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Abstract

A non-overlapping domain decomposition method for elasticity equations based on an optimal control formulation is presented. The existence of a solution is proved and the convergence of a subsequence of the approximate solutions to a solution of the continuous problem is shown. The implementation based on lagrangian method is discussed. Finally, numerical results showing the efficiency of our approach and confirming the convergence result are given.

Key words: Convergence, Domain decomposition, Elasticity equations, Optimal control formulation.

1 Introduction

Domain decomposition methods is divided into two classes, those that use overlapping domain, and those that use non-overlapping domains, which we refer to as substructuring. Various substructuring methods with non-overlapping can be encountered in literature and fruitful references can be found from [17].

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A study of elasticity equations by domain decomposition method was treated from [4,5,11,15]. In [15], the authors have presented the techniques for the algebraic approximation of Dirichlet to Neumann maps for linear elasticity. This techniques are based on the local condensation of the degree of freedom belonging to a small area-defined inside the sub-domain- on a small patch defined on the interface. In [11], the domain decomposition method with Lagrange multipliers is introduced by reformulating the preconditioned system of the FETI algorithm as a saddle point problem with both primal and dual variables as unknowns.

In this paper, we consider a linear elasticity material which occupies an open bounded domain $\Omega \subset \mathbb{R}^2$ where the boundary is denoted by $\Gamma = \partial\Omega$. The linear elasticity problem [12] is given, for $i = 1, 2$, by

$$\begin{cases} -\sum_{j=1}^{2} \dfrac{\partial\, \sigma_{ij}(u)}{\partial\, x_j} = f_i & \text{in} \quad \Omega \\ u_i = 0 & \text{on} \quad \Gamma. \end{cases} \tag{1}$$

where $u = (u_1, u_2)$ is the displacement vector, $f = (f_1, f_2)$ the volume force vector, σ_{ij} is the stress tensor. The traction vector t is defined by for $i = 1, 2$, $t_i = \sum_{j=1}^{2} \sigma_{ij}(u) n_j$ where n is the outward normal unitary vector of the domain Ω along boundary Γ. The strain tensor ε_{ij} is given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\, u_i}{\partial\, x_j} + \frac{\partial\, u_j}{\partial\, x_i}\right) \tag{2}$$

These tensors are related by

$$\sigma_{ij}(u) = 2G\left(\varepsilon_{ij}(u) + \frac{\nu}{1-2\nu}\sum_{k=1}^{2} \varepsilon_{kk}(u)\delta_{ij}\right) \tag{3}$$

with G and ν are the shear modulus and Poisson ratio, respectively, and δ_{ij} is the Kronecker delta tensor.

We wish to determine the solution of (1) by a domain decomposition method. To this end and for simplicity we consider here only the case where Ω is partitioned into two open subdomains $\Omega^{(1)}$ and $\Omega^{(2)}$ such that $\overline{\Omega} = \overline{\Omega^{(1)}} \cup \overline{\Omega^{(2)}}$. The interface between two domains is denoted γ so that $\gamma = \overline{\Omega^{(1)}} \cap \overline{\Omega^{(2)}}$. Let $\Gamma_1 = \overline{\Omega^{(1)}} \cap \Gamma$ and $\Gamma_2 = \overline{\Omega^{(2)}} \cap \Gamma$. Let us denote by $f_i^{(k)} = f_i|_{\Omega^{(k)}}$, for $k = 1, 2$. We consider the problems defined over the subdomains

$$\begin{cases} -\sum_{j=1}^{2} \frac{\partial\, \sigma_{ij}(u^{(1)})}{\partial\, x_j} = f_i^{(1)} \text{ in } \Omega^{(1)} \\ \qquad u_i^{(1)} = 0 \quad \text{on } \Gamma_1 \qquad (4) \\ \sum_{j=1}^{2} \sigma_{ij}(u^{(1)}) n_j^{(1)} = \psi_i \text{ on } \gamma \end{cases} \qquad \begin{cases} -\sum_{j=1}^{2} \frac{\partial\, \sigma_{ij}(u^{(2)})}{\partial\, x_j} = f_i^{(2)} \text{ in } \Omega^{(2)} \\ \qquad u_i^{(2)} = 0 \quad \text{on } \Gamma_2 \qquad (5) \\ \sum_{j=1}^{2} \sigma_{ij}(u^{(2)}) n_j^{(2)} = -\psi_i \text{ on } \gamma \end{cases}$$

where $n^{(i)}$ is the outward normal unitary vector of the subdomain $\Omega^{(i)}$ along the interface γ, for $i = 1, 2$.

In this work, we are interesting to combine the optimization techniques and non-overlapping domain decomposition to solve problem (1). This combination is obtained as a constrained minimization problem for which the cost functional is the $L^2(\gamma)$-norm of the difference between the dependent variables $u^{(1)}$, $u^{(2)}$ across the common boundaries γ and the constraints are the problems (4) and (5). At this stage its must be noted that a similar idea of this combination was already used for Laplace operator in [6,7], for coupled stokes flows [10], for nonlinear sedimentary basin problem [9]. Here, we extend this idea for the study of elasticity equations. Furthermore, we prove the convergence of approximate optimal solutions to continuous one and we give an algorithm based on gradient conjugate with variable steeps.

The paper is organized as follows. In section 2 we provide an optimal control formulation equivalent to the model problem (1). In section 3, we prove the existence of optimal solution. The existence of the discrete optimal control problem, obtained by finite element approximation, is given in section 4. The convergence of approximate solutions to the continuous one is shown in section 5. Section 6 deals with the description of our optimization algorithm, in section 7, we report some numerical result.

2 Optimal control formulation

Define the following convex set :

$$K_0 = \{\psi = (\psi_1, \psi_2) \in (L^2(\gamma))^2 \ / \ \ \|\psi_k\|_{L^2(\gamma)} \leq C_0, \quad \text{for } k = 1, 2\}$$

where C_0 is a nonnegative given constant.

For the numerical approximation of the problem (1), we propose the following optimal control formulation

$$(PO)\begin{cases} \text{Minimize } J(u^{(1)}(\psi), u^{(2)}(\psi)) \quad \text{for all } \psi \in K_0 \\ \text{where} \quad J(u^{(1)}(\psi), u^{(2)}(\psi)) = \dfrac{1}{2}\displaystyle\sum_{i=1}^{2}\int_{\gamma}\left(u_i^{(1)} - u_i^{(2)}\right)^2 d\sigma \\ \text{and } u^{(1)}(\psi),\ u^{(2)}(\psi) \text{ are respectively the solution of (4) and (5).} \end{cases} \tag{6}$$

We have the following result

Proposition 1 *Assume that f and Ω are smooth enough. Then the problem (1) is equivalent to (6).*

Proof.
Let u_e be the solution of (1) and let us denote by $u_e^{(k)} = u_e|_{\Omega^{(k)}}$, for $k = 1, 2$. Assume that f and Ω are smooth enough, such that $\psi_{i,e} = \displaystyle\sum_{j=1}^{2}\sigma_{ij}(u^{(1)})n_j^{(1)}$ is in $L^2(\gamma)$, for $i = 1, 2$. One can choose the constant C_0, defining K_0, such that $\psi_e = (\psi_{1,e}, \psi_{2,e}) \in K_0$; this means that $\max\left(\|\psi_{1,e}\|_{L^2(\gamma)}\,;\, \|\psi_{2,e}\|_{L^2(\gamma)}\right) \leq C_0$. This implies that $(u_e^{(1)}(\psi_e), u_e^{(2)}(\psi_e))$ is a solution of (6).
Conversely, let $(u_*^{(1)}(\psi^*), u_*^{(2)}(\psi^*))$ be a solution of (6) for $\psi^* \in K_0$, then we have $J(u_*^{(1)}(\psi^*), u_*^{(2)}(\psi^*)) \leq J(u^{(1)}(\psi), u^{(2)}(\psi))$ for all $\psi \in K_0$. In particular, we have $0 \leq J(u_*^{(1)}(\psi^*), u_*^{(2)}(\psi^*)) \leq J(u_e^{(1)}(\psi_e), u_e^{(2)}(\psi_e)) = 0$, this involves that $u_* = \begin{cases} u_*^{(1)} & \text{in} \quad \Omega^{(1)} \\ u_*^{(2)} & \text{in} \quad \Omega^{(2)} \end{cases}$ is a solution of (1) and achieves the equivalence result.

3 Existence of optimal solution

We first give some notations and definitions which can be useful in the following. We define the spaces, for $i = 1, 2$,

$$H_{i,D}(\Omega^{(i)}) = \{\upsilon \in (H^1(\Omega^{(i)}))^2 \ / \ \upsilon|_{\Gamma_i} = 0\}$$

where $(H^1(\Omega^{(i)}))^2$ is the Sobolev space equipped with the norm $\|\cdot\|_{1,\Omega^{(i)}}$ defined by

$$\|\upsilon\|_{1,\Omega^{(i)}} = \left(\sum_{l=1}^{2}\left(\|\upsilon_l\|^2_{0,\Omega^{(i)}} + \|\nabla\upsilon_l\|^2_{0,\Omega^{(i)}}\right)\right)^{\frac{1}{2}},\ \|\upsilon_l\|_{0,\Omega^{(i)}} = \left(\int\limits_{\Omega^{(i)}} |\upsilon_l|^2\ dx\right)^{\frac{1}{2}}.$$

$H_{i,D}(\Omega^{(i)})$ are equipped with the following norm $|\upsilon|_{1,\Omega^{(i)}} = \left(\sum_{l=1}^{2}\|\nabla\upsilon_l\|^2_{0,\Omega^{(i)}}\right)^{\frac{1}{2}}$.

For $\psi \in K_0$, we consider the weak formulation of equation (4) and (5) given, for $k = 1, 2$, by

$$\begin{cases} \text{Find } u^{(k)}(\psi) \in H_{k,D}(\Omega^{(k)}) \quad \forall \upsilon = (\upsilon_1, \upsilon_2) \in H_{k,D}(\Omega^{(k)}) \\ a^{(k)}(u^{(k)}, v) = \sum\limits_{i,j=1}^{2}\int\limits_{\Omega^{(k)}} \sigma_{ij}(u^{(k)})\,\varepsilon_{ij}(\upsilon) = \sum\limits_{i=1}^{2}\int\limits_{\Omega^{(k)}} f_i^{(k)}\,\upsilon_i\,dx + (-1)^k\sum\limits_{i=1}^{2}\int\limits_{\gamma}\psi_i\,\upsilon_i\,d\sigma. \end{cases} \tag{7}$$

We define the space of admissible solutions U_{ad} by :

$$U_{ad} = \{(u^{(1)}(\psi), u^{(2)}(\psi))\ \text{solution of (7) / } \psi \in K_0\}.$$

The optimal control problem (6) can be rewritten as:

$$(PO) \quad \text{Minimize } J((u^{(1)}(\psi), u^{(2)}(\psi)) \quad \text{for all } (u^{(1)}(\psi), u^{(2)}(\psi)) \in U_{ad}.$$

We define the convergence of the sequence $(\psi_n)_n = ((\psi_{1,n}, \psi_{2,n}))_n$ in K_0 to $\psi = (\psi_1, \psi_2) \in K_0$ by

$$\psi_n \longrightarrow \psi \iff \psi_{k,n} \rightharpoonup \psi_k \text{ weakly in } L^2(\gamma), \quad \text{for } k = 1, 2. \tag{8}$$

We can then equip U_{ad} with the topology defined by the following convergence: let $((u_n^{(1)}, u_n^{(2)}))_n$ be a sequence of U_{ad} and $(u^{(1)}, u^{(2)}) \in U_{ad}$ then:

$$(u_n^{(1)}, u_n^{(2)}) \longrightarrow (u^{(1)}, u^{(2)}) \iff \begin{cases} u_{k,n}^{(1)} \rightharpoonup u_k^{(1)} \text{ weakly in } H^1(\Omega^{(1)}) \\ u_{k,n}^{(2)} \rightharpoonup u_k^{(2)} \text{ weakly in } H^1(\Omega^{(2)}), \quad \text{for } k = 1, 2. \end{cases} \tag{9}$$

We have then the following result.

Theorem 2 *The problem (PO) is well posed and admits a solution in U_{ad}.*

Proof.
For all ψ in K_0, the result of the existence and unicity of the solution of (7) is ensured by the Lax-Milgram theorem, this involves that the problem (PO) is well posed. The proof of the existence of a solution of (PO) is now reduced to show that U_{ad} is compact for the topology defined by (9) and that J is lower semi-continuous on U_{ad}.
In order to show that U_{ad} is compact, we consider $((u_n^{(1)}, u_n^{(2)}))_n$ a sequence of U_{ad}, i.e. $u_n^{(k)} = u^{(k)}(\psi_n)$ is the solution of (7) for $\psi_n \in K_0$. Since for all n and $k = 1, 2$, we have $\|\psi_{k,n}\|_{L^2(\gamma)} \leq C_0$, we can extract from $(\psi_n)_n$ a subsequence denoted again $(\psi_n)_n$, such that $\psi_{k,n}$ converges weakly in $L^2(\gamma)$ to ψ_k^* and $\psi^* = (\psi_1^*, \psi_2^*)$ is in K_0. The sequence $((u_n^{(1)}, u_n^{(2)}))_n$ converges weakly to $(u_*^{(1)}, u_*^{(2)})$ and $(u_*^{(1)}, u_*^{(2)})$ is such that $(u_*^{(1)}, u_*^{(2)}) = (u^{(1)}(\psi^*), u^{(2)}(\psi^*)) \in U_{ad}$. Indeed, for all n, $u_n^{(k)} = u^{(k)}(\psi_n) \in H_{k,D}(\Omega^{(k)})$ is the solution of

$$a^{(k)}(u_n^{(k)}, \upsilon) = \sum_{i=1}^{2} \int_{\Omega^{(k)}} f_i^{(k)}\, \upsilon_i \, dx + (-1)^k \sum_{i=1}^{2} \int_{\gamma} \psi_{i,n}\, \upsilon_i \, d\sigma \;\forall \upsilon \in H_{k,D}(\Omega^{(k)}) \quad (10)$$

Taking $\upsilon = u_n^{(k)}$ in (10) and using the inequality $\|\psi_{i,n}\|_{L^\infty(\gamma)} \leq C_0$, for $i = 1, 2$, and the Korn's inequality, we obtain that $\|u_n^{(k)}\|_{1,\Omega^{(k)}} \leq \beta$, where β is a nonnegative constant independent of n. Thus we can extract a subsequence denoted again $(u_n^{(k)})_n$, such that $u_{i,n}^{(k)}$ is weakly convergent to $u_{i,*}^{(k)}$ in $H^1(\Omega^{(k)})$, for $i = 1, 2$. Since $\Omega^{(k)}$ is smooth enough, the trace operator from $H^1(\Omega^{(k)})$ to $L^2(\Gamma_k)$ is compact, this implies that $u_*^{(k)} \in H_{k,D}(\Omega^{(k)})$. It remains to show that $u_*^{(k)}$ is solution of

$$a^{(k)}(u_*^{(k)}), \upsilon) = \sum_{i=1}^{2} \int_{\Omega^{(k)}} f_i^{(k)}\, \upsilon_i \, dx + (-1)^k \sum_{i=1}^{2} \int_{\gamma} \psi_i^*\, \upsilon_i \, d\sigma \;\forall \upsilon \in H_{k,D}(\Omega^{(k)}) \quad (11)$$

This is obtained by using the weak convergence of $\frac{\partial\, u_{i,n}^{(k)}}{\partial\, x_j}$ to $\frac{\partial\, u_{i,*}^{(k)}}{\partial\, x_j}$ in $L^2(\Omega^{(k)})$, for $i, j = 1, 2$, and by passing to the limit in equation (10). Consequently $(u_*^{(1)}, u_*^{(2)}) = (u^{(1)}(\psi^*), u^{(2)}(\psi^*)) \in U_{ad}$. This achieves the proof of the compactness of U_{ad} for the topology defined by the convergence (9).
To show the continuity of the functional J in U_{ad}, let us consider a sequence $((u_n^{(1)}, u_n^{(2)}))_n \subset U_{ad}$ which is convergent to $(u^{(1)}, u^{(2)}) \in U_{ad}$. We have

$$\begin{aligned} J(u_n^{(1)}, u_n^{(2)}) - J(u^{(1)}, u^{(2)}) &= \frac{1}{2} \sum_{i=1}^{2} \left(\int_{\gamma} (u_{i,n}^{(1)} - u_{i,n}^{(2)})^2 \, d\sigma - \int_{\gamma} (u_i^{(1)} - u_i^{(2)})^2 \, d\sigma \right) \\ &\leq \frac{1}{2} \left(\sum_{i=1}^{2} (I_{i,n})^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{2} (L_{i,n})^2 \right)^{\frac{1}{2}} \end{aligned}$$

where $I_{i,n} = \left(\int_\gamma \left(u_{i,n}^{(1)} - u_i^{(1)} + u_i^{(2)} - u_{i,n}^{(2)} \right)^2 d\sigma \right)^{\frac{1}{2}}$

and $L_{i,n} = \left(\int_\gamma \left(u_{i,n}^{(1)} - u_{i,n}^{(2)} + u_i^{(1)} - u_i^{(2)} \right)^2 d\sigma \right)^{\frac{1}{2}}$. Since $u_n^{(k)}$ is uniformly bounded in $(H^1(\Omega^{(k)}))^2$ with respect to n, we have that $L_{i,n}$ is uniformly bounded. The use of the compactness of the trace operator from $H^1(\Omega^{(k)})$ to $L^2(\gamma)$ gives $\lim_{n \longrightarrow \infty} I_{i,n} = 0$. Thus $\lim_{n \longrightarrow \infty} J(u_n^{(1)}, u_n^{(2)}) - J(u^{(1)}, u^{(2)}) = 0$. This end the proof.

4 Approximation of the problem

In this section, we use the linear finite element method for the approximation of (PO). We show the existence of the solution of the discrete problem and we study the convergence of a subsequence of these solutions to a solution of the continuous problem. Finally, to confirm the convergence result, we give some numerical results.
For the seek of simplicity, we reduce our study, in this section, to the case where the boundary part γ is assumed to be defined as follows:

$$\gamma = \{(b, x) \ / \ x \in [0, a]\} \tag{12}$$

where $a > 0$ and b are two given constants.

In the following, we need additional regularity assumptions on K_0, namely:

$$\begin{array}{l} K_0 = \{\psi = (\psi_1, \psi_2) \in (C^0(\gamma))^2 \ / \ |\psi_k(b, x) - \psi_k(b, x')| \leq C \, |x - x'| \\ \forall x, x' \in [0, a] \ \text{ and } \|\psi_k\|_{L^\infty(\gamma)} \leq C_0 \quad \text{for} \quad k = 1, 2\} \end{array} \tag{13}$$

where C and C_0 are nonnegative given constants. The convergence of a sequence $(\psi_n)_n = ((\psi_{1,n}, \psi_{2,n}))_n$ in K_0 to $\psi = (\psi_1, \psi_2) \in K_0$ is defined in this case by

$$\psi_n \longrightarrow \psi \Longleftrightarrow \psi_{k,n}(b, .) \longrightarrow \psi_k(b, .) \text{ uniformly in } [0, a], \text{ for } k = 1, 2 \tag{14}$$

Remark 3 *Note that the existence result shown in section 3, remains valid in K_0 with the above convergence. In this case, the compactness of K_0 is ensured by the use of Ascoli-Arzelà theorem's (see [2]).*

4.1 Discretization of the problem

Let us consider an uniform partition $(a_i)_{i=0}^{N-1}$ of the interval $[0, a]$, such that:

$$0 = a_0 < a_1 < \ldots < a_{N-1} = a, \quad a_i - a_{i-1} = h \quad \text{for } i = 1, \ldots, N-1.$$

We define the discrete space associated to K_0 by

$$\begin{aligned} K_0^h = \{&\psi_h = (\psi_{1,h}, \psi_{2,h}) \in (C(\gamma))^2 \ / \ \psi_{k,h}(b,.)|_{[a_{i-1},a_i]} \in P_1([a_{i-1}, a_i]) \\ &i = 1, \ldots, N-1, \ |\tfrac{\psi_{k,h}(b,a_i) - \psi_{k,h}(b,a_{i-1})}{a_i - a_{i-1}}| \leq C, \ i = 1, \ldots, N-1 \\ &\text{and } \|\psi_{k,h}\|_{L^\infty(\gamma)} \leq \frac{C}{2} h + C_0, \text{ for } k = 1, 2\} \end{aligned}$$

with the same constants C and C_0, as in the definition of K_0.
Let $H(\Omega)$ be the finite dimensional space given by

$$H(\Omega^{(k)}) = \{\upsilon_h \in \mathcal{C}(\overline{\Omega^{(k)}}) \ / \ \upsilon_h|_K \in P_1(K), \ \forall K \in \mathcal{T}_h\}$$

where $\mathcal{T}_h$ is a regular triangulation of $\overline{\Omega^{(k)}}$, for $k = 1, 2$. Let

$$H_{k,D}^h(\Omega^{(k)}) = \{\upsilon_h \in (H(\Omega^{(k)}))^2 \ / \ \upsilon_h|_{\Gamma_k} = 0\}$$

be the finite dimensional spaces associated respectively to $H_{k,D}(\Omega^{(k)})$.
For $\psi_h \in K_0^h$, we consider the following discrete problem of (6), for $k = 1, 2$:

$$\begin{cases} \text{Find } u_h^{(k)}(\psi_h) \in H_{k,D}^h(\Omega^{(k)}) \ \forall \upsilon_h \in H_{k,D}^k(\Omega^{(k)}) \\ a_h^{(k)}(u_h^{(k)}, \upsilon_h) = \displaystyle\sum_{i,j=1}^{2} \int_{\Omega^{(k)}} \sigma_{ij}(u_h^{(k)})\, \varepsilon_{ij}(\upsilon_h) = \sum_{i=1}^{2} \int_{\Omega^{(k)}} f_{i,h}^{(k)}\, \upsilon_{i,h}\, dx + (-1)^k \sum_{i=1}^{2} \int_{\gamma} \psi_{i,h}\, \upsilon_{i,h}\, d\sigma \end{cases} \quad (15)$$

where $f_{i,h}^{(k)}$ is an approximation of $f_i^{(k)}$ such that

$$f_{i,h}^{(k)} \text{ is uniformly bounded and converges to } f_i^{(k)} \text{ almost every where.} \quad (16)$$

The discrete space of the admissible solutions is given by

$$U_{ad}^h = \{(u_h^{(1)}(\psi_h), u_h^{(2)}(\psi_h)) \text{ solution of } (15) \ / \ \psi_h \in K_0^h\}$$

We approach the cost functional by the following discrete one :

$$J_h(u_h^{(1)}(\psi_h), u_h^{(2)}(\psi_h)) = \frac{1}{2}\sum_{i=1}^{2}\int_{\gamma}\left(u_{i,h}^{(1)}(\psi_h) - u_{i,h}^{(2)}(\psi_h)\right)\Big)^2\, d\sigma,$$

and we state our discrete optimization problem as follows

$$(PO^h)\begin{cases}\inf\limits_{(u_h^{(1)},u_h^{(2)})\in U_{ad}^h} J_h(u_h^{(1)}, u_h^{(2)}) \\ \text{where } u_h^{(k)} = u_h^{(k)}(\psi_h) \text{ is solution of (15)}, \quad \text{for k=1,2.}\end{cases}$$

Note that the set K_0^h can be identified with the following subset of $\mathbb{R}^{2N}$

$$\begin{aligned}&\mathcal{K}_0 = \Big\{\{X\} = (X_{1,0},\, \ldots,\, X_{1,N-1},\, X_{2,0},\, \ldots,\, X_{2,N-1}) \in \mathbb{R}^{2N} \;/ \\ &-C\,h \le X_{l,i} - X_{l,i-1} \le C\,h,\; i = 1,\ldots,N-1,\; l = 1,2 \\ &\text{and } |X_{l,i}| \le \tfrac{C}{2}\,h + C_0, \quad i = 0,\ldots,N-1,\; l = 1,2\Big\}.\end{aligned}$$

We denote by $M_{\Omega^{(k)}}(h)$ and $M_\gamma(h)$ the set of nodes lying respectively on $\Omega^{(k)}$ and γ. Let $m^{(k)}$ be the number of elements of $M_{\Omega^{(k)}}(h)$, and define $NT^{(k)} = N + m^{(k)}$, for $k = 1, 2$. Let us now introduce in $H(\Omega^{(k)})$ the canonical basis $(p_i^{(k)})_{i=1}^{NT_k}$ such that $p_i^{(1)} = p_i^{(2)} = p_i$, for all $i \in M_\gamma(h)$. For the vector $P^{(k)} = [p_1^{(k)}, p_2^{(k)}, \ldots, p_{NT^{(k)}}^{(k)}]$, we define the following matrix $[\mathcal{P}^{(k)}] = \begin{pmatrix} P^{(k)} & 0 \\ 0 & P^{(k)} \end{pmatrix}$ Then $u_h^{(k)}$ can be written $u_h^{(k)} = [\mathcal{P}^{(k)}]\,\{u_T^{(k)}\}$ where $\{u_T^{(k)}\} = {}^t[u_{1,1}^{(k)}, u_{1,2}^{(k)}, \ldots, u_{1,NT^{(k)}}^{(k)}, u_{2,1}^{(k)}, u_{2,2}^{(k)}, \ldots, u_{2,NT^{(k)}}^{(k)}]$ is the vector of the components of $u_h^{(k)}$ in the basis $P^{(k)}$. Let us denote by

$$D\,P^{(k)} = \begin{pmatrix} \frac{\partial p_1^{(k)}}{\partial x} & \frac{\partial p_2^{(k)}}{\partial x} & \cdots & \frac{\partial p_{NT^{(k)}}^{(k)}}{\partial x} \\ \frac{\partial p_1^{(k)}}{\partial y} & \frac{\partial p_2^{(k)}}{\partial y} & \cdots & \frac{\partial p_{NT^{(k)}}^{(k)}}{\partial y} \end{pmatrix} \text{ and } [\mathcal{D}\,\mathcal{P}^{(k)}] = \begin{pmatrix} D\,P^{(k)} & 0 \\ 0 & D\,P^{(k)} \end{pmatrix}$$

the gradient of $u_h^{(k)}$, $D\,u_h^{(k)}$ can be written in term of $[\mathcal{D}\,\mathcal{P}^{(k)}]$ and $\{u_T^{(k)}\}$ by

$$D\,u_h^{(k)} = {}^t\left(\frac{\partial u_{1,h}^{(k)}}{\partial x}, \frac{\partial u_{1,h}^{(k)}}{\partial y}, \frac{\partial u_{2,h}^{(k)}}{\partial x}, \frac{\partial u_{2,h}^{(k)}}{\partial y}\right) = [\mathcal{D}\,\mathcal{P}^{(k)}]\,\{u_T^{(k)}\}$$

The tensors ε and σ can be read $\{\varepsilon\} = {}^t(\varepsilon_{11}, \varepsilon_{22}, 2\,\varepsilon_{12})$ and $\{\sigma\} = {}^t(\sigma_{11}, \sigma_{22}, \sigma_{12})$

$\{\varepsilon\}$ can be written in term of $D\,u_h^{(k)}$

$$\{\varepsilon\} = \begin{pmatrix} 1\ 0\ 0\ 0 \\ 0\ 0\ 0\ 1 \\ 0\ 1\ 1\ 0 \end{pmatrix} \begin{pmatrix} \frac{\partial u_{1,h}^{(k)}}{\partial x} \\ \frac{\partial u_{1,h}^{(k)}}{\partial y} \\ \frac{\partial u_{2,h}^{(k)}}{\partial x} \\ \frac{\partial u_{2,h}^{(k)}}{\partial y} \end{pmatrix}$$

If we denote by $[\mathcal{D}]$ the above matrix, we have

$$\{\varepsilon\} = [\mathcal{D}]\, D\, u_h^{(k)} = [\mathcal{D}]\, [\mathcal{D}\, \mathcal{P}^{(k)}]\, \{u_T^{(k)}\}$$

Using equation (3), we can write $\{\sigma\}$ in term of $\{\varepsilon\}$ as follows $\{\sigma\} = [\mathcal{E}]\,\{\varepsilon\}$ thus $\{\sigma\} = [\mathcal{E}]\,[\mathcal{D}]\,[\mathcal{D}\,\mathcal{P}^{(k)}]\,\{u_T^{(k)}\}$ where $[\mathcal{E}]$ is a 3×3 symmetric matrix. Using the above notations we have

$$\sum_{i,j=1}^{2} \int_{\Omega^{(k)}} \sigma_{ij}(u_h^{(k)})\, \varepsilon_{ij}(\upsilon_h) = {}^t\{\upsilon_T\} \left(\int_{\Omega^{(k)}} {}^t[\mathcal{D}\,\mathcal{P}^{(k)}]\, {}^t[\mathcal{D}][\mathcal{E}]\, [\mathcal{D}]\, [\mathcal{D}\,\mathcal{P}^{(k)}]\, dx \right) \{u_T^{(k)}\},$$

and $\displaystyle\sum_{i=1}^{2} \int_{\Omega^{(k)}} f_{i,h}^{(k)}\, \upsilon_{i,h}\, dx = {}^t\{\upsilon_T\} \left(\int_{\Omega^{(k)}} {}^t[\mathcal{P}^{(k)}]\, f_h^{(k)}\, dx \right)$.

Setting now the matrix $\mathcal{A}^{(k)} = \displaystyle\int_{\Omega^{(k)}} {}^t[\mathcal{D}\,\mathcal{P}^{(k)}]\, {}^t[\mathcal{D}][\mathcal{E}]\, [\mathcal{D}]\, [\mathcal{D}\,\mathcal{P}^{(k)}]\, dx$, the vectors $\{\mathcal{B}^{(k)}\} = \displaystyle\int_{\Omega^{(k)}} {}^t[\mathcal{P}^{(k)}]\, f_h^{(k)}\, dx$ and $\{G^{(k)}(X)\} = (G_i(X))_{i=1}^{2NT^{(k)}}$ with

$$G_i^{(k)}(X) = (-1)^k \sum_{l=1}^{2} \sum_{j \in M_\gamma(h)} X_{l,j} \int_{\gamma} \left({}^t[\mathcal{P}^{(k)}]\, [\mathcal{P}^{(k)}] \right)_{ij} d\sigma, \tag{17}$$

it is easy to see that problem (15) can be rewritten, for $k = 1, 2$, as

$$\begin{cases} \text{Find } \{u_T^{(k)}(X)\} \in \mathbb{R}^{2NT^{(k)}} \text{ such that} \\ \mathcal{A}^{(k)}\, \{u_T^{(k)}(X)\} = \{\mathcal{B}^{(k)}\} + \{G^{(k)}(X)\} \end{cases} \tag{18}$$

We can identify the set U_{ad}^h with the following subset of $\mathbb{R}^{4NT^{(k)}}$

$$\mathcal{U} = \{(\{u_T^{(1)}\}, \{u_T^{(2)}\}) \text{ solution of (18) / } \{X\} \in \mathcal{K}_0\}.$$

Then the discrete cost functional reads :

$$J_h(u_h^{(1)}, u_h^{(2)}) = J(\{u_T^{(1)}\}, \{u_T^{(2)}\}) = \frac{1}{2}\left\langle [\mathcal{R}]\left(\{u_T^{(1)}\} - \{u_T^{(2)}\}\right), \left(\{u_T^{(1)}\} - \{u_T^{(2)}\}\right)\right\rangle$$

where $\langle ., . \rangle$ is the inner product in $\mathbb{R}^{2NT^{(k)}}$ and the matrix $[\mathcal{R}]$ is defined by

$$[\mathcal{R}] = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$$

where $R = (r_{ij})_{1 \leq i,j \leq 2NT^{(k)}}$ is given by

$$r_{ij} = \begin{cases} \widehat{r}_{ij} = \int\limits_{\gamma} p_i \, p_j \, d\sigma & \text{if } i, j \in M_\gamma(h) \\ 0 & \text{otherwise .} \end{cases}$$

The matrix form of the optimization problem reads:

$$(PM) \begin{cases} \inf\limits_{(\{u_T^{(1)}\}, \{u_T^{(2)}\}) \in \mathcal{U}} J(\{u_T^{(1)}\}, \{u_T^{(2)}\}) \\ \text{s/c} \quad \mathcal{A}^{(k)}\left\{u_T^{(k)}(X)\right\} = \{\mathcal{B}^{(k)}\} + \{G^{(k)}(X)\} \quad \text{for } k = 1, 2 \end{cases} \tag{19}$$

4.2 Existence of the solution of the discrete problem

It is easy to see that (PO^h) is equivalent to (PM), thus we show that (PM) has a solution in $\mathcal{U}$.

Theorem 4 *The problem (PM) admits a solution on $\mathcal{U}$, for all $h > 0$.*

Proof.
Let us consider a minimizing sequence $((\{u_T^{(1)}\}_n, \{u_T^{(2)}\}_n))_n$ of J in $\mathcal{U}$, such that

$$\lim_{n \longrightarrow \infty} J(\{u_T^{(1)}\}_n, \{u_T^{(2)}\}_n)) = \inf_{(w^{(1)}, w^{(2)}) \in \mathcal{U}} J(w^{(1)}, w^{(2)}).$$

We have that for all n and $k = 1, 2$, $\{u_T^{(k)}\}_n = \{u_T^{(k)}\}(X_n)$ is the solution of $\mathcal{A}^{(k)}\left\{u_T^{(k)}(X_n)\right\} = \{\mathcal{B}^{(k)}\} + \{G^{(k)}(X_n)\}$. Using the fact that $\mathcal{K}_0$ is bounded and closed (compact) in $\mathbb{R}^{2N}$, we can extract from $(\{X\}_n)_n$ a subsequence denoted again $(\{X\}_n)_n$ which converges in $\mathbb{R}^{2N}$ to $\{X^*\} \in \mathcal{K}_0$. From the definition of $\{G^{(k)}\}$ in equation (17), we can show that the sequence $(\{G^{(k)}(X_n)\})_n$

converges to $\{G^{(k)}(X^*)\}$ in $\mathbb{R}^{2NT^{(k)}}$. Let $\{u_*^{(k)}\}$ be the solution of $\mathcal{A}^{(k)}\{v\} = \{\mathcal{B}^{(k)}\} + \{G^{(k)}(X^*)\}$, we show that the sequence $(\{u_T^{(k)}\}_n)_n$ converges to $\{u_*^{(k)}\}$ in $\mathbb{R}^{2NT^{(k)}}$, for $k = 1, 2$. Indeed, we have that

$$\begin{aligned}\left\langle \mathcal{A}^{(k)}\{u_T^{(k)}\}_n, \{u_T^{(k)}\}_n - \{u_*^{(k)}\}\right\rangle = &\left\langle \{\mathcal{B}^{(k)}\}, \{u_T^{(k)}\}_n - \{u_*^{(k)}\}\right\rangle \\ &+ \left\langle \{G^{(k)}(X_n)\}, \{u_T^{(k)}\}_n - \{u_*^{(k)}\}\right\rangle\end{aligned} \tag{20}$$

and

$$\begin{aligned}\left\langle \mathcal{A}^{(k)}\{u_*^{(k)}\}, \{u_T^{(k)}\}_n - \{u_*^{(k)}\}\right\rangle = &\left\langle \{\mathcal{B}^{(k)}\}, \{u_T^{(k)}\}_n - \{u_*^{(k)}\}\right\rangle \\ &+ \left\langle \{G^{(k)}(X^*)\}, \{u_T^{(k)}\}_n - \{u_*^{(k)}\}\right\rangle.\end{aligned} \tag{21}$$

Subtracting equation (21) from (20), and using the fact that the matrix $\mathcal{A}^{(k)}$ is symmetric and positive definite, we obtain that there exists a nonnegative constant α such that

$$\begin{aligned}\alpha \|\{u_T^{(k)}\}_n - \{u_*^{(k)}\}\|_{2NT^{(k)}}^2 \leq &\|\{G^{(k)}(X_n)\} - \{G^{(k)}(X^*)\}\|_{2NT^{(k)}} \\ &\times \|\{u_T^{(k)}\}_n - \{u_*^{(k)}\}\|_{2NT^{(k)}}\end{aligned} \tag{22}$$

the result is obtained by passing to the limit in (22).
The main result of this theorem follows from the fact that $J(\{u_T^{(1)}\}_n, \{u_T^{(2)}\}_n)$ converges to $J(\{u_*^{(1)}\}, \{u_*^{(2)}\})$, which is obtained by passing to the limit in the following equation

$$\begin{aligned}&J(\{u_T^{(1)}\}_n, \{u_T^{(2)}\}_n)) - J(\{u_*^{(1)}\}, \{u_*^{(2)}\}) \\ &= \left\langle [\mathcal{R}](\{u_T^{(1)}\}_n - \{u_T^{(2)}\}_n), (\{u_T^{(1)}\}_n - \{u_T^{(2)}\}_n)\right\rangle \\ &\quad - \left\langle [\mathcal{R}](\{u_*^{(1)}\} - \{u_*^{(2)}\}), (\{u_*^{(1)}\} - \{u_*^{(2)}\})\right\rangle \\ &= \left\langle [\mathcal{R}](\{u_T^{(1)}\}_n - \{u_T^{(2)}\}_n), (\{u_T^{(1)}\}_n - \{u_T^{(2)}\}_n)\right\rangle \\ &\quad - \left\langle [\mathcal{R}](\{u_*^{(1)}\} - \{u_*^{(2)}\}), (\{u_T^{(1)}\}_n - \{u_T^{(2)}\}_n)\right\rangle \\ &\quad + \left\langle [\mathcal{R}](\{u_*^{(1)}\} - \{u_*^{(2)}\}), (\{u_T^{(1)}\}_n - \{u_T^{(2)}\}_n)\right\rangle \\ &\quad - \left\langle [\mathcal{R}](\{u_*^{(1)}\} - \{u_*^{(2)}\}), (\{u_*^{(1)}\} - \{u_*^{(2)}\})\right\rangle \\ &= \left\langle [\mathcal{R}](\{u_T^{(1)}\}_n - \{u_T^{(2)}\}_n) - (\{u_*^{(1)}\} - \{u_*^{(2)}\})), (\{u_T^{(1)}\}_n - \{u_T^{(2)}\}_n)\right\rangle \\ &\quad + \left\langle [\mathcal{R}](\{u_*^{(1)}\} - \{u_*^{(2)}\}), ((\{u_T^{(1)}\}_n - \{u_T^{(2)}\}_n) - (\{u_*^{(1)}\} - \{u_*^{(2)}\}))\right\rangle.\end{aligned}$$

5 Convergence result

In this section, we are interested in showing the existence of a subsequence of the solutions of the discrete problems which converges to a solution of the continuous one. For this we introduce the following definitions:
Let $(\psi_h)_h$ be a sequence such that $\psi_h \in K_0^h$ for all h, we define the convergence of $(\psi_h)_h$ to $\psi \in K_0$ as $h \longrightarrow 0$ by

$$\psi_h \longrightarrow \psi \iff \psi_{i,h}(b,.) \longrightarrow \psi_i(b,.) \text{ uniformly in } [0,a] \quad \text{for } i=1,2. \tag{23}$$

For a sequence $((u_h^{(1)}, u_h^{(2)}))_h$ such that $(u_h^{(1)}, u_h^{(1)}) \in U_{ad}^h$, the convergence of the sequence $((u_h^{(1)}, u_h^{(2)}))_h$ to $(u^{(1)}, u^{(2)}) \in U_{ad}$, as $h \longrightarrow 0$, is defined by

$$(u_h^{(1)}, u_h^{(2)}) \longrightarrow (u^{(1)}, u^{(2)}) \Longleftrightarrow \begin{cases} u_{i,h}^{(1)} \rightharpoonup u_i^{(1)} \text{ weakly in } H^1(\Omega^{(1)}) \\ u_{i,h}^{(2)} \rightharpoonup u_i^{(2)} \text{ weakly in } H^1(\Omega^{(2)}) \quad \text{for } i=1,2. \end{cases} \tag{24}$$

Our convergence result is based on the following lemma.

Lemma 5 (**i**) *For any $(u^{(1)}, u^{(2)}) \in U_{ad}$, such that $u^{(k)} = u^{(k)}(\psi)$ for $\psi \in K_0$, there exists a sequence $((u_h^{(1)}, u_h^{(2)}))_h$ such that $u_h^{(k)} = u_h^{(k)}(\psi_h)$ for $\psi_h \in K_0^h$ and $(u_h^{(1)}, u_h^{(2)}) \longrightarrow (u^{(1)}, u^{(2)})$.*
(**ii**) *Let $((u_h^{(1)}, u_h^{(2)}))_h$ be a sequence of U_{ad}^h such that $u_h^{(k)} = u_h^{(k)}(\psi_h)$ for $\psi_h \in K_0^h$. Then there exists a subsequence of $((u_h^{(1)}, u_h^{(2)}))_h$ denoted again by $((u_h^{(1)}, u_h^{(2)}))_h$ and an element $(u^{(1)}, u^{(2)}) \in U_{ad}$ such that $u^{(k)} = u^{(k)}(\psi)$ for $\psi \in K_0$ and $(u_h^{(1)}, u_h^{(2)}) \longrightarrow (u^{(1)}, u^{(2)})$.*
(**iii**) *If $((u_h^{(1)}, u_h^{(2)}))_h$ is a sequence such that $(u_h^{(1)}, u_h^{(2)}) \in U_{ad}^h$, and $(u^{(1)}, u^{(2)}) \in U_{ad}$ such that $(u_h^{(1)}, u_h^{(2)}) \longrightarrow (u^{(1)}, u^{(2)})$.*
Then $J_h((u_h^{(1)}, u_h^{(2)}) \longrightarrow J(u^{(1)}, u^{(2)})$ as $h \longrightarrow 0$.

Proof.
In order to show (**i**), let $(u^{(1)}, u^{(2)}) \in U_{ad}$ such that such that $u^{(k)} = u^{(k)}(\psi)$ for $\psi \in K_0$. For $h > 0$ and $k = 1, 2$, we construct the sequence $(\psi_h)_h = (\psi_{1,h}, \psi_{2,h})$ as follows:

$$\psi_{k,h} \in C(\gamma) \text{ such that } \psi_{k,h}(b,.)|_{[a_{i-1},a_i]} \in P_1 \text{ for } \; i = 1, \ldots, N-1,$$
$$\psi_{k,h}(b,a_i) = \frac{1}{h} \int_{(i-\frac{1}{2})h}^{(i+\frac{1}{2})h} \psi_k(b,\tau)\, d\tau \quad \text{for } \; i = 1, \ldots, N-2,$$

$$\psi_{k,h}(b,0) = \frac{2}{h}\int_0^{\frac{h}{2}} \psi_k(b,\tau)\,d\tau \quad \text{and} \quad \psi_{k,h}(b,a) = \frac{2}{h}\int_{a-\frac{h}{2}}^{a} \psi_k(b,\tau)\,d\tau.$$

It is easy to see that

$$|\psi_{k,h}(b,a_i) - \psi_{k,h}(b,a_{i-1})| \leq C\,h \quad \text{for } i = 1,\ldots,N \tag{25}$$

which leads, with some elementary calculations to the following estimate

$$\|\psi_{k,h} - \psi_k\|_{L^\infty(\gamma)} \leq \frac{C}{2}\,h. \tag{26}$$

We deduce from this that

$$\|\psi_{k,h}\|_{L^\infty(\gamma)} \leq \frac{C}{2}\,h + C_0. \tag{27}$$

Then $\psi_h \in {K_0}^h$ and ψ_h converges to ψ. Let $((u_h^{(1)}, u_h^{(2)}))_h$ be in U_{ad}^h such that $u_h^{(k)} = u_h^{(k)}(\psi_h)$, this means that $u_h^{(k)} \in H_{1k,D}^h(\Omega^{(k)})$ is the solution of

$$a(u_h^{(k)}, \upsilon_h) = \sum_{i=1}^{2} \int_{\Omega^{(k)}} f_{i,h}^{(k)}\,\upsilon_{i,h}\,dx + (-1)^k \sum_{i=1}^{2} \int_\gamma \psi_{i,h}\,\upsilon_{i,h}\,d\sigma \;\forall \upsilon_h \in H_{k,D}^k(\Omega^{(k)}). \tag{28}$$

Using equations (27) and (28), we can show that $(u_h^{(k)})_h$ is uniformly bounded in $(H^1(\Omega^{(k)}))^2$ and thus we can extract a subsequence denoted again $(u_h^{(k)})_h$, such that $u_{i,h}^{(k)}$ is weakly convergent to $V_i^{(k)}$ in $H^1(\Omega^{(k)})$, for $i = 1, 2$. From the compactness of the trace operator from $H^1(\Omega^{(k)})$ to $L^2(\Gamma_k)$ we have that $V^{(k)} = (V_1^{(k)}, V_2^{(k)}) \in H_{k,D}(\Omega^{(k)})$. To conclude that $V^{(k)} = u^{(k)}$, it suffices to show that $V^{(k)}$ is solution of the equation:

$$a(V^{(k)}, \upsilon) = \sum_{i=1}^{2} \int_{\Omega^{(k)}} f_i^{(k)}\,\upsilon_i\,dx + (-1)^k \sum_{i=1}^{2} \int_\gamma \psi_i\,\upsilon_i\,d\sigma \;\forall \upsilon \in H_{k,D}(\Omega^{(k)}). \tag{29}$$

Let υ in $H_{k,D}(\Omega^{(k)})$, and denote by $\Phi_h = \Pi_h \upsilon \in H_{k,D}^h(\Omega^{(k)})$ the piecewise linear interpolant of υ, we have:

$$a(u_h^{(k)}, \Phi_h) = \sum_{i=1}^{2} \int_{\Omega^{(k)}} f_{i,h}^{(k)}\,\Phi_{i,h}\,dx + (-1)^k \sum_{i=1}^{2} \int_\gamma \psi_{i,h}\,\Phi_{i,h}\,d\sigma \tag{30}$$

By passing to the limit in equation (30) as $h \longrightarrow 0$, we obtain that $V^{(k)}$ is a solution of equation (29). Indeed, we have

$$\sum_{i,j=1}^{2} \int\limits_{\Omega^{(k)}} \left(\sigma_{ij}(u_h^{(k)})\, \varepsilon_{ij}(\Phi_h) - \sigma_{ij}(V^{(k)})\, \varepsilon_{ij}(\upsilon)\right) = I_1 + I_2$$

where $I_1 = \sum_{i,j=1}^{2} \int\limits_{\Omega^{(k)}} \left(\sigma_{ij}(u_h^{(k)}) - \sigma_{ij}(V^{(k)})\right) \varepsilon_{ij}(\upsilon)$

and $I_2 = \sum_{i,j=1}^{2} \int\limits_{\Omega^{(k)}} \sigma_{ij}(u_h^{(k)})\ (\varepsilon_{ij}(\Phi_h) - \varepsilon_{ij}(\upsilon))$

From the weak convergence in $H^1(\Omega^{(k)})$ of $u_{i,h}^{(k)}$ to $V_i^{(k)}$, for $i = 1, 2$, we have that I_1 converges to 0 as $h \longrightarrow 0$. By virtue of the convergence result of $\Phi_h = \Pi_h \upsilon$ to υ in $(H^1(\Omega^{(k)}))^2$, as $h \longrightarrow 0$ (see [3]) and since $u_h^{(k)}$ is uniformly bounded in $(H^1(\Omega^{(k)}))^2$, we get that I_2 converges to 0. In similar fashion using the convergence (16) and (23), we can show that

$$\lim_{h\longrightarrow 0} \sum_{i=1}^{2} \left(\int\limits_{\Omega^{(k)}} f_{i,h}^{(k)}\, \Phi_{i,h}\, dx - \int\limits_{\Omega^{(k)}} f_i^{(k)}\, \upsilon_i\, dx \right)$$

$$= \lim_{h\longrightarrow 0} \sum_{i=1}^{2} \left(\int\limits_{\Omega^{(k)}} (f_{i,h}^{(k)} - f_i^{(k)})\, \upsilon_i\, dx + \int\limits_{\Omega^{(k)}} (\Phi_{i,h} - \upsilon_i)\, f_{i,h}^{(k)}\, dx \right) = 0$$

$$\lim_{h\longrightarrow 0} \sum_{i=1}^{2} \left(\int\limits_{\gamma} \psi_{i,h}\, \Phi_{i,h}\, dx\sigma - \int\limits_{\gamma} \psi_i\, \upsilon_i\, d\sigma \right)$$

$$= \lim_{h\longrightarrow 0} \sum_{i=1}^{2} \left(\int\limits_{\gamma} (f_{i,h}^{(k)} - f_i^{(k)})\, \upsilon_i\, d\sigma + \int\limits_{\gamma} (\Phi_{i,h} - \upsilon_i)\, f_{i,h}^{(k)}\, d\sigma \right) = 0$$

This achieve the proof of assertion (**i**).
To show (**ii**), Let $((u_h^{(1)}, u_h^{(2)}))_h$ be a sequence of U_{ad}^h such that $u_h^{(k)} = u_h^{(k)}(\psi_h)$, for $\psi_h \in K_0^h$. We have that for all h and $i = 1, 2$, $\psi_{i,h} \in T$, where T is the space defined by

$$T = \{\chi \in C(\gamma) \ /\ |\chi(b, x) - \chi(b, x')| \leq C\,|x - x'| \quad \forall x, x' \in [0, a] \text{ and } \|\chi\|_{L^\infty(\gamma)} \leq \frac{C}{2} + C0\}.$$

According to the Ascoli-Arzelà theorem's, we can extract a subsequence noted again $(\psi_h)_h$, such that $\psi_{i,h}$ converges in T to $\psi_i \in T$, for $i = 1, 2$. Furthermore, by passing to the limit in equation (27), we have that $\psi = (\psi_1, \psi_2) \in K_0$. Using the same techniques as in the proof of (**i**), we show that for $k = 1, 2$, $u_{i,h}^{(k)}$ converges weakly in $H^1(\Omega^{(k)})$ to $u_i^{(k)} = u_i^{(k)}(\psi)$ and that $u^{(k)}$ is solution of (15).
The proof of the assertion (**iii**) uses mainly the same technique as in the proof of continuity of J in Theorem 1. This ends the proof of the lemma.

We can now prove our main result of convergence stated in the following theorem

Theorem 6 *Let $((u_{*,h}^{(1)}, u_{*,h}^{(2)}))_h$ be a sequence such that $(u_{*,h}^{(1)}, u_{*,h}^{(2)})$ is solution of (PO^h) and and $(u_{*,h}^{(1)}, u_{*,h}^{(2)}) \in U_{ad}^h$. Then, there exists a subsequence denoted again $((u_{*,h}^{(1)}, u_{*,h}^{(2)}))_h$ and an element $(u_*^{(1)}, u_*^{(2)}) \in U_{ad}$ such that*

$$(u_{*,h}^{(1)}, u_{*,h}^{(2)}) \longrightarrow (u_*^{(1)}, u_*^{(2)})$$

furthermore $(u_*^{(1)}, u_*^{(2)})$ is solution of (PO).

Proof.
Let $(u^{(1)}, u^{(2)})$ be an element of U_{ad}, from the assertion (**i**) of Lemma 1, there exists a sequence $((u_h^{(1)}, u_h^{(2)}))_h$ such that $(u_h^{(1)}, u_h^{(2)}) \in U_{ad}^h$ and

$$(u_h^{(1)}, u_h^{(2)}) \longrightarrow (u^{(1)}, u^{(2)})$$

According to the assertion (**iii**), we have that

$$J_h(u_h^{(1)}, u_h^{(2)}) \longrightarrow J(u^{(1)}, u^{(2)}) \text{ as } h \longrightarrow 0$$

Now, Let $((u_{*,h}^{(1)}, u_{*,h}^{(2)}))_h$ be a sequence such that is solution of (PO^h) and $(u_{*,h}^{(1)}, u_{*,h}^{(2)}) \in U_{ad}^h$. From the assertion (**ii**) of Lemma 1, there exists a subsequence denoted again $((u_{*,h}^{(1)}, u_{*,h}^{(2)}))_h$ and an element $(u_*^{(1)}, u_*^{(2)}) \in U_{ad}$ such that

$$(u_{*,h}^{(1)}, u_{*,h}^{(2)}) \longrightarrow (u_*^{(1)}, u_*^{(2)})$$

According to the assertion (**iii**), we have that

$$J_h(u_{*,h}^{(1)}, u_{*,h}^{(2)}) \longrightarrow J(u_*^{(1)}, u_*^{(2)}) \text{ as } h \longrightarrow 0$$

however, we have that

$$J(u^{(1)}_{*,h}, u^{(2)}_{*,h}) \leq J(u^{(1)}_h, u^{(2)}_h) \quad \text{for all } h \tag{31}$$

The main result is then obtained by passing to the limit in equation (31), as $h \longrightarrow 0$.

6 Optimization algorithm

We use the Lagrange multiplier rule to derive an optimality system of equations from which solutions of the optimization problem (PO) may be determined.

Let $u^{(i)}, \lambda^{(i)} \in H_{i,D}(\Omega^{(i)})$, for $i = 1, 2$, and $\psi \in (L^2(\gamma))^2$ we define the Lagrangian

$$\mathcal{L}(u^{(1)}, u^{(2)}, \psi, \lambda^{(1)}, \lambda^{(2)}) = J(\psi, u^{(1)}, u^{(2)}) - \int\limits_{\Omega^{(1)}} \sigma_{ij}(u^{(1)}(x))\varepsilon_{ij}(\lambda^{(1)}(x))\, dx$$

$$+ \int\limits_{\Omega^{(1)}} f^{(1)}(x)\lambda^{(1)}(x)\, dx + \int\limits_{\gamma} \psi(x)\lambda^{(1)}(x)\, dx - \int\limits_{\Omega^{(2)}} \sigma_{ij}(u^{(2)}(x))\varepsilon_{ij}(\lambda^{(2)}(x))\, dx$$

$$+ \int\limits_{\Omega^{(2)}} f^{(2)}(x)\lambda^{(2)}(x)\, dx - \int\limits_{\gamma} \psi(x)\lambda^{(2)}(x)\, dx$$

Setting to zero the first variations with respect to the multipliers λ_1 ans λ_2 yields the constraints (7). Setting to zero the first variations with respect to $u^{(1)}$ and $u^{(2)}$ yield the adjoint equations

$$a^{(1)}(v, \lambda^{(1)}) = (u^{(1)} - u^{(2)}, v)_\gamma \ \ \forall v \in H_{1,D}(\Omega^{(1)}) \tag{32}$$

and

$$a^{(2)}(v, \lambda^{(2)}) = -(u^{(1)} - u^{(2)}, v)_\gamma \ \ \forall v \in H_{2,D}(\Omega^{(2)}) \tag{33}$$

respectively.

Then the adjoint equations is given by

$$\begin{cases} \sum_{j=1}^{2} \frac{\partial\, \sigma_{ij}(\lambda^{(1)})}{\partial\, x_j} = 0 \text{ in } \Omega^{(1)} \\ \lambda^{(1)} = 0 \quad \text{on } \Gamma_1 \\ \sum_{j=1}^{2} \sigma_{ij}(\lambda^{(1)}) n_j = u^{(1)} - u^{(2)} \text{ on } \gamma \end{cases} \tag{34}$$

$$\begin{cases} \sum_{j=1}^{2} \frac{\partial\, \sigma_{ij}(\lambda^{(2)})}{\partial\, x_j} = 0 \text{ in } \Omega^{(2)} \\ \lambda^{(2)} = 0 \quad \text{on } \Gamma_2 \\ \sum_{j=1}^{2} \sigma_{ij}(\lambda^{(2)}) n_j = -(u^{(1)} - u^{(2)}) \text{ on } \gamma \end{cases} \tag{35}$$

Let $\mathcal{J}(\psi) = J(\psi, u^{(1)}, u^{(2)})$ where, for given ψ,

$$u^{(i)} : \psi \in (L^2(\gamma))^2 \rightarrow H_{i,D}(\Omega^{(i)}) \text{ for } i = 1, 2$$

are defined as the solution of (4) and (5) respectively. Then, the minimization problem is equivalent to the problem of determining $\psi \in (L^2(\gamma))^2$ such that $\mathcal{J}(\psi)$ is minimized. Now, the first derivative of $\mathcal{J}$ is defined through its action on variations $\tilde{\psi}$ by

$$\langle \frac{d\mathcal{J}}{d\psi}, \tilde{\psi} \rangle = (u^{(1)} - u^{(2)}, \tilde{u}^{(1)} - \tilde{u}^{(2)})_\gamma \quad \forall \tilde{\psi} \in (L^2(\gamma))^2 \tag{36}$$

where $\tilde{u}^{(1)} \in H_{1,D}(\Omega^{(1)})$ and $\tilde{u}^{(2)} \in H_{2,D}(\Omega^{(2)})$ are the solution of

$$a^{(1)}(\tilde{u}^{(1)}, v) = (\tilde{\psi}, v)_\gamma \ \ \forall v \in H_{1,D}(\Omega^{(1)}) \tag{37}$$

and

$$a^{(2)}(\tilde{u}^{(2)}, v) = -(\tilde{\psi}, v)_\gamma \ \ \forall v \in H_{2,D}(\Omega^{(2)}) \tag{38}$$

respectively. Set $v = \lambda_1^{(1)}$ in (37), $v = \lambda^{(1)}$ in (38), $v = \tilde{u}^{(1)}$ in (32) and $v = \tilde{u}^{(2)}$ in (33). Combning the results yields that

$$\frac{d\mathcal{J}}{d\psi} = \lambda^{(1)} - \lambda^{(2)} \text{ on } \gamma. \tag{39}$$

we now present our domain decomposition algorithm

Algorithm 1 *$k = 0$ and $\psi_{,0}$ is given*
For $k = 0, \ldots$

$$\textit{Solve} \quad \begin{cases} \sum_{j=1}^{2} \dfrac{\partial\, \sigma_{ij}(u_{,k}^{(1)})}{\partial\, x_j} = f_i^{(1)} \ \textit{in}\ \Omega^{(1)} \\ u_{,k}^{(1)} = 0 \quad \textit{on}\ \Gamma_1 \\ \sum_{j=1}^{2} \sigma_{ij}(u_{,k}^{(1)}) n_j = \psi_{i,k} \ \textit{on}\ \gamma \end{cases} \tag{40}$$

$$\begin{cases} \sum_{j=1}^{2} \dfrac{\partial\, \sigma_{ij}(u_{,k}^{(2)})}{\partial\, x_j} = f_i^{(2)} \ \textit{in}\ \Omega^{(2)} \\ u_{,k}^{(2)} = 0 \quad \textit{on}\ \Gamma_2 \\ \sum_{j=1}^{2} \sigma_{ij}(u_{,k}^{(2)}) n_j = -\psi_{i,k} \ \textit{on}\ \gamma \end{cases} \tag{41}$$

$$\textit{Solve} \quad \begin{cases} \sum_{j=1}^{2} \dfrac{\partial\, \sigma_{ij}(\lambda_{,k}^{(1)})}{\partial\, x_j} = 0 \ \textit{in}\ \Omega^{(1)} \\ \lambda^{(1)} = 0 \quad \textit{on}\ \Gamma_1 \\ \sum_{j=1}^{2} \sigma_{ij}(\lambda_{,k}^{(1)}) n_j = u_{,k}^{(1)} - u_{,k}^{(2)} \ \textit{on}\ \gamma \end{cases} \tag{42}$$

$$\begin{cases} \sum_{j=1}^{2} \dfrac{\partial\, \sigma_{ij}(\lambda_{,k}^{(2)})}{\partial\, x_j} = 0 \ \textit{in}\ \Omega^{(2)} \\ \lambda_{,k}^{(2)} = 0 \quad \textit{on}\ \Gamma_2 \\ \sum_{j=1}^{2} \sigma_{ij}(\lambda_{,k}^{(2)}) n_j = -(u_{,k}^{(1)} - u_{,k}^{(2)}) \ \textit{on}\ \gamma \end{cases} \tag{43}$$

Compute $\nabla J(\psi_{,k}) = \lambda_{,k}^{(1)}(\psi_{,k}) - \lambda_{,k}^{(2)}(\psi_{,k})$
Update

$$\gamma^k = \frac{\|\nabla J(\psi_{,k})\|}{\|\nabla J(\psi_{,k-1})\|}$$

$$d_{,k} = \nabla J(\psi_{,k}) + \gamma^k d_{,k-1}$$

$$\textit{Solve} \quad \begin{cases} \sum_{j=1}^{2} \dfrac{\partial\, \sigma_{ij}(D_{,k}^{(1)})}{\partial\, x_j} = 0 \ \textit{in}\ \Omega^{(1)} \\ D_{,k}^{(1)} = 0 \quad \textit{on}\ \Gamma_1 \\ \sum_{j=1}^{2} \sigma_{ij}(D_{,k}^{(1)}) n_j = d_{i,k} \ \textit{on}\ \gamma \end{cases} \tag{44}$$

$$\begin{cases} \sum_{j=1}^{2} \dfrac{\partial\, \sigma_{ij}(D_{,k}^{(2)})}{\partial\, x_j} = 0 \ \textit{in}\ \Omega^{(2)} \\ D_{,k}^{(2)} = 0 \quad \textit{on}\ \Gamma_2 \\ \sum_{j=1}^{2} \sigma_{ij}(D_{,k}^{(2)}) n_j = -d_{i,k} \ \textit{on}\ \gamma \end{cases} \tag{45}$$

Compute

$$\rho_k = \frac{(u_{,k}^{(1)} - u_{,k}^{(2)}, D_{,k}^{(1)} - D_{,k}^{(2)})}{\|D_{,k}^{(1)} - D_{,k}^{(2)}\|^2}$$

$$\psi_{,k+1} = \psi_{,k} - \rho_k d_{,k}$$

End For

7 Numerical results

In order to illustrate the performance of the numerical method described above, we solve the linear elasticity problem (1), in two-dimensional domain $\Omega = (0,1) \times (0,1)$, with $u = u^{an}$ on Γ and $f = 0$. We assume that the boundary is split into two parts $\Gamma_1 = [0, 0.5] \times \{0\} \cup [0, 0.5] \times \{1\} \cup \{0\} \times [0, 1]$ and $\Gamma_2 = [0.5, 1] \times \{0\} \cup \{1\} \times [0, 1] \cup [0.5, 1] \times \{1\}$. For these data, the analytical

solution is given by

$$u_1^{an}(x,y) = \frac{1-\nu}{2G}\sigma_0 xy,\ u_2^{an}(x,y) = -\frac{1}{4G}\sigma_0((1-\nu)(x^2-1)+\nu y^2) \quad (46)$$

$$t_1^{an}(x,y) = \sigma_0 y n_1,\ t_2^{an}(x,y) = 0 \quad (47)$$

with $\sigma_0 = 1.5 \times 10^{10}$, $G = 3.35 \times 10^{10}$ and $\nu = 0.34$.

This example consists to split the domain Ω into two rectangular subdomains $\Omega^{(1)} = (0., 0.5) \times (0, 1)$ and $\Omega^{(2)} = (0.5, 1) \times (0, 1)$ with interface $\gamma = \{0.5\} \times [0, 1]$.

In this section we investigate the convergence of the proposed method by the evaluation at every iteration the accuracy errors denoted for $i, j = 1, 2$ by

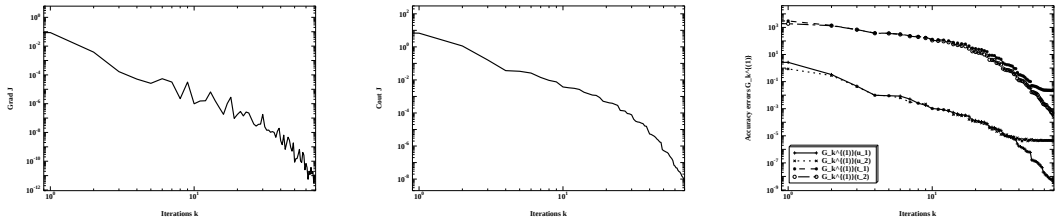
$$G_k^{(i)}(u_j) = \|u_{j,k}^{(i)} - u_j^{(i)^{an}}\|^2_{L^2(\gamma)},\ G_k^{(i)}(t_j) = \|t_{j,k}^{(i)} - t_j^{(i)^{an}}\|^2_{L^2(\gamma)}. \quad (48)$$

The following stopping criterion is considered

$$\|\nabla J(\psi_{,k})\|^2 < \eta \|\nabla J(\psi_{,0})\|^2 \quad (49)$$

where η is a small prescribed positive quantity. For all numerical experiments, we take $\eta = 10^{-11}$.

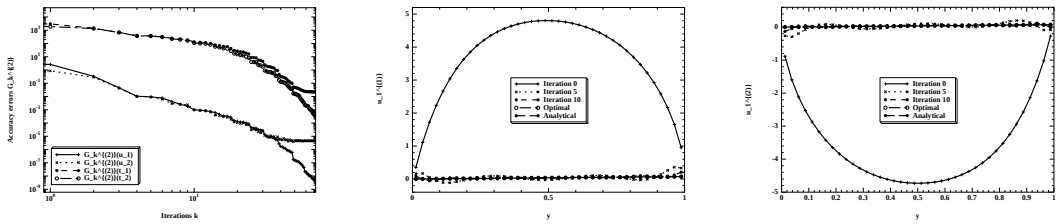
The mesh of discretization is taken as $h = 1/40$. The initial guess $\psi_{i,0}$ on γ has been chosen as $\psi_{i,0} = 100$. When starting with this initial guess, which is not too close to the exact traction, a sequence of displacements $\left\{(u_{,k}^{(1)})_h\right\}_{k\geq 0}$ and $\left\{(u_{,k}^{(2)})_h\right\}_{k\geq 0}$ of approximation functions for $u_{|\gamma}$ is obtained and this sequences converge to the exact solution. We observe from Figure 1(a), (b) that the



(a) (b) (c)

Fig. 1. Computed norm of gradient (a), cost functional (b) and the accuracy errors (c) given by (48) as a function of the number of iterations k.

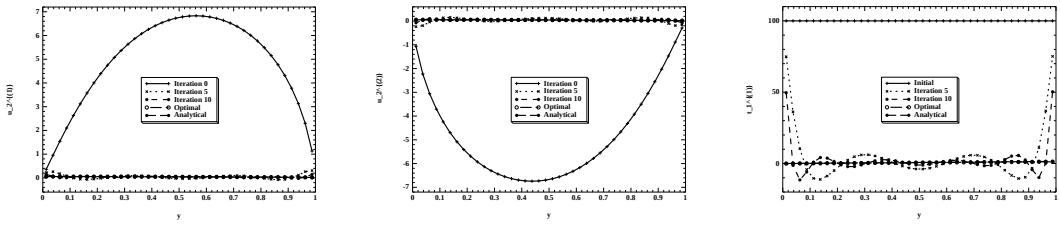
norm of gradient and the cost decrease as a function of number of iterations. Figure 1(c) and Figure 2(a) shows the evaluation of accuracy errors as function of number of iterations. The discrepancy $\|u_{1,opt}^{(1)} - u_1^{(1)^{an}}\|^2_{L^2(\gamma)}$ between the



(a) (b) (c)

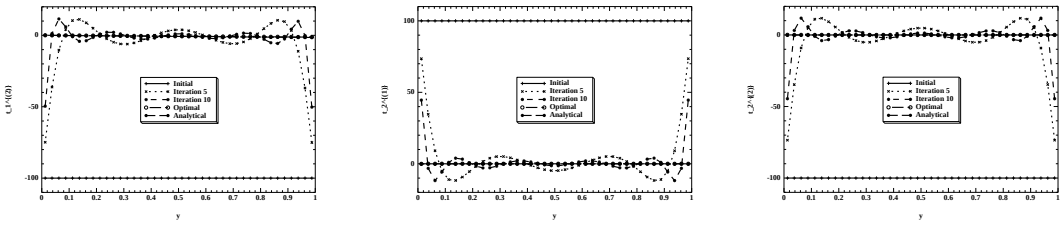
Fig. 2. The accuracy errors (a) given by (48) as a function of the number of iterations k, results of $u_1^{(1)}$ (b) and $u_1^{(2)}$ (c) on interface γ

optimal x_1-displacement and the exact one is equal to 3.35×10^{-09} and the discrepancy $\|t_{2,opt}^{(2)} - t_2^{(2)^{an}}\|^2_{L^2(\gamma)}$ between the optimal x_2-traction and the exact one is equal to 2.92×10^{-04}. Figure 1(c) and Figure 2(a) shows the evaluation of accuracy errors as function of number of iterations. Figure 2- 4 proves the well convergence of the proposed optimal control algorithm.



(a) (b) (c)

Fig. 3. Results of $u_1^{(2)}$ (a), $u_2^{(2)}$ (b) and $t_1^{(1)}$ (c) on interface γ



(a) (b) (c)

Fig. 4. Results of $t_2^{(1)}$ (a), $t_1^{(2)}$ (b) and $t_2^{(2)}$ (c) on interface γ

8 Conclusion

In this paper, the Problem of linear elasticity equations is formulated into an optimal control problem. The linear finite element is used for the approximation of this problem. The convergence of the solutions of discrete problems to a solution of the continuous one is proved. The numerical results obtained were found to be good in agreement with the exact solution.

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