Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: the critical case H=1/4

Ivan Nourdin, Anthony Réveillac

Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: the critical case $H = 1/4$

Ivan Nourdin[1] and Anthony Réveillac[2]

Abstract: We derive the asymptotic behavior of weighted quadratic variations of fractional Brownian motion B with Hurst index $H = 1/4$. This completes the only missing case in a very recent work by I. Nourdin, D. Nualart and C.A. Tudor. Moreover, as an application, we solve a recent conjecture of K. Burdzy and J. Swanson on the asymptotic behavior of the Riemann sums with alternating signs associated to B.

Key words: Fractional Brownian motion; quartic process; change of variable formula; weighted quadratic variations; Malliavin calculus; weak convergence.

Present version: February 2008

1 Introduction

Let B^H be a fractional Brownian motion with Hurst index $H \in (0,1)$. Since the seminal works by Breuer and Major [1], Dobrushin and Major [4], Giraitis and Surgailis [5] or Taqqu [24], it is well-known that

- if $H \in (0, \frac{3}{4})$ then

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left[n^{2H} (B^H_{(k+1)/n} - B^H_{k/n})^2 - 1 \right] \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0, C_H^2); \tag{1.1}$$

- if $H = \frac{3}{4}$ then

$$\frac{1}{\sqrt{n \log n}} \sum_{k=0}^{n-1} \left[n^{3/2} (B^{3/4}_{(k+1)/n} - B^{3/4}_{k/n})^2 - 1 \right] \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0, C^2_{\frac{3}{4}}); \tag{1.2}$$

- if $H \in (\frac{3}{4}, 1)$ then

$$n^{1-2H} \sum_{k=0}^{n-1} \left[n^{2H} (B^H_{(k+1)/n} - B^H_{k/n})^2 - 1 \right] \xrightarrow[n\to\infty]{\text{Law}} \text{“Rosenblatt r.v.”}. \tag{1.3}$$

[1]Université Pierre et Marie Curie Paris VI, Laboratoire de Probabilités et Modèles Aléatoires, Boîte courrier 188, 4 place Jussieu, 75252 Paris Cedex 05, France, inourdin@gmail.com

[2]Université de La Rochelle Laboratoire Mathématiques, Image et Applications, Avenue Michel Crépeau, 17042 La Rochelle Cedex, France anthony.reveillac@univ-lr.fr

Here, $C_H > 0$ denotes a constant depending only on H and which can be computed explicitly. Moreover, the term "Rosenblatt r.v." denotes a random variable whose distribution is the same as that of the Rosenblatt process Z at time one, see (1.9) below.

Now, let f be a real function assumed to be regular enough. Very recently, the asymptotic behavior of

$$\sum_{k=0}^{n-1} f(B^H_{k/n})\big[n^{2H}(B^H_{(k+1)/n} - B^H_{k/n})^2 - 1\big] \tag{1.4}$$

received a lot of attention, see [6, 12, 13, 14, 16] (see also the related works [17, 20, 21, 23]). The initial motivation of such a study was to derive the exact rates of convergence of some approximation schemes associated to scalar stochastic differential equations driven by B^H, see [6, 12, 13] for precise statements. But it turned out that it was also interesting for itself because it highlighted new phenomena with respect to (1.1)-(1.3). Indeed, in the study of the asymptotic behavior of (1.4), a new critical value ($H = \frac{1}{4}$) has appeared. More precisely:

- if $H < \frac{1}{4}$ then

$$n^{2H-1}\sum_{k=0}^{n-1} f(B^H_{k/n})\big[n^{2H}(B^H_{(k+1)/n} - B^H_{k/n})^2 - 1\big] \xrightarrow[n\to\infty]{L^2} \frac{1}{4}\int_0^1 f''(B^H_s)ds; \tag{1.5}$$

- if $\frac{1}{4} < H < \frac{3}{4}$ then

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(B^H_{k/n})\big[n^{2H}(B^H_{(k+1)/n} - B^H_{k/n})^2 - 1\big] \xrightarrow[n\to\infty]{\text{Law}} C_H\int_0^1 f(B^H_s)dW_s \tag{1.6}$$

  for W a standard Brownian motion independent of B^H;

- if $H = \frac{3}{4}$ then

$$\frac{1}{\sqrt{n\,\log n}}\sum_{k=0}^{n-1} f(B^{3/4}_{k/n})\big[n^{3/2}(B^{3/4}_{(k+1)/n} - B^{3/4}_{k/n})^2 - 1\big] \xrightarrow[n\to\infty]{\text{Law}} C_{3/4}\int_0^1 f(B^{3/4}_s)dW_s \tag{1.7}$$

  for W a standard Brownian motion independent of $B^{3/4}$;

- if $H > \frac{3}{4}$ then

$$n^{1-2H}\sum_{k=0}^{n-1} f(B^H_{k/n})\big[n^{2H}(B^H_{(k+1)/n} - B^H_{k/n})^2 - 1\big] \xrightarrow[n\to\infty]{L^2} \int_0^1 f(B^H_s)dZ_s \tag{1.8}$$

  for Z the Rosenblatt process defined by

$$Z_s = I_2^X(L_s), \tag{1.9}$$

  where I_2^X denotes the double stochastic integral with respect to the Wiener process X given by the transfer equation (2.3) and where, for every $s \in [0,1]$, L_s is the symmetric square integrable kernel given by

$$L_s(y_1, y_2) = \frac{1}{2}\mathbf{1}_{[0,s]^2}(y_1, y_2)\int_{y_1\vee y_2}^s \frac{\partial K_H}{\partial u}(u, y_1)\frac{\partial K_H}{\partial u}(u, y_2)du.$$

Even if it is not completely obvious at first glance, convergences (1.1) and (1.5) well agree. Indeed, since $2H-1<-\frac{1}{2}$ if and only if $H<\frac{1}{4}$, (1.5) is actually a particular case of (1.1) when $f\equiv 1$. The convergence (1.5) is proved in [14] while the other cases (1.6)-(1.8) are proved in [16]. On the other hand, notice that the relations (1.5)-(1.8) do not cover the critical case $H=\frac{1}{4}$. Our first main result completes this important (see why just below) missing case:

Theorem 1.1. *If $H=\frac{1}{4}$ then*

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(B^{1/4}_{k/n})\big[\sqrt{n}(B^{1/4}_{(k+1)/n}-B^{1/4}_{k/n})^2-1\big] \xrightarrow[n\to\infty]{\text{Law}} C_{1/4}\int_0^1 f(B^{1/4}_s)dW_s+\frac{1}{4}\int_0^1 f''(B^{1/4}_s)ds \tag{1.10}$$

for W a standard Brownian motion independent of $B^{1/4}$ and where

$$C^2_{1/4}=\frac{1}{2}\sum_{p=-\infty}^{\infty}\Big(\sqrt{|p+1|}+\sqrt{|p-1|}-2\sqrt{|p|}\Big)^2<\infty.$$

Here, it is interesting to compare the obtained limit in (1.10) with those obtained in the recent work [17]. In [17], the authors also studied the asymptotic behavior of (1.4) but when the fractional Brownian motion B^H is replaced by an *iterated Brownian motion* Z, that is the process defined by $Z_t=X(Y_t)$, $t\in[0,1]$, with X and Y two independent standard Brownian motions. Iterated Brownian motion Z is self-similar of index $\frac{1}{4}$ and has stationary increments. Thus, although if it is not Gaussian, Z is "close" to the fractional Brownian motion $B^{1/4}$. For Z instead of $B^{1/4}$, it is proved in [17] that the correctly renormalized weighted quadratic variation (which is note exactly defined as in (1.4), but rather by means of a *random* partition composed of Brownian hitting times) converges in law towards the so-called *weighted Brownian motion in random scenery* at time one, defined as

$$\sqrt{2}\int_{-\infty}^{+\infty} f(X_x)L^x_1(Y)dW_x,$$

compare with the right-hand side of (1.10). Here, $\{L^x_t(Y)\}_{x\in\mathbb{R},t\in[0,1]}$ stands for the jointly continuous version of the local time process of Y, while W denotes a two-sided standard Brownian motion independent of X and Y.

For now, we take $B^H=B^{1/4}$ to be a fractional Brownian motion with Hurst index $H=\frac{1}{4}$. This particular value of H is important because the fractional Brownian motion with Hurst index $H=\frac{1}{4}$ has a remarkable physical interpretation in terms of particle systems. Indeed, if one consider an infinite number of particles, initially placed on the real line according to a Poisson distribution, performing independent Brownian motions and undergoing "elastic" collisions, then the trajectory of a fixed particle (after rescaling) converges to a fractional Brownian motion with Hurst index $H=\frac{1}{4}$. This striking fact has been first pointed out by Harris in [10], and then rigorously proven in [3] (see also references therein).

Now, let us explain an interesting consequence of a slight modification of Theorem 1.1 towards a first step in the construction of a stochastic calculus with respect to $B^{1/4}$. As it is nicely explained by Swanson in [23], there are at least two kinds of Stratonovitch-type

Riemann sums that one can consider in order to define $\int_0^1 f(B_s^{1/4}) \circ dB_s^{1/4}$ when f is a real smooth function. The first corresponds to the so-called "trapezoid rule" and is given by

$$S_n(f) = \sum_{k=0}^{n-1} \frac{f(B_{k/n}^{1/4}) + f(B_{(k+1)/n}^{1/4})}{2} \big(B_{(k+1)/n}^{1/4} - B_{k/n}^{1/4}\big).$$

The second corresponds to the so-called "midpoint rule" and is given by

$$T_n(f) = \sum_{k=1}^{\lfloor n/2 \rfloor} f(B_{(2k-1)/n}^{1/4}) \big(B_{(2k)/n}^{1/4} - B_{(2k-2)/n}^{1/4}\big).$$

By Theorem 3 in [16] (see also [2, 7, 8]), we have that

$$\int_0^1 f'(B_s^{1/4}) d^\circ B_s^{1/4} := \lim_{n\to\infty} S_n(f') \quad \text{exists in probability}$$

and verifies the following *classical* change of variable formula:

$$\int_0^1 f'(B_s^{1/4}) d^\circ B_s^{1/4} = f(B_1^{1/4}) - f(0). \tag{1.11}$$

On the other hand, it is quoted in [23] that Burdzy and Swanson *conjectured*[3] that

$$\int_0^1 f'(B_s^{1/4}) d^\star B_s^{1/4} := \lim_{n\to\infty} T_n(f') \quad \text{exists in law}$$

and verifies, this time, the following *non classical* change of variable formula:

$$\int_0^1 f'(B_s^{1/4}) d^\star B_s^{1/4} \overset{\text{Law}}{=} f(B_1^{1/4}) - f(0) - \frac{\kappa}{2} \int_0^1 f''(B_s^{1/4}) dW_s, \tag{1.13}$$

where κ is an explicit universal constant and W denotes a standard Brownian motion independent of $B^{1/4}$. Our second main result is the following:

[3] In reality, Burdzy and Swanson conjectured (1.13) not for the fractional Brownian motion $B^{1/4}$ but for process F defined by

$$F_t = u(t,0),\ t \in [0,1], \tag{1.12}$$

where

$$u_t = \frac{1}{2} u_{xx} + \dot{W}(t,x),\ t \in [0,1],\ x \in \mathbb{R}, \quad \text{with initial condition } u(0,x) = 0.$$

(Here, as usual, $\dot{W}$ denotes the space-time white noise on $[0,1] \times \mathbb{R}$). It is immediately checked that F is a centered Gaussian process with covariance function

$$E(F_s F_t) = \frac{1}{\sqrt{2\pi}} \big(\sqrt{t+s} - \sqrt{|t-s|}\big).$$

so that F is actually a *bifractional Brownian motion* of indices $\frac{1}{2}$ and $\frac{1}{2}$ in the sense of Houdré and Villa [9]. Using the main result of [11], we have that $B^{1/4}$ and F actually differ only from a process with absolutely continuous trajectories. As a direct consequence, using a Girsanov type transformation, we immediately see that it is equivalent to prove (1.13) either for $B^{1/4}$ or for F.

As quoted in [22, Remark 12], notice finally that the change of variable formula (1.11) also holds for F.

Theorem 1.2. *The conjecture of Burdzy and Swanson is true. More precisely, (1.13) holds for any real function $f : \mathbb{R} \to \mathbb{R}$ verifying $(\mathbf{H}_9)$ (see Section 3 below).*

Finally, we would mention that the strategy we used in this paper can also be derived in order to obtain the following analogue of Theorem 1.2, that we propose (in order to keep the length of the present paper within limit) to prove in a forthcoming paper:

Theorem 1.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be smooth enough. Then $\int_0^1 f'(B_s^{1/6}) d^\circ B_s^{1/6}$ exists in law and verifies*

$$\int_0^1 f'(B_s^{1/6}) d^\circ B_s^{1/6} \overset{\text{Law}}{=} f(B_1^{1/6}) - f(0) - \frac{\widetilde{\kappa}}{6} \int_0^1 f'''(B_s^{1/6}) dW_s,$$

where $\widetilde{\kappa}$ is an explicit universal constant and W denotes a standard Brownian motion independent of $B^{1/6}$.

The rest of the paper is organized as follows. In Section 2, we recall some notion concerning fractional Brownian motion. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2.

2 Preliminaries and notations

We begin by briefly recalling some basic facts about stochastic calculus with respect to a fractional Brownian motion. We refer to [18] for further details. Let $B^H = (B_t^H)_{t\in[0,1]}$ be a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$ defined on a probability space $(\Omega, \mathcal{A}, P)$. We mean that B^H is a centered Gaussian process with the covariance function

$$R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{2.1}$$

We denote by $\mathscr{E}$ the set of step $\mathbb{R}-$valued functions on $[0,1]$. Let $\mathfrak{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathfrak{H}} = R_H(t,s).$$

The covariance kernel $R_H(t,s)$ introduced in (2.1) can be written as

$$R_H(t,s) = \int_0^{s\wedge t} K_H(s,u) K_H(t,u) du,$$

where $K_H(t,s)$ is the square integrable kernel defined by

$$K_H(t,s) = c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right], \quad 0 < s < t, \tag{2.2}$$

where ${c_H}^2 = 2H(1-2H)^{-1}\beta(1-2H, H+1/2)^{-1}$ and β denotes the Beta function. By convention, we set $K_H(t,s) = 0$ if $s \geq t$.

Let $\mathcal{K}_H^* : \mathscr{E} \to \mathrm{L}^2([0,1])$ be the linear operator defined by:

$$\mathcal{K}_H^*\left(\mathbf{1}_{[0,t]}\right) = K_H(t,\cdot).$$

The following equality holds for any $s, t \in [0,1]$:

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = \langle \mathcal{K}_H^* \mathbf{1}_{[0,t]}, \mathcal{K}_H^* \mathbf{1}_{[0,s]} \rangle_{\mathrm{L}^2([0,1])} = \mathrm{E}\left(B_t^H B_s^H\right)$$

and then $\mathcal{K}_H^*$ provides an isometry between the Hilbert spaces $\mathfrak{H}$ and a closed subspace of $\mathrm{L}^2([0,1])$. The process $X = (X_t)_{t\in[0,1]}$ defined by

$$X_t = B^H\big((\mathcal{K}_H^*)^{-1}(\mathbf{1}_{[0,t]})\big) \tag{2.3}$$

is a Wiener process, and the process B^H has an integral representation of the form

$$B_t^H = \int_0^t K_H(t,s) dX_s.$$

Let $\mathscr{S}$ be the set of all smooth cylindrical random variables, *i.e.* of the form

$$F = \psi(B_{t_1}^H, \ldots, B_{t_m}^H) \tag{2.4}$$

where $m \geq 1$, $\psi : \mathbb{R}^m \to \mathbb{R} \in \mathscr{C}_b^\infty$ and $0 \leq t_1 < \ldots < t_m \leq 1$. The Malliavin derivative of F with respect to B^H is the element of $\mathrm{L}^2(\Omega, \mathfrak{H})$ defined by

$$D_s F \;=\; \sum_{i=1}^m \frac{\partial \psi}{\partial x_i}(B_{t_1}^H, \ldots, B_{t_m}^H) \mathbf{1}_{[0,t_i]}(s), \quad s \in [0,1].$$

In particular $D_s B_t^H = \mathbf{1}_{[0,t]}(s)$. For any integer $k \geq 1$, we denote by $\mathbb{D}^{k,2}$ the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{k,2}^2 \;=\; \mathrm{E}\left[F^2\right] + \sum_{j=1}^k \mathrm{E}\left[|D^j F|_{\mathfrak{H}^{\otimes j}}^2\right].$$

The Malliavin derivative D verifies the chain rule: if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is $\mathscr{C}_b^1$ and if $(F_i)_{i=1,\ldots,n}$ is a sequence of elements of $\mathbb{D}^{1,2}$ then $\varphi(F_1, \ldots, F_n) \in \mathbb{D}^{1,2}$ and we have, for any $s \in [0,1]$:

$$D_s\, \varphi(F_1, \ldots, F_n) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F_1, \ldots, F_n) D_s F_i.$$

The divergence operator I is the adjoint of the derivative operator D. If a random variable $u \in \mathrm{L}^2(\Omega, \mathfrak{H})$ belongs to the domain of the divergence operator, that is if it verifies

$$|\mathrm{E}\langle DF, u\rangle_{\mathfrak{H}}| \leq c_u \|F\|_{\mathrm{L}^2} \quad \text{for any } F \in \mathscr{S},$$

then $I(u)$ is defined by the duality relationship

$$\mathrm{E}\big(FI(u)\big) = \mathrm{E}\big(\langle DF, u\rangle_{\mathfrak{H}}\big),$$

for every $F \in \mathbb{D}^{1,2}$.

For every $n \geq 1$, let $\mathcal{H}_n$ be the nth Wiener chaos of B^H, that is, the closed linear subspace of $L^2(\Omega, \mathcal{A}, P)$ generated by the random variables $\{H_n\left(B^H(h)\right), h \in H, |h|_{\mathfrak{H}} = 1\}$, where H_n is the nth Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n! H_n\left(B^H(h)\right)$ provides a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot n}$ and $\mathcal{H}_n$. For $H = \frac{1}{2}$, I_n coincides with the multiple stochastic integral. The following duality formula holds

$$E\left(F I_n(h)\right) = E\left(\langle D^n F, h\rangle_{\mathfrak{H}^{\otimes n}}\right), \tag{2.5}$$

for any element $h \in \mathfrak{H}^{\odot n}$ and any random variable $F \in \mathbb{D}^{n,2}$. Let $\{e_k,\, k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \ldots, p \wedge q$, the rth contraction of f and g is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined as

$$f \otimes_r g = \sum_{i_1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \ldots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \ldots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}}.$$

Note that $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $p = q$, $f \otimes_p g = \langle f, g\rangle_{\mathfrak{H}^{\otimes p}}$. Finally, we mention the useful following multiplication formula: if $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then

$$I_p(f) I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}(f \otimes_r g). \tag{2.6}$$

3 Proof of Theorem 1.1

In all this section, $B = B^{1/4}$ denotes a fractional Brownian motion with Hurst index $H = 1/4$.

Let

$$G_n := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n})[\sqrt{n}(B_{(k+1)/n} - B_{k/n})^2 - 1], \quad n \geq 1.$$

For $k = 0, \ldots, n-1$ and $t \in [0,1]$, we set

$$\delta_{k/n} := \mathbf{1}_{[k/n,(k+1)/n]} \quad \text{and} \quad \varepsilon_t := \mathbf{1}_{[0,t]}.$$

The relations between Hermite polynomials and multiple stochastic integrals (see Section 2) allow to write

$$\sqrt{n}(B_{(k+1)/n} - B_{k/n})^2 - 1 = \sqrt{n}\, I_2(\delta_{k/n}^{\otimes 2}).$$

As a consequence:

$$G_n = \sum_{k=0}^{n-1} f(B_{k/n}) I_2(\delta_{k/n}^{\otimes 2}).$$

In the sequel, for $f : \mathbb{R} \to \mathbb{R}$, we will need assumption of the type:

Hypothesis ($\mathbf{H}_\mathrm{q}$):
The function $f : \mathbb{R} \to \mathbb{R}$ belongs to $\mathscr{C}^q$ and is such that

$$\sup_{t\in[0,1]} E\left(|f^{(i)}(B_t)|^p\right) < \infty$$

for any $p \geq 1$ and $i \in \{0, \ldots, q\}$.

We begin by the following technical lemma:

Lemma 3.1. *Let $n \geq 1$ and $k = 0, \ldots, n-1$. We have*

(i) $|E\left(B_r(B_t - B_s)\right)| \leq \sqrt{t-s}$ *for any* $r \in [0,1]$ *and* $0 \leq s < t \leq 1$,

(ii) $\displaystyle\sup_{t\in[0,1]} \sum_{k=0}^{n-1} \left|\left\langle \varepsilon_t, \delta_{k/n} \right\rangle_{\mathfrak{H}}\right| \underset{n\to\infty}{=} O(1)$,

(iii) $\displaystyle\sum_{k,j=0}^{n-1} \left|\left\langle \varepsilon_{j/n}, \delta_{k/n} \right\rangle_{\mathfrak{H}}\right| \underset{n\to\infty}{=} O(n)$,

(iv) $\displaystyle\left|\left\langle \varepsilon_{k/n}, \delta_{k/n} \right\rangle_{\mathfrak{H}}^2 - \frac{1}{4n}\right| \leq \frac{\sqrt{k+1}-\sqrt{k}}{4n}$; *consequently* $\displaystyle\sum_{k=0}^{n-1} \left|\left\langle \varepsilon_{k/n}, \delta_{k/n} \right\rangle_{\mathfrak{H}}^2 - \frac{1}{4n}\right| \underset{n\to\infty}{\longrightarrow} 0.$

Proof of Lemma 3.1.

(i) We have

$$E(B_r(B_t - B_s)) = \frac{1}{2}(\sqrt{t} - \sqrt{s}) + \frac{1}{2}\left(\sqrt{|s-r|} - \sqrt{|t-r|}\right).$$

Using the classical inequality $\left|\sqrt{|b|} - \sqrt{|a|}\right| \leq \sqrt{|b-a|}$, the desired result follows.

(ii) Observe that

$$\left\langle \varepsilon_t, \delta_{k/n} \right\rangle_{\mathfrak{H}} = \frac{1}{2\sqrt{n}}\left(\sqrt{k+1} - \sqrt{k} - \sqrt{|k+1-nt|} + \sqrt{|k-nt|}\right).$$

Consequently, we have

$$\begin{aligned}\sum_{k=0}^{n-1} \left|\left\langle \varepsilon_t, \delta_{k/n} \right\rangle_{\mathfrak{H}}\right| \;\leq\; & \frac{1}{2} + \frac{1}{2\sqrt{n}}\left(\sum_{k=0}^{\lfloor nt\rfloor -1} \sqrt{nt-k} - \sqrt{nt-k-1}\right. \\ & \left. + \sqrt{\lfloor nt\rfloor + 1 - nt} - \sqrt{nt - \lfloor nt\rfloor} + \sum_{k=\lfloor nt\rfloor+1}^{n-1} \sqrt{nt-k} - \sqrt{nt-k-1}\right).\end{aligned}$$

The desired conclusion follows easily.

(iii) It is a direct consequence of *(ii)*:

$$\begin{aligned}\sum_{k,j=0}^{n-1} \left|\left\langle \varepsilon_{j/n}, \delta_{k/n} \right\rangle_{\mathfrak{H}}\right| \;&\leq\; n \sup_{j=0,\ldots,n-1} \sum_{k=0}^{n-1} \left|\left\langle \varepsilon_{j/n}, \delta_{k/n} \right\rangle_{\mathfrak{H}}\right| \\ &\underset{n\to\infty}{=}\; O(n).\end{aligned}$$

(iv) We have

$$\left|\left\langle \varepsilon_{k/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}^2 - \frac{1}{4n}\right| = \frac{1}{4n}\left(\sqrt{k+1}-\sqrt{k}\right)\left|\sqrt{k+1}-\sqrt{k}-2\right|.$$

Thus, the desired bound is immediately checked by using $0 \leq \sqrt{x+1}-\sqrt{x} \leq 1$ available for $x \geq 0$.

□

The main result of the current section is the following:

Theorem 3.2. *Under Hypothesis* $(\mathbf{H}_4)$*, we have*

$$G_n \xrightarrow[n\to\infty]{Law} C_{1/4}\int_0^1 f(B_s)dW_s + \frac{1}{4}\int_0^1 f''(B_s)ds,$$

where $W = (W_t)_{t\in[0,1]}$ *is a standard Brownian motion independent of* B *and*

$$C_{1/4} := \sqrt{\frac{1}{2}\sum_{p=-\infty}^{\infty}\left(\sqrt{|p+1|}+\sqrt{|p-1|}-2\sqrt{|p|}\right)^2} < \infty.$$

Proof. This proof is mainly inspired by the first draft of [15]. During all the proof, C will denote a constant depending only on $\|f^{(a)}\|_\infty$, $a = 0, 1, 2, 3, 4$, which can differ from one line to another.

Step 1.- We begin the proof by showing the following limits:

$$\lim_{n\to\infty} E\left(G_n\right) = \frac{1}{4}\int_0^1 E\left(f''(B_s)\right)ds, \tag{3.1}$$

and

$$\lim_{n\to\infty} E\left(G_n^2\right) = C_{1/4}^2\int_0^1 E\left(f^2(B_s)\right)ds + \frac{1}{16}E\left(\int_0^1 f''(B_s)ds\right)^2. \tag{3.2}$$

Proof of (3.1): we can write

$$\begin{aligned}
E\left(G_n\right) &= \sum_{k=0}^{n-1} E\left(f(B_{k/n})I_2(\delta_{k/n}^{\otimes 2})\right)\\
&= \sum_{k=0}^{n-1} E\left(\left\langle D^2(f(B_{k/n})), \delta_{k/n}^{\otimes 2}\right\rangle_{\mathfrak{H}}\right)\\
&= \sum_{k=0}^{n-1} E\left(f''(B_{k/n})\right)\left\langle \varepsilon_{k/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}^2\\
&= \frac{1}{4n}\sum_{k=0}^{n-1} E\left(f''(B_{k/n})\right) + \sum_{k=0}^{n-1} E\left(f''(B_{k/n})\right)\left(\left\langle \varepsilon_{k/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}^2 - \frac{1}{4n}\right)
\end{aligned}$$

$$\underset{n\to\infty}{\longrightarrow} \quad \frac{1}{4}\int_0^1 E\big(f''(B_s)\big)ds, \text{ by Lemma 3.1}(iv) \text{ and under } (\mathbf{H}_4).$$

Proof of (3.2):
By the multiplication formula (2.6), we have

$$I_2(\delta_{j/n}^{\otimes 2})I_2(\delta_{k/n}^{\otimes 2}) = I_4(\delta_{j/n}^{\otimes 2}\otimes\delta_{k/n}^{\otimes 2}) + 4\, I_2(\delta_{j/n}\otimes\delta_{k/n})\left\langle \delta_{j/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}} + 2\left\langle \delta_{j/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}^2. \quad (3.3)$$

Thus

$$\begin{aligned}
E\left(G_n^2\right) &= \sum_{j,k=0}^{n-1} E\left(f(B_{j/n})f(B_{k/n})I_2(\delta_{j/n}^{\otimes 2})I_2(\delta_{k/n}^{\otimes 2})\right) \\
&= \sum_{j,k=0}^{n-1} E\left(f(B_{j/n})f(B_{k/n})I_4(\delta_{j/n}^{\otimes 2}\otimes\delta_{k/n}^{\otimes 2})\right) \\
&+ 4\sum_{j,k=0}^{n-1} E\left(f(B_{j/n})f(B_{k/n})I_2(\delta_{j/n}\otimes\delta_{k/n})\right)\left\langle \delta_{j/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}} \\
&+ 2\sum_{j,k=0}^{n-1} E\left(f(B_{j/n})f(B_{k/n})\right)\left\langle \delta_{j/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}^2 \\
&= A_n + B_n + C_n.
\end{aligned}$$

Using Malliavin integration by parts formula (2.5), A_n can be expressed as follows:

$$\begin{aligned}
A_n &= \sum_{j,k=0}^{n-1} E\left(\left\langle D^4(f(B_{j/n})f(B_{k/n})), \delta_{j/n}^{\otimes 2}\otimes\delta_{k/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 4}}\right) \\
&= 24\sum_{j,k=0}^{n-1}\sum_{a+b=4} E\left(f^{(a)}(B_{j/n})f^{(b)}(B_{k/n})\right)\left\langle \varepsilon_{j/n}^{\otimes a}\widetilde{\otimes}\varepsilon_{k/n}^{\otimes b}, \delta_{j/n}^{\otimes 2}\otimes\delta_{k/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 4}}.
\end{aligned}$$

In fact, in the previous sum, each term is negligible except

$$\begin{aligned}
&\sum_{j,k=0}^{n-1} E\left(f''(B_{j/n})f''(B_{k/n})\right)\left\langle \varepsilon_{j/n}, \delta_{j/n}\right\rangle_{\mathfrak{H}}^2\left\langle \varepsilon_{k/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}^2 \\
= \quad & E\left(\left[\sum_{k=0}^{n-1} f''(B_{k/n})\left\langle \varepsilon_{k/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}^2\right]^2\right) \\
= \quad & E\left(\left[\frac{1}{4n}\sum_{k=0}^{n-1} f''(B_{k/n}) + \sum_{k=0}^{n-1} f''(B_{k/n})\big(\left\langle \varepsilon_{k/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}^2 - \frac{1}{4n}\big)\right]^2\right) \\
\underset{n\to\infty}{\longrightarrow} \quad & E\left(\left[\frac{1}{4}\int_0^1 f''(B_s)ds\right]^2\right), \text{by Lemma 3.1}(iv) \text{ and under } (\mathbf{H}_4).
\end{aligned}$$

The other terms appearing in A_n make no contribution to the limit. Indeed, they have the form

$$\sum_{j,k=0}^{n-1} E\left(f^{(a)}(B_{j/n})f^{(b)}(B_{k/n})\right)\left\langle \varepsilon_{j/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}} \prod_{i=1}^{3}\left\langle \varepsilon_{x_i/n}, \delta_{y_i/n}\right\rangle_{\mathfrak{H}}$$

(where x_i and y_i are for j or k) and, from Lemma 3.1 *(i) (iii)*, we have that

$$\begin{cases} \sup_{j,k=0,\ldots,n-1}\prod_{i=1}^{3}\left|\left\langle \varepsilon_{x_i/n}, \delta_{y_i/n}\right\rangle_{\mathfrak{H}}\right| \underset{n\to\infty}{=} O(n^{-3/2}), \\ \sum_{j,k=0}^{n-1}\left|\left\langle \varepsilon_{j/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \underset{n\to\infty}{=} O(n). \end{cases}$$

Still using Malliavin integration by parts formula (2.5), we can bound B_n as follows:

$$\begin{aligned} |B_n| &\leq 8\sum_{j,k=0}^{n-1}\sum_{a+b=2}\left|E\left(f^{(a)}(B_{j/n})f^{(b)}(B_{k/n})\right)\left\langle \varepsilon_{j/n}^{\otimes a}\widetilde{\otimes}\varepsilon_{k/n}^{\otimes b}, \delta_{j/n}\otimes\delta_{k/n}\right\rangle_{\mathfrak{H}^{\otimes 2}}\left\langle \delta_{j/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \\ &\leq Cn^{-1}\sum_{j,k=0}^{n-1}\left|\left\langle \delta_{j/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}\right|, \text{ by Lemma 3.1 } \textit{(i)} \text{ and under } (\mathbf{H}_4) \\ &= Cn^{-3/2}\sum_{j,k=0}^{n-1}|\rho(j-k)| \leq Cn^{-1/2}\sum_{r=-\infty}^{\infty}|\rho(r)| \underset{n\to\infty}{=} O(n^{-1/2}), \end{aligned}$$

where

$$\rho(r) := \sqrt{|r+1|} + \sqrt{|r-1|} - 2\sqrt{|r|}, \quad r\in\mathbb{Z}. \tag{3.4}$$

Observe that the serie $\sum_{r=-\infty}^{\infty}|\rho(r)|$ is convergent since $|\rho(r)| \underset{|r|\to\infty}{\sim} \frac{1}{2}|r|^{-\frac{3}{2}}$.

Finally, we consider the term C_n:

$$\begin{aligned} C_n &= \frac{1}{2n}\sum_{j,k=0}^{n-1} E\left(f(B_{j/n})f(B_{k/n})\right)\rho^2(j-k) \\ &= \frac{1}{2n}\sum_{r=-\infty}^{\infty}\sum_{j=0\vee -r}^{(n-1)\wedge(n-1-r)} E\left(f(B_{j/n})f(B_{(j+r)/n})\right)\rho^2(r) \\ &\underset{n\to\infty}{\longrightarrow} \frac{1}{2}\int_0^1 E\left(f^2(B_s)\right)ds\sum_{r=-\infty}^{\infty}\rho^2(r) = C_{1/4}^2\int_0^1 E\left(f^2(B_s)\right)ds. \end{aligned}$$

The desired convergence (3.2) follows.

Step 2.- Since the sequence (G_n) is bounded in L^1, the sequence $\left(G_n, (B_t)_{t\in[0,1]}\right)$ is tight in $\mathbb{R}\times\mathscr{C}([0,1])$. Assume that $\left(G_\infty, (B_t)_{t\in[0,1]}\right)$ denotes the limit in law of a certain subsequence of $\left(G_n, (B_t)_{t\in[0,1]}\right)$, denoted again by $\left(G_n, (B_t)_{t\in[0,1]}\right)$.

We have to prove that

$$G_\infty \overset{Law}{=} C_{1/4}\int_0^1 f(B_s)dW_s + \frac{1}{4}\int_0^1 f''(B_s)ds,$$

where W denotes a standard Brownian motion independent of B, or equivalently that

$$E\left(e^{i\lambda G_\infty}|\,(B_t)_{t\in[0,1]}\right)=\exp\left\{i\frac{\lambda}{4}\int_0^1 f''(B_s)ds-\frac{\lambda^2}{2}\,C_{1/4}^2\int_0^1 f^2(B_s)ds\right\}. \tag{3.5}$$

This will be done by showing that for every random variable ξ of the form (2.4) and every real number λ, we have

$$\lim_{n\to\infty}\phi_n'(\lambda)=E\left\{e^{i\lambda G_\infty}\xi\left(\frac{i}{4}\int_0^1 f''(B_s)ds-\lambda C_{1/4}^2\int_0^1 f^2(B_s)ds\right)\right\} \tag{3.6}$$

where

$$\phi_n'(\lambda):=\frac{d}{d\lambda}E\left(e^{i\lambda G_n}\xi\right)=iE\left(G_n e^{i\lambda G_n}\xi\right),\quad n\geq 1.$$

Let us make precise this argument. Because $\left(G_\infty,(B_t)_{t\in[0,1]}\right)$ is the limit in law of $\left(G_n,(B_t)_{t\in[0,1]}\right)$ and (G_n) is bounded in L^1, we have that

$$E(G_\infty\,\xi\,e^{i\lambda G_\infty})=\lim_{n\to\infty}E\left(G_n\,\xi\,e^{i\lambda G_n}\right),\quad\forall\lambda\in\mathbb{R},$$

for every ξ of the form (2.4). Furthermore, because convergence (3.6) holds for every ξ of the form (2.4), the conditional characteristic function $\lambda\mapsto E\left(e^{i\lambda G_\infty}|(B_t)_{t\in[0,1]}\right)$ satisfies the following linear ordinary differential equation:

$$\frac{d}{d\lambda}E\left(e^{i\lambda G_\infty}|(B_t)_{t\in[0,1]}\right)=E\left(e^{i\lambda G_\infty}|(B_t)_{t\in[0,1]}\right)\left[\frac{i}{4}\int_0^1 f''(B_s)ds-\lambda C_{1/4}^2\int_0^1 f^2(B_s)ds\right].$$

By solving it, we obtain (3.5), which yields the desired conclusion.

Thus, it remains to show (3.6). By the duality between the derivative and divergence operators, we have

$$E\left(f(B_{k/n})I_2(\delta_{k/n}^{\otimes 2})e^{i\lambda G_n}\xi\right)=E\left(\left\langle D^2\left(f(B_{k/n})e^{i\lambda G_n}\xi\right),\delta_{k/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right). \tag{3.7}$$

The first and second derivatives of $f(B_{k/n})e^{i\lambda G_n}\xi$ are given by

$$D\left(f(B_{k/n})e^{i\lambda G_n}\xi\right)=f'(B_{k/n})e^{i\lambda G_n}\xi\,\varepsilon_{k/n}+i\lambda f(B_{k/n})e^{i\lambda G_n}\xi\,DG_n+f(B_{k/n})e^{i\lambda G_n}D\xi$$

and

$$\begin{aligned}D^2\left(f(B_{k/n})e^{i\lambda G_n}\xi\right)\;=\;&f''(B_{k/n})e^{i\lambda G_n}\xi\,\varepsilon_{k/n}^{\otimes 2}+2i\lambda f'(B_{k/n})e^{i\lambda G_n}\xi\left(\varepsilon_{k/n}\widetilde{\otimes}DG_n\right)\\&+2f'(B_{k/n})e^{i\lambda G_n}\left(\varepsilon_{k/n}\widetilde{\otimes}D\xi\right)-\lambda^2 f(B_{k/n})e^{i\lambda G_n}\xi\,DG_n^{\otimes 2}\\&+2i\lambda f(B_{k/n})e^{i\lambda G_n}\left(DG_n\widetilde{\otimes}D\xi\right)\\&+i\lambda f(B_{k/n})e^{i\lambda G_n}\xi D^2G_n+f(B_{k/n})e^{i\lambda G_n}D^2\xi.\end{aligned}$$

Hence, taking expectation and multiplying by $\delta_{k/n}^{\otimes 2}$ yields

$$E\left(\left\langle D^2\left(f(B_{k/n})e^{i\lambda G_n}\xi\right),\delta_{k/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right)$$

$$
\begin{aligned}
&= E\left(f''(B_{k/n})e^{i\lambda G_n}\xi\right)\left\langle \varepsilon_{k/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}^2 + 2i\lambda E\left(f'(B_{k/n})e^{i\lambda G_n}\xi\left\langle DG_n,\delta_{k/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{k/n},\delta_{k/n}\right\rangle_{\mathfrak{H}} \\
&\quad +2E\left(f'(B_{k/n})e^{i\lambda G_n}\ \left\langle D\xi,\delta_{k/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{k/n},\delta_{k/n}\right\rangle_{\mathfrak{H}} - \lambda^2 E\left(f(B_{k/n})e^{i\lambda G_n}\xi\left\langle DG_n,\delta_{k/n}\right\rangle_{\mathfrak{H}}^2\right) \\
&\quad +2i\lambda E\left(f(B_{k/n})e^{i\lambda G_n}\ \left\langle D\xi,\delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle DG_n,\delta_{k/n}\right\rangle_{\mathfrak{H}}\right) \\
&\quad +i\lambda E\left(f(B_{k/n})e^{i\lambda G_n}\xi\left\langle D^2G_n,\delta_{k/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right) + E\left(f(B_{k/n})e^{i\lambda G_n}\left\langle D^2\xi,\delta_{k/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right). \qquad (3.8)
\end{aligned}
$$

We also need explicit expressions for $\left\langle DG_n,\delta_{k/n}\right\rangle_{\mathfrak{H}}$ and for $\left\langle D^2G_n,\delta_{k/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}$. Differentiating G_n we obtain

$$
DG_n = \sum_{l=0}^{n-1}\left[f'(B_{l/n})I_2(\delta_{l/n}^{\otimes 2})\varepsilon_{l/n} + 2f(B_{l/n})\Delta B_{l/n}\delta_{l/n}\right] \qquad (3.9)
$$

and, as a consequence,

$$
\left\langle DG_n,\delta_{k/n}\right\rangle_{\mathfrak{H}} = \sum_{l=0}^{n-1} f'(B_{l/n})I_2(\delta_{l/n}^{\otimes 2})\left\langle \varepsilon_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}} + 2\sum_{l=0}^{n-1} f(B_{l/n})\Delta B_{l/n}\left\langle \delta_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}. \qquad (3.10)
$$

Also

$$
D^2G_n = \sum_{l=0}^{n-1}\left[f''(B_{l/n})I_2(\delta_{l/n}^{\otimes 2})\varepsilon_{l/n}^{\otimes 2} + 4f'(B_{l/n})\ \Delta B_{l/n}\left(\varepsilon_{l/n}\widetilde{\otimes}\delta_{l/n}\right) + 2f(B_{l/n})\delta_{l/n}^{\otimes 2}\right],
$$

and, as a consequence,

$$
\begin{aligned}
\left\langle D^2G_n,\delta_{k/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}} &= \sum_{l=0}^{n-1}\Big[f''(B_{l/n})I_2(\delta_{l/n}^{\otimes 2})\left\langle \varepsilon_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}^2 \\
&\quad +4f'(B_{l/n})\ \Delta B_{l/n}\left\langle \varepsilon_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle \delta_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}} + 2f(B_{l/n})\left\langle \delta_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}^2\Big]. \qquad (3.11)
\end{aligned}
$$

Substituting (3.11) into (3.8) yields the following decomposition for $\phi_n'(\lambda) = i\,E(G_ne^{i\lambda G_n}\xi)$:

$$
\begin{aligned}
\phi_n'(\lambda) &= -2\lambda\sum_{k,l=0}^{n-1} E\left(f(B_{k/n})f(B_{l/n})e^{i\lambda G_n}\xi\right)\left\langle \delta_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}^2 \\
&\quad + \ i\sum_{k=0}^{n-1} E\left(f''(B_{k/n})e^{i\lambda G_n}\xi\right)\left\langle \varepsilon_{k/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}^2 + i\sum_{k=0}^{n-1} r_{k,n} \qquad (3.12)
\end{aligned}
$$

where $r_{k,n}$ is given by

$$
\begin{aligned}
r_{k,n} &= 2i\lambda E\left(f'(B_{k/n})e^{i\lambda G_n}\xi\left\langle DG_n,\delta_{k/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{k/n},\delta_{k/n}\right\rangle_{\mathfrak{H}} \\
&\quad +2E\left(f'(B_{k/n})e^{i\lambda G_n}\ \left\langle D\xi,\delta_{k/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{k/n},\delta_{k/n}\right\rangle_{\mathfrak{H}} - \lambda^2 E\left(f(B_{k/n})e^{i\lambda G_n}\xi\left\langle DG_n,\delta_{k/n}\right\rangle_{\mathfrak{H}}^2\right)
\end{aligned}
$$

$$+2i\lambda E\left(f(B_{k/n})e^{i\lambda G_n}\ \left\langle D\xi,\delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle DG_n,\delta_{k/n}\right\rangle_{\mathfrak{H}}\right)$$
$$+i\lambda\sum_{l=0}^{n-1}E\left(f(B_{k/n})e^{i\lambda G_n}\xi\ \ f''(B_{l/n})I_2(\delta_{l/n}^{\otimes 2})\ \right)\left\langle\varepsilon_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}^2$$
$$+4i\lambda\sum_{l=0}^{n-1}E\left(f(B_{k/n})e^{i\lambda G_n}\xi f'(B_{l/n})\ \Delta B_{l/n}\right)\left\langle\varepsilon_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle\delta_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}$$
$$+E\left(f(B_{k/n})e^{i\lambda G_n}\left\langle D^2\xi,\delta_{k/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right)=\sum_{j=1}^{7}R_{k,n}^{(j)}. \tag{3.13}$$

Remark that the first sum in the right hand side of (3.12) is very similar to C_n presented in Step 1. In fact, similar computations give

$$\lim_{n\to\infty}-2\lambda\sum_{k,l=0}^{n-1}\mathbb{E}[f(B_{k/n})f(B_{l/n})e^{i\lambda G_n}\xi]\left\langle\delta_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}^2$$
$$=-C_{1/4}^2\lambda\int_0^1 E\left(f^2(B_s)e^{i\lambda G_\infty}\xi\right)ds. \tag{3.14}$$

Furthermore, the second term of (3.12) is very similar to $E\left(G_n\right)$. In fact, using the arguments presented in Step 1, we obtain here that

$$\lim_{n\to\infty}i\sum_{k=0}^{n-1}E\left(f''(B_{k/n})e^{i\lambda G_n}\xi\right)\left\langle\varepsilon_{k/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}^2=\frac{i}{4}\int_0^1 E\left(f''(B_s)e^{i\lambda G_\infty}\xi\right)ds. \tag{3.15}$$

Consequently, (3.6) will be shown as soon as we will prove that $\lim_{n\to\infty}\sum_{k=0}^{n-1}r_{k,n}=0$. This will be done in several steps.

Step 3.- In this step, we state and prove some estimates which will be crucial in the rest of the proof. First, we will show that

$$\left|E\left(f'(B_{k/n})f'(B_{l/n})e^{i\lambda G_n}\xi\, I_2(\delta_{l/n}^{\otimes 2})\right)\right|\leq\frac{C}{n}\quad\text{for any } 0\leq k,l\leq n-1. \tag{3.16}$$

Then we will prove that

$$\left|E\left(f(B_{k/n})f'(B_{j/n})f'(B_{l/n})e^{i\lambda G_n}\xi\, I_4(\delta_{j/n}^{\otimes 2}\otimes\delta_{l/n}^{\otimes 2})\right)\right|\leq\frac{C}{n^2}\quad\text{for any } 0\leq k,j,l\leq n-1. \tag{3.17}$$

Proof of (3.16):
Let $\zeta_{\xi,k,n}$ denotes any random variable of the form $f^{(a)}(B_{k/n})f^{(b)}(B_{l/n})e^{i\lambda G_n}\xi$ with a and b two positive integers less or equal to four. From the Malliavin integration by parts formula (2.5) we have

$$E\left(f'(B_{k/n})f'(B_{l/n})e^{i\lambda G_n}\xi\, I_2(\delta_{l/n}^{\otimes 2})\right)=E\left(\left\langle D^2\left(f'(B_{k/n})f'(B_{l/n})e^{i\lambda G_n}\xi\right),\delta_{l/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right).$$

When computing the RHS, three types of terms appear. First, we have some terms of the form:

$$\begin{cases} E\left(\zeta_{\xi,k,n}\right)\left\langle \varepsilon_{k/n},\delta_{l/n}\right\rangle_{\mathfrak{H}}^{2}, \text{ or} \\ E\left(\zeta_{\xi,k,n}\left\langle D\xi,\delta_{l/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{k/n},\delta_{l/n}\right\rangle_{\mathfrak{H}}, \text{ or} \\ E\left(\zeta_{\xi,k,n}\left\langle D^{2}\xi,\delta_{l/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right), \end{cases} \tag{3.18}$$

where $D\xi$ and $D^2\xi$ are given by:

$$\begin{cases} D\xi=\sum_{i=1}^{m}\frac{\partial\psi}{\partial x_i}(B_{t_1},\ldots,B_{t_m})\,\varepsilon_{t_i}, \\ D^{2}\xi=\sum_{i,j=1}^{m}\frac{\partial^{2}\psi}{\partial x_j\partial x_i}(B_{t_1},\ldots,B_{t_m})\,\varepsilon_{t_j}\otimes\varepsilon_{t_i}. \end{cases}$$

From Lemma 3.1 *(i)* and under $(\mathbf{H}_4)$, we have that each of the three terms in (3.18) is less or equal to Cn^{-1}. The second type of terms we have to deal with is

$$\begin{cases} E\left(\zeta_{\xi,k,n}\left\langle DG_n,\delta_{l/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{k/n},\delta_{l/n}\right\rangle_{\mathfrak{H}}, \text{ or} \\ E\left(\zeta_{\xi,k,n}\left\langle DG_n,\delta_{l/n}\right\rangle_{\mathfrak{H}}\left\langle D\xi,\delta_{l/n}\right\rangle_{\mathfrak{H}}\right). \end{cases} \tag{3.19}$$

By Cauchy-Schwarz inequality, under $(\mathbf{H}_4)$ and by using (4.20) in [15], that is

$$E\left(\left\langle DG_n,\delta_{l/n}\right\rangle_{\mathfrak{H}}^{2}\right)\leq Cn^{-1},$$

we have that both expressions in (3.19) are also less or equal to Cn^{-1}.

The last type of terms which has to be taken into account is the term

$$-\lambda^{2}E\left(f'(B_{k/n})f'(B_{l/n})e^{i\lambda G_n}\xi\left\langle D^{2}G_n,\delta_{k/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right).$$

Again, by using Cauchy-Schwarz inequality and the estimate

$$E\left(\left\langle D^{2}G_n,\delta_{k/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}^{2}\right)\leq Cn^{-2}$$

(which can be obtained by mimicing the proof of (4.20) in [15]), we can conclude that

$$\left|-\lambda^{2}E\left(f'(B_{k/n})f'(B_{l/n})e^{i\lambda G_n}\xi\left\langle D^{2}G_n,\delta_{k/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right)\right|\leq\frac{C}{n}.$$

As a consequence (3.16) is shown.

Proof of (3.17):
By the Malliavin integration by parts formula (2.5), we have

$$E\left(\zeta_{\xi,k,n}f'(B_{j/n})f'(B_{l/n})I_4(\delta_{j/n}^{\otimes 2}\otimes\delta_{l/n}^{\otimes 2})\right)=E\left(\left\langle D^{4}\left(\zeta_{\xi,k,n}f'(B_{j/n})f'(B_{l/n})\right),\delta_{j/n}^{\otimes 2}\otimes\delta_{l/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 4}}\right).$$

When computing the RHS, we have to deal with the same type of terms as in the proof of (3.16) plus two additional types of terms containing

$$E\left(\left\langle D^{3}G_n,\delta_{j/n}^{\otimes 2}\otimes\delta_{l/n}\right\rangle_{\mathfrak{H}^{\otimes 3}}^{2}\right)\quad\text{and}\quad E\left(\left\langle D^{4}G_n,\delta_{j/n}^{\otimes 2}\otimes\delta_{l/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 4}}^{2}\right).$$

In fact, by mimicing the proof of (4.20) in [15], we can obtain the following bounds:

$$E\left(\left\langle D^3G_n, \delta_{j/n}^{\otimes 2}\otimes\delta_{l/n}\right\rangle_{\mathfrak{H}^{\otimes 3}}^2\right) \leq Cn^{-3} \quad \text{and} \quad E\left(\left\langle D^4G_n, \delta_{j/n}^{\otimes 2}\otimes\delta_{l/n}^{\otimes 2}\right\rangle_{\mathfrak{H}^{\otimes 4}}^2\right) \leq Cn^{-4}.$$

This allows us to obtain (3.17).

Step 4.- We compute the terms corresponding to $R_{k,n}^{(1)}$, $R_{k,n}^{(4)}$ and $R_{k,n}^{(6)}$ in (3.13). The derivative DG_n is given by (3.9), so that

$$\begin{aligned}
\sum_{k=0}^{n-1} R_{k,n}^{(1)} &= 2i\lambda \sum_{k,l=0}^{n-1} E\left(f'(B_{k/n})f'(B_{l/n})e^{i\lambda G_n}\xi I_2(\delta_{l/n}^{\otimes 2})\right)\left\langle \varepsilon_{l/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle \varepsilon_{k/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}} \\
&+ 2\sum_{k,l=0}^{n-1} E\left(f'(B_{k/n})f(B_{l/n})e^{i\lambda G_n}\xi\Delta B_{l/n}\right)\left\langle \delta_{l/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle \varepsilon_{k/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}} \\
&= T_1^{(1)} + T_2^{(1)}.
\end{aligned}$$

From (3.16), Lemma 3.1 *(i)*, *(iii)* and under $(\mathbf{H}_4)$, we have that

$$\left|T_1^{(1)}\right| \leq Cn^{-3/2}\sum_{k,l=0}^{n-1}\left|\left\langle \varepsilon_{l/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \leq Cn^{-1/2}.$$

For $T_2^{(1)}$, remark first that Cauchy-Schwarz inequality and hypothesis $(\mathbf{H}_4)$ yield

$$\left|E\left(f'(B_{k/n})e^{i\lambda G_n}\xi f(B_{l/n})\Delta B_{l/n}\right)\right| \leq Cn^{-1/4}. \tag{3.20}$$

Thus, by Lemma 3.1 (i),

$$\begin{aligned}
\left|T_2^{(1)}\right| &\leq Cn^{-3/4}\sum_{k,l=0}^{n-1}\left|\left\langle \delta_{l/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}\right| = Cn^{-5/4}\sum_{k,l=0}^{n-1}|\rho(k-l)| \\
&\leq Cn^{-1/4}\sum_{r=-\infty}^{\infty}|\rho(r)| = Cn^{-1/4},
\end{aligned}$$

where ρ has been defined in (3.4).

The term corresponding to $R_{k,n}^{(4)}$ is very similar to $R_{k,n}^{(1)}$. Indeed, by (3.9), we have

$$\begin{aligned}
\sum_{k=0}^{n-1} R_{k,n}^{(4)} &= 2i\lambda\sum_{i=1}^{m}\sum_{k,l=0}^{n-1} E\left(f(B_{k/n})f'(B_{l/n})e^{i\lambda G_n}\frac{\partial\psi}{\partial x_i}(B_{t_1},\ldots,B_{t_m})I_2(\delta_{l/n}^{\otimes 2})\right) \\
&\qquad\qquad \times\left\langle \varepsilon_{l/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle \varepsilon_{t_i}, \delta_{k/n}\right\rangle_{\mathfrak{H}} \\
&+ 4i\lambda\sum_{i=1}^{m}\sum_{k,l=0}^{n-1} E\left(f(B_{k/n})f(B_{l/n})e^{i\lambda G_n}\Delta B_{l/n}\frac{\partial\psi}{\partial x_i}(B_{t_1},\ldots,B_{t_m})\right) \\
&\qquad\qquad \times\left\langle \delta_{l/n}, \delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle \varepsilon_{t_i}, \delta_{k/n}\right\rangle_{\mathfrak{H}}
\end{aligned}$$

$$= T_1^{(4)} + T_2^{(4)}$$

and we can proceed for $T_i^{(4)}$ as for $T_i^{(1)}$.

The term corresponding to $R_{k,n}^{(6)}$ is very similar to $T_2^{(1)}$. More precisely, we have

$$\begin{aligned}\left|\sum_{k=0}^{n-1} R_{k,n}^{(6)}\right| &\leq Cn^{-3/4}\sum_{k,l=0}^{n-1}\left|\left\langle \delta_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\right| = Cn^{-5/4}\sum_{k,l=0}^{n-1}|\rho(k-l)| \\ &\leq Cn^{-1/4}\sum_{r=-\infty}^{\infty}|\rho(r)| = Cn^{-1/4}.\end{aligned}$$

Step 5.- Estimation of $R_{k,n}^{(3)}$. Let $\zeta_{\xi,k,n} := \lambda^2 f(B_{k/n})e^{i\lambda G_n}\xi$. Using (3.9), we have

$$\begin{aligned}\langle DG_n,\delta_{k/n}\rangle_{\mathfrak{H}}^2 &= \sum_{j,l=0}^{n-1} f'(B_{l/n})f'(B_{j/n})I_2(\delta_{l/n}^{\otimes 2})I_2(\delta_{j/n}^{\otimes 2})\langle\varepsilon_{j/n},\delta_{k/n}\rangle_{\mathfrak{H}}\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathfrak{H}} \\ &\quad + \sum_{j,l=0}^{n-1} f(B_{j/n})f(B_{l/n})\Delta B_{j/n}\Delta B_{l/n}\langle\delta_{j/n},\delta_{k/n}\rangle_{\mathfrak{H}}\langle\delta_{l/n},\delta_{k/n}\rangle_{\mathfrak{H}}\end{aligned}$$

and, consequently:

$$\begin{aligned}\left|\sum_{k=0}^{n-1} R_{k,n}^3\right| &\leq \sum_{k=0}^{n-1}\left|E\left(\zeta_{\xi,k,n}\langle DG_n,\delta_{k/n}\rangle_{\mathfrak{H}}^2\right)\right| \\ &\leq 2\sum_{k,j,l=0}^{n-1}\left|E\left(\zeta_{\xi,k,n}f'(B_{j/n})f'(B_{l/n})I_2(\delta_{j/n}^{\otimes 2})I_2(\delta_{l/n}^{\otimes 2})\right)\langle\varepsilon_{j/n},\delta_{k/n}\rangle_{\mathfrak{H}}\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathfrak{H}}\right| \\ &+ 8\sum_{k,j,l=0}^{n-1}\left|E\left(\zeta_{\xi,k,n}f(B_{j/n})f(B_{l/n})\Delta B_{j/n}\Delta B_{l/n}\right)\langle\delta_{j/n},\delta_{k/n}\rangle_{\mathfrak{H}}\langle\delta_{l/n},\delta_{k/n}\rangle_{\mathfrak{H}}\right|.\end{aligned}$$

Using the product formula (3.3), we have

$$\begin{aligned}\left|\sum_{k=0}^{n-1} R_{k,n}^3\right| &\leq 2\sum_{k,j,l=0}^{n-1}\left|E\left(\zeta_{\xi,k,n}f'(B_{j/n})f'(B_{l/n})I_4(\delta_{j/n}^{\otimes 2}\otimes\delta_{l/n}^{\otimes 2})\right)\right|\left|\left\langle\varepsilon_{j/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle\varepsilon_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \\ &+ 8\sum_{k,j,l=0}^{n-1}\left|E\left(\zeta_{\xi,k,n}f'(B_{j/n})f'(B_{l/n})I_2(\delta_{j/n}\otimes\delta_{l/n})\right)\right|\left|\left\langle\delta_{j/n},\delta_{l/n}\right\rangle_{\mathfrak{H}}\left\langle\varepsilon_{j/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle\varepsilon_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \\ &+ 4\sum_{k,j,l=0}^{n-1}\left|E\left(\zeta_{\xi,k,n}f'(B_{j/n})f'(B_{l/n})\right)\right|\left|\left\langle\delta_{j/n},\delta_{l/n}\right\rangle_{\mathfrak{H}}^2\left\langle\varepsilon_{j/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle\varepsilon_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \\ &+ 8\sum_{k,j,l=0}^{n-1}\left|E\left(\zeta_{\xi,k,n}f(B_{j/n})f(B_{l/n})\Delta B_{j/n}\Delta B_{l/n}\right)\right|\left|\left\langle\delta_{j/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle\delta_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \\ &= \sum_{i=1}^{4} T_i^{(3)}.\end{aligned}$$

From (3.17), we have

$$\begin{aligned}|T_1^{(3)}| &\leq Cn^{-1/2}\sum_{k,j,l=0}^{n-1}\left|E\left(\zeta_{\xi,k,n}f'(B_{j/n})f'(B_{l/n})I_4(\delta_{j/n}^{\otimes 2}\otimes\delta_{l/n}^{\otimes 2})\right)\right|\left|\left\langle\varepsilon_{j/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \\ &\leq Cn^{-5/2}n^2\sup_{j=0,\ldots n-1}\sum_{k=0}^{n-1}\left|\left\langle\varepsilon_{j/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \leq Cn^{-1/2} \text{ by Lemma 3.1 } (ii).\end{aligned}$$

Now, let us consider $T_2^{(3)}$. Using (3.16) and Lemma 3.1 *(ii)*, we deduce that

$$\begin{aligned}\left|T_2^{(3)}\right| &\leq Cn^{-3/2}\sum_{j,l=0}^{n-1}\left|\left\langle\delta_{j/n},\delta_{l/n}\right\rangle_{\mathfrak{H}}\right|\sup_{j=0,\ldots,n-1}\sum_{k=0}^{n-1}\left|\left\langle\varepsilon_{j/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \\ &\leq Cn^{-1/2}\sum_{r=-\infty}^{\infty}|\rho(r)| = Cn^{-1/2}.\end{aligned}$$

For $T_3^{(3)}$, we have

$$\begin{aligned}\left|T_3^{(3)}\right| &\leq Cn^{-1/2}\sum_{j,l=0}^{n-1}\left\langle\delta_{j/n},\delta_{l/n}\right\rangle_{\mathfrak{H}}^2\sup_{j=0,\ldots,n-1}\sum_{k=0}^{n-1}\left|\left\langle\varepsilon_{j/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \\ &\leq Cn^{-1/2}\sum_{r=-\infty}^{\infty}\rho^2(r) = Cn^{-1/2}.\end{aligned}$$

Finally, by Cauchy-Schwarz inequality and under $(\mathbf{H}_4)$, we have

$$\left|E\left(\zeta_{\xi,k,n}f(B_{j/n})f(B_{l/n})\Delta B_{j/n}\Delta B_{l/n}\right)\right| \leq Cn^{-1/2}.$$

Consequently:

$$\begin{aligned}\left|T_4^{(3)}\right| &\leq Cn^{-1/2}\sum_{k,j,l=0}^{n-1}\left|\left\langle\delta_{j/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle\delta_{k/n},\delta_{l/n}\right\rangle_{\mathfrak{H}}\right| \\ &\leq Cn^{-3/2}\sum_{k,j,l=0}^{n-1}|\rho(k-l)\rho(k-j)| \leq Cn^{-1/2}\left(\sum_{r=-\infty}^{\infty}|\rho(r)|\right)^2 = Cn^{-1/2}.\end{aligned}$$

Step 6.- Estimation of $R_{k,n}^{(5)}$. From (3.16) and Lemma 3.1 *(iii)*, we have,

$$\left|\sum_{k=0}^{n-1}R_{k,n}^{(5)}\right| \leq Cn^{-3/2}\sum_{k,l=0}^{n-1}\left|\left\langle\varepsilon_{l/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \leq Cn^{-1/2}.$$

Step 7.- Estimation of $R_{k,n}^{(2)}$ and $R_{k,n}^{(7)}$. We recall that

$$0 \leq \sqrt{x+1}-\sqrt{x} \leq 1 \quad \text{for any } x \geq 0.$$

Thus, under $(\mathbf{H}_4)$ and using Lemma 3.1, we have:

$$\begin{aligned}\left|\sum_{k=0}^{n-1} R_{k,n}^{(2)}\right| &\leq 2\sum_{i=1}^{m}\sum_{k=0}^{n-1}\left|E\left(f'(B_{k/n})e^{i\lambda G_n}\frac{\partial\psi}{\partial x_i}(B_{t_1},\ldots,B_{t_m})\right)\left\langle\varepsilon_{t_i},\delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle\varepsilon_{k/n},\delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \\ &\leq C(f,\psi)\,n^{-\frac{1}{2}}\sup_{t\in[0,1]}\sum_{k=0}^{n-1}\left|\left\langle\varepsilon_t,\delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \leq Cn^{-1/2}.\end{aligned}$$

Similarly, the following bound holds:

$$\begin{aligned}\left|\sum_{k=0}^{n-1} R_{k,n}^{(7)}\right| &\leq \sum_{i,j=1}^{m}\sum_{k=0}^{n-1}\left|E\left(f(B_{k/n})e^{i\lambda G_n}\frac{\partial^2\psi}{\partial x_j\partial x_i}(B_{t_1},\ldots,B_{t_m})\right)\left\langle\varepsilon_{t_i},\delta_{k/n}\right\rangle_{\mathfrak{H}}\left\langle\varepsilon_{t_j},\delta_{k/n}\right\rangle_{\mathfrak{H}}\right| \\ &\leq Cn^{-1/2}.\end{aligned}$$

The proof of Theorem 3.2 is done. □

4 Proof of Theorem 1.2

Once again, $B = B^{1/4}$ denotes a fractional Brownian motion with Hurst index $H = 1/4$. Moreover, we recall that we note $\Delta B_{k/n}$ (resp. $\delta_{k/n}$; $\varepsilon_{k/n}$) instead of $B_{(k+1)/n} - B_{k/n}$ (resp. $\mathbf{1}_{[k/n,(k+1)/n]}$; $\mathbf{1}_{[0,k/n]}$). The aim of this section is to prove Theorem 1.2, or equivalently:

Theorem 4.1. *(Itô's formula) Let $f:\mathbb{R}\to\mathbb{R}$ verifying $(\mathbf{H}_9)$. Then*

$$\int_0^1 f'(B_s)d^\star B_s := \lim_{n\to\infty}\sum_{k=1}^{\lfloor n/2\rfloor} f'(B_{(2k-1)/n})\left(B_{(2k)/n}-B_{(2k-2)/n}\right) \quad \textit{exists in law}$$

and we have

$$\int_0^1 f'(B_s)d^\star B_s \overset{\text{Law}}{=} f(B_1)-f(0)-\frac{\kappa}{2}\int_0^1 f''(B_s)dW_s,$$

with κ defined by

$$\kappa = \sqrt{2+\sum_{r=1}^{\infty}(-1)^r\rho^2(r)} = 1,290\ldots. \tag{4.1}$$

and where W denotes a standard Brownian motion independent of B.

Proof. In [23], identity (1.6), it is proved that

$$\begin{aligned}&\sum_{k=1}^{\lfloor n/2\rfloor} f'(B_{(2k-1)/n})\left(B_{(2k)/n}-B_{(2k-2)/n}\right) \\ \approx\ & f(B_1)-f(0)-\frac{1}{2}\sum_{k=1}^{\lfloor n/2\rfloor} f''(B_{(2k-1)/n})\left[(\Delta B_{(2k-1)/n})^2-(\Delta B_{(2k-2)/n})^2\right]\end{aligned}$$

$$-\frac{1}{6}\sum_{j=1}^{\lfloor n/2\rfloor} f'''(B_{(2j-1)/n})\big[(\Delta B_{(2j-2)/n})^3+(\Delta B_{(2j-1)/n})^3\big]$$

where "$\approx$" means the difference goes to zero in L^2. Therefore, Theorem 4.1 is a direct consequence of Lemmas 4.2 and 4.3 below. □

Lemma 4.2. *Let $f:\mathbb{R}\to\mathbb{R}$ verifying $(\mathbf{H}_6)$. Then*

$$\sum_{j=1}^{\lfloor n/2\rfloor} f(B_{(2j-1)/n})\big[(\Delta B_{(2j-2)/n})^3+(\Delta B_{(2j-1)/n})^3\big] \xrightarrow[n\to\infty]{L^2} 0. \tag{4.2}$$

Proof. Let $H_3(x)=x^3-3x$ be the third Hermite polynomial. Using the relation between Hermite polynomial and multiple integral (see Section 2), remark that

$$\begin{aligned}(\Delta B_{(2j-2)/n})^3+(\Delta B_{(2j-1)/n})^3 &= n^{-\frac{3}{4}}\Big[H_3\big(n^{\frac14}\Delta B_{(2j-2)/n}\big)+H_3\big(n^{\frac14}\Delta B_{(2j-1)/n}\big)\\ &\qquad +\frac{3}{\sqrt{n}}\big(B_{(2j-2)/n}-B_{(2j)/n}\big)\Big]\\ &= I_3\big(\delta^{\otimes 3}_{(2j-2)/n}\big)+I_3\big(\delta^{\otimes 3}_{(2j-1)/n}\big)+\frac{3}{\sqrt{n}}\,I_1\big(\mathbf{1}_{[(2j-2)/n,(2j)/n]}\big)\end{aligned}$$

so that (4.2) can be shown by successively proving that

$$E\left|\frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor n/2\rfloor} f(B_{(2j-1)/n})\,I_1\big(\mathbf{1}_{[(2j-2)/n,(2j)/n]}\big)\right|^2 \underset{n\to+\infty}{\longrightarrow} 0; \tag{4.3}$$

$$E\left|\sum_{j=1}^{\lfloor n/2\rfloor} f(B_{(2j-1)/n})\,I_3\big(\delta^{\otimes 3}_{(2j-2)/n}\big)\right|^2 \underset{n\to+\infty}{\longrightarrow} 0; \tag{4.4}$$

$$E\left|\sum_{j=1}^{\lfloor n/2\rfloor} f(B_{(2j-1)/n})\,I_3\big(\delta^{\otimes 3}_{(2j-1)/n}\big)\right|^2 \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.5}$$

Let us first proceed with the proof of (4.3). We can write, using in particular (2.6):

$$\begin{aligned}&E\left|\frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor n/2\rfloor} f(B_{(2j-1)/n})\,I_1\big(\mathbf{1}_{[(2j-2)/n,(2j)/n]}\big)\right|^2\\ =\;&\frac{1}{n}\left|\sum_{j,k=1}^{\lfloor n/2\rfloor} E\big\{f(B_{(2j-1)/n})f(B_{(2k-1)/n})\,I_1\big(\mathbf{1}_{[(2j-2)/n,(2j)/n]}\big)\,I_1\big(\mathbf{1}_{[(2k-2)/n,(2k)/n]}\big)\big\}\right|\\ \le\;&\frac{1}{n}\sum_{j,k=1}^{\lfloor n/2\rfloor}\big|E\big\{f(B_{(2j-1)/n})f(B_{(2k-1)/n})\,I_2\big(\mathbf{1}_{[(2j-2)/n,(2j)/n]}\otimes\mathbf{1}_{[(2k-2)/n,(2k)/n]}\big)\big\}\big|\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{1}{n\sqrt{n}} \sum_{j,k=1}^{\lfloor n/2 \rfloor} \left| E\big\{ f(B_{(2j-1)/n}) f(B_{(2k-1)/n}) \rho(2j-2k) \big\} \right| \\
= \;& \frac{2}{n} \sum_{a+b=2} \sum_{j,k=1}^{\lfloor n/2 \rfloor} \left| E\big\{ f^{(a)}(B_{(2j-1)/n}) f^{(b)}(B_{(2k-1)/n}) \big\} \right| \\
&\qquad \times \left| \left\langle \varepsilon_{(2j-1)/n}^{\otimes a} \otimes \varepsilon_{(2k-1)/n}^{\otimes b}, \mathbf{1}_{[(2j-2)/n,(2j)/n]} \otimes \mathbf{1}_{[(2k-2)/n,(2k)/n]} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right| \\
&\quad + \frac{1}{n\sqrt{n}} \sum_{j,k=1}^{\lfloor n/2 \rfloor} \left| E\big\{ f(B_{(2j-1)/n}) f(B_{(2k-1)/n}) \rho(2j-2k) \big\} \right| .
\end{aligned}
$$

But, by Lemma 3.1 *(i)*, we have

$$
\begin{aligned}
& \left| \left\langle \varepsilon_{(2j-1)/n}^{\otimes a} \otimes \varepsilon_{(2k-1)/n}^{\otimes b}, \mathbf{1}_{[(2j-2)/n,(2j)/n]} \otimes \mathbf{1}_{[(2k-2)/n,(2k)/n]} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right| \\
\leq \;& \frac{1}{\sqrt{n}} \Big(\left| \left\langle \varepsilon_{(2j-1)/n}, \mathbf{1}_{[(2j-2)/n,(2j)/n]} \right\rangle_{\mathfrak{H}} \right| + \left| \left\langle \varepsilon_{(2k-1)/n}, \mathbf{1}_{[(2k-2)/n,(2k)/n]} \right\rangle_{\mathfrak{H}} \right| \Big) \\
= \;& \frac{1}{n} \big(\sqrt{2j} - \sqrt{2j-2} + \sqrt{2k} - \sqrt{2k-2} \big).
\end{aligned}
$$

Thus, under $(\mathbf{H}_6)$:

$$
\begin{aligned}
&\sum_{a+b=2} \sum_{j,k=1}^{\lfloor n/2 \rfloor} \left| E\big\{ f^{(a)}(B_{(2j-1)/n}) f^{(b)}(B_{(2k-1)/n}) \big\} \right| \\
&\qquad \times \left| \left\langle \varepsilon_{(2j-1)/n}^{\otimes a} \otimes \varepsilon_{(2k-1)/n}^{\otimes b}, \mathbf{1}_{[(2j-2)/n,(2j)/n]} \otimes \mathbf{1}_{[(2k-2)/n,(2k)/n]} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right| = O(\sqrt{n}).
\end{aligned}
$$

Moreover

$$
\sum_{j,k=1}^{\lfloor n/2 \rfloor} \left| E\big\{ f(B_{(2j-1)/n}) f(B_{(2k-1)/n}) \rho(2j-2k) \big\} \right| \leq C \sum_{j,k=1}^{\lfloor n/2 \rfloor} \left| \rho(2j-2k) \right| = O(n).
$$

Finally, convergence (4.3) holds.

Now, let us only proceed with the proof of (4.4), the proof of (4.5) being similar. We have

$$
\begin{aligned}
& E \left| \sum_{j=1}^{\lfloor n/2 \rfloor} f(B_{(2j-1)/n}) \, I_3\big(\delta_{(2j-2)/n}^{\otimes 3}\big) \right|^2 \\
= \;& \sum_{j,k=1}^{\lfloor n/2 \rfloor} E\big\{ f(B_{(2j-1)/n}) f(B_{(2k-1)/n}) I_3\big(\delta_{(2j-2)/n}^{\otimes 3}\big) I_3\big(\delta_{(2k-2)/n}^{\otimes 3}\big) \big\} \\
= \;& \sum_{r=0}^{3} r! \binom{3}{r}^2 n^{-\frac{3-r}{2}} \sum_{j,k=1}^{\lfloor n/2 \rfloor} E\big\{ f(B_{(2j-1)/n}) f(B_{(2k-1)/n}) I_{2r}\big(\delta_{(2j-2)/n}^{\otimes r} \otimes \delta_{(2k-2)/n}^{\otimes r}\big) \big\} \rho^{3-r}(2j-2k).
\end{aligned}
$$

To obtain (4.4), it is then sufficient to prove that, for every fixed $r \in \{0,1,2,3\}$, the quantities

$$R_n^{(r)} = n^{-\frac{3-r}{2}} \sum_{j,k=1}^{\lfloor n/2 \rfloor} E\big\{ f(B_{(2j-1)/n}) f(B_{(2k-1)/n}) I_{2r}\big(\delta^{\otimes r}_{(2j-2)/n} \otimes \delta^{\otimes r}_{(2k-2)/n}\big)\big\} \rho^{3-r}(2j-2k)$$

tend to zero as $n \to \infty$. We have, by Lemma 3.1 *(i)* and under $(\mathbf{H}_6)$:

$$\begin{aligned}
& \sup_{j,k=1,\ldots,\lfloor n/2 \rfloor} \Big| E\big\{ f(B_{(2j-1)/n}) f(B_{(2k-1)/n}) I_{2r}\big(\delta^{\otimes r}_{(2j-2)/n} \otimes \delta^{\otimes r}_{(2k-2)/n}\big)\big\} \Big| \\
= & \sup_{j,k=1,\ldots,\lfloor n/2 \rfloor} (2r)! \Bigg| \sum_{a+b=2r} E\Big\{ f^{(a)}(B_{(2j-1)/n}) f^{(b)}(B_{(2k-1)/n}) \Big\} \\
& \qquad \times \Big\langle \varepsilon^{\otimes a}_{(2j-1)/n} \widetilde{\otimes} \varepsilon^{\otimes b}_{(2j-1)/n}, \mathbf{1}^{\otimes r}_{[(2j-2)/n,(2j/n)]} \otimes \mathbf{1}^{\otimes r}_{[(2k-2)/n,(2k/n)]} \Big\rangle_{\mathfrak{H}^{\otimes 2}} \Bigg| \\
\leq & C \sup_{j,k=1,\ldots,\lfloor n/2 \rfloor} \sup_{a+b=2r} \Big| \Big\langle \varepsilon^{\otimes a}_{(2j-1)/n} \widetilde{\otimes} \varepsilon^{\otimes b}_{(2j-1)/n}, \mathbf{1}^{\otimes r}_{[(2j-2)/n,(2j/n)]} \otimes \mathbf{1}^{\otimes r}_{[(2k-2)/n,(2k/n)]} \Big\rangle_{\mathfrak{H}^{\otimes 2}} \Big| \\
= & O(n^{-r}).
\end{aligned}$$

Consequently, when $r \neq 3$, we deduce

$$\big| R_n^{(r)} \big| \leq C\, n^{-\frac{r+3}{2}} \sum_{j,k=1}^{\lfloor n/2 \rfloor} \big| \rho(2j-2k) \big| = O(n^{-\frac{r+1}{2}}) \underset{n \to +\infty}{\longrightarrow} 0$$

while, when $r = 3$, we deduce

$$\big| R_n^{(3)} \big| \leq C\, n^{-1} \underset{n \to +\infty}{\longrightarrow} 0.$$

The proof of (4.4) is done. Since the proof of (4.5) follows the same lines, we finally proved (4.2). □

Lemma 4.3. *Let $f : \mathbb{R} \to \mathbb{R}$ verifying $(\mathbf{H}_4)$. Set*

$$F_n = \sum_{k=1}^{\lfloor n/2 \rfloor} f(B_{(2k-1)/n}) \big[(\Delta B_{(2k-1)/n})^2 - (\Delta B_{(2k-2)/n})^2 \big].$$

Then

$$F_n \quad \xrightarrow[n \to \infty]{\text{stably}} \quad \kappa \int_0^1 f(B_s) dW_s, \tag{4.6}$$

with κ defined by (4.1), and where W denotes a standard Brownian motion independent of B. Here, the stable convergence (4.6) has to be understood in the following sense: for any real number λ and any $\sigma\{B\}$-measurable and integrable random variable ξ, we have that

$$E\left(e^{i\lambda F_n}\, \xi \right) \underset{n \to \infty}{\longrightarrow} E\left(e^{-\frac{\lambda^2 \kappa^2}{2} \int_0^1 f^2(B_s) ds}\, \xi \right).$$

Proof. Since we follow exactly the proof of Theorem 3.2, we only describe the main ideas. First, observe that

$$F_n = \sum_{k=1}^{\lfloor n/2 \rfloor} f\left(B_{(2k-1)/n}\right)\left(I_2(\delta^{\otimes 2}_{(2k-1)/n}) - I_2(\delta^{\otimes 2}_{(2k-2)/n})\right).$$

Here, the analogue of Lemma 3.1 is:

$$\sup_{t\in[0,1]} \sum_{k=1}^{\lfloor n/2 \rfloor} \left|\left\langle \varepsilon_t, \delta_{(2k-1)/n}\right\rangle_{\mathfrak{H}}\right| \underset{n\to\infty}{=} O(1), \quad \sup_{t\in[0,1]} \sum_{k=1}^{\lfloor n/2 \rfloor} \left|\left\langle \varepsilon_t, \delta_{(2k-2)/n}\right\rangle_{\mathfrak{H}}\right| \underset{n\to\infty}{=} O(1), \tag{4.7}$$

$$\left|\left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-1)/n}\right\rangle^2_{\mathfrak{H}} - \frac{1}{4n}\right| \leq \frac{\sqrt{2k} - \sqrt{2k-1}}{4n} \tag{4.8}$$

and

$$\left|\left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-2)/n}\right\rangle^2_{\mathfrak{H}} - \frac{1}{4n}\right| \leq \frac{\sqrt{2k-1} - \sqrt{2k-2}}{2n}. \tag{4.9}$$

In fact, the bounds (4.7) are obtained by following the arguments presented in the proof of Lemma 3.1. The only difference is that, in order to bound sums of the type $\sum_{k=1}^{\lfloor n/2 \rfloor} \sqrt{2k} - \sqrt{2k-1}$ (which are no more telescopic), we use

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \sqrt{2k} - \sqrt{2k-1} \leq \sum_{k=1}^{\lfloor n/2 \rfloor} \sqrt{2k} - \sqrt{2k-2} = \sqrt{2\lfloor n/2 \rfloor} \leq \sqrt{n}.$$

As in Step 1 of the proof of Theorem 3.2, here we also have that (F_n) is bounded in L^2. Consequently the sequence $\left(F_n, (B_t)_{t\in[0,1]}\right)$ is tight in $\mathbb{R} \times \mathscr{C}([0,1])$. Assume that $\left(F_\infty, (B_t)_{t\in[0,1]}\right)$ denotes the limit in law of a certain subsequence of $\left(F_n, (B_t)_{t\in[0,1]}\right)$, denoted again by $\left(F_n, (B_t)_{t\in[0,1]}\right)$.

We have to prove that

$$E\left(e^{i\lambda F_\infty} | \, (B_t)_{t\in[0,1]}\right) = \exp\left\{-\frac{\lambda^2}{2}\kappa^2 \int_0^1 f^2(B_s)ds\right\}. \tag{4.10}$$

We proceed as in Step 2 of the proof of Theorem 3.2. That is, (4.10) will be obtained by showing that for every random variable ξ of the form (2.4) and every real number λ, we have

$$\lim_{n\to\infty} \phi_n'(\lambda) = -\lambda\kappa^2 \, E\left(e^{i\lambda F_\infty}\xi \int_0^1 f^2(B_s)ds\right)$$

where

$$\phi_n'(\lambda) := \frac{d}{d\lambda} E\left(e^{i\lambda F_n}\xi\right) = iE\left(F_n e^{i\lambda F_n}\xi\right), \quad n \geq 1.$$

By the duality formula (2.5) we have that

$$\phi_n'(\lambda) = \sum_{k=1}^{\lfloor n/2 \rfloor} E\left(\left\langle D^2\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\xi\right), \delta^{\otimes 2}_{(2k-1)/n} - \delta^{\otimes 2}_{(2k-2)/n}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right).$$

The analogue of (3.8) is here:

$$\begin{aligned}
&\left\langle E\left(D^2\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\xi\right)\right), \delta^{\otimes 2}_{(2k-1)/n} - \delta^{\otimes 2}_{(2k-2)/n}\right\rangle_{\mathfrak{H}^{\otimes 2}} \\
= \;& E\left(f''(B_{(2k-1)/n})e^{i\lambda F_n}\xi\right)\left[\left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-1)/n}\right\rangle^2_{\mathfrak{H}} - \left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-2)/n}\right\rangle^2_{\mathfrak{H}}\right] \\
&+2i\lambda E\left(f'(B_{(2k-1)/n})e^{i\lambda F_n}\xi\left\langle DF_n, \delta_{(2k-1)/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-1)/n}\right\rangle_{\mathfrak{H}} \\
&-2i\lambda E\left(f'(B_{(2k-1)/n})e^{i\lambda F_n}\xi\left\langle DF_n, \delta_{(2k-2)/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-2)/n}\right\rangle_{\mathfrak{H}} \\
&+2E\left(f'(B_{(2k-1)/n})e^{i\lambda F_n}\left\langle D\xi, \delta_{(2k-1)/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-1)/n}\right\rangle_{\mathfrak{H}} \\
&-2E\left(f'(B_{(2k-1)/n})e^{i\lambda F_n}\left\langle D\xi, \delta_{(2k-2)/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-2)/n}\right\rangle_{\mathfrak{H}} \\
&-\lambda^2 E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\xi\left\langle DF_n, \delta_{(2k-1)/n}\right\rangle^2_{\mathfrak{H}}\right) \\
&+\lambda^2 E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\xi\left\langle DF_n, \delta_{(2k-2)/n}\right\rangle^2_{\mathfrak{H}}\right) \\
&+2i\lambda E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\left\langle D\xi, \delta_{(2k-1)/n}\right\rangle_{\mathfrak{H}}\left\langle DF_n, \delta_{(2k-1)/n}\right\rangle_{\mathfrak{H}}\right) \\
&-2i\lambda E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\left\langle D\xi, \delta_{(2k-2)/n}\right\rangle_{\mathfrak{H}}\left\langle DF_n, \delta_{(2k-2)/n}\right\rangle_{\mathfrak{H}}\right) \\
&+i\lambda E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\xi\left\langle D^2F_n, \delta^{\otimes 2}_{(2k-1)/n} - \delta^{\otimes 2}_{(2k-2)/n}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right) \\
&+E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\left\langle D^2\xi, \delta^{\otimes 2}_{(2k-1)/n} - \delta^{\otimes 2}_{(2k-2)/n}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\phi_n'(\lambda) \;=\; & -2\lambda\sum_{k,l=0}^{n-1} E\left(f(B_{(2k-1)/n})f(B_{(2l-1)/n})e^{i\lambda F_n}\xi\right) \\
& \times\left\langle \delta^{\otimes 2}_{(2l-1)/n} - \delta^{\otimes 2}_{(2l-2)/n}, \delta^{\otimes 2}_{(2k-1)/n} - \delta^{\otimes 2}_{(2k-2)/n}\right\rangle_{\mathfrak{H}^{\otimes 2}} + i\sum_{k=0}^{n-1} r_{k,n} \qquad (4.11)
\end{aligned}$$

where $r_{k,n}$ is given by

$$\begin{aligned}
r_{k,n} \;=\; & E[f''(B_{(2k-1)/n})e^{i\lambda F_n}\xi]\left[\left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-1)/n}\right\rangle^2_{\mathfrak{H}} - \left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-2)/n}\right\rangle^2_{\mathfrak{H}}\right] \\
&+2i\lambda E\left(f'(B_{(2k-1)/n})e^{i\lambda F_n}\xi\left\langle DF_n, \delta_{(2k-1)/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-1)/n}\right\rangle_{\mathfrak{H}} \\
&-2i\lambda E\left(f'(B_{(2k-1)/n})e^{i\lambda F_n}\xi\left\langle DF_n, \delta_{(2k-2)/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-2)/n}\right\rangle_{\mathfrak{H}} \\
&+2E\left(f'(B_{(2k-1)/n})e^{i\lambda F_n}\left\langle D\xi, \delta_{(2k-1)/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-1)/n}\right\rangle_{\mathfrak{H}} \\
&-2E\left(f'(B_{(2k-1)/n})e^{i\lambda F_n}\left\langle D\xi, \delta_{(2k-2)/n}\right\rangle_{\mathfrak{H}}\right)\left\langle \varepsilon_{(2k-1)/n}, \delta_{(2k-2)/n}\right\rangle_{\mathfrak{H}} \\
&-\lambda^2 E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\xi\left\langle DF_n, \delta_{(2k-1)/n}\right\rangle^2_{\mathfrak{H}}\right) \\
&+\lambda^2 E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\xi\left\langle DF_n, \delta_{(2k-2)/n}\right\rangle^2_{\mathfrak{H}}\right)
\end{aligned}$$

$$
\begin{aligned}
&+2i\lambda E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\ \left\langle D\xi,\delta_{(2k-1)/n}\right\rangle_{\mathfrak{H}}\left\langle DF_n,\delta_{(2k-1)/n}\right\rangle_{\mathfrak{H}}\right)\\
&-2i\lambda E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\ \left\langle D\xi,\delta_{(2k-2)/n}\right\rangle_{\mathfrak{H}}\left\langle DF_n,\delta_{(2k-2)/n}\right\rangle_{\mathfrak{H}}\right)\\
&+E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\left\langle D^2\xi,\delta^{\otimes 2}_{(2k-1)/n}-\delta^{\otimes 2}_{(2k-2)/n}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right)\\
&+i\lambda\sum_{l=1}^{\lfloor n/2\rfloor}E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\xi\ \ f''(B_{(2l-1)/n})(I_2(\delta^{\otimes 2}_{(2l-1)/n})-I_2(\delta^{\otimes 2}_{(2l-2)/n})\ \right)\\
&\qquad\qquad\times\left\langle\varepsilon^{\otimes 2}_{(2l-1)/n},\delta^{\otimes 2}_{(2k-1)/n}-\delta^{\otimes 2}_{(2k-2)/n}\right\rangle_{\mathfrak{H}^{\otimes 2}}\\
&+4i\lambda\sum_{l=1}^{\lfloor n/2\rfloor}E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\xi f'(B_{(2l-1)/n})\Delta B_{(2l-1)/n}\right)\\
&\qquad\qquad\times\left\langle\varepsilon_{(2l-1)/n}\widetilde{\otimes}\delta_{(2l-1)/n},\delta^{\otimes 2}_{(2k-1)/n}-\delta^{\otimes 2}_{(2k-2)/n}\right\rangle_{\mathfrak{H}^{\otimes 2}}\\
&-4i\lambda\sum_{l=1}^{\lfloor n/2\rfloor}E\left(f(B_{(2k-1)/n})e^{i\lambda F_n}\xi f'(B_{(2l-1)/n})\Delta B_{(2l-2)/n}\right)\\
&\qquad\qquad\times\left\langle\varepsilon_{(2l-2)/n}\widetilde{\otimes}\delta_{(2l-2)/n},\delta^{\otimes 2}_{(2k-1)/n}-\delta^{\otimes 2}_{(2k-2)/n}\right\rangle_{\mathfrak{H}^{\otimes 2}}\\
=&\ \sum_{j=1}^{13}R^j_{k,n}. \qquad (4.12)
\end{aligned}
$$

The only difference with respect to (3.12) is that, this time, the term

$$
i\sum_{k=0}^{n-1}E[f''(B_{(2k-1)/n})e^{i\lambda F_n}\xi]\left[\left\langle\varepsilon_{(2k-1)/n},\delta_{(2k-1)/n}\right\rangle^2_{\mathfrak{H}}-\left\langle\varepsilon_{(2k-1)/n},\delta_{(2k-2)/n}\right\rangle^2_{\mathfrak{H}}\right]
$$

corresponding to (3.15) is negligible. Indeed, we can write

$$
\begin{aligned}
&\sum_{k=0}^{n-1}E[f''(B_{(2k-1)/n})e^{i\lambda F_n}\xi]\left[\left\langle\varepsilon_{(2k-1)/n},\delta_{(2k-1)/n}\right\rangle^2_{\mathfrak{H}}-\left\langle\varepsilon_{(2k-1)/n},\delta_{(2k-2)/n}\right\rangle^2_{\mathfrak{H}}\right]\\
=&\ \sum_{k=0}^{n-1}E\left(f''(B_{(2k-1)/n})e^{i\lambda F_n}\xi\right)\left[\left\langle\varepsilon_{(2k-1)/n},\delta_{(2k-1)/n}\right\rangle^2_{\mathfrak{H}}-\frac{1}{4n}\right]\\
&\qquad-\sum_{k=0}^{n-1}E\left(f''(B_{(2k-1)/n})e^{i\lambda F_n}\xi\right)\left[\left\langle\varepsilon_{(2k-1)/n},\delta_{(2k-2)/n}\right\rangle^2_{\mathfrak{H}}-\frac{1}{4n}\right]\\
\underset{n\to\infty}{\longrightarrow}&\ 0\quad\text{by (4.8)-(4.9), under }(\mathbf{H}_4).
\end{aligned}
$$

Moreover, exactly as in the proof of Theorem 3.2, we can show that $\lim_{n\to\infty}\sum_{k=1}^{\lfloor n/2\rfloor}r_{k,n}=0$. Consequently, we have

$$
\lim_{n\to\infty}\phi_n'(\lambda)
$$

$$
\begin{aligned}
&= -2\lambda \lim_{n\to\infty} \sum_{k,l=1}^{\lfloor n/2\rfloor} E\left(f(B_{(2k-1)/n}) f(B_{(2l-1)/n}) e^{i\lambda F_n}\xi\right) \left\langle \delta^{\otimes 2}_{(2l-1)/n} - \delta^{\otimes 2}_{(2l-2)/n}, \delta^{\otimes 2}_{(2k-1)/n} - \delta^{\otimes 2}_{(2k-2)/n} \right\rangle_{\mathfrak{H}^{\otimes 2}} \\
&= -\frac{\lambda}{2} \lim_{n\to\infty} \frac{1}{n} \sum_{k,l=1}^{\lfloor n/2\rfloor} E\left(f(B_{(2k-1)/n}) f(B_{(2l-1)/n}) e^{i\lambda F_n}\xi\right) \\
&\qquad\qquad \times \big(2\rho^2(2k-2l) - \rho^2(2l-2k+1) - \rho^2(2l-2k-1)\big) \\
&= -\frac{\lambda}{4} \sum_{r=-\infty}^{\infty} \big(2\,\rho^2(2r) - \rho^2(2r+1) - \rho^2(2r-1)\big) \\
&\qquad\qquad \times \lim_{n\to\infty} \frac{2}{n} \sum_{k=1\vee(1-r)}^{\lfloor n/2\rfloor \wedge (\lfloor n/2\rfloor - r)} E\left(f(B_{(2k-1)/n}) f(B_{(2k-1-2r)/n}) e^{i\lambda F_n}\xi\right) \\
&= -\lambda\,\kappa^2 \int_0^1 E\left(f^2(B_s) e^{i\lambda F_\infty}\xi\right) ds,
\end{aligned}
$$

where κ is defined by (4.1). In other words, (4.10) is shown and the proof of Lemma 4.3 is done. □

Acknowledgments

Some of our computations are inspired by the first draft of [15]. We are grateful to David Nualart for letting us use them freely.

References

[1] P. Breuer and P. Major (1983): *Central limit theorems for nonlinear functionals of Gaussian fields.* J. Multivariate Anal. **13** (3), 425-441.

[2] P. Cheridito and D. Nualart (2005): *Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter H in $(0, 1/2)$.* Ann. Inst. H. Poincaré Probab. Statist. **41**, 1049-1081.

[3] D. Dürr, S. Goldstein and J.L. Lebowitz (1985): *Asymptotics of particle trajectories in infinite one-dimensional systems with collisions.* Comm. Pure Appl. Math. **38**, 573-597.

[4] R.L. Dobrushin and P. Major (1979): *Non-central limit theorems for nonlinear functionals of Gaussian fields.* Z. Wahrsch. verw. Gebiete **50**, 27-52.

[5] L. Giraitis and D. Surgailis (1985): *CLT and other limit theorems for functionals of Gaussian processes.* Z. Wahrsch. verw. Gebiete **70**, 191-212.

[6] M. Gradinaru and I. Nourdin (2007): *Convergence of weighted power variations of fractional Brownian motion.* Preprint.

[7] M. Gradinaru, I. Nourdin, F. Russo and P. Vallois (2005): *m-order integrals and Itô's formula for non-semimartingale processes; the case of a fractional Brownian motion with any Hurst index.* Ann. Inst. H. Poincaré Probab. Statist. **41**, 781-806.

[8] M. Gradinaru, F. Russo and P. Vallois (2001): *Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index* $H \geq \frac{1}{4}$. Ann. Probab. **31**, 1772-1820.

[9] C. Houdré and J. Villa (2003): *An Example of Infinite Dimensional Quasi-helix Stochastic Models.* Contemp. Math. **336**, 195-201.

[10] T.E. Harris (1965): *Diffusions with collisions between particles.* J. Appl. Probab. **2**, 323-338.

[11] P. Lei and D. Nualart (2008): *A decomposition of the bifractional Brownian motioin and some applications.* Preprint.

[12] A. Neuenkirch and I. Nourdin (2007): *Exact rate of convergence of some approximation schemes associated to SDEs driven by a fractional Brownian motion.* J. Theor. Probab. **20**, no. 4, 871-899.

[13] I. Nourdin (2008): *A simple theory for the study of SDEs driven by a fractional Brownian motion, in dimension one.* Séminaire de Probabilités **XLI**, to appear.

[14] I. Nourdin (2008): *Asymptotic behavior of weighted quadratic and cubic variations of fractional Brownian motion.* Ann. Probab, to appear.

[15] I. Nourdin and D. Nualart (2008): *Central limit theorems for multiple Skorohod integrals.* Preprint.

[16] I. Nourdin, D. Nualart and C.A. Tudor (2007). *Central and non-central limit theorems for weighted power variations of fractional Brownian motion.* Preprint.

[17] I. Nourdin and G. Peccati (2007). *Weighted power variations of iterated Brownian motion.* Preprint.

[18] D. Nualart (2003): *Stochastic calculus with respect to the fractional Brownian motion and applications.* Contemp. Math. **336**, 3-39.

[19] D. Nualart (2006): *The Malliavin calculus and related topics of Probability and Its Applications.* Springer Verlag, Berlin, Second edition, 2006.

[20] A. Réveillac (2007): *Estimation of quadratic variation for two-parameter diffusions.* Preprint.

[21] A. Réveillac (2007): *Convergence of finite-dimensional laws of the weighted quadratic variations process for some fractional Brownian sheets.* Preprint.

[22] F. Russo and C.A. Tudor (2006): *On the bifractional Brownian motion.* Stoch. Proc. Appl. **116**, no. 5, 830-856.

[23] J. Swanson (2007): *Variation of the solution to a stochastic heat equation.* Ann. Probab. **35**, no. 6, 2122-2159.

[24] M. Taqqu (1979): *Convergence of integrated processes of arbitrary Hermite rank.* Z. Wahrsch. verw. Gebiete **50**, 53-83.