HAL
open science

# Polyakov soldering and second order frames : the role of the Cartan connection

Serge Lazzarini, Carina Tidei

## To cite this version:

Serge Lazzarini, Carina Tidei. Polyakov soldering and second order frames : the role of the Cartan connection. Letters in Mathematical Physics, 2008, 85 (1), pp.27-37. 10.1007/s11005-008-0253-8 . hal-00258919v2

**HAL Id: hal-00258919**

**https://hal.science/hal-00258919v2**

Submitted on 3 Jul 2008

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Polyakov soldering and second order frames : the role of the Cartan connection

Serge Lazzarini * and Carina Tidei †

Centre de Physique Théorique ‡
CNRS Luminy – case postale 907,
F–13288 Marseille Cedex 9, France.

**Abstract**

The so-called '*soldering*' procedure performed by A.M. Polyakov in [1] for a $SL(2,\mathbb{R})$-gauge theory is geometrically explained in terms of a Cartan connection on second order frames of the projective space $\mathbb{R}\mathrm{P}^1$. The relationship between a Cartan connection and the usual (Ehresmann) connection on a principal bundle allows to gain an appropriate insight into the derivation of the genuine '*diffeomorphisms out of gauge transformations*' given by Polyakov himself.

PACS-2006 number : 02.40.Dr Euclidean and projective geometries, 02.40.Hw Classical differential geometry, 11.15.-q Gauge field theories, 11.25.Hf Conformal field theory, algebraic structures.
MSC-2000 number : 57R25 Vector fields, frame fields.

Keywords : Higher order frames, Cartan connection, Polyakov soldering.

CPT-P013-2008

*e-mail : `sel@cpt.univ-mrs.fr`

†e-mail : `tidei@cpt.univ-mrs.fr` ; allocataire de recherche MESR.

‡Unité Mixte de Recherche (UMR 6207) du CNRS et des Universités Aix-Marseille I, Aix-Marseille II et de l'Université du Sud Toulon-Var. Unité affiliée à la FRUMAM Fédération de Recherche 2291.

# 1 Introduction

More than fifteen years ago, in a paper by A.M. Polyakov [1], diffeomorphism transformations for the 2-d conformal geometry were explicitly derived by a '*soldering procedure*' from a partial gauge fixing of two components out of three of a chirally split $SL(2,\mathbb{R})$-connection in the light-cone formulation. In doing so, Polyakov ended with a residual gauge transformation which exactly reproduced the Virasoro group action on the effective energy-momentum tensor. This striking result was raised by Polyakov himself as '*a geometrical surprise*' and in addition some of the gauge parameters were noticed to be gauge field dependent.

Subsequently, some work *e.g.* [2,3] directly referred to the Polyakov partial gauge fixing, which we shall call in the sequel as the '*Polyakov soldering*'. It has to be said that in [3] this intriguing result obtained by Polyakov was also emphasized as such.

In the present paper a proper differential geometrical framework is proposed in order to explain this '*geometrical surprise*' [1,3]. It is essentially grounded on the use of the Cartan connection on the second order frame bundle [4] and all the differential algebraic setup which goes with.

In order to have a concise and efficient geometrical writing, we shall use the BRS differential algebra for treating the gauge symmetry aspects.

One of the main ingredients will be the so-called solder forms on frame bundles [5], well known objects in mathematics and intuitively figured out by Polyakov under the phrasing '*soldering*' procedure.

# 2 Second order frames

Let us briefly introduce the notion of the second order frame bundle. Most of the time, we shall adopt the viewpoint that physics often requires, the use of local expressions over a manifold. In other words, local comparison of a $n$-dimensional manifold $M$ with $\mathbb{R}^n$ will be of constant use.

According to [4,6], consider the principal bundle $F_2M$ of second order frames, or shortly 2-frames [1], over $M$. The elements of $F_2M$ are 2-jets $j_2(f)(0)$ of local diffeomorphisms $f$ from a neighborhood of 0 in $\mathbb{R}^n$ to $M$ such that $f(0)=x$. Given local coordinate systems $\{x^\mu\}_{\mu=1}^n$ on $M$ and $\{u^a\}_{a=1}^n$ on $\mathbb{R}^n$, $F_2M$ is equipped with local coordinates[2] $(x^\mu, e^\mu{}_a, e^\mu{}_{ab})$ corresponding to the successive derivatives of $f$ at 0. Hence $e^\mu{}_a \neq 0$ and $e^\mu{}_{ab} = e^\mu{}_{ba}$. When the composition of maps makes sense, the jet product is given by the chain rule derivation $j_2(f)\cdot j_2(f') = j_2(f\circ f')$.

Accordingly 2-frame fields are sections of the bundle $F_2M$ over $M$, $e_2(x) = \big(x^\mu, e^\mu{}_a(x), e^\mu{}_{ab}(x)\big)$ with inverse given by $e_2{}^{-1}(x) = \big(x^\mu, e^a{}_\mu(x), e^a{}_{\mu\nu}(x)\big)$ where

$$e^a{}_\mu e^\mu{}_b = \delta^a_b, \qquad\qquad e^a{}_{\mu\nu} = -e^\lambda{}_{bc} e^b{}_\mu e^c{}_\nu e^a{}_\lambda\,. \tag{1}$$

In the $u$-coordinates, a 2-frame $e_2$ at $x\in M$ has the polynomial representative $f$,

$$f^\mu(u) = x^\mu + e^\mu{}_a u^a + e^\mu{}_{ab} u^a u^b, \qquad u\in\mathbb{R}^n\ ,$$

[1] It can be shown that they are torsion free linear frames over the linear frame bundle itself [7]

[2] Usual notation for gravitation will be constantly used [8].

such that $e_2 = j_2(f)(0)$, and the coordinates $e^\mu{}_{ab}$ and $e^\mu{}_a$ are independent variables. Moreover, note that the 2-frame at $x$ associated to the local $x$-coordinates themselves are obtained as 2-jets of translations $(x^\mu, \delta^\mu{}_a, 0) = j_2(u \mapsto x + u)(0)$. This induces a *natural* 2-frame at $x$ [9].

The structure group of $F_2M$ is the so-called differential group of order two $G_2$ [10] locally given by the 2-jets $g_2 = (g^a{}_{a'}, g^a{}_{a'b'})$ at 0 of diffeomorphisms $g$ of $\mathbb{R}^n$ fixing the origin[3] 0. The right action given by jet product $e_2' = e_2 \cdot g_2 = j_2(f \circ g)(0)$ reads

$$f(g(0)) = f(0) = x, \qquad e^\mu{}_{a'} = e^\mu{}_a\, g^a{}_{a'}, \qquad e^\mu{}_{a'b'} = e^\mu{}_{ab}\, g^a{}_{a'} g^b{}_{b'} + e^\mu{}_a\, g^a{}_{a'b'}.$$

It is worthwhile to note the semi-direct product decomposition of $G_2 = GL_0 \ltimes GL_1$ with respect to the jet product, namely $g_2 = (g^a{}_{b'}, 0) \cdot (\delta^{b'}{}_{a'}, g^{b'}{}_b g^b{}_{a'b'})$, with $(g^a{}_{a'}) \in GL_0 := GL(n, \mathbb{R})$ [4].

To the local $x$-coordinates on $M$, there corresponds a *natural gauge* in which the local coordinates of any 2-frame field $e_2 : x \mapsto e_2(x) \in G_2$ can be considered as an element of the gauge group, thanks to the pointwise identification $\big(e^\mu{}_a(x), e^\mu{}_{ab}(x)\big) = (\delta^\mu{}_c, 0) \cdot (g^c{}_a(x), g^c{}_{ab}(x))$.

Let us now introduce a family of solder 1-forms on $F_2M$ (also called canonical forms) which are invariant under diffeomorphisms of $M$ by taking the Maurer-Cartan like 1-form "$f^{-1} \circ df$" in powers of $u \in \mathbb{R}^n$ [11],

$$(f^{-1} \circ df)^a(u) = \theta^a + \theta^a{}_b\, u^b + \tfrac{1}{2}\theta^a{}_{bc} u^b u^c + \cdots .$$

Then, in terms of the local coordinates $(x^\mu, e^\mu{}_a, e^\mu{}_{ab})$ on $F_2M$ the solder 1-forms read [5]

$$\theta^a = e^a{}_\mu dx^\mu, \qquad \theta^a{}_b = e^a{}_\mu de^\mu{}_b + e^a{}_{\mu\nu} e^\mu{}_b dx^\nu . \tag{2}$$

In particular, they fulfil the torsion free condition $d\theta^a + \theta^a{}_b \wedge \theta^b = 0$.

## 3 Cartan connection on second order frames

Adding translations denoted as $g^a \in GL_{-1} \simeq \mathbb{R}^n$, consider now the structure group $G_2$ as a Lie subgroup of the Lie group $G := GL_{-1} \times G_2$. Denoting $\mathfrak{gl}_k :=$ Lie $GL_k, k = 0, \pm 1$, the Lie algebra $\mathfrak{g}$ of $G$ admits the decomposition $\mathfrak{g} = \mathfrak{gl}_{-1} \oplus \mathfrak{gl}_0 \oplus \mathfrak{gl}_1$. In the sequel, we set $\mathfrak{g}_2 :=$ Lie $G_2 = \mathfrak{gl}_0 \oplus \mathfrak{gl}_1$. Since $G$ consists of 2-jets at 0 of diffeomorphisms $g_t(u) = u + tX(u)$ of $\mathbb{R}^n$, with $g_0 = id_{\mathbb{R}^n}$, $X(u) = \big(X^a + X^a{}_b u^b + X^a{}_{bc} u^b u^c + \cdots\big)\, \partial_a$, its Lie algebra $\mathfrak{g}$ consists of tangent vector fields at the identity $\mathbf{1} := j_2(id_{\mathbb{R}^n})(0) = (\delta^a{}_b, 0)$, given by the 2-jet at 0 of the vector field $X(u)$ on $\mathbb{R}^n$,

$$X_2 := j_2(X)(0) = \left.\frac{d}{dt}\right|_{t=0} j_2(g_t)(0) = (X^a, X^a{}_b, X^a{}_{bc}).$$

The algebraic bracket is then defined to be *minus* the Lie bracket of vector fields [12],

$$\begin{aligned} [X,Y]^a &= X^a{}_b Y^b - Y^a{}_b X^b \\ [X,Y]^a{}_b &= X^a{}_c Y^c{}_b + X^a{}_{bc} Y^c - (X \leftrightarrow Y) \\ [X,Y]^a{}_{bc} &= X^a{}_d Y^d{}_{bc} + X^a{}_{dc} Y^d{}_b + X^a{}_{bd} Y^d{}_c - (X \leftrightarrow Y). \end{aligned} \tag{3}$$

[3] The differential group is the little group relative to the origin.

[4] The motivation for the lower indices attached to $GL$ will become clear below in the main text.

[5] $e^a{}_\mu$ denotes the usual tetrad [8].

Then, $[\mathfrak{gl}_{-1}, \mathfrak{gl}_{-1}] = 0$ for the translation part and $[\mathfrak{gl}_k, \mathfrak{gl}_\ell] \subset \mathfrak{gl}_{k+\ell}, k, \ell = 0, \pm 1$. So, the Lie algebra $\mathfrak{g}$ turns out to be a graded Lie algebra [4] with respect to the dilatation generator $u^a \partial_a$.

Moreover, considering the differential group of order three $G_3 \ni \; g_3 \; = \; \left( g^{a'}_{\;a}, g^{a'}_{\;ab}, g^{a'}_{\;abc} \right)$, an adjoint type action on $\mathfrak{g}$ can be defined by [5]

$$\mathrm{Ad}(g_3)\, X_2 = \left. \frac{d}{dt} \right|_{t=0} j_2(g' \circ g_t \circ g'^{-1})(0) = \left( X^{a'}, X^{a'}_{\;b'}, X^{a'}_{\;b'c'} \right) ,$$

where the transformed components are given by chain rule derivations as,

$$\begin{aligned} X^{a'} &= g^{a'}_{\;a} X^a \; , \\ X^{a'}_{\;b'} &= g^{a'}_{\;ab} X^b g^a_{\;b'} + g^{a'}_{\;a} X^a_{\;b} g^b_{\;b'} \; , \\ X^{a'}_{\;b'c'} &= g^{a'}_{\;abc} X^c g^b_{\;b'} g^a_{\;c'} + g^{a'}_{\;ab} X^b_{\;c} g^c_{\;b'} g^a_{\;c'} + g^{a'}_{\;ac} g^c_{\;b'} X^a_{\;b} g^b_{\;c'} \\ &\quad + g^{a'}_{\;a} X^a_{\;bc} g^c_{\;b'} g^b_{\;c'} + g^{a'}_{\;ab} X^b g^a_{\;b'c'} + g^{a'}_{\;a} X^a_{\;b} g^b_{\;b'c'} \, . \end{aligned} \tag{4}$$

Now, we are in position to introduce a Cartan connection on $F_2M$ as a $\mathfrak{g}$-valued 1-form satisfying:

- $\forall e_2 \in F_2M$, $\omega_{|_{e_2}} : T_{e_2}(F_2M) \to \mathfrak{g}$ is a linear isomorphism (absolute parallelism),
- $\forall g_2 \in G_2 \subset G_3$, $R^*_{g_2}\omega = \mathrm{Ad}(g_2{}^{-1})\,\omega$, where $g_2 = \left( g^{a'}_{\;a}, g^{a'}_{\;ab}, 0 \right)$ ($G_2$-equivariance),
- $\forall \hat{X} \in V(F_2M)$, $\omega(\hat{X}) \in \mathfrak{g}_2$.

According to the grading of $\mathfrak{g}$, the Cartan connection $\omega$ can be chosen to be [4]

$$\omega := \omega_{-1} + \omega_0 + \omega_1 = \theta^a + \theta^a_{\;b} + \omega^a_{\;bc} \, ,$$

where $\theta^a$ and $\theta^a_{\;b}$ are the solder forms on $F_2M$ which are respectively $\mathfrak{gl}_{-1}$-valued, and $\mathfrak{gl}_0$-valued 1-forms, and $\omega^a_{\;bc}$ is a $\mathfrak{gl}_1$-valued 1-form on $F_2M$. Its curvature is defined by

$$K = d\omega + \frac{1}{2}[\omega, \omega] = K_{-1} + K_0 + K_1$$

where the bracket (3) has to be used. More explicitly one finds the torsion free condition $K_{-1} = d\omega_{-1} + \omega_0 \wedge \omega_{-1} = 0$ and the (generalized) curvature $K_0 = d\omega_0 + \frac{1}{2}[\omega_0, \omega_0] + [\omega_{-1}, \omega_1]$.

For the sequel, it is of use to note that, from a Cartan connection on $F_2M$ it is possible to recover the Yang-Mills context by constructing a (Ehresmann) connection on the principal $G$-bundle $F_2M \times_{G_2} G$, see [13, 14]. Actually, at a point $(e_2, g) \in F_2M \times G$, one can construct the $\mathfrak{g}$-valued 1-form

$$\mathcal{A}_{|(e_2,g)} = \mathrm{Ad}(g^{-1})(\pi_{F_2M})^*\omega + (\pi_G)^*\Theta_G,$$

where $\Theta_G$ is the Maurer-Cartan form on $G$ and $\pi_{F_2M}$ (resp. $\pi_G$) is the canonical projection on $F_2M$ (resp. $G$). $\mathcal{A}$ turns out to be a connection 1-form on the principal bundle $F_2M \times_{G_2} G$ [13].

Since we are concerned with local expressions, consider the local connection 1-form $A = \sigma^* \mathcal{A}$ (gauge field on $M$) obtained as pull-back of the connection 1-form by a (local) section $\sigma(x)$ of $F_2M \times_{G_2} G$. When we restrict ourselves to $F_2M \simeq F_2M \times_{G_2} \{\mathbf{1}\}$, and take $\sigma(x) = (e_2(x), \mathbf{1})$, the gauge field reduces to $A = e_2^* \omega$, namely the local expression of the Cartan connection on $M$. As it will be shown, this is the gauge field considered by Polyakov.

As outlined in [15], let us first consider, in the natural gauge, the "local expression" on $F_2M$ of the Cartan connection $\omega$ (a gauge like redefinition which to some extent is field dependent)

$$\Gamma(e_2, \omega) = \mathrm{Ad}\,(\ell(e_2))\ \omega + e_2 \cdot de_2^{-1}, \tag{5}$$

where $\ell(e_2) = \left(e^\mu{}_a, e^\mu{}_{ab}, e^\mu{}_{abc}\right)$ is the necessary lift of an element $e_2 \in G_2$ into $G_3$. It can be shown that the local connection 1-form $\Gamma$ generalizes the Christoffel symbols to second order frames [7]. Moreover by extending the proof for an affine connection [9] it can be checked that the generalized Christoffel symbols depend only on the $x$-coordinate . A direct computation [6] gives for the components of the torsion free Cartan connection

$$\begin{aligned}
\Gamma^\mu &= e^\mu{}_a \theta^a = dx^\mu\ , \\
\Gamma^\mu{}_\nu &= e^\mu{}_a \theta^a{}_b e^b{}_\nu + e^\mu{}_{ab} \theta^a e^b{}_\nu + e^\mu{}_a de^a{}_\nu = 0\ , \\
\Gamma^\mu{}_{\nu\rho} &= e^\mu{}_{abc} \theta^c e^b{}_\nu e^a{}_\rho + e^\mu{}_{ac} \theta^c e^a{}_{\nu\rho} + e^\mu{}_{ab} \theta^b{}_d e^d{}_\nu e^a{}_\rho + e^\mu{}_{ac} e^c{}_\nu \theta^a{}_b e^b{}_\rho \\
&\quad + e^\mu{}_a \theta^a{}_b e^b{}_{\nu\rho} + e^\mu{}_a \omega^a{}_{bc} e^c{}_\nu e^b{}_\rho + e^\mu{}_a de^a{}_{\nu\rho} + e^\mu{}_{ab} d(e^b{}_\nu e^a{}_\rho) = \Gamma^\mu{}_{\rho\nu}\ .
\end{aligned} \tag{6}$$

## 4 Reduction to the projective case

Having in mind the implementation of a $SL(2,\mathbb{R})$ gauge symmetry, we reduce right away the structure group $G = GL_{-1} \times G_2$ to $SL(2,\mathbb{R}) = SL_{-1} \times (SL_0 \ltimes SL_1)$ with the parametrization

$$\mathbb{R} \simeq SL_{-1} \ni \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad SL_0 \ltimes SL_1 \ni \begin{pmatrix} a & 0 \\ c & 1/a \end{pmatrix}, c \in \mathbb{R}, a \in \mathbb{R} \setminus \{0\}\,.$$

Accordingly the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ inherits the graded splitting $\mathfrak{sl}_{-1} \oplus \mathfrak{sl}_0 \oplus \mathfrak{sl}_1$.

This reduction amounts to restricting ourselves to the 1-dimensional (real) projective space $M = \mathbb{R}P^1$ with local coordinate $x$. Let $u$ denote the local coordinate on $\mathbb{R}$. The bundle $P_2$ of projective 2-frames $(x, e^x{}_u, e^x{}_{uu})$ is a principal sub-bundle of $F_2M$ with the reduced structure group $SL_0 \ltimes SL_1 \subset G_2$ (the so-called $G$-structure [4]). More precisely, elements $g_2$ of the structure group are 2-jets at $u = 0$ of the projective maps fixing $u = 0$ given by

$$g(u) = \frac{au}{cu + 1/a} \iff \begin{pmatrix} g(u) \\ 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 1/a \end{pmatrix} \begin{pmatrix} u \\ 1 \end{pmatrix}.$$

[6] In the course of the computation using (1) and (4) it is easier to consider $e_2 \cdot de_2^{-1} = -de_2 \cdot e_2^{-1}$.

In particular, the identification of $SL_0 \ltimes SL_1$ as subgroup of $G_2$ is made through

$$\begin{pmatrix} \left(g^{u'}_{\ u}\right)^{1/2} & 0 \\ -\frac{1}{2}\, g^{u'}_{\ uu} \left(g^{u'}_{\ u}\right)^{-3/2} & \left(g^{u'}_{\ u}\right)^{-1/2} \end{pmatrix}.$$

The Cartan connection $\omega$ on $F_2M$ restricts to $P_2$ as the $\mathfrak{sl}(2,\mathbb{R})$-valued 1-form whose local expression (see (2)) reads

$$\omega_{-1} = \theta^u = e^u_{\ x}\, dx, \qquad \omega_0 = \theta^u_{\ u} = e^x_{\ u} e^u_{\ xx}\, dx + e^u_{\ x}\, de^x_{\ u}, \qquad \omega_1 = \omega^u_{\ uu}\ . \tag{7}$$

In the natural gauge, a projective 2-frame field $e_2(x) = j_2(f)(0)$ with $f(u) = x + g(u)$ is viewed as an element $G_2$. Moreover it is well known that projective transformations $y = f(u)$ are solutions of the third order differential equation $y''' = \frac{3}{2}(y'')^2/y'$. This induces a relation between the local coordinates of the whole projective frame $j_3(f)(0) = (e^x_{\ u}, e^x_{\ uu}, e^x_{\ uuu}) \in G_3$, where the third order jet has the unique form

$$e^x_{\ uuu} = \tfrac{3}{2}\,(e^x_{\ uu})^2\, e^u_{\ x}. \tag{8}$$

The relationship (8) gives rise to the unique lift $\ell(e_2)$ of the projective 2-frames into $G_3$ dictated by the projective structure itself on $\mathbb{R}P^1$.

By restricting the local expression (5) to the projective case, and taking into account the expressions (1), (7) and (8) the components (6) of the Cartan connection 1-form on $P_2$ read :

$$\begin{aligned} \Gamma^x &= dx\ , \\ \Gamma^x_{\ x} &= 0\ , \\ \Gamma^x_{\ xx} &= e^u_{\ x}\omega^u_{\ uu} + d\,(e^u_{\ xx}e^x_{\ u}) - \tfrac{1}{2}\,(e^u_{\ xx}e^x_{\ u})^2\, dx\ , \end{aligned} \tag{9}$$

where the special combination $\chi_x := e^u_{\ xx}e^x_{\ u}$ occurs in a Miura like expression [16]. Under a change of local coordinates $x \mapsto x'$ on $M$, the frame coordinates transform as

$$e^x_{\ u} \to e^{x'}_{\ u} = \frac{dx'}{dx}\, e^x_{\ u}, \qquad e^x_{\ uu} \to e^{x'}_{\ uu} = \frac{dx'}{dx}\, e^x_{\ uu} + \frac{d^2x'}{dx^2}\,(e^x_{\ u})^2$$

while for the inverse

$$e^u_{\ x'} = \frac{dx}{dx'}\, e^u_{\ x}, \qquad e^u_{\ x'x'} = \left(\frac{dx}{dx'}\right)^2 e^u_{\ xx} - \left(\frac{dx}{dx'}\right)^3 \frac{d^2x'}{dx^2}\, e^u_{\ x}.$$

It can be checked, on the one hand, that $\chi_x$ behaves as an affine connection, and, on the other hand, that the third component $\Gamma^x_{\ xx}$ behaves as a projective connection, namely

$$\Gamma^x_{\ xx} - \frac{dx'}{dx}\,\Gamma^{x'}_{\ x'x'} = \left(\frac{d^3x'}{dx^3}\Big/\frac{dx'}{dx} - \frac{3}{2}\left(\frac{d^2x'}{dx^2}\Big/\frac{dx'}{dx}\right)^2\right) dx\ , \tag{10}$$

where the right hand side is recognized as the Schwarzian derivative.

So, locally on $P_2$ the connection $\Gamma$ as a $\mathfrak{sl}(2,\mathbb{R})$-valued 1-form is parametrized as

$$\Gamma = \begin{pmatrix} 0 & dx \\ \Gamma^x{}_{xx} & 0 \end{pmatrix} . \tag{11}$$

Given a projective 2-frame field $e_2(x)$, let us denote the local representative of the Cartan connection on $M$ by $e_2^*\omega = (\theta^u{}_{,x} + \theta^u{}_{u,x} + \omega^u{}_{uu,x})dx$. Thus (11) is pulled-back to $M$ as

$$\begin{pmatrix} 0 & 1 \\ \Gamma^x{}_{xx,x}(x) & 0 \end{pmatrix} dx\,, \quad \text{with } \Gamma^x{}_{xx,x}(x) = e^u{}_x\omega^u{}_{uu,x} + \partial_x\chi_x - \tfrac{1}{2}\left(\chi_x\right)^2 . \tag{12}$$

Hence, the local expression of the Cartan connection $\omega$ on the projective 2-frame bundle gives rise directly to a '*Polyakov soldering*' procedure but in an appropriate geometrical framework.

## 5 The BRS-structure

Let us now turn to the infinitesimal gauge aspect related to the $SL(2,\mathbb{R})$ Yang-Mills counterpart of the Cartan connection. The infinitesimal $SL(2,\mathbb{R})$-gauge transformations can be recast in the more powerful and elegant BRS (graded) differential algebra [17,18] by turning the gauge parameters to Faddeev-Popov ghost fields $\gamma$.

The infinitesimal gauge transformation is usually written in terms of a nilpotent $s$-operation as

$$s\omega = -d\gamma - [\omega,\gamma], \qquad s\gamma = -\tfrac{1}{2}[\gamma,\gamma], \qquad s^2 = 0, \tag{13}$$

where the graded bracket is understood with respect to the de Rham degree and the ghost number (or BRS grading). The Lie algebra content of the graded bracket is given by algebraic bracket (3). With respect to the bigrading, one has the nilpotency properties $(d+s)^2 = 0$, namely $d^2 = s^2 = ds + sd = 0$. Recall that the nilpotent algebra (13) can be compactly encapsulated into one formula only, the so-called Russian formula for the field strength (the curvature)

$$d\omega + \tfrac{1}{2}[\omega,\omega] = (d+s)(\omega+\gamma) + \tfrac{1}{2}[\omega+\gamma,\omega+\gamma], \tag{14}$$

where $\omega+\gamma$ acquires the status of algebraic connection [19].

Let us introduce now the projective parametrization [20] as the redefinition [7]

$$\Gamma + c = \mathrm{Ad}(\ell(e_2))(\omega+\gamma) + e_2\cdot(d+s)e_2^{-1}\ , \tag{15}$$

where the ghost field is redefined by $c = \mathrm{Ad}(\ell(e_2))\gamma + e_2\cdot s e_2^{-1}$. It can be checked that

$$s\Gamma = -dc - [\Gamma, c], \qquad sc = -\tfrac{1}{2}[c,c].$$

Recalling that the local representative $e_2^*\omega$ of the Cartan connection is a $(\mathfrak{sl}_{-1}\oplus\mathfrak{sl}_0\oplus\mathfrak{sl}_1)$-valued 1-form on $M$, an obvious $\mathfrak{sl}(2,\mathbb{R})$-ghost parameter can be chosen to be

$$\gamma = \left(e_2^*\omega\right)(\xi) = \omega(e_{2*}\xi)$$

[7] Limiting ourselves to 1-frames $\mathrm{Ad}(\ell(e_1))(\omega+\gamma) + e_1\cdot(d+s)e_1^{-1}$ where $\ell(e_1) = (e^\mu{}_a, 0) \in G_2$ exactly yields the conformal parametrization given in [20].

where $\xi = \xi^x \partial_x$ is the ghost vector field defined on $M$. More explicitly the components of that particular gauge ghost read

$$\gamma^u = \theta^u{}_{,x}\xi^x = e^u{}_x\xi^x, \quad \gamma^u{}_u = \theta^u{}_{u,x}\xi^x = \left(e^u{}_{xx}e^x{}_u + e^u{}_x\partial_x e^x{}_u\right)\xi^x, \quad \gamma^u{}_{uu} = \omega^u{}_{uu,x}\xi^x. \tag{16}$$

Then, performing the BRS transformations (13) on the components of the Cartan connection, with respect to that particular ghost parametrization, one gets

$$\begin{aligned} s\theta^u{}_{,x} &= \partial_x\gamma^u + \theta^u{}_{u,x}\gamma^u - \gamma^u{}_u\theta^u{}_{,x} = \partial_x\left(\theta^u{}_{,x}\xi^x\right) \ , \\ s\theta^u{}_{u,x} &= \partial_x\gamma^u{}_u + \omega^u{}_{uu,x}\gamma^u - \gamma^u{}_{uu}\theta^u{}_{,x} = \partial_x\left(\theta^u{}_{u,x}\xi^x\right) \ , \\ s\omega^u{}_{uu,x} &= \partial_x\gamma^u{}_{uu} + \omega^u{}_{uu,x}\gamma^u{}_u - \gamma^u{}_{uu}\theta^u{}_{u,x} = \partial_x\left(\omega^u{}_{uu,x}\xi^x\right) \ ; \end{aligned} \tag{17}$$

these variations can be compactly gathered as $s\omega = -d(\omega(\xi)) = (i_\xi d - d i_\xi) =: L_\xi\omega$, which is nothing but the Lie derivative[8] expressing the action of diffeomorphisms on the Cartan connection 1-form.

Using the local expressions (7) of $\theta^u$ and $\theta^u{}_u$, the variations (17) infer the lift of the diffeomorphisms on 2-frame fields as BRS transformations given by

$$\begin{aligned} se^u{}_x &= s\theta^u{}_{,x} = \partial_x\left(e^u{}_x\xi^x\right) \\ se^u{}_{xx} &= e^u{}_x\, s\theta^u{}_{u,x} + e^x{}_u e^u{}_{xx}\, se^u{}_x + e^x{}_u\partial_x e^u{}_x se^u{}_x - \left(e^u{}_x\right)^2\partial_x\left(se^x{}_u\right) \\ &= \partial_x e^u{}_{xx}\xi^x + 2e^u{}_{xx}\partial_x\xi^x + e^u{}_x\partial_x^2\xi^x \end{aligned} \tag{18}$$

Accordingly, in each sector of the graded algebra $\mathfrak{sl}_{-1} \oplus \mathfrak{sl}_0 \oplus \mathfrak{sl}_1$ we are respectively left with the following ghosts

$$\begin{aligned} c^x &= e^x{}_u\gamma^u = \xi^x, \\ c^x{}_x &= e^x{}_{uu}e^u{}_x\gamma^u + \gamma^u{}_u + e^x{}_u se^u{}_x = \partial_x\xi^x \\ c^x{}_{xx} &= \tfrac{1}{2}\left(e^u{}_{xx}\right)^2\left(e^x{}_u\right)^3\gamma^u + e^x{}_{uu}\left(e^u{}_x\right)^2\gamma^u{}_u + \gamma^u{}_{uu}e^u{}_x + 2e^x{}_{uu}e^u{}_x se^u{}_x + e^x{}_u se^u{}_{xx} \\ &= \partial^2\xi^x + \left(\partial_x\chi_x - \tfrac{1}{2}\left(\chi_x\right)^2 + e^u{}_x\omega^u{}_{uu,x}\right)\xi^x \ = \ \partial_x^2\xi^x + \Gamma^x{}_{xx,x}\xi^x \ , \end{aligned} \tag{19}$$

which are exactly the ghost version of the gauge parameters found by Polyakov in [1]. With the help of (18) the BRS transformations on the connection form $\Gamma$ (9) are computed to be

$$s\Gamma^x{}_{,x} = 0, \qquad s\Gamma^x{}_{x,x} = 0, \qquad s\Gamma^x{}_{xx,x} = \partial_x^3\xi^x + \xi^x\partial_x\Gamma^x{}_{xx,x} + 2\partial_x\xi^x\Gamma^x{}_{xx,x}. \tag{20}$$

These variations derive from the geometry, while the first two were imposed in [1] as constraints in order to keep the gauge choice. The third variation can be rewritten as,

$$s\Gamma^x{}_{xx,x} = \partial_x^3 c^x + c^x\partial_x\Gamma^x{}_{xx,x} + 2\partial_x c^x\Gamma^x{}_{xx,x}, \tag{21}$$

where $c^x$ is the ghost vector field with $sc^x = c^x\partial_x c^x$ and $s^2 = 0$.

As claimed in [1], this residual BRS algebra exactly reproduces the well known Virasoro algebra action on the energy-momentum tensor suitably identified with $\Gamma^x{}_{xx,x}$.

[8] Remind that $\xi$ carries a ghost degree and that here $M$ is one dimensional.

## 6 Conclusions

Cartan geometry provides a proper framework which amounts to explain both the 'geometrical surprise' expressed by Polyakov and the field dependence of the constraint on the gauge parameters. As shown, on the one hand, it turns out that the specific Polyakov partial gauge fixing is just the local expression (9) of the Cartan connection for which the first two components with respect to a graded Lie algebra are those of the solder 1-form on the second order frame bundle, while the untouched third component of the Cartan connection acquires a geometrical status, namely that of a projective connection. Recall that the latter shares exactly the same geometrical feature of the effective energy-momentum tensor in 2D-conformal chiral theory. The approach has the advantage of shedding some light on the Polyakov soldering which is the main basis of the so-called $W$-gravity [3, 20]. Incidentally, the framework of Cartan geometry gives a significant improvement of some manipulations performed in [20].

On the other hand, the relationship between a Cartan connection and the usual (Ehresmann) connection on a principal bundle has allowed us to gain some insight into the derivation of "*diffeomorphisms out of gauge transformations*" given by Polyakov in [1]. This has been achieved by lifting the diffeomorphisms from the base manifold to the second order frame bundle, see (18). All geometrical manipulations were carried out in terms of projective 2-frame fields.

Furthermore, the use of Cartan connections has been somewhat put aside in spite of a few attempts in the past [21] as well as the use of second order frames [22]. However there is a revival interest in the use of the concept of Cartan connections, especially in (conformal) gravity or in the study of PDE's, see for instance [14, 23–30]. An extensive study of the Cartan connections and the related geometries has been also performed by the mathematical Czech school see *e.g.* [31] and references therein. In particular, a possible route in relation with the so-called $W$-symmetries as extended conformal symmetries could have been opened on the mathematical side by the study of parabolic geometries and their Cartan connections *e.g.* [32, 33].

The Polyakov soldering provides another physical situation in which Cartan connections appear naturally. They allow to lift the action of diffeomorphisms to higher order differential structures which ought to be of interest for physical models of gravitation. They offer also a proper differential geometrical framework which allows a generalization of the Polyakov soldering (6) to higher dimensions [7], for instance when third order frames can be expressed in terms of the lower ones (this is the case for projective and conformal structures [4]).

**Acknowledgements.** We would like to thank T. Schücker for comments and for a careful reading of the manuscript. We are indebted to one of the Referees for pointing out to us some references about the mathematical Czech school.

## References


[1] A.M. Polyakov. "Gauge transformations and diffeomorphisms". *Int. Journ. Math. Phys.* **A5**, pages 833–842, (1990).

[2] Verlinde H. "Conformal field theory, two dimensional quantum gravity and quantization of Teichmüller space ". *Nucl. Phys.* **B337**, pages 652–680, (1990).

[3] H. Ooguri, K. Schoutens, A. Sevrin, and P. van Nieuwenhuizen. The induced action of $W_3$ gravity. *Comm. Math. Phys. 145*, pages 515–539, (1992).

[4] S. Kobayashi. *Transformation groups in differential geometry.* Classics in Mathematics, vol. **70**. Spinger-Verlag, Berlin, 1972.

[5] S. Kobayashi. Canonical forms on frame bundles of higher order contact. *Proc. Symposia in Pure Math.*, 3:186.

[6] D.J. Saunders. *The geometry of jet bundles*, volume 142 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, 1989.

[7] M. Grasseau. PhD thesis, Université de la Méditerranée - Aix-Marseille II, July 2006. unpublished, in french.

[8] T. Egushi, P.B. Gilkey, and A.J. Hanson. Gravitation, gauge theories and differential geometry. *Phys. Rep.*, **66**:213–393, (1980).

[9] S.S. Chern, W.H. Chen, and K.S. Lam. *Lectures on differential geometry*, volume 1 of *Series on University Mathematics.* World Scientific, Singapore, 1999.

[10] R.V. Gamkrelidze, editor. *Geometry I*, volume 28 of *Encyclopaedia of Mathematical Sciences.* Springer-Verlag, 1991.

[11] D. Perrot. On the topological interpretation of gravitational anomalies. *J. Geom. Phys.*, **39**:81–95, (2001).

[12] Milnor J. Remarks on infinite-dimensional Lie groups. In B.S. DeWitt and R. Stora, editors, *Relativity, groups and topology II*, Les Houches, pages 1009–1057. Elsevier Science Publishers, 1984. Session XL, 1983.

[13] R.W. Sharpe. *Differential geometry: Cartan's generalization of Klein's erlangen program*, volume 166 of *Graduate texts in mathematics.* Springer, New-York, Berlin, Heidelberg, 1997.

[14] D.K. Wise. *MacDowell-Mansouri gravity and Cartan geometry.* `arXiv:gr-qc/0611154`, November 2006.

[15] M. Daniel and C.M. Viallet. The geometrical setting of gauge theories of the yang-mills type. *Rev. Mod. Phy.*, 52.

[16] P. Mathieu. Super Miura transformations, super Schwarzian derivatives and super Hill operators. In B. A. Kupershmidt, editor, *Integrable and Superintegrable Systems*, pages 352–388. World Scientific, 1990.

[17] Becchi C., Rouet A., and Stora R. Renormalization of gauge theories. *Ann. Phys.* **98**, pages 287–321, (1976).

[18] R. Stora. Continuum gauge theories. In M. Lévy and P. Mitter, editors, *New Developments in Quantum Field Theory and Statistical Mechanics, Cargèse 1976*, NATO ASI Ser.B, Vol.26. Plenum Press, 1977.

[19] M. Dubois-Violette. The Weil-BRS algebra of a Lie algebra and the anomalous terms in gauge theory. *J. Geom. Phys.*, **3**:525–565, (1986).

[20] D. Garajeu, R. Grimm, and S. Lazzarini. "W-gauge structures and their anomalies: an algebraic approach". *Journ. Math. Phys.* **36**, pages 7043–7072, (1995).

[21] W. Drechsler. Gauge theory of strong and electromagnetic interactions formulated on a fiber bundle of Cartan type. In A. Böhm and J.D. Dollard, editors, *Fiber bundle techniques in gauge theory*, volume 67 of *Lecture Notes in Physics*, page 145. Lectures in Mathematical Physics at the University of Texas at Austin, Springer-Verlag, 1977.

[22] J. Harnad and J. Pettitt. *Gauge Theories for Space-Time Symmetries. II. Second order conformal structures.* CRM-745, February (1978). Unpublished.

[23] J. Grant and B. Lackey. On Galilean connections and the first jet bundle. math.DG/9909148.

[24] S. Frittelli, C. Kozameh, and E.T. Newman. Differential geometry from differential equations. *Commun.Math.Phys.*, 223:383–408, (2001). gr-qc/0012058.

[25] S. Frittelli, N. Kamran, and E.T. Newman. Differential equations and conformal geometry. *J. Geom. Phys.*, 43:133–145, (2002).

[26] S. Frittelli, C. Kozameh, E.T. Newman, and P. Nurowski. Cartan conformal connections from differential equations. *Class.Quant.Grav.*, **19**:5235–5247, (2002).

[27] M. Korzynski and J. Lewandowski. Conformal Cartan connection and the Bach tensor. *Class.Quant.Grav. 20 (2003) 3745-3764.*

[28] E.T. Newman and P. Nurowski. Projective connections associated with second order ODE's. *Class. Quant. Grav. 20 (2003) 2325-2336*, gr-qc/0302016.

[29] C. Kozameh, E.T. Newman, and P. Nurowski. Conformal einstein equations and cartan conformal connection. *Class.Quant.Grav. 20 (2003) 3029-3036*, gr-qc/0302080.

[30] E. Gallo, C. Kozameh, E. T. Newman, and K. Perkins. Cartan normal conformal connections from pairs of 2nd order PDE's. *Class. Quant. Grav. 21 (2004) 4063-4986*, arXiv:gr-qc/0404072.

[31] Čap A. Overdetermined systems, conformal geometry, and the BGG complex. In *Symmetries and overdetermined systems of partial differential equations*, IMA Vol. Math. Appl. **144**, pages 1–24. Springer, 2008. `arXiv:math/0610225`.

[32] Čap A. and Schilchl H. Parabolic geometries and canonical Cartan connections. *Hokkaido Math. J.*, **29**:453–505, (2000).

[33] Čap A. Two constructions with parabolic geometries. In *Proc. of the 25th Winter School* Geometry and Physics*, Srni, January 15-22, 2005,*, Rend. Circ. Mat. Palermo Suppl. No. 79, pages 11–37, 2006. `arXiv:math/0504389`.