - |\beta/\alpha|^t) \leq |\alpha|^t(1 - |\beta/\alpha|^s) \\ &\leq |\alpha|^t(1 - |\beta/\alpha|)^{A_s s} \\ &\leq |\alpha|^{A_s t}(1 - |\beta/\alpha|)^{A_s t} \\ &= |\alpha - \beta|^{A_s t} \end{aligned} \quad \blacksquare_{4.1}$$

Lemma 4.2. *For all $\gamma, t \in [0, 1]$: $t(1 - \gamma) + \gamma^t \leq 1$ (where $0^0 = 1$).*

Proof. This is clear for $t \in \{0, 1\}$. So let $t \in (0, 1)$, and we will show that for all $\gamma \in [0, 1]$: $t(1 - \gamma) + \gamma^t - 1 \leq 0$. For $\gamma = 1$ this is true, so it will suffice to show that the derivative is

positive for all $\gamma \in (0, 1)$. Indeed, for $0 < \gamma < 1$ and $t - 1 < 0$ we have $\gamma^{t-1} > 1$ whereby:

$$\frac{d}{d\gamma}(t(1 - \gamma) + \gamma^t - 1) = -t + t\gamma^{t-1} > -t + t = 0. \quad \blacksquare_{4.2}$$

For $1 \leq s \leq r$ and $0 \leq x \leq 2$, define:

$$\eta_s(x) = \begin{cases} x^{A_s/s} & x \leq 1 \\ x^s & 1 < x \leq 2 \end{cases}$$

$$\hat{\eta}_s(x) = 2^{1-A_s} B_s \eta_s(x) / A_s.$$

Lemma 4.3. *As $s \rightarrow 1$, the functions η_s converge uniformly to the identity. As a consequence, $\hat{\eta}_s \rightarrow \text{id}$ uniformly as $s \rightarrow 1$.*

Proof. For η_s , one verifies uniform convergence separately for $x \mapsto x^{A_s/s}$ on $[0, 1]$ and for $x \mapsto x^s$ on $[1, 2]$. Uniform convergence of $\hat{\eta}_s$ follows. $\blacksquare_{4.3}$

For $1 \leq s \leq r$, define:

$$C_s^1 = \sup \{|x - x^s| : x \in [0, 1]\}$$

$$C_s^2 = \sup \{|x - \hat{\eta}_s(x)| : x \in [0, 2]\}.$$

$$C_s = \max\{C_s^1, C_s^2\}.$$

Then $\lim_{s \rightarrow 1} C_s = 0$.

Lemma 4.4. *Let $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be convex, $f(0) = 0$, and let $\alpha \geq |\beta| \geq 0$. Then*

$$f\left(\frac{\alpha + \beta}{2}\right) \leq \frac{f(\alpha) + \text{sgn}(\beta)f(|\beta|)}{2}.$$

Proof. If $\beta \geq 0$ this is by definition of convexity, so assume $\beta < 0$. As f is convex it is differentiable in all but countably many points, $f'(x)$ is increasing, and $f(x) = \int_0^x f'(t) dt$. Thus:

$$\begin{aligned} \frac{f(\alpha) - f(-\beta)}{2} &= \frac{1}{2} \int_{-\beta}^{\alpha} f'(t) dt \geq \frac{1}{2} \int_0^{\alpha+\beta} f'(t) dt \\ &\geq \int_0^{\frac{\alpha+\beta}{2}} f'(t) dt = f\left(\frac{\alpha + \beta}{2}\right). \end{aligned} \quad \blacksquare_{4.4}$$

Lemma 4.5. *For all $\alpha, \beta \in [-1, 1]$ and $t \in [1/s, s]$:*

$$\left| \text{sgn}\left(\frac{\alpha+\beta}{2}\right) \left|\frac{\alpha+\beta}{2}\right|^t - \frac{\text{sgn}(\alpha)|\alpha|^t + \text{sgn}(\beta)|\beta|^t}{2} \right| \leq C_s.$$

Proof. We use the fact that $x \mapsto x^{ts}$ is convex while $x \mapsto x^{t/s}$ is concave.

Up to changing α and β we may assume that $|\alpha| \geq |\beta|$, and up to changing the signs we may assume that $\alpha = |\alpha| \geq 0$. Then $\alpha \geq |\beta|$, and regardless of $\text{sgn}(\beta)$ we have:

$$\alpha^{ts} + \text{sgn}(\beta)|\beta|^{ts} \leq \alpha^t + \text{sgn}(\beta)|\beta|^t \leq \alpha^{t/s} + \text{sgn}(\beta)|\beta|^{t/s}.$$

We now use Lemma 4.4 and obtain:

$$\begin{aligned} \left(\frac{\alpha+\beta}{2}\right)^t - \frac{\alpha^t + \text{sgn}(\beta)|\beta|^t}{2} &\leq \left(\left(\frac{\alpha+\beta}{2}\right)^{ts} + C_s\right) - \frac{\alpha^{ts} + \text{sgn}(\beta)|\beta|^{ts}}{2} \leq C_s \\ \left(\frac{\alpha+\beta}{2}\right)^t - \frac{\alpha^t + \text{sgn}(\beta)|\beta|^t}{2} &\geq \left(\left(\frac{\alpha+\beta}{2}\right)^{t/s} - C_s\right) - \frac{\alpha^{t/s} + \text{sgn}(\beta)|\beta|^{t/s}}{2} \geq -C_s. \end{aligned} \quad \blacksquare_{4.5}$$

4.2. Perturbing the exponent.

Definition 4.6. Let (X, \mathfrak{B}, μ) be a measure space and $p, q: X \rightarrow [1, r]$ measurable. We define $\mathcal{E}_{p,q}: L_0(X, \mathfrak{B}, \mu) \rightarrow L_0(X, \mathfrak{B}, \mu)$ by:

$$(\mathcal{E}_{p,q}f)(x) = \text{sgn}(f(x))|f(x)|^{p(x)/q(x)}.$$

Lemma 4.7. *We continue with the assumptions of Definition 4.6. Let $(N, \Theta) = (L_{p(\cdot)}(X, \mathfrak{B}, \mu), \Theta_{p(\cdot)})$ and $(N', \Theta') = (L_{q(\cdot)}(X, \mathfrak{B}, \mu), \Theta_{q(\cdot)})$.*

- (i) *For each $f \in L_0(X, \mathfrak{B}, \mu)$ we have $\Theta(f) = \Theta'(\mathcal{E}_{p,q}f)$. Thus in particular $\mathcal{E}_{p,q}$ sends N into N' and the unit ball of N into the unit ball of N' .*
- (ii) *The mapping $\mathcal{E}_{p,q}$ is bijective, its inverse being $\mathcal{E}_{q,p}$. It restricts to a bijection between N and N' , as well as to a bijection between their respective unit balls.*
- (iii) *The mapping $\mathcal{E}_{p,q}$ commutes with measure density change. More precisely, assume ν is another measure on (X, \mathfrak{B}) , equivalent to μ , say $d\nu(x) = \zeta(x)d\mu(x)$. Let $M = L_{p(\cdot)}(X, \mathfrak{B}, \nu)$, $M' = L_{q(\cdot)}(X, \mathfrak{B}, \nu)$. Let $D_{\mu,\nu}^p: N \rightarrow M$ and $D_{\mu,\nu}^q: N' \rightarrow M'$ be the respective density change mappings. Then $D_{\mu,\nu}^q \circ \mathcal{E}_{p,q} = \mathcal{E}_{p,q} \circ D_{\mu,\nu}^p: N \rightarrow M'$.*

Proof. For the first item we calculate that:

$$\Theta'(\mathcal{E}_{p,q}f) = \int |f(x)|^{p(x)} d\mu = \Theta(f).$$

The second item follows. Finally, we calculate:

$$\begin{aligned} (D_{\mu,\nu}^q \mathcal{E}_{p,q}f)(x) &= \zeta(x)^{-1/q(x)} (\mathcal{E}_{p,q}f)(x) \\ &= \zeta(x)^{-1/q(x)} \text{sgn}(f(x)) |f(x)|^{p(x)/q(x)} \\ &= \text{sgn}(\zeta(x)^{-1/p(x)} f(x)) |\zeta(x)^{-1/p(x)} f(x)|^{p(x)/q(x)} \\ &= \text{sgn}((D_{\mu,\nu}^p f)(x)) |(D_{\mu,\nu}^p f)(x)|^{p(x)/q(x)} \\ &= (\mathcal{E}_{p,q} D_{\mu,\nu}^p f)(x). \end{aligned} \quad \blacksquare_{4.7}$$

Proposition 4.8. *We continue with the notation and assumptions of Lemma 4.7. Assume that s is such that $1/s \leq q(x)/p(x) \leq s$ (for example, we can take $s = r$). Then for every $f, g \in N_1$ (the unit ball of N): $\|\mathcal{E}_{p,q}f - \mathcal{E}_{p,q}g\| \leq \hat{\eta}_s(\|f - g\|)$ and $\|f - g\| \leq \hat{\eta}_s(\|\mathcal{E}_{p,q}f - \mathcal{E}_{p,q}g\|)$.*

Proof. Let $f, g \in N_1$. By Lemma 4.7(iii) we may assume that $|f| \vee |g| = \chi_S$ for some set $S \in \mathfrak{B}$, so $f(x), g(x) \in [-1, 1]$. Let

$$\begin{aligned} h(x) &= |\operatorname{sgn}(f(x))|f(x)|^{p(x)/q(x)} - \operatorname{sgn}(g(x))|g(x)|^{p(x)/q(x)}| \\ S_1 &= \{x \in S : h(x) \leq |f(x) - g(x)|^{A_s p(x)/q(x)}\} \\ S_2 &= S \setminus S_1 \subseteq \{x \in S : h(x) \leq B_s |f(x) - g(x)|^{p(x)/q(x)}\}. \end{aligned}$$

We observe that as $\|f\|, \|g\| \leq 1$ we have $\mu(S) \leq 2$. Observe also that $A_s q(x)/s \leq q(x)/s \leq p(x)$ and that $sq(x)/A_s \geq sq(x) \geq p(x)$. It follows that if $\|f - g\| \leq 1$ then:

$$\begin{aligned} \|f - g\|^{p(x)} &\leq \eta_s(\|f - g\|)^{q(x)/A_s} = \|f - g\|^{q(x)/s} \\ &\leq \eta_s(\|f - g\|)^{q(x)} = \|f - g\|^{A_s q(x)/s}. \end{aligned}$$

Otherwise $1 < \|f - g\| \leq 2$, and:

$$\begin{aligned} \|f - g\|^{p(x)} &\leq \eta_s(\|f - g\|)^{q(x)} = \|f - g\|^{sq(x)} \\ &\leq \eta_s(\|f - g\|)^{q(x)/A_s} = \|f - g\|^{sq(x)/A_s}. \end{aligned}$$

Let $\gamma = \int_{S_1} \frac{|f(x) - g(x)|^{p(x)}}{\|f - g\|^{p(x)}} d\mu(x)$ and $a = \hat{\eta}_s(\|f - g\|) = 2^{1-A_s} B_s \eta_s(\|f - g\|)/A_s$. Then:

$$\begin{aligned} \Theta' \left(\frac{\mathcal{E}_{p,q} f - \mathcal{E}_{p,q} g}{a} \right) &= \int_S \frac{h(x)^{q(x)}}{a^{q(x)}} d\mu(x) \\ &\leq \int_{S_1} \frac{|f(x) - g(x)|^{A_s p(x)}}{a^{q(x)}} d\mu(x) + \int_{S_2} \frac{B_s^{q(x)} |f(x) - g(x)|^{p(x)}}{a^{q(x)}} d\mu(x) \end{aligned}$$

We will work on each integral separately.

$$\begin{aligned} \int_{S_1} \frac{|f(x) - g(x)|^{A_s p(x)}}{a^{q(x)}} d\mu(x) &= \int_{S_1} \frac{\mu(S_1) A_s^{q(x)}}{(2^{1-A_s} B_s)^{q(x)}} \left(\frac{|f(x) - g(x)|^{p(x)}}{\eta_s(\|f - g\|)^{q(x)/A_s}} \right)^{A_s} \frac{d\mu(x)}{\mu(S_1)} \\ &\leq \frac{\mu(S_1)}{2^{1-A_s}} \int_{S_1} \left(\frac{|f(x) - g(x)|^{p(x)}}{\|f - g\|^{p(x)}} \right)^{A_s} \frac{d\mu(x)}{\mu(S_1)} \\ &\leq \frac{\mu(S_1)}{2^{1-A_s}} \left(\int_{S_1} \frac{|f(x) - g(x)|^{p(x)}}{\|f - g\|^{p(x)}} \frac{d\mu(x)}{\mu(S_1)} \right)^{A_s} \\ &= \frac{\mu(S)^{1-A_s}}{2^{1-A_s}} \gamma^{A_s} \leq \gamma^{A_s}. \end{aligned}$$

And:

$$\begin{aligned} \int_{S_2} \frac{B_s^{q(x)} |f(x) - g(x)|^{p(x)}}{a^{q(x)}} d\mu(x) &= \int_{S_2} \frac{(A_s B_s)^{q(x)} |f(x) - g(x)|^{p(x)}}{(2^{1-A_s} B_s)^{q(x)} \eta_s(\|f - g\|)^{q(x)}} d\mu(x) \\ &\leq A_s \int_{S_2} \frac{|f(x) - g(x)|^{p(x)}}{\|f - g\|^{p(x)}} d\mu(x) = A_s (1 - \gamma). \end{aligned}$$

Thus:

$$\Theta' \left(\frac{\mathcal{E}_{p,q}f - \mathcal{E}_{p,q}g}{a} \right) \leq \gamma^{A_s} + A_s(1 - \gamma) \leq 1$$

We conclude that $\|\mathcal{E}_{p,q}f - \mathcal{E}_{p,q}g\| \leq a = \hat{\eta}_s(\|f - g\|)$. As $1/s \leq p(x)/q(x) \leq s$ as well we have

$$\|f - g\| = \|\mathcal{E}_{q,p}\mathcal{E}_{p,q}f - \mathcal{E}_{q,p}\mathcal{E}_{p,q}g\| \leq \hat{\eta}_s(\|\mathcal{E}_{p,q}f - \mathcal{E}_{p,q}g\|).$$

■_{4.8}

Corollary 4.9. *The mapping $\mathcal{E}_{p,q}: N_1 \rightarrow N'_1$ is uniformly continuous, the modulus of uniform continuity depending solely on r .*

Proof. Define $\Delta_r(\varepsilon) = \min \left\{ (2^{A_r-1}A_r\varepsilon/B_r)^{r/A_r}, 1 \right\}$. Then for all $\varepsilon > 0$ we have $\Delta_r(\varepsilon) > 0$ and $\|f - g\| < \Delta_r(\varepsilon) \implies \|\mathcal{E}_{p,q}f - \mathcal{E}_{p,q}g\| \leq \varepsilon$. ■_{4.9}

Proposition 4.10. *Let $\mathcal{E}_{p,q}: N \rightarrow N'$ be as in Definition 4.6, and let $f, g \in N_1$. Then:*

- (i) $\mathcal{E}_{p,q}0 = 0$; $-\mathcal{E}_{p,q}f = \mathcal{E}_{p,q}(-f)$; $\mathcal{E}_{p,q}(f \wedge g) = \mathcal{E}_{p,q}f \wedge \mathcal{E}_{p,q}g$; and $\mathcal{E}_{p,q}(f \vee g) = \mathcal{E}_{p,q}f \vee \mathcal{E}_{p,q}g$.
- (ii) $\left| \|f - g\| - \|\mathcal{E}_{p,q}f - \mathcal{E}_{p,q}g\| \right| \leq C_s$.
- (iii) $\left\| \mathcal{E}_{p,q} \frac{f+g}{2} - \frac{\mathcal{E}_{p,q}f + \mathcal{E}_{p,q}g}{2} \right\| \leq 2C_s$.

Proof. The first item is clear. For the second we use Proposition 4.8:

$$\begin{aligned} \|\mathcal{E}_{p,q}f - \mathcal{E}_{p,q}g\| - \|f - g\| &\leq \hat{\eta}_s(\|f - g\|) - \|f - g\| \leq C_s, \\ \|f - g\| - \|\mathcal{E}_{p,q}f - \mathcal{E}_{p,q}g\| &\leq \hat{\eta}_s(\|\mathcal{E}_{p,q}f - \mathcal{E}_{p,q}g\|) - \|\mathcal{E}_{p,q}f - \mathcal{E}_{p,q}g\| \leq C_s. \end{aligned}$$

For the third we may assume that $|f| \vee |g| = \chi_S$ for some measurable set S , so $\mu(S) \leq 2$.

By Lemma 4.5 we have $\left| \mathcal{E}_{p,q} \frac{f+g}{2}(x) - \frac{\mathcal{E}_{p,q}f + \mathcal{E}_{p,q}g}{2}(x) \right| \leq C_s$ for $x \in S$, and we get:

$$\Theta \left(\frac{\mathcal{E}_{p,q} \frac{f+g}{2} - \frac{\mathcal{E}_{p,q}f + \mathcal{E}_{p,q}g}{2}}{2C_s} \right) \leq \int_S 2^{-q(x)} d\mu(x) \leq \frac{\mu(S)}{2} \leq 1. \quad \blacksquare_{4.10}$$

We now wish to define a perturbation system \mathfrak{p} for \mathcal{L}_{BI} -structures. We do this by defining a $\mathfrak{p}(\varepsilon)$ -perturbation of structures N and N' directly as a bijection $\theta: N \rightarrow N'$ such that for all $f, g, h \in N$:

$$\begin{aligned} \theta 0 &= 0, \\ \theta(-f) &= -\theta f, \\ \left| d\left(\frac{f+g}{2}, h\right) - d\left(\frac{\theta f + \theta g}{2}, \theta h\right) \right| &\leq \varepsilon, \\ \left| d\left(\frac{f \wedge g}{2}, h\right) - d\left(\frac{\theta f \wedge \theta g}{2}, \theta h\right) \right| &\leq \varepsilon, \\ \left| d\left(\frac{f \vee g}{2}, h\right) - d\left(\frac{\theta f \vee \theta g}{2}, \theta h\right) \right| &\leq \varepsilon, \\ \left| \|f\| - \|\theta f\| \right| &\leq \varepsilon, \end{aligned}$$

and:

$$e^{-\varepsilon e^\varepsilon} d(f, g)^{e^\varepsilon} \leq d(\theta f, \theta g) \leq e^\varepsilon d(f, g)^{e^{-\varepsilon}}.$$

(While for most symbols we can just allow to “change by ε ”, we need to take special care with the distance symbol.) This indeed defines a perturbation system, as it clearly verifies the following characterisation:

Fact 4.11. *Let T be a theory, and assume that for each $r \in \mathbb{R}^+$ and $M, N \in \text{Mod}(T)$, $\text{Pert}'_r(M, N)$ is a set of bijections of M with N satisfying the following properties:*

- (i) *Monotonicity:* $\text{Pert}'_r(M, N) = \bigcap_{s>r} \text{Pert}'_s(M, N)$.
- (ii) *Non-degenerate reflexivity:* $\text{Pert}'_0(M, N)$ is the set of isomorphisms of M with N .
- (iii) *Symmetry:* $f \in \text{Pert}'_r(M, N)$ if and only if $f^{-1} \in \text{Pert}'_r(N, M)$.
- (iv) *Transitivity:* if $f \in \text{Pert}'_r(M, N)$ and $g \in \text{Pert}'_s(N, L)$ then $g \circ f \in \text{Pert}'_{r+s}(M, L)$.
- (v) *Uniform continuity:* for each $r \in \mathbb{R}^+$, all members of $\text{Pert}'_r(M, N)$, where M, N vary over all models of T , satisfy a common modulus of uniform continuity.
- (vi) *Ultraproducts:* If $f_i \in \text{Pert}'_r(M_i, N_i)$ for $i \in I$, and \mathcal{U} is an ultrafilter on I then $\prod_{\mathcal{U}} f_i \in \text{Pert}'_r(\prod_{\mathcal{U}} M_i, \prod_{\mathcal{U}} N_i)$. (Note that $\prod_{\mathcal{U}} f_i$ exists by the uniform continuity assumption).
- (vii) *Elementary substructures:* If $f \in \text{Pert}'_r(M, N)$, $M_0 \preceq M$, and $N_0 = f(M_0) \preceq N$ then $f \upharpoonright_{M_0} \in \text{Pert}'_r(M_0, N_0)$.

Then there exists a unique perturbation system \mathfrak{p} for T such that $\text{Pert}'_r(M, N) = \text{Pert}_{\mathfrak{p}(r)}(M, N)$ for all r, M and N .

Proof. [Benc, Theorem 4.4]. ■_{4.11}

Recall that given two n -types p, q we say that $d_{\mathfrak{p}}(p, q) \leq \varepsilon$ if there are \mathcal{L}_{BI} -structures N, N' and an ε -perturbation $\theta: N \rightarrow N'$ sending a realisation of p to one of q .

Lemma 4.12. *For every $\varepsilon > 0$ there exists $s > 1$ such that if $N = L_{p(\cdot)}(X, \mathfrak{B}, \mu)$, $N' = L_{q(\cdot)}(X, \mathfrak{B}, \mu)$ and $\mathcal{E}_{p,q}: N \rightarrow N'$ is as in Definition 4.6 (so in particular $1/s \leq p(x)/q(x) \leq s$ for almost all $x \in X$), then $\mathcal{E}_{p,q}$ is a $\mathfrak{p}(\varepsilon)$ -perturbation.*

Proof. By Proposition 4.8, Proposition 4.10 and the fact that $\lim_{s \rightarrow 1} C_s = 0$. ■_{4.12}

Lemma 4.13. *Fix a compact $K \subseteq [1, r]$ and $s > 1$. Then there is a finite set $K_s \subseteq [0, 1]$ such that for every atomless measure space (X, \mathfrak{B}, μ) and $p: X \rightarrow [1, r]$ with $\text{ess rng}(p) = K$ there exists $q: X \rightarrow [1, r]$ such that $\text{ess rng}(q) = K_s$ and for almost all $x \in X$: $1 \leq q(x)/p(x) \leq s$.*

Proof. By compactness we can cover K with finitely many open intervals $[1, r] \subseteq \bigcup \{(a_i, b_i) : i < n\}$, with $1 < b_i/a_i \leq s$. We may assume that $K \cap (a_i, b_i) \neq \emptyset$ for all $i < n$. We then define $K_s = \{b_i : i < n\}$.

Assume now that (X, \mathfrak{B}, μ) is atomless and $p: X \rightarrow [1, r]$ satisfies $\text{ess rng}(p) = K$. We can then split X into a finite disjoint union of positive measure sets $X = \bigcup_{i < n} X_i$

such that the essential range of $p_i = p|_{X_i}$ is contained in (a_i, b_i) . Define $q(x) = b_i$ when $x \in X_i$. Then q is as required. $\blacksquare_{4.13}$

Fact 4.14. *For K consisting of a single point, the theory $\text{Th}(\mathcal{AN}_K)$ is \aleph_0 -categorical and \aleph_0 -stable.*

Proof. [BBH]. $\blacksquare_{4.14}$

Lemma 4.15. *Let $K \subseteq [1, r]$ be finite. Then $\text{Th}(\mathcal{AN}_K)$ is \aleph_0 -categorical and \aleph_0 -stable.*

Proof. Let $K = \{p_i : i < n\}$, $p_0 < \dots < p_{n-1}$. If $N = L_{p(\cdot)}(X, \mathfrak{B}, \mu) \in \mathcal{AN}_K$ then X can be written as a disjoint union $X = \bigcup_{i < n} X_i$ where $X_i \in \mathfrak{B}$, $\mu(X_i) > 0$ and $p|_{X_i} \equiv p_i$ a.e. For $i < n$ let N_i be the Banach lattice $\chi_{X_i}N$. Thus the N_i are orthogonal bands in N and $N = \bigoplus_{i < n} N_i$. Since we can recover Θ from the norm on each N_i we can recover Θ on N , and thus we can recover the norm on N . Similarly, as the N_i are orthogonal bands we can recover the lattice structure on N from that of N_i .

Now, if N is separable (and atomless), each N_i is separable and atomless, and thus uniquely determined by p_i up to isomorphism, whereby N is uniquely determined by K . This proves \aleph_0 -categoricity.

Similarly, let $N' \preceq N$ be a separable elementary sub-model and let $N'_i = N' \cap N_i$. By \aleph_0 -stability of $\text{Th}(N_i)$, $S_\ell^{N_i}(N'_i)$ is metrically separable for each i . Now let $\bar{f} = f^0, \dots, f^{\ell-1} \in N$, and let $f^j = \sum_{i < n} f_k^j$ where $f_k^j \in N_i$. Naming Θ and using quantifier elimination we see that $\text{tp}^N(\bar{f}/N')$ is uniquely determined by $(\text{tp}^{N_i}(\bar{f}_i/N'_i) : i < n)$, and we might as well write $\text{tp}^N(\bar{f}/N') = \sum_{i < n} \text{tp}^{N_i}(\bar{f}_i/N'_i)$. If $q = \sum_{i < n} q_i$ and $q' = \sum_{i < n} q'_i$ are two such decompositions then we have $d(q, q') \leq \sum_{i < n} d(q_i, q'_i)$. Thus $S_\ell^N(N')$ is metrically separable. $\blacksquare_{4.15}$

We can now conclude:

Theorem 4.16. *The theory $\text{Th}(\mathcal{AN}_{\subseteq[1,r]})$ is \mathfrak{p} - \aleph_0 -stable, and every completion thereof (which is of the form $\text{Th}(\mathcal{AN}_K)$) is \mathfrak{p} - \aleph_0 -categorical.*

Proof. Combining Lemma 4.12 and Lemma 4.13 we see that for every $\varepsilon > 0$ there is a finite set $K' \subseteq [1, r]$ such that every separable $N, N' \in \mathcal{AN}_K$ admit $\mathfrak{p}(\varepsilon/2)$ -perturbations with separable $\tilde{N}, \tilde{N}' \in \mathcal{AN}_{K'}$, respectively. But $\tilde{N} \cong \tilde{N}'$ by Lemma 4.15, so N and N' admit a $\mathfrak{p}(\varepsilon)$ -perturbation.

Similarly for \mathfrak{p} - \aleph_0 -stability. $\blacksquare_{4.16}$

Corollary 4.17. *The theory $\text{Th}(\mathcal{AN}_{\subseteq[1,r]})$ is stable.*

Proof. By [Benc, Proposition 4.11] λ - \mathfrak{p} -stability implies stability. (See [Benc, Section 4.3] for more properties and characterisations of \aleph_0 -stability up to perturbation.) $\blacksquare_{4.17}$

Remark. It is in fact also true that the theory $\text{Th}(\mathcal{N}_{\{p_0\}})$ (i.e., constant p , but possibly with atoms) is \aleph_0 -stable, and remarks to that effect may be added to [BBH] before publications. By the same reasoning, the theory $\text{Th}(\mathcal{N}_{\subseteq[1,r]})$ is \mathfrak{p} - \aleph_0 -stable and in particular stable.

APPENDIX A. SOME BASIC CONTINUOUS MODEL THEORY

Theorem A.1 (Beth's definability theorem for continuous logic). *Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be continuous signatures and T an \mathcal{L} -theory such that every \mathcal{L}_0 -structure M_0 admits at most a single expansion to an \mathcal{L} -structure M which is a model of T . Then every symbol in \mathcal{L} admits an explicit \mathcal{L}_0 -definition in T . That is to say that for every predicate symbol $P(\bar{x}) \in \mathcal{L}$ is equal in all models of T to some \mathcal{L}_0 -definable predicate $\varphi_P(\bar{x})$, and for every function symbol $f(\bar{x}) \in \mathcal{L}$ the predicate $d(f(\bar{x}), y)$ is equal in all models of T to some \mathcal{L}_0 -definable predicate $\varphi_f(\bar{x})$.*

Proof. Let $P \in \mathcal{L}$ be an n -ary function symbol, and consider the mapping $\theta_n: S_n(T) \rightarrow S_n(\mathcal{L}_0)$, the latter being the space of all complete n -types in the language \mathcal{L}_0 . It is known that θ_n is continuous, and we claim it is injective.

Indeed, let $p, p' \in S_n(T)$ be such that $\theta_n(p) = \theta_n(p') = q$. Let $M \models p(\bar{a})$ and $M' \models p'(\bar{a}')$, so Then $\text{tp}^{\mathcal{L}_0}(\bar{a}) = \text{tp}^{\mathcal{L}_0}(\bar{a}') = q$.

Claim. There exists an elementary extension $M \preceq M_1$ and an \mathcal{L}_0 -elementary embedding $M' \hookrightarrow M_1$ sending \bar{a}' to \bar{a} .

Proof of claim. We need to verify that $\text{Th}_{\mathcal{L}(M)}(M) \cup \text{Th}_{\mathcal{L}_0}(M') \cup \{\bar{a} = \bar{a}'\}$ is consistent. But the assumptions on the types tell us precisely that $\text{Th}_{\mathcal{L}_0}(M') \cup \{\bar{a} = \bar{a}'\}$ is approximately finitely satisfiable in (M, \bar{a}) . ■ *Claim*

We will identify M' as a set with its image in M_1 , and in particular assume that $\bar{a} = \bar{a}'$.

Claim. Let N and N' be two \mathcal{L} -structures, and assume that $N \preceq_{\mathcal{L}_0} N'$ (but needn't even be an \mathcal{L} -substructure). Then there exists $N'' \succeq N$ such that $N' \preceq_{\mathcal{L}_0} N''$.

Proof of claim. The assumption $N \preceq_{\mathcal{L}_0} N'$ implies that $\text{Th}_{\mathcal{L}_0(N')}(N')$ is approximately finitely satisfiable in N , so $\text{Th}_{\mathcal{L}(N)}(N) \cup \text{Th}_{\mathcal{L}_0(N')}(N')$ is consistent. ■ *Claim*

Using the claim we can extend the pair $M' = M'_0 \preceq_{\mathcal{L}_0} M_1$ to a chain of \mathcal{L} -structures. We now construct a sequence of structures $M'_0 \preceq_{\mathcal{L}_0} M_1 \preceq_{\mathcal{L}_0} M'_1 \preceq_{\mathcal{L}_0} M_2 \preceq_{\mathcal{L}_0} M'_2 \dots$ such that $M_i \preceq M_{i+1}$ and $M'_i \preceq M'_{i+1}$.

Let $M_\omega = \bigcup M_i$, $M'_\omega = \bigcup M'_i$. Then both M_ω and M'_ω are models of T and have the same \mathcal{L}_0 -reduct, and are therefore the same. It follows that $p = \text{tp}_{M_\omega}(\bar{a}) = \text{tp}_{M'_\omega}(\bar{a}) = p'$.

Once we have established that θ_n is an injective continuous mapping between compact Hausdorff spaces it is necessarily an embedding (i.e., a homeomorphism with its image). We may identify the predicate P with a continuous function $P: S_n(T) \rightarrow [0, 1]$. By Tietze's extension theorem there exists a continuous function $\varphi_P: S_n(\mathcal{L}_0) \rightarrow [0, 1]$ such that $P = \varphi_P \circ \theta_n$. Then φ_P is the required \mathcal{L}_0 -definable predicate.

If f is a function symbol, apply the preceding argument to $d(f(\bar{x}), y)$. ■ **A.1**

Definition A.2. Let T be a theory, $\varphi(\bar{x})$ a definable predicate. We say that φ is *increasing* (*decreasing*) in T if whenever $M \subseteq N$ are both models of T and $\bar{a} \in M$ we have $\varphi(\bar{a})^M \leq \varphi(\bar{a})^N$ ($\varphi(\bar{a})^M \geq \varphi(\bar{a})^N$). We say that φ is *constant* in T if it is both increasing and decreasing.

Definition A.3. A *sup-formula* is a formula of the form $\sup_{\bar{y}} \varphi(\bar{x}, \bar{y})$ where φ is quantifier-free.

A *sup-definable predicate* is a definable predicate which can be written syntactically as $\mathcal{F}\lim \varphi_n(\bar{x})$ where each φ_n is a sup-formula. Notice that every such predicate is equal to a uniform limit of sup-formulae.

We make the analogous definitions for inf.

Theorem A.4. *Let T be a theory, $\varphi(\bar{x})$ a definable predicate. Then φ is increasing (decreasing) in T if and only if φ is equivalent modulo T to a sup-definable (inf-definable) predicate.*

Proof. Clearly it suffices to prove the case of increasing definable predicates. Right to left being immediate, we will prove left to right.

Assume therefore that $\varphi(\bar{x})$ is increasing in T . Let Ψ be the collection of all sup-formulae $\psi(\bar{x}) = \sup_{\bar{y}} \tilde{\psi}(\bar{x}, \bar{y})$ such that $T \vdash \psi(\bar{x}) \leq \varphi(\bar{x})$. Notice that the latter means that $T \vdash \tilde{\psi}(\bar{x}, \bar{y}) \leq \varphi(\bar{x})$. If for every $n < \omega$ there is $\psi_n \in \Psi$ such that $T \vdash \varphi(\bar{x}) \div 2^{-n} \leq \psi_n(\bar{x})$ then $\varphi = \mathcal{F}\lim \psi_n$ and we are done. In order to conclude we will assume the converse and obtain a contradiction.

We assume then that there is $n < \omega$ such that $T \cup \{\varphi(\bar{x}) \div \psi(\bar{x}) \geq 2^{-n}\}$ is consistent for all $\psi \in \Psi$. As Ψ is closed under \vee and $\varphi \div (\psi \vee \psi') \geq 2^{-n} \implies \varphi \div \psi \geq 2^{-n}$, the set $\Sigma = T \cup \{\varphi \div \psi \geq 2^{-n} : \psi \in \Psi\}$ is consistent. Let (M, \bar{a}) be a model for it, and let $r = \varphi(\bar{a})^M$.

Let $\Sigma' = T \cup \text{Diag}_a(M) \cup \{\varphi(\bar{a}) \leq r - 2^{-n}\}$ (Diag_a denoting the atomic diagram). If Σ' were consistent we would get a contradiction to φ being increasing, so Σ' is contradictory. By compactness there exists a quantifier-free formula $\chi(\bar{x}, \bar{y})$ and $\bar{b} \in M$ such that $\chi(\bar{a}, \bar{b})^M = 0$ and $T \cup \{\chi(\bar{x}, \bar{y}) = 0\} \cup \{\varphi(\bar{x}) \leq r - 2^{-n}\}$ is contradictory. It follows there is some m such that $T \cup \{\chi(\bar{x}, \bar{y}) \leq 2^{-m}\} \cup \{\varphi(\bar{x}) \leq r - 2^{-n}\}$ is contradictory. Let $r' \in (r - 2^{-n}, r)$ be a dyadic number, and let $\tilde{\psi} = r' \div 2^m \chi$. Then $\tilde{\psi}$ is a quantifier-free formula, and we claim that $T \vdash \tilde{\psi}(\bar{x}, \bar{y}) \leq \varphi(\bar{x})$. indeed, for any model $N \models T$ and any $\bar{c}, \bar{d} \in N$:

$$\begin{aligned} \varphi(\bar{c})^N \geq r' &\implies \varphi(\bar{c})^N \geq r' \geq \tilde{\psi}(\bar{c}, \bar{d})^N \\ \varphi(\bar{c})^N \leq r' &\implies \chi(\bar{c}, \bar{d})^N \geq 2^{-m} \implies \varphi(\bar{c})^N \geq 0 = \tilde{\psi}(\bar{c}, \bar{d})^N. \end{aligned}$$

Thus $\psi(\bar{x}) = \sup_{\bar{y}} \tilde{\psi}(\bar{x}, \bar{y}) \in \Psi$, whereby $\varphi(\bar{a})^M \div \psi(\bar{a})^M \geq 2^{-n}$. But $\chi(\bar{a}, \bar{b})^M = 0$, so $\psi(\bar{a}) \geq r'$ whereby $\varphi(\bar{a})^M \geq r' + 2^{-n} > r$, a contradiction. This concludes the proof. ■_{A.4}

Corollary A.5. *A continuous theory T is model complete if and only if every formula (definable predicate) is equivalent modulo T to an inf-definable predicate.*

Proof. Left to right is by Theorem A.4. For right to left, every formula φ is decreasing in T , and considering $\neg\varphi$ every formula is increasing as well, and therefore constant in T , which means precisely that T is model complete. ■_{A.5}

APPENDIX B. A CONVERGENCE RATE FOR APPROXIMATIONS OF THE MODULAR FUNCTIONAL

We conclude with a result that was used in earlier versions of this paper in Section 3, later superseded by a more direct approach. A naïve way to try to approximate the modular functional is by $\Theta(f) \approx \sum \|f_k\|^{p_k}$ where $f = \sum f_k$ consists of cutting the domain of f into chunks such that the exponent function $p(\cdot)$ is almost constant p_k on each chunk. We show here that these approximations do converge to $\Theta(f)$ at a uniform rate: the difference is always smaller than $C\sqrt{\Delta}$ where Δ is the maximum of diameters of the range of p on the chunks and C is a constant.

Lemma B.1. *Let $(N, \Theta) = L_{p(\cdot)}(X, \mathfrak{B}, \mu)$, and assume that $\text{ess\,rng } p \subseteq [s, s + \varepsilon]$ where $1 \leq s < s + \varepsilon \leq r$. Let $f \in N$, and assume that $\|f\| \leq 1$. Then $|\Theta(f) - \|f\|^{s+\varepsilon}| \leq \frac{\varepsilon}{s} |\ln \Theta(f)| \Theta(f)$.*

Proof. We may assume that $f \geq 0$ and $\|f\| > 0$. Let $a = \|f\|$, so $\Theta(f/a) = 1$, and for all t :

$$a^t = a^t \left(\int (f/a)^p d\mu \right) = \int f^p a^{t-p} d\mu,$$

Notice that for all x we have $s - p(x) \leq 0 \implies a^{s-p(x)} \geq 1$ while $s + \varepsilon - p(x) \geq 0 \implies a^{s+\varepsilon-p(x)} \leq 1$, so:

$$a^{s+\varepsilon} = \int f^p a^{s+\varepsilon-p} d\mu \leq \int f^p d\mu \leq \int f^p a^{s-p} d\mu = a^s.$$

In other words: $a^{s+\varepsilon} \leq \Theta(f) \leq a^s$. It follows that $\Theta(f)^{1+\frac{\varepsilon}{s}} \leq a^{s+\varepsilon} \leq \Theta(f)$. Thus:

$$|\Theta(f) - a^{s+\varepsilon}| \leq |\Theta(f) - \Theta(f)^{1+\frac{\varepsilon}{s}}| \leq \frac{\varepsilon}{s} |\ln \Theta(f)| \Theta(f). \quad \blacksquare_{\text{B.1}}$$

Lemma B.2. *There is a constant C such that for every $n < \omega$ and every sequence $(a_k : k < \omega)$ such that $a_k \geq 0$ and $\sum a_k \leq 1$:*

$$\sum \frac{1}{k+n} a_k |\ln a_k| \leq \frac{C}{\sqrt{n}}.$$

(We follow the convention that $0 \ln 0 = 0$.)

Proof. At first we will assume that $a_k \leq \frac{1}{e}$ for all k , noting that $\theta(x) = -x \ln x$ is increasing on $[0, \frac{1}{e}]$.

We may assume that the sequence is ordered so that $a_k |\ln a_k|$ is decreasing. It follows that $(a_k : k < \omega)$ is a decreasing sequence. Since $\sum a_k \leq 1$ we have $a_k \leq \frac{1}{k+1}$ for all k . Again, as $\theta(x)$ is increasing we get $a_k |\ln a_k| = \theta(a_k) \leq \theta((k+1)^{-1}) = \frac{1}{k+1} \ln(k+1)$. Let

$C_0 = \sum \frac{\ln(k+1)}{(k+1)^{3/2}} < \infty$. Then we have:

$$\begin{aligned} \sum \frac{1}{k+n} a_k |\ln a_k| &\leq \sum \frac{1}{(k+n)(k+1)} \ln(k+1) \\ &\leq \sum \frac{1}{n^{1/2}(k+1)^{1/2}(k+1)} \ln(k+1) \\ &= \frac{1}{n^{1/2}} \sum \frac{\ln(k+1)}{(k+1)^{3/2}} = \frac{C_0}{\sqrt{n}}. \end{aligned}$$

In the general case there may be at most 2 indices k such that $a_k > \frac{1}{e}$. By removing them we change the sum we wish to bound by at most $2/n$. Thus $\sum \frac{1}{k+n} \theta(a_k) a_k |\ln a_k| \leq \frac{C}{\sqrt{n}}$ where $C = C_0 + 2$. ■_{B.2}

Lemma B.3. *Let $(N, \Theta) = L_{p(\cdot)}(X, \mathfrak{B}, \mu)$ be a Nakano space and let $0 < n < \omega$ be fixed. Let $K_k = [\frac{n+k}{n}, \frac{n+k+1}{n})$, and let ℓ be large enough so that $\bigcup_{k < \ell} K_k \supseteq [1, r]$. Let C be the constant from Lemma B.2.*

Then every $f \in N$ can be expressed as $f = \sum_{k < \ell} f_k$ where $f_k = f|_{K_k} \in L_{p|_{K_k}(\cdot)}(K_k, \mathfrak{B}|_{K_k}, \mu|_{K_k})$. If $\|f\| \leq 1$ then we have:

$$\left| \Theta_{p(\cdot)}(f) - \sum_{k < \ell} \|f_k\|^{\frac{n+k+1}{n}} \right| \leq \frac{C}{\sqrt{n}}.$$

Proof. We have $\sum_{k < \ell} \Theta(f_k) = \Theta(f) \leq 1$, whereby:

$$\begin{aligned} \left| \Theta(f) - \sum_{k < \ell} \|f_k\|^{\frac{n+k+1}{n}} \right| &\leq \sum_{k < \ell} \left| \Theta(f_k) - \|f_k\|^{\frac{n+k+1}{n}} \right| \\ &\leq \sum_{k < \ell} \frac{1/n}{(n+k)/n} |\ln(\Theta(f_k))| \Theta(f_k) \\ &= \sum_{k < \ell} \frac{1}{n+k} |\ln(\Theta(f_k))| \Theta(f_k) \\ &\leq \frac{C}{\sqrt{n}}. \end{aligned} \quad \text{■}_{B.3}$$

REFERENCES

- [BBH] Itai Ben Yaacov, Alexander Berenstein, and C. Ward Henson, *Model-theoretic independence in the Banach lattices $L^p(\mu)$* , submitted.
- [BBHU] Itai Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Newton Institute MAA programme proceedings, to appear.
- [Bena] Itai Ben Yaacov, *Definability of groups in \aleph_0 -stable metric structures*, submitted.
- [Benb] ———, *On perturbations of continuous structures*, submitted.
- [Benc] ———, *Topometric spaces and perturbations of metric structures*, submitted.

- [BU] Itaï Ben Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*, Transactions of the AMS, to appear.
- [Fre03] D. H. Fremlin, *Measure theory volume 2: Further topics in the general theory*, Torres Fremlin, 25 Ireton Road, Colchester CO3 3AT, England, 2003, <http://www.essex.ac.uk/maths/staff/fremlin/mt2.2003/index.htm>.
- [Mac73] Hilton Vieira Machado, *A characterization of convex subsets of normed spaces*, Kōdai Mathematical Seminar Reports **25** (1973), 307–320.
- [Poi06] L. Pedro Poitevin, *Model theory of Nakano spaces*, Ph.D. thesis, University of Illinois at Urbana-Champaign, 2006.
- [PR] L. Pedro Poitevin and Yves Raynaud, *Ranges of positive contractive projections in Nakano spaces*, submitted.

ITAÏ BEN YAACOV, UNIVERSITÉ DE LYON, UNIVERSITÉ LYON 1, INSTITUT CAMILLE JORDAN, UMR 5208 CNRS, 43 BOULEVARD DU 11 NOVEMBRE 1918, F-69622 VILLEURBANNE CEDEX, FRANCE
URL: <http://math.univ-lyon1.fr/~begnac/>