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# MATHEMATICAL STUDY OF RESONANT WIND-DRIVEN OCEANIC MOTIONS

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ABSTRACT. We are interested here in describing the linear response of the ocean to some wind forcing, which admits fast time oscillations and may be resonant with the Coriolis force. In addition to the usual Ekman layer, we exhibit another - much larger - boundary layer, and some global vertical profile. That means in particular that the wind effect is no longer localized in the vicinity of the surface.

From a mathematical point of view, the main novelty here is to introduce some systematic approach for the study of boundary effects.

The goal of this paper is to understand the influence of a wind - depending on time - on the oceanic circulation. More precisely, we are interested in the effects of a resonant forcing, i.e. of a wind oscillating with the same period as the rotation of the Earth.

In the non-resonant case, the works by Desjardins and Grenier [4] then by Masmoudi [14] show that the wind forcing creates essentially some boundary layer in the vicinity of the surface, which contributes to the mean motion by a source term, known as the Ekman pumping. For a precise description of the method leading to such convergence results, we refer to the book [2] by Chemin, Desjardins, Gallagher and Grenier.

Here the situation is much more complicated since the resonant part of the wind will be proved to generate another boundary layer with a different typical size, and overall to destabilize the whole fluid with the apparition of a vertical profile. We give here a precise description of these (linear) effects of the Coriolis force in presence of resonant wind.

## 1. DESCRIPTION OF THE WIND-DRIVEN CIRCULATION

Let us first present the mathematical framework of our study and state precisely our main results.

### 1.1. A Navier-Stokes model with suitable boundary conditions.

• Our starting point is the **homogeneous incompressible Navier-Stokes system in a rotating frame**

$$(1.1) \quad \begin{aligned} \partial_t u + (u \cdot \nabla)u + \nabla p &= \mathcal{F} + u \wedge \Omega, \\ \nabla \cdot u &= 0, \end{aligned}$$

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where  $\mathcal{F}$  denotes the frictional force acting on the fluid,  $\Omega$  is the (vertical component of the) Earth rotation vector, and  $p$  is the pressure defined as the Lagrange multiplier associated with the incompressibility constraint.

We assume that the movement to be studied occurs at midlatitudes. At such latitudes, the Coriolis acceleration can in a crude approximation be considered as a constant. In other words, we can **neglect the variations of the Coriolis parameter**  $\Omega$  and use the  $f$ -plane approximation, which makes the analysis much simpler than in the case of the full model.

The observed persistence over several days of large-scale waves in the oceans shows that **frictional forces**  $\mathcal{F}$  are weak, almost everywhere, when compared with the Coriolis acceleration and the pressure gradient, but large when compared with the kinematic viscous dissipation of water. One common but not very precise notion is that small-scale motions, which appear sporadic or on longer time scales, act to smooth and mix properties on the larger scales by processes analogous to molecular, diffusive transports. For the present purposes it is only necessary to note that one way to estimate the dissipative influence of smaller-scale motions is to retain the same representation of the frictional force

$$\mathcal{F} = A_h \Delta_h u + A_z \partial_{zz} u$$

where  $A_z$  and  $A_h$  are respectively the vertical and horizontal turbulent viscosities, of much larger magnitude than the molecular value, supposedly because of the greater efficiency of momentum transport by macroscopic chunks of fluid. Notice that  $A_z \neq A_h$  is therefore natural in geophysical framework (see [16]).

This paper is thus devoted to the analysis of the so-called “rotating fluid equations”, consisting in the three-dimensional anisotropic Navier-Stokes system in which a constant coefficient penalization operator has been added to account for the Earth rotation.

- We consider the motion **in a bounded domain**

$$\omega = \omega_h \times [0, 1]$$

where the bottom and upper surface of the ocean are assumed to be flat at  $z = 0$  and  $z = 1$  and where  $\omega_h$  is the horizontal domain to be studied.

As boundary conditions on the upper surface, we enforce

$$(1.2) \quad \begin{aligned} u_3|_{z=1} &= 0, \\ \partial_z u_h|_{z=1} &= \Sigma, \end{aligned}$$

where  $\Sigma$  is a given stress tensor, describing the **wind on the surface of the ocean**. Notice that (1.2) is a rigid lid approximation since we assume that the upper surface of the ocean remains at  $z = 1$ . This is a drastic, but standard simplification, since it is actually a free surface, and a moving interface between air and water, which has its own self consistent motion. The justification of (1.2) starting from a free surface is open from a mathematical point of view. Nevertheless, from a physical point of view, the simplification

does not appear so dramatic, since in any case the free surface is so turbulent with waves and foam, that only modelization is tractable and meaningful. Condition (1.2) is a simple modelization which already catches most of the physical phenomena (see [16]).

At the bottom we use the Dirichlet boundary condition

$$(1.3) \quad u|_{z=0} = 0,$$

which is a **braking condition at the interface ocean/earth's crust**. Note that, for the sake of simplicity, we do not take into account here the topography of the bottom. The topographic effects should actually modify the Ekman boundary layer and consequently the limit equations, even if the variations of the bottom are small (see [4] and [8] for instance).

In order to simplify the study and focus on the wind influence, we also **neglect the effects of the lateral boundary conditions** by considering the case when  $\omega_h$  is the two-dimensional torus. Of course such an assumption is not physically relevant. It is well known for instance that the lateral boundary layers, called Munk layers, play a crucial role in the oceanic circulation, in particular in the western intensification of currents.

## 1.2. The dominating influence of the Coriolis force : a formal asymptotics.

- We will study the following asymptotic behaviour

$$(1.4) \quad \begin{aligned} \frac{U}{L\Omega} &= \epsilon \rightarrow 0, \\ \frac{A_h}{\rho UL} &\sim 1, \quad \frac{LA_z}{\rho UD^2} = \nu \rightarrow 0 \end{aligned}$$

where  $L$  and  $D$  are the typical horizontal and vertical length scales, and  $\rho$  denotes the (constant) density of the fluid. In other words, we will consider the following system

$$(1.5) \quad \begin{aligned} \partial_t u + (u \cdot \nabla)u + \frac{1}{\epsilon} e_3 \wedge u - \Delta_h u - \nu \partial_{zz} u + \nabla p &= 0 \\ \nabla \cdot u &= 0, \end{aligned}$$

Such a **scaling of parameters** seems convenient for instance for the mesoscale eddies that have been observed in western Atlantic. One has indeed

$$U \sim 5 \text{ cm/s}, \quad L \sim 100 \text{ km}, \quad D \sim 4 \text{ km} \quad \text{and} \quad \Omega \sim 10^{-4} \text{ s}^{-1}$$

which leads to the following value for  $\epsilon$

$$\epsilon = 5 \times 10^{-3}.$$

Possible values for the turbulent viscosities given in [16] are

$$A_h \sim 10^7 \text{ cm}^2/\text{s} \quad \text{and} \quad A_z \sim 10 \text{ cm}^2/\text{s}$$

so that

$$\nu = 10^{-3}.$$

As mentioned above, we will supplement this equation with the following boundary condition

$$\partial_z u|_{z=1} = \beta \sigma,$$

where  $\beta$  is the typical size of the scaled forcing and  $\sigma$  is of order one.

- The first step in the study of rotating fluids is to verify that the only way to control the Coriolis force as  $\epsilon \rightarrow 0$  is to balance it with the pressure gradient term. Hence in the limit,  $e_3 \wedge u$  must be a gradient

$$e_3 \wedge \bar{u}_{mean}^{int} = -\nabla p$$

which leads to

$$\bar{u}_{mean}^{int} = \nabla_h^\perp p$$

where the limit pressure and thus the limit velocity are independent of  $z$ . In particular,  $\bar{u}_{mean}^{int}$  is a two-dimensional, horizontal, divergence-free vector-field. The fluid being limited by rigid boundaries, from above and below, the divergence-free condition leads indeed to  $u_3 = 0$  (at least to first order in  $\epsilon$ ). In other words, all the particles which have the same  $x_h$  have the same velocity. The particles of fluid move in vertical columns, called **Taylor-Proudman columns**. That is the main effect of rotation and a very strong constraint on the fluid motion.

As the domain evolution is limited by two parallel planes, the height of Taylor-Proudman columns is constant as time evolves, which is compatible with the incompressibility constraint. We can then prove that the columns move freely and in the limit of high rotation the fluid behaves like a **two-dimensional incompressible fluid**. Integrating the motion equation (1.5) with respect to  $z$  and taking formal limits as  $\epsilon \rightarrow 0$  leads indeed to

$$(1.6) \quad \begin{aligned} \partial_t \bar{u}_{mean}^{int} + (\bar{u}_{mean}^{int} \cdot \nabla_h) \bar{u}_{mean}^{int} + \nabla_h p &= \Delta_h \bar{u}_{mean}^{int}, \\ \nabla_h \cdot \bar{u}_{mean}^{int} &= 0. \end{aligned}$$

Note however that on the boundary of the domain, where the velocity is prescribed, the  $z$  independence is violated. That leads to **vertical boundary layers** modifying the limit equation (1.6), which will be investigated in the rest of the paper.

- Before starting with the precise study of these boundary layers, let us now describe what happens for the three-dimensional part of the initial data, i.e. the part of the initial data that does not satisfy the geostrophic constraint. The dominant process is then governed by the linear Coriolis operator

$$L : u \in V_0 \mapsto \mathbb{P}(e_3 \wedge u) \in V_0,$$

where  $V_0$  denotes the subspace of  $L^2(\omega)$  of divergence-free vector fields having zero flux both through the bottom and through the surface

$$V_0 = \{u \in L^2([0, 1] \times \mathbf{T}^2) \ / \ \nabla \cdot u = 0 \text{ and } u_3|_{z=0} = u_3|_{z=1} = 0\},$$

and  $\mathbb{P}$  denotes the orthogonal projection onto  $V_0$  in  $L^2(\omega)$ . Notice that in general,  $\mathbb{P}$  is different from the Leray projector on divergence-free vector fields in  $L^2$ .

The equation

$$\epsilon \partial_t u + Lu = 0$$

turns out to describe the propagation of waves, called **Poincaré waves**, with the following dispersion law relating the pulsation  $\lambda_k$  to the wavenumber  $k$

$$\lambda_k = -\frac{k_3 \pi}{\sqrt{|k_h|^2 + (\pi k_3)^2}}.$$

For the sake of completeness, we recall here - essentially without proof - some fundamental properties of the Coriolis operator  $L$ . For a detailed study of these spectral properties we refer for instance to [2].

Extending any  $u \in V_0$  on  $[-1, 1] \times \mathbf{T}^2$  as follows

$$(1.7) \quad u_h(x_h, z) = u_h(x_h, -z) \quad \text{and} \quad u_3(x_h, z) = -u_3(x_h, -z)$$

(which is compatible with the incompressibility constraint  $\nabla \cdot u = 0$ ) we obtain a periodic function, so that it is possible to use some Fourier decomposition.

Setting

$$\begin{cases} n_1(k) = \frac{1}{2\pi|k_h|}(ik_2 + k_1\lambda_k) \\ n_2(k) = \frac{1}{2\pi|k_h|}(-ik_1 + k_2\lambda_k) & \text{if } k_h \neq 0, \\ n_3(k) = i\frac{|k_h|}{2\pi\sqrt{|k_h|^2 + (\pi k_3)^2}} \end{cases}$$

and

$$\begin{cases} n_1(k) = \frac{\text{sign}(k_3)}{2\pi} \\ n_2(k) = \frac{i}{2\pi} \\ n_3(k) = 0 \end{cases} \quad \text{else,}$$

what can be proved actually is that the family  $(N_k)$  defined by

$$N_k = \exp(ik_h \cdot x_h) \begin{pmatrix} n_1(k) \cos(\pi k_3 z) \\ n_2(k) \cos(\pi k_3 z) \\ n_3(k) \sin(\pi k_3 z) \end{pmatrix}$$

is an hilbertian basis of  $V_0$  constituted of eigenvectors of the linear penalization

$$(1.8) \quad \mathbb{P}(e_3 \wedge N_k) = i\lambda_k N_k \quad \text{with} \quad \lambda_k = -\frac{k_3 \pi}{\sqrt{|k_h|^2 + (\pi k_3)^2}}.$$

Thus the three-dimensional part of the initial data generates waves, which propagate very rapidly in the domain (with a speed of order  $\epsilon^{-1}$ ). The time average of these waves vanish, like their weak limit, but they carry a non-zero energy.

To handle with waves we therefore use some filtering method introduced independently by Grenier [10] and Schochet [20]. Conjugating (1.5) by the Poincaré group leads to

$$\partial_t \left( \exp \left( \frac{tL}{\epsilon} \right) u \right) + \exp \left( \frac{tL}{\epsilon} \right) (Q(u, u) - \Delta_h u - \nu \partial_{zz} u) = 0,$$

or equivalently, setting  $u_L = \exp \left( \frac{tL}{\epsilon} \right) u$ ,

$$\partial_t u_L - \Delta_h u_L - \nu \partial_{zz} u_L + \exp \left( \frac{tL}{\epsilon} \right) Q \left( \exp \left( -\frac{tL}{\epsilon} \right) u_L, \exp \left( -\frac{tL}{\epsilon} \right) u_L \right) = 0.$$

Note that  $u$  can be retrieved from  $u_L$  thanks to the formula

$$u = \exp \left( -\frac{tL}{\epsilon} \right) u_L = \sum \langle N_k | u_L \rangle \exp \left( -i \frac{t}{\epsilon} \lambda_k \right) N_k.$$

The first order approximation, i.e. **the envelope equation**, is then obtained by taking limits in that filtered system :

$$(1.9) \quad \partial_t \bar{u}_L^{int} - \Delta_h \bar{u}_L^{int} + \tilde{Q}(\bar{u}_L^{int}, \bar{u}_L^{int}) = 0$$

where  $\tilde{Q}$  is defined as some projection of  $Q$  on the resonant modes of the linear penalization  $L$ . The oscillatory profile  $u_L^{int}$  thus satisfies a three-dimensional parabolic equation, with special properties because of the rare occurrence of wave interactions (see [1] for a detailed study of the limiting nonlinearity and the global wellposedness of the limit system.)

Note that the interaction between two Poincaré waves does not create  $z$  independent field, meaning that the limit  $z$  independent field  $\bar{u}_{mean}^{int}$  is completely decoupled from the waves (see [7] for a simple proof of this result based on compensated compactness).

## 2. GENERAL STRATEGY AND MAIN RESULTS

The goal of this paper is to study the influence of the wind forcing on the motion, in particular when this forcing has fast oscillations.

As usual in singular perturbation problems, the first step is to consider the linear penalization problem, or in other words study the waves produced by the Coriolis force. Our objective in this paper is to determine particular solutions satisfying the boundary conditions, and to understand if the effects of the forcing are local (in the vicinity of the surface) or global (in the whole fluid).

We are thus interested in the following linear system

$$(2.1) \quad \begin{aligned} \partial_t u + \frac{1}{\epsilon} \mathbb{P}(e_3 \wedge u) - \Delta_h u - \nu \partial_{zz} u &= 0, \\ \nabla \cdot u &= 0, \end{aligned}$$

supplemented with the boundary conditions

$$(2.2) \quad \begin{aligned} u|_{z=0} &= 0, \\ u_3|_{z=1} &= 0, \quad \partial_z u_h|_{z=1} = \beta \sigma, \end{aligned}$$

where  $\beta$  is a positive number and  $\sigma$  is some smooth function, periodic in  $x_h$  and almost periodic in  $t/\epsilon$ .

In order to simplify the presentation, we will assume in all the sequel that  $\sigma$  has only a finite number of Fourier modes in space and time, which is not a restriction in the linear case since one can use a superposition principle. We assume that the parameter  $\beta$  satisfies the following condition

$$(2.3) \quad \exists(\alpha_0, \alpha_1) \in \mathbf{R}^2, \quad \alpha_0 < 5/9, \quad \alpha_1 > 2/9, \quad \beta = O(\nu^{-\alpha_0} \epsilon^{\alpha_1}).$$

Our main result is the following:

**Theorem 2.1.** *Let  $\gamma \in L^2(\omega)$ . Let  $\sigma \in L^\infty(\mathbf{R}_+; L^2(\omega))$  such that*

$$\sigma(t, x_h) = \sum_{|k_h| \leq N} \sum_{\mu \in M} \hat{\sigma}(\mu, k_h) e^{i\mu \frac{t}{\epsilon}}$$

where  $M \subset \mathbf{R}$  is a finite set.

Let  $u$  be the solution of (2.1), (2.2). Assume that (2.3) is satisfied.

Then as  $\epsilon, \nu$  vanish, we have

$$u(t) - \left[ \exp\left(-\frac{t}{\epsilon} L\right) \bar{u}_L^{int} + u^{stat} \right] \rightarrow 0$$

in  $L_{loc}^\infty(\mathbf{R}_+, L^2(\omega))$ , where  $\bar{u}_L^{int}$  is the solution of the envelope equation (5.14), and  $u^{sing}$  is a singular profile, due to resonant forcing on the modes  $\mu = \pm 1$ ,  $k_h = 0$ ; by definition,  $u^{sing} = u^{sing,0} + u^{sing,1}$ , where  $u^{sing,0}$  (resp.  $u^{sing,1}$ ) is given by (5.20) (resp. (4.10)).

**Remark 2.2. (i)** *That result extends previous works by Masmoudi [14] and Chemin, Desjardins, Gallagher and Grenier [2]. They have indeed studied analogous boundary problems for rotating fluids, but have used in a crucial way a spectral assumption on the forcing modes, which ensures that the forcing is non-resonant, or in other words that the boundary layers remain stable.*

**(ii)** *If the horizontal viscosity, say  $\alpha$ , did also depend on  $\epsilon$ , it would change neither the method of proof, nor the nature of the convergence, at least in the linear case : we would just get a limit system with (possibly evanescent) horizontal viscosity  $\lim_{\epsilon \rightarrow 0} \alpha$ .*

*Indeed, as we consider some horizontal domain  $\omega_h$  without boundary and the linear operator has constant coefficients, there is no coupling between horizontal modes. We can then work with fixed  $k_h$ . The horizontal viscosity induces then some dissipation mechanism with rate  $\alpha|k_h|^2$ , the limit of which is trivial.*

*On the contrary, because of the boundary conditions at  $z = 0$  and  $z = 1$ , there is a strong coupling between the vertical modes. The Ekman pumping obtained in the asymptotics  $\epsilon \sim \nu$  is the expression of this coupling. It corresponds in some sense to a result of non commutation between the penalization  $\frac{1}{\epsilon} L$  and the vertical dissipation  $\nu \partial_{zz}$  supplemented with suitable*



boundary conditions at  $z = 0$  and  $z = 1$ . More precisely, it is obtained as the limit

$$\lim_{\substack{\epsilon, \nu \rightarrow 0 \\ \epsilon \sim \nu}} \left( e^{\frac{tL}{\epsilon}} \nu \partial_{zz} e^{-\frac{tL}{\epsilon}} \right).$$

(iii) If  $\nu = O(\epsilon)$ , then condition (2.3) can be relaxed into

$$\exists (\alpha_0, \alpha_1) \in \mathbf{R}^2, \quad \alpha_0 < 5/7, \quad \alpha_1 > 2/7, \quad \beta = O(\nu^{-\alpha_0} \epsilon^{\alpha_1}),$$

for technical reasons explained in the proof page 25.

The approximation of the function  $u$  obtained in the proof of Theorem 2.1 is actually much more precise than the mere function  $\bar{u}_L^{int}$ . Indeed, we will need to build boundary and corrector terms, which are all small in  $L^2$  norm, and thus do not play a role in the final convergence result, but are necessary in order that equation (2.1) is approximately satisfied.

• We have seen in Remark 2.2(ii) that, at least in the linear case, the horizontal viscosity is essentially a silent variable, in the sense that one has just to take limits in the dissipation factor  $\alpha|k_h|^2$ . Then, by a simple **change of time variable**, one can always come down to the case when the vertical viscosity is of the same order as the Rossby number. Setting

$$\tilde{u}(t, x) = u \left( \left( \frac{\epsilon}{\nu} \right)^{1/2} t, x \right)$$

we get indeed

$$\partial_t \tilde{u} + \frac{1}{(\epsilon\nu)^{1/2}} L \tilde{u} - (\epsilon\nu)^{1/2} \partial_{zz} \tilde{u} - \left( \frac{\epsilon}{\nu} \right)^{1/2} \Delta_h \tilde{u} = 0$$

Therefore the case when  $\nu \sim \epsilon$  is the most relevant. Formally both other cases can be deduced from this one, by considering either the asymptotics  $t \rightarrow \infty$  (case  $\epsilon \ll \nu$ ), or the asymptotics  $t \rightarrow 0$  (case  $\epsilon \gg \nu$ ). That means that, in the case when  $\epsilon \gg \nu$ , the effects of the boundary terms, even damped by the penalization, remain localized in the vicinity of the surface and thus do not contribute to the mean motion

$$\lim_{\substack{\epsilon, \nu \rightarrow 0 \\ \epsilon \sim \nu}} \left( e^{\frac{tL}{\epsilon}} \nu \partial_{zz} e^{-\frac{tL}{\epsilon}} \right) = 0.$$

In the case when  $\nu \gg \epsilon$ , the vertical dissipation damped by the penalization induces a strong relaxation mechanism, so that we expect the solution to be well approximated, outside from some initial layer, by a “stationary” solution to the wind-driven system. That initial layer should be of size  $O(\sqrt{\frac{\epsilon}{\nu}})$  and the relaxation should be governed by the Ekman dissipation process (5.14). Note that in that case

$$\lim_{\substack{\epsilon, \nu \rightarrow 0 \\ \epsilon \sim \nu}} \left( e^{\frac{tL}{\epsilon}} \nu \partial_{zz} e^{-\frac{tL}{\epsilon}} \right)$$

does not make sense.

**Remark 2.3.** *In the nonlinear case, the same argument cannot be directly applied since the contribution of the convection term and of the horizontal dissipation will be different depending on the asymptotic value of  $\epsilon/\nu$ . Nevertheless, we expect to have the same balance between the two mechanisms, or in other words the same limit for*

$$\lim_{\substack{\epsilon, \nu \rightarrow 0 \\ \epsilon \sim \nu}} \left( e^{\frac{tL}{\epsilon}} \nu \partial_{zz} e^{-\frac{tL}{\epsilon}} \right).$$

*In the case when  $\nu \gg \epsilon$ , that limit is not defined and the interaction of both mechanisms produces a strong relaxation process leading to some initial layer.*

• As the evolution equation is linear, we will further use **some superposition principle**, meaning that we will deal separately with the forcing and with the initial condition. More precisely, we will consider on the one hand the wind-driven system

$$(2.4) \quad \begin{aligned} \partial_t u + \frac{1}{\epsilon} \mathbb{P}(e_3 \wedge u) - \Delta_h u - \nu \partial_{zz} u &= 0, \\ \nabla \cdot u &= 0, \\ u|_{t=0} &= 0, \\ u|_{z=0} &= 0, \\ u_3|_{z=1} = 0, \quad \partial_z u_h|_{z=1} &= \beta \sigma. \end{aligned}$$

For that system, we will construct an approximate solution constituted of a boundary term  $u^{BL,1}$  localized near the surface, and of a singular term  $u^{sing,1}$  in the case when the forcing is resonant (in a sense that will be defined in the next section). The convergence of the modes such that  $|\mu| \neq 1$  is then proved using a somewhat soft argument, which can be applied with a crude approximation. Concerning the quasi-resonant modes, for which  $|\mu| = 1$  and  $k_h \neq 0$ , the situation is more complicated, and we have to build several correctors before reaching the adequate order of approximation.

On the other hand, we will study the initial value problem

$$(2.5) \quad \begin{aligned} \partial_t u + \frac{1}{\epsilon} \mathbb{P}(e_3 \wedge u) - \Delta_h u - \nu \partial_{zz} u &= 0, \\ \nabla \cdot u &= 0, \\ u|_{t=0} &= \gamma, \\ u|_{z=0} &= 0, \\ u_3|_{z=1} = 0, \quad \partial_z u_h|_{z=1} &= 0. \end{aligned}$$

Here we will use, following [2], an energy method which requires to obtain a very precise approximation. A quantitative result about the required precision is given in the stopping condition below (Lemma 1). The approximate solution is actually obtained as the sum of two interior terms  $u^{int}$  that we

seek in the form

$$u^{int} = \sum c_l N_l e^{-i\lambda_l \frac{t}{\epsilon}}$$

coming from the analysis of the linear penalization as an operator of  $L^2$ , and two boundary terms  $u^{BL,0}$  the form of which will be determined in the next section by a careful study of the boundary layer operator.

**Remark 2.4.** *Of course, in the nonlinear case the superposition principle does not hold anymore, and both systems (2.4) and (2.5) will be coupled. However the construction of the boundary layers (which are obtained essentially by a linear process) can be extended in that case.*

*The difficult point is to prove the convergence of the approximation, in particular when the (resonant) forcing creates some singular profile which is not localized in the vicinity of the surface, and thus is expected to modify strongly the (nonlinear) convection. For a study on that topics we refer to our forthcoming paper [5].*

The next sections are then devoted to the construction of  $u_{app}$ . We start with a precise description of the boundary layer operator  $\mathcal{B}$  in Section 3. We then build, in Section 4, the approximation and prove the convergence for the (possibly resonant) wind-driven system (2.4). For the sake of completeness, we finally study the system (2.5) which has already been dealt with in a number of mathematical papers. Let us recall that in both cases we need a refined approximation with many orders. We have then to iterate some process giving the successive correctors. Note however that we are not able to really obtain an asymptotic expansion leading to a more accurate approximation (in  $L^2$  sense). At each step of the process the order of the resonances involved in the estimates is indeed increased, so that it is not possible to obtain convergent series. For more precisions regarding that point, we refer to the proof in Section 5.

### 3. THE BOUNDARY LAYER OPERATOR

We recall that the boundary condition at the surface is given by (2.2) :

$$\delta_{1,h}(t, x_h) = \sum_{\mu, k_h} \hat{\delta}_{1,h}(\mu, k_h) \exp\left(i\mu \frac{t}{\epsilon}\right) \exp(ik_h \cdot x_h).$$

In the same way, as the Coriolis operator generates waves oscillating at a frequency  $-\lambda_k/\epsilon$ , from the Dirichlet boundary condition we deduce that the boundary term at the bottom is of the form

$$\delta_{0,h}(t, x_h) = \sum_{\mu, k_h} \hat{\delta}_{0,h}(\mu, k_h) \exp\left(i\mu \frac{t}{\epsilon}\right) \exp(ik_h \cdot x_h).$$

Actually,  $\hat{\delta}_{0,h}$  depends slowly on time, but this will be taken into account later on.

Thus it is natural to seek the boundary terms as a sum of oscillating modes, rapidly decaying in  $z$ . Our goal in this paragraph is to characterize these modes, or in other words to describe the propagation with respect to  $z$  of the boundary conditions

$$v|_{z=0} = \delta_{0,h}, \quad \partial_z v|_{z=1} = \beta \delta_{1,h}.$$

We will use the following Ansatz

$$v(t, x) = v^0(t, x) + v^1(t, x)$$

with

$$v^j(t, x) = \sum_{\mu, k_h} v^j(\mu, k_h; x) \exp\left(i\frac{t}{\epsilon} - \mu x\right)$$

where  $\mu$  and  $k_h$  are the oscillation period and horizontal Fourier mode.

We further seek  $v^0(\mu, k_h)$  and  $v^1(\mu, k_h)$  in the form

$$(3.1) \quad \begin{aligned} v^0(\mu, k_h; x) &= \hat{v}^0(\mu, k_h) \exp(ik_h \cdot x_h) \exp\left(-\lambda \frac{z}{\sqrt{\epsilon\nu}}\right), \\ v^1(\mu, k_h; x) &= \hat{v}^1(\mu, k_h) \exp(ik_h \cdot x_h) \exp\left(-\lambda \frac{(1-z)}{\sqrt{\epsilon\nu}}\right) \end{aligned}$$

so that they are expected to be localized in a neighbourhood of size  $O(\sqrt{\epsilon\nu})$  respectively near the bottom and near the surface. Note in particular that, with such a choice,  $v^0$  (resp.  $v^1$ ) introduces only exponentially small error terms on the surface (resp. at the bottom).

Plugging this Ansatz in the system (2.1) we get actually

$$(3.2) \quad \begin{aligned} i\mu \hat{v}_1 - \lambda^2 \hat{v}_1 + \epsilon k_h^2 \hat{v}_1 - \hat{v}_2 + \epsilon\nu \frac{k_1 k_2 \hat{v}_1 - k_1^2 \hat{v}_2}{\lambda^2 - \epsilon\nu k_h^2} &= 0, \\ i\mu \hat{v}_2 - \lambda^2 \hat{v}_2 + \epsilon k_h^2 \hat{v}_2 + \hat{v}_1 + \epsilon\nu \frac{-k_1 k_2 \hat{v}_2 + k_2^2 \hat{v}_1}{\lambda^2 - \epsilon\nu k_h^2} &= 0, \\ \sqrt{\epsilon\nu}(ik_1 \hat{v}_1 + ik_2 \hat{v}_2) \pm \lambda \hat{v}_3 &= 0. \end{aligned}$$

which expresses the balance between the forcing, the viscosity, the Coriolis force and the pressure.

Denote by  $A_\lambda$  the matrix corresponding to (3.2)

$$A_\lambda(\mu, k_h) = \begin{pmatrix} i\mu - \lambda^2 + \epsilon k_h^2 + \frac{\epsilon\nu k_1 k_2}{\lambda^2 - \epsilon\nu k_h^2} & -1 - \frac{\epsilon\nu k_1^2}{\lambda^2 - \epsilon\nu k_h^2} \\ 1 + \frac{\epsilon\nu k_2^2}{\lambda^2 - \epsilon\nu k_h^2} & i\mu - \lambda^2 + \epsilon k_h^2 - \frac{\epsilon\nu k_1 k_2}{\lambda^2 - \epsilon\nu k_h^2} \end{pmatrix}.$$

Classical results on boundary layers are then based on the fact that  $|\mu| \neq 1$ , which ensures that the matrix

$$\begin{pmatrix} \mu & i \\ -i & \mu \end{pmatrix}$$

is hyperbolic in the sense of dynamical systems, i.e. that its eigenvalues have non zero real parts. This feature, as well as general properties of the system, is therefore stable by small perturbation. The method consists then in neglecting the perturbation, i.e. the pressure and horizontal viscosity terms and to compute a solution to

$$\partial_t v + e_3 \wedge v - \nu \partial_{zz} v = 0$$

with suitable boundary conditions.

Now, if  $|\mu| = 1$ , the matrix

$$\begin{pmatrix} \mu & i \\ -i & \mu \end{pmatrix}$$

admits zero as an eigenvalue, and we expect its behaviour to be very sensitive to perturbations. Actually we will distinguish between two cases

- either  $k_h \neq (0, 0)$  and we will prove that the same type of behaviour as previously occurs, with the difference that the decay rate  $\lambda$  of the singular component is anomalously small. We will thus develop a general method, which can be used independently of the size of  $\lambda$  (the classical method fails since the error depends on  $1/\lambda^2$ ).
- or  $k_h = (0, 0)$  and we have a bifurcation. The solution  $v$  is not localized anymore.

**Case when  $k_h \neq (0, 0)$ .**

• Let us first introduce some notations in order to define an abstract framework to deal with. For the sake of simplicity, we omit here all the parameters  $\mu$  and  $k_h$ .

If  $\lambda$  is such that  $\det(A_\lambda) = 0$ , then there exists  $w_\lambda$  such that

$$(3.3) \quad A_\lambda w_\lambda = 0.$$

In other words the vector fields  $W_\lambda^0$  and  $W_\lambda^1$  defined by

$$(3.4) \quad \begin{aligned} W_\lambda^0(t, x) &= \begin{pmatrix} w_\lambda \\ \frac{\sqrt{\epsilon\nu}}{\lambda} ik_h \cdot w_\lambda \end{pmatrix} \exp(ik_h \cdot x_h) \exp(i\mu \frac{t}{\epsilon}) \exp\left(-\lambda \frac{z}{\sqrt{\epsilon\nu}}\right) \\ W_\lambda^1(t, x) &= \begin{pmatrix} \frac{\sqrt{\epsilon\nu}}{\lambda} w_\lambda \\ -\frac{\epsilon\nu}{\lambda^2} ik_h \cdot w_\lambda \end{pmatrix} \exp(ik_h \cdot x_h) \exp(i\mu \frac{t}{\epsilon}) \exp\left(-\lambda \frac{(1-z)}{\sqrt{\epsilon\nu}}\right) \end{aligned}$$

are exact solutions to (2.1) satisfying respectively the horizontal boundary condition

$$\begin{aligned} W_{\lambda,h|z=0}^0 &= w_\lambda \exp(ik_h \cdot x_h) \exp(i\mu \frac{t}{\epsilon}), \\ \partial_z W_{\lambda,h|z=1}^0 &= -\frac{\lambda}{\sqrt{\epsilon\nu}} w_\lambda \exp(ik_h \cdot x_h) \exp(i\mu \frac{t}{\epsilon}) \exp\left(-\frac{\lambda}{\sqrt{\epsilon\nu}}\right), \end{aligned}$$

and

$$\begin{aligned} \partial_z W_{\lambda,h|z=1}^1 &= w_\lambda \exp(ik_h \cdot x_h) \exp(i\mu \frac{t}{\epsilon}), \\ W_{\lambda,h|z=0}^1 &= \frac{\sqrt{\epsilon\nu}}{\lambda} w_\lambda \exp(ik_h \cdot x_h) \exp(i\mu \frac{t}{\epsilon}) \exp\left(-\frac{\lambda}{\sqrt{\epsilon\nu}}\right). \end{aligned}$$

We have moreover the following estimates (provided that  $\frac{\lambda}{\sqrt{\epsilon\nu}} \gg 1$ )

$$(3.5) \quad \begin{aligned} W_\lambda^0 &= O(1)_{L^\infty(\mathbf{R}^+, L^\infty(\Omega))}, & W_\lambda^0 &= O\left(\left(\frac{\epsilon\nu}{\lambda^2}\right)^{1/4}\right)_{L^\infty(\mathbf{R}^+, L^2(\Omega))}, \\ W_\lambda^1 &= O\left(\left(\frac{\epsilon\nu}{\lambda^2}\right)^{1/2}\right)_{L^\infty(\mathbf{R}^+, L^\infty(\Omega))}, & W_\lambda^1 &= O\left(\left(\frac{\epsilon\nu}{\lambda^2}\right)^{3/4}\right)_{L^\infty(\mathbf{R}^+, L^2(\Omega))}. \end{aligned}$$

We intend to build one particular solution to (2.1) satisfying the horizontal boundary condition

$$\begin{aligned} v_h|_{z=0} &= \delta_{0,h}, \\ \partial_z v_h|_{z=1} &= \beta \delta_{1,h}. \end{aligned}$$

Hence, we only have to find (for all  $\mu$  and  $k_h$ ) some  $w_{\lambda^-}$  and  $w_{\lambda^+}$  constituting a basis of  $\mathbf{C}^2$ .

• In order to determine some suitable  $w_{\lambda^-}$  and  $w_{\lambda^+}$ , we have to get some asymptotic expansions of the eigenvalues and eigenvectors of  $A_\lambda(\mu, k_h)$ .

In view of the previous paragraph, at leading order, we have

$$A_\lambda = \begin{pmatrix} i\mu - \lambda^2 & -1 \\ 1 & i\mu - \lambda^2 \end{pmatrix} + o(1)$$

so that

$$\det(A_\lambda) = (i\mu - \lambda^2)^2 + 1 + o(1) = 0$$

for  $(\lambda^-)^2 = i(\mu + 1) + o(1)$  or  $(\lambda^+)^2 = i(\mu - 1) + o(1)$ . We further have

$$w_{\lambda^-} = (1, -i) + o(1) \text{ and } w_{\lambda^+} = (1, i) + o(1)$$

For  $|\mu| \neq 1$ , we choose  $\lambda^-$  and  $\lambda^+$  to be the roots of  $\det(A_\lambda) = 0$  with non-negative real parts. The previous asymptotic equivalences are then enough to prove that

$$\det(w_{\lambda^-}, w_{\lambda^+}) = 2i + o(1)$$

from which we deduce that  $(w_{\lambda^-}, w_{\lambda^+})$  is a (quasi-orthogonal) basis of  $\mathbf{C}^2$ , and that we have uniform bounds (with respect to  $\epsilon$  and  $\nu$  sufficiently small) on the transition matrix  $P$  and its inverse.

For  $\mu = 1$  we expect  $\lambda^-$  to be given by  $(\lambda^-)^2 = 2i + \eta^-$  with  $\eta^- = o(1)$ , and  $\lambda^+$  to be given by  $(\lambda^+)^2 = \eta^+$  with  $\eta^+ = o(1)$

$$\begin{aligned} \det(A_\lambda) &= \left( i\mu - \lambda^2 + \epsilon k_h^2 + \frac{\epsilon\nu k_1 k_2}{\lambda^2 - \epsilon\nu k_h^2} \right) \left( i\mu - \lambda^2 + \epsilon k_h^2 - \frac{\epsilon\nu k_1 k_2}{\lambda^2 - \epsilon\nu k_h^2} \right) \\ &\quad - \left( -1 - \frac{\epsilon\nu k_1^2}{\lambda^2 - \epsilon\nu k_h^2} \right) \left( 1 + \frac{\epsilon\nu k_2^2}{\lambda^2 - \epsilon\nu k_h^2} \right) \\ &= \left( -i - \eta^- + \epsilon k_h^2 + \frac{\epsilon\nu k_1 k_2}{2i} \right) \left( -i - \eta^- + \epsilon k_h^2 - \frac{\epsilon\nu k_1 k_2}{2i} \right) \\ &\quad - \left( -1 - \frac{\epsilon\nu k_1^2}{2i} \right) \left( 1 + \frac{\epsilon\nu k_2^2}{2i} \right) + o(\epsilon) \end{aligned}$$

and

$$\begin{aligned} \det(A_\lambda) &= \left( i - \eta^+ + \epsilon k_h^2 + \frac{\epsilon\nu k_1 k_2}{\eta^+} \right) \left( i - \eta^+ + \epsilon k_h^2 - \frac{\epsilon\nu k_1 k_2}{\eta^+} \right) \\ &\quad - \left( -1 - \frac{\epsilon\nu k_1^2}{\eta^+} \right) \left( 1 + \frac{\epsilon\nu k_2^2}{\eta^+} \right) + O(\epsilon^2\nu^2/(\eta^+)^2) \end{aligned}$$

from which we deduce that

$$\begin{aligned} \eta^- &= \epsilon k_h^2 + \frac{1}{4}\epsilon\nu k_h^2 + o(\epsilon) \\ \eta^+ &= \epsilon k_h^2 + \frac{\epsilon\nu k_h^2}{2i\eta^+} + o(\sqrt{\epsilon\nu}) + o(\epsilon), \end{aligned}$$

and thus there exists a constant  $C$ , depending only on  $k_h$ , such that

$$(3.6) \quad \begin{aligned} C(k_h)^{-1} &\leq |\lambda^-(1, k_h)| \leq C(k_h), \\ C(k_h)^{-1}(\epsilon + \sqrt{\epsilon\nu})^{-1/2} &\leq |\lambda^+(1, k_h)| \leq C(k_h)(\epsilon + \sqrt{\epsilon\nu})^{1/2}. \end{aligned}$$

Plugging these expansions in the formula of  $A_\lambda$  leads then to

$$\begin{aligned} w_{\lambda^-} &= (1, -i + O(\epsilon)), \\ w_{\lambda^+} &= (1, i + O(\sqrt{\epsilon\nu}) + O(\epsilon)) \end{aligned}$$

In particular we have

$$\det(w_{\lambda^-}, w_{\lambda^+}) = 2i + O(\epsilon) + O(\sqrt{\epsilon\nu})$$

from which we deduce that  $(w_{\lambda^-}, w_{\lambda^+})$  is a (quasi-orthogonal) basis of  $\mathbf{C}^2$ , and that we have uniform bounds (with respect to  $\epsilon$  and  $\nu$  sufficiently small) on the transition matrix  $P$  and its inverse.

For  $\mu = -1$  we have in the same way

$$(3.7) \quad \begin{aligned} C(k_h)^{-1}(\epsilon + \sqrt{\epsilon\nu})^{-1/2} &\leq |\lambda^-(-1, k_h)| \leq C(k_h)(\epsilon + \sqrt{\epsilon\nu})^{1/2}, \\ C(k_h)^{-1} &\leq |\lambda^+(-1, k_h)| \leq C(k_h) \end{aligned}$$

and

$$w_{\lambda^-} = (1, -i + O(\sqrt{\epsilon\nu}) + O(\epsilon)), \quad w_{\lambda^+} = (1, i + O(\epsilon))$$

from which we deduce uniform bounds (with respect to  $\epsilon$  and  $\nu$  sufficiently small) on the transition matrix  $P$  and its inverse.

- We then define  $v^0(\mu, k_h)$  and  $v^1(\mu, k_h)$  by

$$(3.8) \quad v^j(\mu, k_h; x) \exp\left(i\mu\frac{t}{\epsilon}\right) = \beta^j(\alpha_-^j W_{\lambda^-}^j(t, x) + \alpha_+^j W_{\lambda^+}^j(t, x))$$

where  $W_{\lambda}^j$  is defined in terms of  $w_{\lambda}$  by (3.4) and the coefficients  $\alpha_-^j$  and  $\alpha_+^j$  are defined by

$$(3.9) \quad (\alpha_-^j, \alpha_+^j) = P^{-1} \hat{\delta}_{j,h}(\mu, k_h).$$

**Case when  $k_h = (0, 0)$ .**

That case is strongly different since there is no term of higher order in (3.2) :

$$A_{\lambda} = \begin{pmatrix} i\mu - \lambda^2 & -1 \\ 1 & i\mu - \lambda^2 \end{pmatrix}$$

meaning that when  $|\mu| = 1$  we cannot find a basis of eigenvectors ( $w_{\lambda^-}, w_{\lambda^+}$ ) with  $\Re(\lambda^-) > 0$  and  $\Re(\lambda^+) > 0$ . One of the eigenvalue is necessarily 0, and thus the corresponding solution has no decay in  $z$ . In other words we do not expect the boundary terms to be localized in the vicinity of the boundary uniformly in time.

For  $|\mu| \neq 1$  we use exactly the same arguments as previously and define  $\hat{v}^j(\mu, 0)$  by formulas (3.8)(3.9).

If  $\mu = 1$  we have

$$\lambda^- = 2i \text{ and } \lambda^+ = 0$$

with

$$w_{\lambda^-} = (1, -i) \text{ and } w_{\lambda^+} = (1, i).$$

Define as previously  $W_{\lambda^-}^j$  by (3.4), and  $\alpha_{\pm}^j$  by (3.9). Setting

$$\begin{aligned} & [v^0(1, 0; x) + v^1(1, 0; x)] \exp\left(i\frac{t}{\epsilon}\right) \\ &= \alpha_-^0 W_{\lambda^-}^0(t, x) + \beta \alpha_-^1 W_{\lambda^-}^1(t, x) + (\alpha_+^0 + \beta \alpha_+^1 z) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \exp\left(i\frac{t}{\epsilon}\right) \end{aligned}$$

we can check that it is an exact solution to (2.1), which further satisfies the required horizontal boundary condition.

We then define

$$W_{\lambda^+(1,0)}^0 = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \exp\left(i\frac{t}{\epsilon}\right), \quad W_{\lambda^+(1,0)}^1 = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} z \exp\left(i\frac{t}{\epsilon}\right)$$

so that formula (3.8) is still satisfied.

Note that the vertical component of the linear profile is identically zero.



If  $\mu = -1$  we have in a similar way

$$\lambda^- = 0 \text{ and } \lambda^+ = -2i$$

with

$$w_{\lambda^-} = (1, -i) \text{ and } w_{\lambda^+} = (1, i).$$

Denoting by  $W_{\lambda^+}^0$  and  $W_{\lambda^+}^1$  the vector fields defined by (3.4), by

$$W_{\lambda^-(-1,0)}^0 = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \exp\left(-i\frac{t}{\epsilon}\right), \quad W_{\lambda^-(-1,0)}^1 = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} z \exp\left(-i\frac{t}{\epsilon}\right)$$

we define  $\hat{v}^0(-1,0)$  and  $\hat{v}^1(-1,0)$  by formulas (3.8)(3.9) and we can check that it is an exact solution to (2.1), which further satisfies the required horizontal boundary condition.

With that definition of the boundary layer operator  $\mathcal{B}$ , we immediately get the following continuity estimates :

**Lemma 3.1.** *Let  $\delta_{0,h}, \delta_{1,h}$  be some rapidly oscillating 2D fields (with finite numbers of horizontal Fourier modes and oscillating modes). Define*

$$(3.10) \quad v = \mathcal{B}(\delta_{0,h}, \delta_{1,h}) = \sum_{\mu, k_h} \sum_{j \in \{0,1\}} \sum_{\pm} \alpha_{\pm}^j(\mu, k_h) W_{\lambda^{\pm}(\mu, k_h)}^j$$

where  $\alpha_{\pm}^j(\mu, k_h)$  is given by (3.9).

Then  $v$  can be decomposed as

$$v = \bar{v} + \tilde{v} + v^{sing}$$

with

$$(3.11) \quad \begin{aligned} \|\bar{v}_h\|_{L^2(\omega)} + (\epsilon\nu)^{-\frac{1}{2}} \|\tilde{v}_3\|_{L^2(\omega)} &\leq C \left[ (\epsilon\nu)^{\frac{1}{4}} \|\delta_{0,h}\|_{H^1(\omega_h)} + \beta (\epsilon\nu)^{\frac{3}{4}} \|\delta_{1,h}\|_{H^1(\omega_h)} \right], \\ \|\tilde{v}_h\|_{L^2(\omega)} &\leq C \left[ \left( \frac{\epsilon\nu}{\epsilon + \sqrt{\epsilon\nu}} \right)^{\frac{1}{4}} \|\delta_{0,h}\|_{H^1(\omega_h)} + \beta \left( \frac{\epsilon\nu}{\epsilon + \sqrt{\epsilon\nu}} \right)^{\frac{3}{4}} \|\delta_{1,h}\|_{H^1(\omega_h)} \right], \\ \|\tilde{v}_3\|_{L^2(\omega)} &\leq C \left[ \left( \frac{\epsilon\nu}{\epsilon + \sqrt{\epsilon\nu}} \right)^{\frac{3}{4}} \|\delta_{0,h}\|_{H^1(\omega_h)} + \beta \left( \frac{\epsilon\nu}{\epsilon + \sqrt{\epsilon\nu}} \right)^{\frac{5}{4}} \|\delta_{1,h}\|_{H^1(\omega_h)} \right], \\ \|v_h^{sing}\|_{L^2(\omega)} &\leq C (\|\delta_{0,h}\|_{H^1(\omega_h)} + \beta \|\delta_{1,h}\|_{H^1(\omega_h)}), \quad v_3^{sing} \equiv 0. \end{aligned}$$

**Remark 3.2.** *The constant  $C$  in the estimates above depends on the values of  $\mu$  and  $k_h$ . In the special case when  $(\mu, k_h) \in \{(-\lambda_{(k_h, k_3)}, k_h), k = (k_h, k_3) \in \mathbf{Z}^3\}$ , the inequality for  $\bar{v}$  can be slightly refined. Precisely, using*

estimate (6.3) in the Appendix, we obtain in this case that for all  $\alpha > 1$ , there exists a constant  $C_\alpha > 0$  (independent of  $k_h, k_3$ ) such that

$$\begin{aligned} \|\bar{v}_h\|_{L^2(\omega)} &\leq C(\epsilon\nu)^{1/4} \left[ \sum_{k \in \mathbf{Z}^3} |k|^{\alpha+1} \left| \hat{\delta}_{0,h}(-\lambda_k, k_h) \right|^2 \right]^{1/2} \\ &+ C\beta(\epsilon\nu)^{3/4} \left[ \sum_{k \in \mathbf{Z}^3} |k|^{\alpha+3} \left| \hat{\delta}_{1,h}(-\lambda_k, k_h) \right|^2 \right]^{1/2}, \end{aligned}$$

and

$$\begin{aligned} \|\bar{v}_3\|_{L^2(\omega)} &\leq C(\epsilon\nu)^{1/4} \left[ \sum_{k \in \mathbf{Z}^3} |k|^{\alpha+3} \left| \hat{\delta}_{0,h}(-\lambda_k, k_h) \right|^2 \right]^{1/2} \\ &+ C\beta(\epsilon\nu)^{3/4} \left[ \sum_{k \in \mathbf{Z}^3} |k|^{\alpha+5} \left| \hat{\delta}_{1,h}(-\lambda_k, k_h) \right|^2 \right]^{1/2}. \end{aligned}$$

*Proof.* The profile  $v^{sing}$  defined by

$$\begin{aligned} v^{sing} &= \sum_j \sum_{\mu\sigma=1} \beta^j \alpha_\sigma^j(\mu, 0) W_{\lambda^\sigma(\mu, 0)}^j \\ &= (\alpha_+^0(1, 0) + \beta\alpha_+^1(1, 0)z) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \exp\left(i\frac{t}{\epsilon}\right) \\ &+ (\alpha_+^0(-1, 0) + \beta\alpha_+^1(-1, 0)z) \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \exp\left(-i\frac{t}{\epsilon}\right) \end{aligned}$$

obviously satisfied the third estimate in (3.11).

We then split  $v - v^{sing}$  according to the size of the boundary layers

$$\begin{aligned} \bar{v} &= \sum_{k_h} \sum_j \sum_{\mu\sigma \neq 1} \beta^j \alpha_\sigma^j(\mu, k_h) W_{\lambda^\sigma(\mu, k_h)}^j, \\ \tilde{v} &= \sum_{k_h \neq 0} \sum_j \sum_{\mu\sigma=1} \beta^j \alpha_\sigma^j(\mu, k_h) W_{\lambda^\sigma(\mu, k_h)}^j \end{aligned}$$

By definition of  $\alpha_\sigma^j(\mu, k_h)$  and  $W_{\lambda^\sigma(\mu, k_h)}^j$ , we then obtain the estimates

$$\|\bar{v}_h\|_{L^2} + (\epsilon\nu)^{-1/2} \|\bar{v}_3\|_{L^2} \leq C \left[ (\epsilon\nu)^{1/4} \|\delta_{0,h}\|_{H^1(\omega_h)} + \beta(\epsilon\nu)^{3/4} \|\delta_{1,h}\|_{H^1(\omega_h)} \right]$$

for the classical boundary layer, and

$$\|\tilde{v}_h\|_{L^2(\omega)} + \frac{\sqrt{\epsilon} + (\epsilon\nu)^{1/4}}{(\epsilon\nu)^{1/2}} \|\tilde{v}_3\|_{L^2(\omega)} \leq C \left[ \left( \frac{\epsilon\nu}{\epsilon + \sqrt{\epsilon\nu}} \right)^{1/4} \|\delta_{0,h}\|_{H^1(\omega_h)} + \beta \left( \frac{\epsilon\nu}{\epsilon + \sqrt{\epsilon\nu}} \right)^{3/4} \|\delta_{1,h}\|_{H^1(\omega_h)} \right]$$

for the quasi-resonant boundary layer.  $\square$

**Definition 3.3.** Assume that the wind-stress  $\sigma$  is given by

$$\sigma(t, x_h) = \sum_{k_h} \sum_{\mu} \hat{\sigma}(\mu, k_h) e^{ik_h \cdot x_h} e^{i\mu \frac{t}{\epsilon}}.$$

- We define the **resonant part**  $\sigma_{res}$  of the wind stress by

$$\sigma_{res} := \sigma_+(1, 0) \begin{pmatrix} 1 \\ i \end{pmatrix} \exp\left(i \frac{t}{\epsilon}\right) + \sigma_-(1, 0) \begin{pmatrix} 1 \\ -i \end{pmatrix} \exp\left(-i \frac{t}{\epsilon}\right),$$

where  $(\sigma_-(\mu, k_h), \sigma_+(\mu, k_h)) = P(\mu, k_h)^{-1} \hat{\sigma}(\mu, k_h)$ ,  $P(\mu, k_h)$  is the matrix  $(w_{\lambda^-(\mu, k_h)}, w_{\lambda^+(\mu, k_h)})$ , and  $w_\lambda$  is defined by (3.3).

- We will say that a windstress  $\sigma$  is **non-resonant** if  $\sigma_{res} = 0$ , or in other words, if it does not generate any  $O(\beta)$  singular profile.

Notice that if the wind-stress is non-resonant, then

$$\|\mathcal{B}(v_h^0, \sigma)\|_{L^2} \leq C.$$

Indeed, the singular profile reduces to

$$v^{sing} = \alpha_+^0(1, 0) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \exp\left(i \frac{t}{\epsilon}\right) + \alpha_-^0(-1, 0) \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \exp\left(-i \frac{t}{\epsilon}\right)$$

and the third line in equation (3.11) becomes

$$\|v_h^{sing}\|_{L^2} \leq C \|v_{0,h}\|_{H^1(\omega_h)}.$$

#### 4. STUDY OF THE WIND-DRIVEN PART OF THE MOTION

##### 4.1. Some stability inequality for the wind-driven system (2.4).

As mentioned in Section 2, for the part of the wind-driven system (2.4) corresponding to the modes  $\mu \neq \pm 1$ , we will only need a rather crude approximation of the solution. We have indeed the following

**Proposition 4.1.** *Denote by  $u$  the solution to (2.4) and by  $u_{app}$  any approximate solution in the sense that*

$$(4.1) \quad \begin{aligned} \partial_t u + \frac{1}{\epsilon} \mathbb{P}(e_3 \wedge u) - \Delta_h u - \nu \partial_{zz} u &= \eta, \\ \nabla \cdot u &= 0, \\ u|_{t=0} &= \eta_t, \\ u_{3|z=0} &= 0, \quad u_{h|z=0} = \epsilon \eta_0, \\ u_{3|z=1} &= 0, \quad \partial_z u_{h|z=1} = \beta \sigma + \eta_1, \end{aligned}$$

with  $\eta \rightarrow 0$  in  $L^2([0, T] \times \omega)$ ,  $\eta_t \rightarrow 0$  in  $L^2(\omega)$  and  $\eta_0, \nu^{3/4} \eta_1, \epsilon \partial_t \eta_0 \rightarrow 0$  in  $L^2([0, T] \times \omega_h)$ . Then as  $\epsilon, \nu \rightarrow 0$ ,

$$\begin{aligned} \|u - u_{app}\|_{L^\infty([0, T], L^2(\omega))} &\rightarrow 0, \\ \|\nabla_h(u - u_{app})\|_{L^2([0, T] \times \omega)} + \sqrt{\nu} \|\partial_z(u - u_{app})\|_{L^2([0, T] \times \omega)} &\rightarrow 0. \end{aligned}$$

*Proof.* • The first step consists in building a family  $w$  such that  $\tilde{u}_{app} \stackrel{\text{def}}{=} u_{app} + w$  satisfies

$$(4.2) \quad \begin{aligned} \partial_t u + \frac{1}{\epsilon} \mathbb{P}(e_3 \wedge u) - \Delta_h u - \nu \partial_{zz} u &= \tilde{\eta}, \\ \nabla \cdot u &= 0, \\ u|_{t=0} &= \tilde{\eta}_t, \\ u_{3|z=0} &= 0, \quad u_{h|z=1} = 0, \\ u_{3|z=1} &= 0, \quad \partial_z u_{h|z=1} = \beta \sigma + \eta_1. \end{aligned}$$

with  $\tilde{\eta}_t \rightarrow 0$  in  $L^2(\omega)$  and  $\tilde{\eta} \rightarrow 0$  in  $L^2([0, T] \times \omega_h)$ .

In order to do so, we just apply Lemma 1 in the Appendix with

$$\delta_{0,h} = -\epsilon \eta_0, \quad \delta_{0,3} = 0 \text{ and } \delta_1 = 0.$$

We recall that we work with a finite number of horizontal frequencies  $k_h$ , so that all Sobolev norms in  $\omega_h$  are bequivalent to the  $L^2$  norm.

A simple computation allows then to establish all the properties (4.2).

• The convergence is then obtained by a standard energy estimate. Combining (4.2) and (2.4), and integrating by parts lead indeed to

$$\begin{aligned} \frac{1}{2} \|(u - \tilde{u}_{app})(t)\|_{L^2}^2 + \int_0^t \|\nabla_h(u - \tilde{u}_{app})\|_{L^2}^2 ds + \nu \int_0^t \|\nabla_h(u - \tilde{u}_{app})\|_{L^2}^2 ds \\ \leq \frac{1}{2} \|\tilde{\eta}_t\|_{L^2(\omega)}^2 + \int_0^t \|u - \tilde{u}_{app}\|_{L^2(\omega)} \|\tilde{\eta}\|_{L^2(\omega)} ds \\ + \nu \int_0^t \|(u - \tilde{u}_{app})_{h|z=1}(t)\|_{L^2(\omega_h)} \|\eta_1\|_{L^2(\omega_h)} ds \end{aligned}$$

• To conclude we therefore need to estimate the trace  $(u - \tilde{u}_{app})_{h|z=1}$  in  $L^2(\omega_h)$  in terms of the  $H^1$  norm of  $u - \tilde{u}_{app}$ . By Sobolev embeddings and

the Cauchy-Schwarz inequality, we have

$$\nu^{1/2} \|(u - \tilde{u}_{app})_{h|z=1}\|_{L^2(\omega_h)}^2 \leq C \|u - \tilde{u}_{app}\|_{L^2(\omega)}^2 + \nu \|\partial_z(u - \tilde{u}_{app})\|_{L^2(\omega)}^2$$

Plugging that estimate in the previous energy inequality, we get

$$\begin{aligned} & \frac{1}{2} \|(u - \tilde{u}_{app})(t)\|_{L^2}^2 + \int_0^t \|\nabla_h(u - \tilde{u}_{app})\|_{L^2}^2 ds + \frac{\nu}{2} \int_0^t \|\nabla_h(u - \tilde{u}_{app})\|_{L^2}^2 ds \\ & \leq \frac{1}{2} \|\tilde{\eta}_t\|_{L^2(\omega)}^2 + \frac{1}{2} \int_0^t \|\tilde{\eta}\|_{L^2(\omega)}^2 ds + \nu^{3/2} \int_0^t \|\eta_1\|_{L^2(\omega_h)}^2 ds \\ & \quad + C \int_0^t \|(u - \tilde{u}_{app})(s)\|_{L^2(\omega)}^2 ds \end{aligned}$$

using again the Cauchy-Schwarz inequality. We conclude by Gronwall's lemma

$$\begin{aligned} & \frac{1}{2} \|(u - \tilde{u}_{app})(t)\|_{L^2}^2 + \int_0^t \|\nabla_h(u - \tilde{u}_{app})\|_{L^2}^2 ds + \nu \int_0^t \|\nabla_h(u - \tilde{u}_{app})\|_{L^2}^2 ds \\ & \leq \frac{1}{2} \|\tilde{\eta}_t\|_{L^2(\omega)}^2 e^{2Ct} + \frac{1}{2} \int_0^t \|\tilde{\eta}\|_{L^2(\omega)}^2 e^{2C(t-s)} ds + \nu^{3/2} \int_0^t \|\eta_1\|_{L^2(\omega_h)}^2 e^{2C(t-s)} ds \end{aligned}$$

which proves that  $u - \tilde{u}_{app}$  converges to 0 in  $L_{\text{loc}}^\infty(\mathbf{R}_+, L^2(\omega))$ . Theorem 2.1 will be proved in the case when  $\gamma = 0$  if we are able to build some approximate solution  $u_{app}$  that converges strongly to 0 as  $\epsilon, \nu \rightarrow 0$ .  $\square$

**4.2. The approximate solution in the non-resonant case.** In order to obtain some approximate solution to (2.4) (in the sense (4.1) of the previous paragraph), we will essentially need to construct the boundary layer term and some small corrector to account for the vertical component of the boundary condition. Of course that process will be slightly different if the forcing is resonant, i.e. if the boundary term  $\mathcal{B}(0, \sigma)$  is not localized in the vicinity of the surface. Thus let us first deal with the nonresonant case.

- We define with the notations of Proposition 3.1

$$u^{BL,1} = \mathcal{B}(0, \sigma - \sigma_{res}) = \bar{u}^{BL,1} + \tilde{u}^{BL,1}.$$

Since we assume that  $\sigma$  has a finite number of horizontal Fourier modes  $k_h$  and of oscillating modes  $\mu$ , by Lemma 3.1, we have

$$\begin{aligned} \|\bar{u}_h^{BL,1}\|_{L^2(\omega)} & \leq C \|\sigma\|_{L^2(\omega_h)} \beta(\epsilon\nu)^{3/4}, \\ \|\bar{u}_3^{BL,1}\|_{L^2(\omega)} & \leq C \|\sigma\|_{L^2(\omega_h)} \beta(\epsilon\nu)^{5/4}, \end{aligned}$$

and for the modes with  $|\mu| = 1$ ,

$$\begin{aligned} \|\tilde{u}_h^{BL,1}\|_{L^2(\omega)} & \leq C \|\sigma\|_{L^2(\omega_h)} \beta \left( \frac{\epsilon\nu}{\epsilon + \sqrt{\epsilon\nu}} \right)^{3/4}, \\ \|\tilde{u}_3^{BL,1}\|_{L^2(\omega)} & \leq C \|\sigma\|_{L^2(\omega_h)} \beta \left( \frac{\epsilon\nu}{\epsilon + \sqrt{\epsilon\nu}} \right)^{5/4}. \end{aligned}$$

It can be checked that hypothesis (2.3) ensures that the right-hand sides of the above equation are all  $o(1)$ . Furthermore, using the explicit formula for  $\mathcal{B}$ , we get, for the classical part of the system (i.e. the modes  $\mu \neq \pm 1$ )

$$\begin{aligned} \|\bar{u}_{3|z=1}^{BL,1}\|_{H^s(\omega_h)} &= O(\beta\epsilon\nu) \quad \text{and} \quad \|\partial_t \bar{u}_{3|z=1}^{BL,1}\|_{H^s(\omega_h)} = O(\beta\nu) \\ \|\bar{u}_{3|z=0}^{BL,1}\|_{H^s(\omega_h)} &= O(\beta(\epsilon\nu)^N) \quad \text{and} \quad \|\partial_t \bar{u}_{3|z=0}^{BL,1}\|_{H^s(\omega_h)} = O(\beta(\epsilon\nu)^N), \end{aligned}$$

and for the quasi-resonant part of the system

$$\begin{aligned} \|\tilde{u}_{3|z=1}^{BL,1}\|_{H^s(\omega_h)} &= O(\beta(\epsilon\nu)^{1/2}) \quad \text{and} \quad \|\partial_t \tilde{u}_{3|z=1}^{BL,1}\|_{H^s(\omega_h)} = O\left(\beta\sqrt{\frac{\nu}{\epsilon}}\right) \\ \|\tilde{u}_{3|z=0}^{BL,1}\|_{H^s(\omega_h)} &= O(\beta(\epsilon\nu)^N) \quad \text{and} \quad \|\partial_t \tilde{u}_{3|z=0}^{BL,1}\|_{H^s(\omega_h)} = O(\beta(\epsilon\nu)^N) \end{aligned}$$

for any integer  $N$ . As a consequence,  $\bar{u}_{3|z=1}^{BL,1}$ ,  $\bar{u}_{3|z=0}^{BL,1}$  and  $\tilde{u}_{3|z=0}^{BL,1}$  satisfy the conditions of the stopping Lemma 1 in the Appendix as soon as  $\beta\nu = o(1)$ , which is always ensured by hypothesis (2.3). We denote by  $w^{stop}$  the function defined in Lemma 1 with

$$\begin{aligned} \delta_{0,h} &= 0, \quad \delta_{1,h} = 0 \\ \delta_{0,3} &= -\bar{u}_{3|z=0}^{BL,1} - \tilde{u}_{3|z=0}^{BL,1}, \quad \delta_{1,3} = -\bar{u}_{3|z=1}^{BL,1}. \end{aligned}$$

The term  $\tilde{u}_{3|z=1}^{BL,1}$ , on the other hand, does not match the conditions of Lemma 1. We therefore introduce some corrector  $v^{int,1}$  to restore the zero-flux condition. We first define its vertical component

$$v_3^{int,1} = -\tilde{u}_{3|z=1}^{BL,1} z,$$

then its horizontal component in order that the divergence-free condition is satisfied

$$v_h^{int,1} = \nabla_h(\Delta_h)^{-1} \tilde{u}_{3|z=1}^{BL,1}.$$

Note that for  $k_h = 0$ ,  $v^{int,1}$  is identically zero. The fact that  $v_3^{int,1} \neq 0$  means that a small amount of fluid, of order  $\sqrt{\epsilon\nu}\delta_{j,3}$ , enters the domain (or the boundary layer, depending on the sign of the coefficient). This phenomenon is called Ekman suction and  $v_3^{int,1}$  is called Ekman transpiration velocity. This velocity will be responsible for global circulation in the whole domain, of order  $\beta(\epsilon\nu)^{\frac{1}{2}}$ , but not limited to the boundary layer.

In any case, we get easily that

$$\begin{aligned} \|v^{int,1}\|_{L^\infty([0,\infty), H^s(\omega))} &= O(\beta(\epsilon\nu)^{1/2}), \\ \|\partial_t v^{int,1}\|_{L^\infty([0,\infty), L^2(\omega))} &= O\left(\beta\sqrt{\frac{\nu}{\epsilon}}\right). \end{aligned}$$

• If there is no quasi-resonant mode  $|\mu| = 1$ , we then claim that  $u_{app} = \bar{u}^{BL,1} + w^{stop}$  satisfies the required conditions. We indeed have clearly

$$u_{app,3|z=0} = u_{app,3|z=1} = 0$$

by definition of  $w^{stop}$ . Notice that in this case  $\tilde{u}^{BL} = 0$  and  $v^{int,1} = 0$ . We further have

$$\partial_z u_{app,h}|_{z=1} - \beta\sigma = 0,$$

$$\|u_{app,h}|_{z=0}\|_{L^2(\omega_h)} = O(\beta(\epsilon\nu)^N) \text{ and } \|\partial_t u_{app,h}|_{z=0}\|_{L^2(\omega_h)} = O(\beta(\epsilon\nu)^N)$$

for all  $N$ . We also have for all  $t \geq 0$

$$\|u_{app}(t)\|_{L^2(\omega)} \leq \|\bar{u}^{BL,1}(t)\|_{L^2(\omega)} + \|w^{stop}(t)\|_{L^2(\omega)} = O(\beta(\epsilon\nu)^{3/4}) = o(1).$$

It remains then to check that the evolution equation is approximately satisfied. We have

$$\partial_t u_{app} + \frac{1}{\epsilon} \mathbb{P}(e_3 \wedge u_{app}) - \Delta_h u_{app} - \nu \partial_{zz} u_{app} = O(\nu\beta)_{L^2([0,T] \times \omega)} = o(1)$$

supplemented with some initial condition

$$u_{app}|_{t=0} = O(\beta(\epsilon\nu)^{3/4})_{L^2(\omega)}.$$

We therefore apply Proposition 4.1 and conclude that  $u_{app}$  has the same asymptotic behaviour as the solution of

$$(4.3) \quad \begin{aligned} \partial_t u + \frac{1}{\epsilon} \mathbb{P}(e_3 \wedge u) - \Delta_h u - \nu \partial_{zz} u &= 0, \\ \nabla \cdot u &= 0, \\ u|_{t=0} &= 0, \\ u_3|_{z=0} &= 0, \quad u_h|_{z=0} = 0, \\ u_3|_{z=1} &= 0, \quad \partial_z u_h|_{z=1} = \beta\sigma. \end{aligned}$$

• For the quasi-resonant modes  $|\mu| = 1$ , the influence of the forcing is much more extended inside the domain. In particular, the defect

$$\begin{aligned} \Sigma &= \partial_t v^{int,1} + \frac{1}{\epsilon} e_3 \wedge v^{int,1} - \Delta_h v^{int,1} \\ &= \sum_{\mu=\pm 1} \sum_{k_h} \left( i \frac{\mu}{\epsilon} + |k_h|^2 \right) \hat{v}^{int,1}(\mu, k_h, z) e^{ik_h \cdot x_h} e^{i\mu \frac{t}{\epsilon}} \\ &\quad + \frac{1}{\epsilon} \sum_{\mu=\pm 1} \sum_{k_h} \begin{pmatrix} -\hat{v}_2^{int,1}(\mu, k_h, z) \\ \hat{v}_1^{int,1}(\mu, k_h, z) \\ 0 \end{pmatrix} e^{ik_h \cdot x_h} e^{i\mu \frac{t}{\epsilon}} \end{aligned}$$

does not converge strongly to 0 in  $L^2$  norm. It is however expected to have rapid oscillations, and thus to converge weakly to 0. The standard method to deal with such a problem consists then in building some corrector which will be small in  $L^2$  norm in contrast with its time derivative which has to compensate the previous defect.

More precisely we will use the small divisor estimate stated in Appendix B. For  $K > 0$  arbitrary, denote by  $\delta u_K^{int,1} = \sum_l \hat{w}_l e^{-i \frac{t}{\epsilon} \lambda_l} N_l$  the solution to

$$\partial_t \delta u_K^{int,1} + \frac{1}{\epsilon} \mathbb{P}(e_3 \wedge \delta u_K^{int,1}) - \Delta_h \delta u_K^{int,1} - \nu \partial_{zz} \delta u_K^{int,1} = -\mathbb{P}_K(\Sigma),$$

supplemented with the initial condition

$$\delta u_{K|t=0}^{int,1} = 0.$$

The notation  $\mathbb{P}_K$  stands for the projection onto the vector space generated by  $\{N_l, |l| \leq K\}$ .

In other words, for  $|l| \leq K$ ,  $\hat{w}_l$  is the solution of

$$\partial_t \hat{w}_l + |l_h|^2 \hat{w}_l + \nu' |l_3|^2 \hat{w}_l = -e^{i\lambda_l \frac{t}{\epsilon}} \langle N_l | \Sigma \rangle$$

where  $\nu' = \pi^2 \nu$ . Direct computations give for  $l_h \neq 0$ ,  $\mu = \pm 1$ ,

$$\begin{aligned} \hat{v}_h^{int,1}(\mu, l_h, z) &= i \hat{\delta}_3(\mu, l_h) \frac{l_h}{|l_h|^2}, \\ \hat{v}_3^{int,1}(\mu, l_h, z) &= \hat{\delta}_3(\mu, l_h) z, \end{aligned}$$

where

$$\hat{\delta}_3(\mu, l_h) = i\beta(\epsilon\nu) \frac{\alpha_\mu^1(\mu, l_h) l_h \cdot w_{\lambda\mu}}{(\lambda^\mu(\mu, k_h))^2},$$

with the same notations as in the previous section. We recall that

$$|\lambda^\mu(\mu, k_h)|^{-2} \leq C(\sqrt{\epsilon\nu} + \epsilon)^{-1}.$$

Moreover,

$$(4.4) \quad \begin{aligned} \left\langle N_l \left| \begin{pmatrix} il_1 \\ il_2 \\ |l_h|^2 z \end{pmatrix} e^{il_h \cdot x_h} \right. \right\rangle &= i \frac{|l_h|^3}{2\pi^2 |l| l_3} (-1)^{l_3} \mathbf{1}_{l_3 \neq 0} \\ \left\langle N_l \left| \begin{pmatrix} -il_2 \\ il_1 \\ 0 \end{pmatrix} e^{il_h \cdot x_h} \right. \right\rangle &= \begin{cases} 0 & \text{if } l_3 \neq 0, \\ -\frac{|l_h|}{2\pi} & \text{else,} \end{cases} \end{aligned}$$

where  $|l|^2 = |l_h|^2 + (\pi l_3)^2$ . We thus have

$$(4.5) \quad \begin{aligned} &\partial_t \hat{w}_l + (|l_h|^2 + \nu' |l_3|^2) \hat{w}_l \\ &= \frac{1}{\epsilon} \sum_{\mu=\pm 1} \frac{\hat{\delta}_3(\mu, l_h)}{2\pi} \left( \mathbf{1}_{l_3 \neq 0} \frac{(\mu - i\epsilon |l_h|^2) |l_h|}{\pi |l| l_3} + \frac{\mathbf{1}_{l_3=0}}{|l_h|} \right) e^{i(\lambda_l + \mu) \frac{t}{\epsilon}}. \end{aligned}$$

Notice that by truncating the large frequencies in  $l$ , we have introduced a source term in the equation. Precisely,  $\delta u_K^{int,1} + v^{int,1}$  is a solution of equation (2.1) with a source term equal to

$$(\Sigma - \mathbb{P}\Sigma) + (\mathbb{P}\Sigma - \mathbb{P}_K \Sigma).$$

The term  $\Sigma - \mathbb{P}\Sigma$  belongs to  $V_0^\perp$  by definition of  $\mathbb{P}$ , and thus for all  $u \in V_0$ , we have

$$\int_\omega (\Sigma - \mathbb{P}\Sigma) \cdot u = 0.$$



As for the remainder term  $\mathbb{P}\Sigma - \mathbb{P}_K\Sigma$ , since the horizontal frequency  $k_h$  is bounded, we have, according to formulas (4.4)

$$(4.6) \quad \begin{aligned} \|\mathbb{P}\Sigma - \mathbb{P}_K\Sigma\|_{L^\infty((0,\infty),L^2(\omega))} &\leq C\beta\sqrt{\frac{\nu}{\epsilon}}K^{-3/2}, \\ \frac{1}{\sqrt{\nu}}\|\mathbb{P}\Sigma - \mathbb{P}_K\Sigma\|_{L^\infty((0,\infty),H^{-1}(\omega))} &\leq C\beta\frac{1}{\sqrt{\epsilon}}K^{-5/2}. \end{aligned}$$

The first estimate in (4.6) will be useful when  $\nu$  is not too large, for instance  $\nu = O(\epsilon)$ . On the other hand, we have, for all  $u \in L^2_{\text{loc}}([0, \infty), H^1(\omega) \cap V_0)$

$$\begin{aligned} \int_0^t \int_\omega (\mathbb{P}\Sigma - \mathbb{P}_K\Sigma) \cdot u &\leq \|u\|_{L^2((0,t);H^1(\omega))} \|\mathbb{P}\Sigma - \mathbb{P}_K\Sigma\|_{L^\infty((0,\infty),H^{-1}(\omega))} \\ &\leq C\beta\frac{1}{\sqrt{\epsilon}}K^{-5/2} \cdot \sqrt{\nu}\|u\|_{L^2((0,t);H^1(\omega))}. \end{aligned}$$

Thus, using a slightly modified version of Proposition 4.1, when  $\nu$  is large, it is sufficient to take  $K$  such that

$$\frac{\beta}{\sqrt{\epsilon}}K^{-5/2} = o(1).$$

We now apply Lemma 2; notice that all the terms  $s(\mu, l)$  in  $\mathbb{P}\Sigma$  are such that  $|\mu| = 1$ ,  $k_h \neq 0$  and thus satisfy, with the notations of Lemma 2

$$\begin{aligned} \|s\|_{r,K}^2 &\leq \sum_{|l| \leq K} \sum_{\mu \in \{-1,1\}} |l|^{2(r+2)} |s(\mu, l)|^2 \\ &\leq C\beta^2 \frac{\nu}{\epsilon} \sum_{|l| \leq K} \sum_{\mu \in \{-1,1\}} |l|^{2r} |\hat{\sigma}^\mu(\mu, l_h)|^2 \\ &\leq C\beta^2 \frac{\nu}{\epsilon} K^{2r+1}. \end{aligned}$$

Thus, we get, for all  $r > 0$

$$\|\delta u_K^{\text{int},1}\|_{H^r(\omega)} \leq C\beta(\epsilon\nu)^{\frac{1}{2}} K^{r+\frac{1}{2}}.$$

For further purposes, we have to choose  $K$  such that the  $H^s$  norm of  $w_K^{\text{int}}$  is  $o(\sqrt{\frac{\epsilon}{\nu}})$  for some  $r > 3/2$ , and such that at least one of the right-hand sides in (4.6) is  $o(1)$  as  $\epsilon, \nu \rightarrow 0$ . Indeed, the remainder term  $(\mathbb{P} - \mathbb{P}_K)\Sigma$  will eventually be included in the remainder term  $\eta$  of Proposition 4.1. We distinguish between the cases when  $\nu$  is large (say  $\nu \gg \epsilon$ ) and  $\nu$  is small (say  $\nu = O(\epsilon)$ ), which yield different values for  $K$ .

- If  $\nu = O(\epsilon)$ , we choose  $K$  so that

$$\beta\sqrt{\frac{\nu}{\epsilon}}K^{-3/2} = \beta\sqrt{\frac{\nu}{\epsilon}}(\epsilon\nu)^{1/2}K^{r+\frac{1}{2}}$$

for some  $r > 3/2$ , which yields

$$K = (\epsilon\nu)^{-\frac{1}{2(r+2)}}.$$

With this choice, we have

$$\|\mathbb{P}\Sigma - \mathbb{P}_K\Sigma\|_{L^2}, \sqrt{\frac{\nu}{\epsilon}}\|\delta u_K^{int,1}\|_{H^r(\omega)} \leq C\beta\nu^{1-\frac{r+\frac{1}{2}}{2(r+2)}}\epsilon^{-\frac{r+\frac{1}{2}}{2(r+2)}};$$

hence we can take  $\beta = O(\nu^{-\alpha_0}\epsilon^{\alpha_1})$  as long as  $\alpha_0 < 5/7$  and  $\alpha_1 > 2/7$ . Notice that when  $\nu = O(\epsilon)$ , hypothesis (2.3) ensures that we can find such a couple  $(\alpha_0, \alpha_1)$ . We then choose  $r_0 > 3/2$  such that

$$1 - \frac{r_0 + \frac{1}{2}}{2(r_0 + 2)} - \alpha_0 > 0,$$

$$\alpha_1 - \frac{r + \frac{1}{2}}{2(r + 2)} > 0,$$

and we have

$$(4.7) \quad \|\mathbb{P}\Sigma - \mathbb{P}_K\Sigma\|_{L^2}, \sqrt{\frac{\nu}{\epsilon}}\|\delta u_K^{int,1}\|_{H^{r_0}(\omega)} = o(1).$$

- Else, we choose  $K$  so that

$$\beta\frac{1}{\sqrt{\epsilon}}K^{-5/2} = \beta\sqrt{\frac{\nu}{\epsilon}}(\epsilon\nu)^{1/2}K^{s+\frac{1}{2}}$$

for some  $s > 3/2$ , which yields

$$K = (\nu\sqrt{\epsilon})^{-\frac{1}{s+3}}.$$

In this case, the same calculations as above show that we can take  $\beta = O(\nu^{-\alpha_0}\epsilon^{\alpha_1})$  as long as  $\alpha_0 < 5/9$  and  $\alpha_1 > 2/9$ . We emphasize that this method remains valid when  $\nu = O(\epsilon)$ , but gives a more restrictive condition than the previous one. Moreover, when  $\nu \geq \epsilon$ , hypothesis (2.3) ensures that there exists  $\alpha_0 < 5/9$  and  $\alpha_1 > 2/9$  such that  $\beta = O(\nu^{-\alpha_0}\epsilon^{\alpha_1})$ .

With  $\alpha_0, \alpha_1$  fixed, we then choose  $r_0 > 3/2$  such that

$$1 - \alpha_0 - \frac{r_0 + \frac{1}{2}}{r_0 + 3} > 0,$$

$$\alpha_1 - \frac{r_0 + \frac{1}{2}}{2(r_0 + 3)} > 0,$$

and we have

$$(4.8) \quad \frac{1}{\sqrt{\nu}}\|\mathbb{P}\Sigma - \mathbb{P}_K\Sigma\|_{L^\infty((0,\infty), H^{-1}(\omega))}, \sqrt{\frac{\nu}{\epsilon}}\|\delta u_K^{int,1}\|_{L^\infty((0,\infty), H^{r_0}(\omega))} = o(1).$$

• Because of the terms  $v^{int,1}$  and  $\delta u_K^{int,1}$ , the horizontal boundary conditions are no longer satisfied at  $z = 0$ . Thus, we construct another boundary layer term, which we denote by  $\delta u^{BL,1}$ , such that

$$\delta u^{BL,1} = \mathcal{B}(-v_{h|z=0}^{int,1} - \delta u_{K,h|z=0}^{int,1}, 0).$$

We decompose  $\delta u^{BL,1}$  into  $\delta u^{BL,1} = \delta \tilde{u}^{BL,1} + \delta \bar{u}^{BL,1}$  as in Lemma 3.1; the term  $\delta \tilde{u}^{BL,1}$  is due to the modes  $k_h \neq 0$ ,  $|\mu| = 1$ , and thus depends only on

$v^{int,1}$ , since  $|\lambda_k| < 1$  if  $k_h \neq 0$ . Notice that there is no term  $\delta v^{sing}$  because  $\hat{v}^{int,1}(\mu, l_h, z) = 0$ ,  $\hat{w}_l = 0$  for  $l_h = 0$ .

According to the estimates (3.11) and Remark 3.2, we have, for all  $r > 1$ ,

$$\begin{aligned} \|\delta u_h^{BL,1}\|_{L^2(\omega)} &\leq C \|v^{int,1}\|_{L^2(\omega)} \left( \frac{\epsilon\nu}{\epsilon + (\epsilon\nu)^{1/2}} \right)^{\frac{1}{4}} + C \|\delta u_K^{int,1}\|_{H^r} (\epsilon\nu)^{1/4} \\ &\leq C (\epsilon\nu)^{1/8} + C \beta\nu (\epsilon\nu)^{1/4} (\nu\sqrt{\epsilon})^{-\frac{r+\frac{1}{2}}{r_0+3}}, \\ \|\delta u_3^{BL,1}\|_{L^2(\omega)} &\leq \|v^{int,1}\|_{L^2(\omega)} \left( \frac{\epsilon\nu}{\epsilon + (\epsilon\nu)^{1/2}} \right)^{\frac{3}{4}} + C \|\delta u_K^{int,1}\|_{H^{r+1}} (\epsilon\nu)^{3/4} \\ &\leq C (\epsilon\nu)^{3/8} + C \beta\nu (\epsilon\nu)^{3/4} (\nu\sqrt{\epsilon})^{-\frac{r+\frac{3}{2}}{r_0+3}}. \end{aligned}$$

Thus, choosing for instance  $r = r_0 - 1/2$ , we infer that  $\delta u^{BL,1}$  vanishes in  $L^2$  norm. According to (4.5), there exists a constant  $C$  such that

$$|\partial_t \hat{w}_l| \leq \frac{C}{|l_3|^2} \sqrt{\frac{\nu}{\epsilon}} \beta,$$

so that  $\delta u^{BL,1}$  is an approximate solution of equation (2.1), with an error term of order

$$\sqrt{\frac{\nu}{\epsilon}} \beta (\epsilon\nu)^{1/4} \leq C \epsilon^{\alpha'_1 - \frac{1}{4}} \nu^{\frac{3}{4} - \alpha'_0}$$

according to hypothesis (2.3); we recall that by assumption,  $\alpha'_1 > 1/4$  and  $\alpha'_0 < 7/12$ , so that the error term is always  $o(1)$  in  $L^2$ .

Furthermore,  $\delta u_{h|z=1}^{BL,1}$ ,  $\delta u_{3|z=1}^{BL,1}$  are exponentially small, and thus satisfy the hypotheses of Proposition 4.1 and Lemma 1 respectively. Moreover,

$$\|\delta \bar{u}_{3|z=0}^{BL,1}\|_{L^2(\omega_h)} \leq C \sqrt{\epsilon\nu} \|\delta u_K^{int,1}\|_{H^{r_0}} + C \sqrt{\epsilon\nu} \|v^{int,1}\|_{L^2}.$$

Thus, by definition of the truncation parameter  $K$ ,  $\delta \bar{u}_{3|z=0}^{BL,1}$  satisfies the hypotheses of Lemma 1.

We now consider  $\delta \tilde{u}_{3|z=0}^{BL,1}$ , which is due to the modes  $\mu = \pm 1$ ,  $k_h \neq 0$  in  $v^{int,1}$ ; we have

$$\|\delta \tilde{u}_{3|z=0}^{BL,1}\|_{L^2(\omega_h)} \leq C \frac{\sqrt{\epsilon\nu}}{(\epsilon\nu)^{1/4} + \sqrt{\epsilon}} \|v^{int,1}\|_{L^2} \leq C \beta (\epsilon\nu)^{3/4}.$$

Hence  $\delta \tilde{u}_{3|z=0}^{BL,1}$  does not match the conditions of Lemma 1, and we need to construct additional correctors  $\delta v^{int,1}$ ,  $\delta^2 u_K^{int,1}$ . We do not go into the details of the construction, since it is rigorously analogous to the one of  $v^{int,1}$  and  $\delta u_K^{int,1}$ ; for instance,  $\delta v^{int,1}$  is given by

$$\begin{aligned} \delta v_3^{int,1} &= -(1-z) \delta \tilde{u}_{3|z=0}^{BL,1}, \\ \delta v_h^{int,1} &= -\nabla_h \Delta_h^{-1} \delta \tilde{u}_{3|z=0}^{BL,1}, \end{aligned}$$

so that

$$\|\delta v^{int,1}\|_{L^2(\omega)} \leq C\beta(\epsilon\nu)^{3/4}.$$

Thus a factor  $(\epsilon\nu)^{1/4}$  is gained by replacing  $v^{int,1}$  by  $\delta v^{int,1}$ ; as a consequence, the trace at  $z = 0$  of the vertical component of the atypical boundary term at the next order, namely  $\delta^2 \tilde{u}^{BL,1}$ , is of order

$$\frac{\sqrt{\epsilon\nu}}{(\epsilon\nu)^{1/4} + \sqrt{\epsilon}} \|\delta v^{int,1}\|_{L^2} = O(\beta\epsilon\nu),$$

so that the conditions of Lemma 1 are now met.

Thus, we slightly modify the definition of the function  $w^{stop}$  given by Lemma 1, so that the boundary conditions are now

$$\begin{aligned} \delta_{0,h} &= 0, & \delta_{1,h} &= 0 \\ \delta_{0,3} &= -u_{3|z=0}^{BL,1} - \delta \bar{u}_{3|z=0}^{BL,1} - \delta^2 u_{3|z=0}^{BL,1}, & \delta_{1,3} &= -\bar{u}_{3|z=1}^{BL,1} - \delta u_{3|z=1}^{BL,1} - \delta^2 u_{3|z=0}^{BL,1}, \end{aligned}$$

where we have denoted by  $\delta^2 u^{BL,1}$  the boundary term due to the construction of the additional corrector terms  $\delta v^{int,1} + \delta^2 u_K^{int,1}$ .

- We then claim that

$$u_{app} = u^{BL,1} + w^{stop} + v^{int,1} + \delta u_K^{int,1} + \delta u^{BL,1} + \delta v^{int,1} + \delta^2 u_K^{int,1} + \delta^2 u^{BL,1}$$

satisfies the assumptions of Proposition 4.1. We indeed have clearly

$$u_{app,3|z=0} = u_{app,3|z=1} = 0$$

by definition of  $v^{int,1}$ ,  $\delta v^{int,1}$  and  $w$ . We further have, for all  $N > 0$

$$\begin{aligned} \|\partial_z u_{app,h|z=1} - \beta\sigma\|_{L^2(\omega_h)} &= O((\epsilon\nu)^N), \\ \|u_{app,h|z=0}\|_{L^2(\omega_h)} &= O((\epsilon\nu)^N) \text{ and } \|\partial_t u_{app,h|z=0}\|_{L^2(\omega_h)} = O((\epsilon\nu)^N). \end{aligned}$$

We also have for all  $t \geq 0$

$$\|u_{app}(t)\|_{L^2(\omega)} \leq C \left( (\epsilon\nu)^{1/8} + \beta(\epsilon\nu)^{1/2} (\sqrt{\nu\epsilon})^{-\frac{1}{2r_0+6}} \right) = o(1).$$

By definition of the different terms, the evolution equation is approximately satisfied, up to an error term of order  $o(\sqrt{\nu})$  in  $L^\infty((0, \infty), H^1(\omega))$ , and another one of order  $o(1)$  in  $L^2((0, T) \times \omega)$ .

We therefore apply a variant of Proposition 4.1 and conclude that  $u_{app}$  has the same asymptotic behaviour as the solution of

$$\begin{aligned} (4.9) \quad \partial_t u + \frac{1}{\epsilon} \mathbb{P}(e_3 \wedge u) - \Delta_h u - \nu \partial_{zz} u &= 0, \\ \nabla \cdot u &= 0, \\ u|_{t=0} &= 0, \\ u_{3|z=0} &= 0, \quad u_{h|z=0} = 0, \\ u_{3|z=1} &= 0, \quad \partial_z u_{h|z=1} = \beta\sigma. \end{aligned}$$

Thus the solution of (4.9) vanishes in  $L^2$  norm as  $\epsilon, \nu \rightarrow 0$  with  $(\epsilon, \nu, \beta)$  satisfying (2.3).

**4.3. Construction of the singular term  $u^{sing,1}$ .** We have seen in section 3 that the boundary operator has a very singular behaviour on the modes  $(\mu, k_h, \sigma, j)$  with  $k_h = 0$  and  $\mu\sigma = 1$ . In particular it creates unbounded terms in the whole domain in infinite time ( $t \sim 1/\nu$ .) For finite time, what is observed is a boundary layer of size  $O(\sqrt{\nu t})$ , corresponding to the propagation of boundary data according to the heat equation.

- We thus start by solving exactly the corresponding part of the equation. We proceed in two steps. We first build some profile  $v^{sing,1}$  satisfying the boundary conditions

$$v^{sing,1} = \beta\sigma_+(1,0)z \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \exp\left(i\frac{t}{\epsilon}\right) + \beta\sigma_-(-1,0)z \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \exp\left(-i\frac{t}{\epsilon}\right)$$

where the coefficients  $\sigma_{\pm}(\pm 1, 0)$  are given in terms of  $\sigma$  by (3.9).

The second step is then to determine - using explicit computations - some 2D corrector  $\bar{u}^{sing,1}$  (depending only on  $t$  and  $z$ ) to match the initial condition :

$$(\bar{u}^{sing,1} + v^{sing,1})|_{t=0} = 0$$

Note that  $\bar{u}^{sing,1}$  will be also very large (since  $\beta \gg 1$  in general), meaning that the resonant forcing destabilizes the whole fluid.

Let us first remark that, for the 2D vector field  $\bar{u}^{sing,1}$  (depending only on  $t$  and  $z$ ), (2.1) can be rewritten

$$\partial_t \bar{u}^{sing,1} + \frac{1}{\epsilon} \bar{u}^{sing,1} \wedge e_3 - \nu \partial_{zz} \bar{u}^{sing,1} = 0.$$

As we further have

$$\bar{u}|_{z=0}^{sing,1} = 0, \quad \partial_z \bar{u}|_{z=1}^{sing,1} = 0,$$

we can decompose  $\bar{u}^{sing,1}$  according to the following family

$$M_{k_3}^+ = \begin{pmatrix} \sin(\frac{\pi}{2}k_3z) \\ i \sin(\frac{\pi}{2}k_3z) \\ 0 \end{pmatrix} \quad M_{k_3}^- = \begin{pmatrix} \sin(\frac{\pi}{2}k_3z) \\ -i \sin(\frac{\pi}{2}k_3z) \\ 0 \end{pmatrix} \quad \text{with } k_3 \in 2\mathbf{Z} + 1.$$

We obtain the following decomposition for  $v^{sing,1}$  in  $L^2([0, 1])$

$$\sum_{k_3} \left( \left( \frac{2}{k_3\pi} \right)^2 (-1)^{(k_3-1)/2} \beta \sigma_+(1,0) \right) M_{k_3}^+ + \left( \left( \frac{2}{k_3\pi} \right)^2 (-1)^{(k_3-1)/2} \beta \sigma_-(-1,0) \right) M_{k_3}^-$$

and thus

$$\begin{aligned} \bar{u}^{sing,1}(t, z) = & -\beta \sum_{k_3} \left( \frac{2}{k_3 \pi} \right)^2 (-1)^{\frac{k_3-1}{2}} \sigma_+(1, 0) M_{k_3}^+ \exp\left(-i\frac{t}{\epsilon} - \nu k_3^2 t\right) \\ & + \left( \frac{2}{k_3 \pi} \right)^2 (-1)^{\frac{k_3-1}{2}} \sigma_-(-1, 0) M_{k_3}^- \exp\left(i\frac{t}{\epsilon} - \nu k_3^2 t\right). \end{aligned}$$

We then check that the vector field

$$(4.10) \quad u^{sing,1} = \bar{u}^{sing,1} + v^{sing,1}$$

satisfies exactly both the evolution equation (2.1) and the “bad” part of the boundary conditions.

- We then establish an approximation of “boundary layer” type.

The solution computed in the previous paragraph is exact, but the corresponding formula does not give precise informations on its asymptotic behaviour as  $\epsilon$  and  $\nu$  tend to zero. We are thus interested in deriving some approximation of “boundary layer” type, where the main term does not depend neither on  $\epsilon$ , nor on  $\nu$ .

As mentioned previously, the singular component  $u^{sing,1}$  of the velocity is a 2D vector field (depending only on  $t$  and  $z$ ), so that (2.1) can be rewritten

$$\partial_t u^{sing,1} + \frac{1}{\epsilon} u^{sing,1} \wedge e_3 - \nu \partial_{zz} u^{sing,1} = 0.$$

- As there is no more pressure in the equation, it can be filtered by a simple change of unknown :

$$u_L^{sing,1} = \frac{1}{2} \left\langle \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \middle| u^{sing,1} \right\rangle \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} e^{-i\frac{t}{\epsilon}} + \frac{1}{2} \left\langle \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \middle| u^{sing,1} \right\rangle \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} e^{i\frac{t}{\epsilon}}$$

A straightforward computation leads then to

$$\partial_t u_L^{sing,1} - \nu \partial_{zz} u_L^{sing,1} = 0,$$

which is nothing else than the heat equation with small conductivity  $\nu$ .

It is supplemented with the boundary condition

$$u_{L|z=0}^{sing} = 0, \quad \partial_z u_{L|z=1}^{sing} = \beta \sigma_+(1, 0) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} + \beta \sigma_-(-1, 0) z \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix},$$

and the initial condition

$$u_{L|t=0}^{sing} = 0.$$

- Standard results on the heat kernel show therefore that the boundary effects will remain localized (in  $L^2$  sense) in a layer of size  $O(\sqrt{\nu t})$  near the surface.

Seeking  $\partial_z u_L^{sing,1}$  in the form of a self similar profile

$$\partial_z u_L^{sing,1} = \varphi \left( \frac{1-z}{\sqrt{\nu t}} \right)$$

we get indeed

$$-\frac{1}{2}X\varphi'(X) - \varphi''(X) = 0,$$

from which we deduce that

$$\varphi'(X) = \varphi'(0) \exp \left( -\frac{1}{4}X^2 \right),$$

and

$$\varphi(X) = - \int_X^{+\infty} \varphi'(0) \exp \left( -\frac{1}{4}Y^2 \right) dY.$$

We then have

$$\begin{aligned} \partial_z u_L^{sing,1}(t, z) &= \frac{\beta\sqrt{2}}{\sqrt{\pi}} \left( \sigma_+(1, 0) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} + \sigma_-(-1, 0) z \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \right) \int_{\frac{1-z}{\sqrt{\nu t}}}^{+\infty} e^{-\frac{Y^2}{4}} dY \\ &\sim \beta\sigma_{res} \left( \frac{1-z}{\sqrt{\nu t}} \right)^{-1} \exp \left( -\frac{1}{4} \left( \frac{1-z}{\sqrt{\nu t}} \right)^2 \right) \end{aligned}$$

and thus using the bottom boundary condition

$$u_L^{sing,1}(t, z) = \int_0^z \partial_z u_L^{sing,1}(t, z') dz',$$

we deduce that  $u_L^{sing,1}$  is exponentially small outside from a layer of size  $O(\sqrt{\nu t})$ .

## 5. STUDY OF THE DISSIPATING PART OF THE MOTION

This section is dedicated to the rest of the proof of Theorem 2.1. According to the preceding section, there remains to define the term  $u^{Dirichlet}$ , which is an approximate solution of (2.1), supplemented with the following boundary conditions

$$\begin{aligned} u_h^{Dirichlet}|_{z=0} &= 0, & u_3^{Dirichlet}|_{z=0} &= 0 \\ \partial_z u_h^{Dirichlet}|_{z=1} &= 0, & u_3^{Dirichlet}|_{z=1} &= 0, \\ u|_{t=0}^{Dirichlet} &= \gamma. \end{aligned}$$

This point has already been investigated by several authors, see for instance [2]: the idea is to construct an interior term, denoted by  $u^{int}$ , which satisfies the evolution equation up to error terms which are  $o(1)$ , and a boundary layer term, denoted by  $u^{BL}$ , which restores the horizontal boundary conditions violated by the interior term. We emphasize that in order that the equation and the boundary conditions are satisfied up to sufficiently

small error terms, we need to build some second order terms in both  $u^{int}$  and  $u^{BL}$ .

The organization of the section is as follows: in the spirit of the third section, we first define an operator  $\mathcal{U}$ , which allows us to construct an interior term, given arbitrary vertical boundary conditions. Then we explain how to choose the boundary conditions for the boundary layer term and the interior term in order to retrieve (2.2). In the last paragraph, we build one additional boundary layer term, and we prove Theorem 2.1 thanks to an energy estimate.

Throughout this section, we use repeatedly the following norm: if  $\delta \in L^\infty([0, \infty) \times [0, \infty), L^2(\omega_h))$  is such that

$$\delta(t, \tau, x_h) = \sum_{|k_h| \leq N} \sum_{k_3 \in \mathbf{Z}} \hat{\delta}(-\lambda_k, k_h; t) e^{ik_h \cdot x_h} e^{-i\lambda_k \tau},$$

then

$$\|\delta(t, \cdot)\|_s := \left( \sum_{|k_h| \leq N} \sum_{k_3 \in \mathbf{Z}} |k_3|^{2s} \left| \hat{\delta}(-\lambda_k, k_h; t) \right|^2 \right)^{1/2}.$$

**5.1. Construction of the operator  $\mathcal{U}$ .** Let  $\delta_{1,3}$  and  $\delta_{0,3}$  in  $L^\infty([0, \infty) \times [0, \infty), L^2(\omega_h))$  be such that

$$(5.1) \quad \delta_{i,3}(t, \tau, x_h) = \sum_{|k_h| \leq N} \sum_{k_3 \in \mathbf{Z}} \hat{\delta}_{i,3}(-\lambda_k, k_h; t) e^{ik_h \cdot x_h} e^{-i\lambda_k \tau},$$

and let  $\gamma \in V_0$ . In practice, the functions  $\delta_{1,3}$  and  $\delta_{0,3}$  will not be arbitrary, and will be dictated by the expression of boundary layer operator constructed in the third section. In fact, we will see that  $\delta_{1,3} = 0$ , so that the expression of  $u^{int}$  below is simpler, but we have preferred to keep an arbitrary value for  $\delta_{1,3} = 0$  in order not to anticipate on this result.

We define the operator  $\mathcal{U}$  by

$$\mathcal{U}(\gamma; \delta_{0,3}, \delta_{1,3}) = u^{int},$$

where  $u^{int}$  is an approximate solution of equation (2.1) and satisfies the following boundary conditions

$$(5.2) \quad u_3^{int}|_{z=1} = \sqrt{\epsilon\nu} \delta_{1,3},$$

$$(5.3) \quad u_3^{int}|_{z=0} = \sqrt{\epsilon\nu} \delta_{0,3},$$

$$(5.4) \quad u|_{t=0} = \gamma + o(1).$$

We emphasize that conditions (5.2)-(5.3) will be satisfied exactly (without any error term). Of course the above conditions are not sufficient to define the term  $u^{int}$  unequivocally. We merely define here a particular solution of this system, which is sufficient for our purposes; the construction is very similar to what has been done in section 4.



The explicit construction of  $u^{int}$  requires three steps: first, we exhibit a divergence-free vector field  $v^{int,0}$  which satisfies the vertical boundary conditions (5.2)-(5.3), but not equation (2.1), and then we define a function  $\delta u^{int,0}$ , which satisfies homogeneous boundary conditions, and such that

$$(5.5) \quad u^{int} := \bar{u}^{int} + \delta u^{int,0} + v^{int,0}$$

is an approximate solution of (2.1), supplemented with the initial condition (5.4). As usual in this type of problem, we first assume that  $\bar{u}^{int}$  is the preponderant term in  $u^{int}$ , and thus we begin by deriving an equation for the corrector term  $\delta u^{int}$  involving  $\bar{u}^{int}$ . Ultimately, this will allow us to write an equation for  $\bar{u}^{int}$ . In the third step, we prove that the function  $\delta u^{int}$  thus defined is of order  $O(\sqrt{\nu\epsilon})$  in  $L^2$ .

- A natural choice for  $v^{int,0}$  is

$$(5.6) \quad \begin{cases} v_3^{int,0} = \sqrt{\epsilon\nu} [\delta_{1,3}z + \delta_{0,3}(1-z)], \\ v_h^{int,0} = \sqrt{\epsilon\nu} \nabla_h \Delta_h^{-1} [\delta_{0,3} - \delta_{1,3}]. \end{cases}$$

(Note that  $v^{int,0}$  is not uniquely determined by (5.2)-(5.3)). We denote by  $\hat{v}^{int,0}(\mu, k_h, t, z)$  the Fourier coefficient of  $v^{int,0}$ , that is

$$v^{int,0}(t, x) = \sum_{\mu, k_h} \hat{v}^{int,0}(\mu, k_h, t, z) \exp(ik_h \cdot x_h) \exp\left(\frac{t}{\epsilon} \mu\right).$$

Note that the Ekman suction at the bottom has a very important effect in the energy balance. The order of magnitude of  $\nu \int |\nabla u^{BL}|^2$  in the Ekman layer is indeed  $O(\sqrt{\frac{\nu}{\epsilon}})$ , so that the Ekman layer damps the interior motion, like a friction term. This phenomenon is called **Ekman pumping**. We therefore expect that the limit flow of (1.5) in the high rotation limit is not determined by the formal equations (1.6) and (1.9) but by dissipative versions of these equations.

- As in the previous section, we seek

$$\begin{aligned} \bar{u}^{int} &= \sum_{l \in \mathbf{Z}^3} c_l(t) e^{-i\lambda_l \frac{t}{\epsilon}} N_l, \\ \delta u^{int,0} &= \sum_{l \in \mathbf{Z}^3} \delta c_l(t) e^{-i\lambda_l \frac{t}{\epsilon}} N_l, \end{aligned}$$

so that

$$\begin{aligned} & \left[ \partial_t + \frac{1}{\epsilon} L - \nu \partial_{zz} - \Delta_h \right] (\bar{u}^{int} + \delta u^{int,0}) \\ &= \sum_{l \in \mathbf{Z}^3} \partial_t (c_l(t) + \delta c_l(t)) e^{-i\lambda_l \frac{t}{\epsilon}} N^l \\ &+ \sum_{l \in \mathbf{Z}^3} (|l_h|^2 + \nu' |l_3|^2) (c_l(t) + \delta c_l(t)) e^{-i\lambda_l \frac{t}{\epsilon}} N^l. \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \left[ \partial_t + \frac{1}{\epsilon} e_3 \wedge -\nu \partial_{zz} - \Delta_h \right] v^{int,0} \\
 = & \sum_{\mu, k_h} \left[ \partial_t \hat{v}^{int,0}(\mu, k_h, t, z) + |k_h|^2 \hat{v}^{int,0}(\mu, k_h, t, z) \right] e^{ik_h \cdot x_h} e^{i\mu \frac{t}{\epsilon}} \\
 + & \frac{1}{\epsilon} \sum_{\mu, k_h} i\mu \hat{v}^{int,0}(\mu, k_h, t, z) e^{ik_h \cdot x_h} e^{i\mu \frac{t}{\epsilon}} \\
 + & \sqrt{\frac{\nu}{\epsilon}} \sum_{\mu, k_h} \frac{(\hat{\delta}_{1,3} - \hat{\delta}_{0,3})(\mu, k_h, t)}{|k_h|^2} \begin{pmatrix} -ik_2 \\ ik_1 \\ 0 \end{pmatrix} e^{ik_h \cdot x_h} e^{i\mu \frac{t}{\epsilon}}.
 \end{aligned}$$

In order that  $\bar{u}^{int} + \delta u^{int,0} + v^{int,0}$  is an approximate solution of (2.1), we project both equations on  $N_l$  for  $l \in \mathbf{Z}^3$ , multiply by  $\exp(i\lambda_l \frac{t}{\epsilon})$ , and identify each term. We further apply the following rules in order to determine the equations for  $\delta u^{int,0}$  and  $\bar{u}^{int}$ :

- all the terms which do not have fast oscillations and are of order  $O(\delta_{j,3} \sqrt{\frac{\nu}{\epsilon}})$  become source terms in the equation on  $c_l$ ,
- all the terms which are either  $o(\delta_{j,3} \sqrt{\frac{\nu}{\epsilon}})$  or oscillating at a frequency  $1/\epsilon$  become source terms in the equation on  $\delta c_l$ .

We work with a fixed  $l \in \mathbf{Z}^3$ . Recall that  $v^{int,0}$  has no purely vertical component, i.e.  $\hat{v}^{int,0}(\mu, l_h, t, z) = 0$  if  $l_h = 0$ . Thanks to formulas (4.4), the equation on  $c_l$  reads

$$(5.7) \quad \partial_t c_l + |l_h|^2 c_l + \nu' |l_3|^2 c_l = -\sqrt{\frac{\nu}{\epsilon}} \frac{|l_h|}{2\pi |l|^2} \left[ \hat{\delta}_{0,3}(-\lambda_l, l_h, t) - (-1)^{l_3} \hat{\delta}_{1,3}(-\lambda_l, l_h, t) \right],$$

supplemented with the initial condition

$$(5.8) \quad c_l(0) = \langle N_l | \gamma \rangle$$

and the equation on  $\delta c_l$  is

$$\begin{aligned}
 (5.9) \quad & \partial_t \delta c_l + (|l_h|^2 + \nu' |l_3|^2) \delta c_l \\
 = & - \sum_{\mu \neq -\lambda_l} \left\langle N_l | (\partial_t \hat{v}^{int,0}(\mu, l_h, t, z) + |l_h|^2 \hat{v}^{int,0}(\mu, l_h, t, z)) e^{il_h \cdot x_h} \right\rangle e^{i(\lambda_l + \mu) \frac{t}{\epsilon}} \\
 & - \sqrt{\frac{\nu}{\epsilon}} \sum_{\mu \neq -\lambda_l} \frac{\hat{\delta}_{0,3}(\mu, l_h, t) - (-1)^{l_3} \hat{\delta}_{1,3}(\mu, l_h, t)}{2\pi} \times \\
 & \times \left( \mathbf{1}_{l_3 \neq 0} \frac{\mu |l_h|}{\pi |l| l_3} + \frac{\mathbf{1}_{l_3=0}}{|l_h|} \right) e^{i(\lambda_l + \mu) \frac{t}{\epsilon}}.
 \end{aligned}$$

For the time being, we do not specify an initial condition for  $\delta c_l$ . Indeed, we shall see that it is convenient to choose another condition than  $-\langle N_l, v^{int,0} \rangle$ ,

in order to use the decay of  $\hat{\delta}_{j,3}(\mu, l_h, t)$  with respect to  $t$ . This choice will be precised in paragraph 5.4.

As in the previous section, we truncate the large frequencies in  $\delta_{cl}$ . This creates an error term in the evolution equation, which is of order

$$O\left(\sqrt{\frac{\nu}{\epsilon}} \frac{1}{A^{3/2}}\right)_{L^2},$$

where  $A$  is the truncation parameter, to be chosen later on. We set

$$\delta u_A^{int,0} = \sum_{l_h} \sum_{|l_3| \leq A} \delta_{cl} N_l.$$

• We now apply to  $\delta u_A^{int,0}$  the small divisor estimate stated in Lemma 2 in the Appendix with

$$s(\mu, l, t) = -\sqrt{\frac{\nu}{\epsilon}} \frac{\hat{\delta}_{0,3}(\mu, l_h, t) - (-1)^{l_3} \hat{\delta}_{1,3}(\mu, l_h, t)}{2\pi} \left( \mathbf{1}_{l_3 \neq 0} \frac{\mu |l_h|}{\pi |l| l_3} + \frac{\mathbf{1}_{l_3=0}}{|l_h|} \right),$$

and we merely include the terms

$$\begin{aligned} & - \sqrt{\epsilon \nu} \left[ \hat{\delta}_{0,3}(\mu, l_h, t) - (-1)^{l_3} \hat{\delta}_{1,3}(\mu, l_h, t) \right] \mathbf{1}_{l_3 \neq 0} \frac{|l_h|^3}{\pi |l| l_3} \\ & - \sqrt{\epsilon \nu} \left[ \partial_t \hat{\delta}_{0,3}(\mu, l_h, t) - (-1)^{l_3} \partial_t \hat{\delta}_{1,3}(\mu, l_h, t) \right] \mathbf{1}_{l_3 \neq 0} \frac{|l_h|}{\pi |l| l_3} \end{aligned}$$

in the remainder  $\eta$  of Proposition 4.1. We deduce that if  $r \leq 2$ ,

$$\begin{aligned} \|\delta u_A^{int,0}(t)\|_{H^r(\omega)} & \leq CA^{1/2} \sqrt{\epsilon \nu} \sum_j \{ \|\delta_{j,3}(0)\|_4 + \|\delta_{j,3}(t)\|_4 \} \\ & + CA^{1/2} \sqrt{\epsilon \nu} \sum_j \left\{ \int_0^t \|\partial_s \delta_{j,3}(s)\|_4 ds + \sup_{s \in [0,t]} \|\delta_{j,3}(s)\|_4 \right\} \\ & + \|\delta u_{A|t=0}^{int,0}\|_{H^r(\omega)}. \end{aligned}$$

We now choose  $A$  such that

$$\sqrt{\frac{\nu}{\epsilon}} \frac{1}{A^{3/2}} = A^{1/2} \sqrt{\epsilon \nu},$$

i.e.  $A = \epsilon^{-1/2}$ . We infer that the error term in the evolution equation is of order  $\epsilon^{1/4} \nu^{1/2}$  in  $L^\infty([0, T], L^2(\omega))$ , and that

$$\begin{aligned} \|\delta u_A^{int,0}\|_{L^\infty([0, T], H^2(\omega))} & \leq C \epsilon^{1/4} \nu^{1/2} \sum_j \sup_{t \in [0, T]} \|\delta_{j,3}(t)\|_4 \\ & + C \epsilon^{1/4} \nu^{1/2} \sum_j \int_0^T \|\partial_s \delta_{j,3}(s)\|_4 ds \\ & + \|\delta u_{A|t=0}^{int,0}\|_{H^2(\omega)}. \end{aligned}$$

**5.2. Choice of the boundary conditions for  $u^{BL}$  and  $u^{int}$ .** We now explain how the boundary conditions are chosen. As before, we work with  $k_h$  fixed. Also, since the boundary conditions are all almost-periodic with respect to the fast time variable  $t/\epsilon$ , we work with a fixed frequency  $\mu \in \mathbf{R}$ . Note that this decomposition is allowed by the linearity of the equation.

We set

$$u^{BL} = \mathcal{B}(\delta_{0,h}, \delta_{1,h}),$$

where the boundary conditions  $\delta_{0,h}, \delta_{1,h}$  are yet to be defined.

In order to match the boundary conditions (2.2) with  $\sigma = 0$ , we must take  $u^{BL}$  and  $u^{int}$  such that

$$\begin{aligned} (u_h^{BL} + u_h^{int})|_{z=0} &= o(\delta), \\ \partial_z (u_h^{BL} + u_h^{int})|_{z=1} &= o(\delta), \\ (u_3^{BL} + u_3^{int})|_{z=0} &= o(\sqrt{\epsilon\nu}\delta), \\ (u_3^{BL} + u_3^{int})|_{z=1} &= o(\sqrt{\epsilon\nu}\delta), \end{aligned}$$

denoting by  $\delta$  the order of magnitude of  $\delta_j$ , in a sense to be precised later on.

We now examine each of the boundary conditions independently.

- At  $z = 0$ , the horizontal boundary condition yields

$$(5.10) \quad \hat{\delta}_{0,h}(\mu, k_h, t) + \mathbf{1}_{\mu=-\lambda_k} c_k(t) \begin{pmatrix} n_1(k) \\ n_2(k) \end{pmatrix} = 0.$$

Since  $k_h$  is fixed, note that for all  $\mu \in \mathbf{R}$ , there exists at most one  $k_3 \in \mathbf{Z}$  such that  $\lambda_{k_h, k_3} = -\mu$ , and thus the expression above is well-defined.

- Let us now tackle the vertical boundary condition at  $z = 0$ . According to the third section, the vertical component of  $u^{BL}$  at  $z = 0$  depends on  $\delta_{0,h}$ . Precisely, we recall that

$$\hat{u}_3^{BL}(\mu, k_h)|_{z=0} = \sqrt{\epsilon\nu} \sum_{\pm} \frac{\alpha_{\pm}^0}{\lambda_{\pm}} (ik_1 w_{\lambda_{\pm},1} + ik_2 w_{\lambda_{\pm},2}),$$

(up to exponentially small terms), and

$$(\alpha_{-}^0, \alpha_{+}^0) = P^{-1} \hat{\delta}_{0,h}(\mu, k_h).$$

As a consequence, in order that the vertical boundary condition at  $z = 0$  is approximately satisfied, we choose

$$(5.11) \quad \hat{\delta}_{0,3} = - \sum_{\pm} \frac{\alpha_{\pm}^0}{\lambda_{\pm}} (ik_1 w_{\lambda_{\pm},1} + ik_2 w_{\lambda_{\pm},2}).$$

- At  $z = 1$ ,  $\partial_z u_h^{int}$  is identically zero by construction of the operator  $\mathcal{U}$ , and thus we infer  $\delta_{1,h} = 0$ .
- Concerning the vertical component at  $z = 1$ , the calculation is the same as before. Since  $\delta_{1,h} = 0$ , we deduce that  $\delta_{1,3} = 0$ .

The above relations (5.10)-(5.11) allow us to write  $\delta_0$  in terms of  $\bar{u}_{int}$ . Conversely, the equation (5.7) on  $\bar{u}^{int}$  depends on  $\delta_{0,3}$ , and thus on  $\delta_{0,h}$  through the operator  $\mathcal{B}$ . In other words, there is a coupling between the boundary condition at the bottom for  $u^{BL}$ , and the equation satisfied by  $\bar{u}^{int}$ . Since  $\bar{u}^{int}$  is the only non-vanishing term in  $L^2$  norm, we choose (as is usually done in the rotating fluids literature) to write an explicit equation for  $\bar{u}^{int}$ , and to express  $u^{BL}$  in terms of  $\bar{u}^{int}$ .

**5.3. Derivation of the equation for  $\bar{u}^{int}$ .** We now compute the Ekman pumping term, that is, the right-hand side in the equation satisfied by  $c_k$  (see (5.7)). Notice that if  $k \in \mathbf{Z}^3$  and  $k_h \neq 0$ , then  $|\lambda_k| \neq 1$ . In other words, the source term in (5.7) involves only the part  $\bar{u}^{BL}$  of the boundary layer; precisely, with the notations of section 3, the decay rate of  $u^{BL}(t, \lambda_k, k_h)$  is

$$(\lambda_k^\pm)^2 = i(-\lambda_k \mp 1) + o(1),$$

which yields (remember that  $\Re(\lambda_k^\pm) > 0$ )

$$\lambda_k^\pm = \sqrt{1 \pm \lambda_k} \exp\left(\mp i \frac{\pi}{4}\right) + o(1).$$

Moreover,

$$\begin{aligned} (\alpha_-^0, \alpha_+^0) &= P^{-1}[-c_k(t)(n_1(k), n_2(k))] \\ &= -c_k(t)(n_-(k), n_+(k)), \end{aligned}$$

where

$$\begin{aligned} (n_-(k), n_+(k)) &:= P^{-1}(n_1(k), n_2(k)) \\ &= \frac{1}{2}(n_1(k) + in_2(k), n_1(k) - in_2(k)) + o(1). \end{aligned}$$

Replacing these expressions in the formula giving  $\delta_{0,3}$ , we infer

$$\hat{\delta}_{0,3}(-\lambda_k, k_h, t) = c_k(t) \sum_{\sigma \in \{-1, 1\}} \frac{n_\sigma(k)}{\lambda_k^\sigma} (ik_1 w_{\lambda^\sigma, 1} + ik_2 w_{\lambda^\sigma, 2}).$$

We deduce that  $c_k$  satisfies a linear evolution equation with a damping term, namely

$$(5.12) \quad \frac{dc_k}{dt} + |k_h|^2 c_k + \nu' |k_3|^2 c_k + \sqrt{\frac{\nu}{\epsilon}} A_k c_k(t) = 0,$$

where

$$A_k := \mathbf{1}_{k_h \neq 0} \frac{|k_h|}{2\pi |k|^2} \sum_{\pm} \frac{n_\pm(k)}{\lambda_k^\pm} (ik_1 w_{\lambda^\pm, 1} + ik_2 w_{\lambda^\pm, 2}).$$

An estimate of  $\Re(A_k)$ , where  $\Re(x)$  denotes the real part of a complex number  $x$ , is computed in Remark 5.2 below. Using Duhamel's formula, we deduce that

$$(5.13) \quad |c_k(t)| \leq \exp\left(-t \left(|k_h|^2 + \nu' |k_3|^2 + \sqrt{\frac{\nu}{\epsilon}} \Re(A_k)\right)\right) |\langle N_k, \gamma \rangle|.$$

We deduce the following Lemma:

**Lemma 5.1.** *Assume that  $\gamma \in V_0$ . Then there exists a unique solution  $\bar{u}_L^{int} \in L_{loc}^\infty(\mathbf{R}_+, V_0) \cap L_{loc}^2(\mathbf{R}_+, H_h^1(\omega))$  of the envelope equation*

$$(5.14) \quad \begin{cases} \partial_t \bar{u}_L^{int} - \Delta_h \bar{u}_L^{int} + \sqrt{\frac{\nu}{\epsilon}} S_{Ekman} [\bar{u}_L^{int}] = 0, \\ \bar{u}_L^{int}|_{t=0} = \gamma, \end{cases}$$

where the Ekman pumping term  $S_{Ekman}$  is defined by

$$S_{Ekman} [\bar{u}_L^{int}] = \sum_{k \in \mathbf{Z}^3} A_k \langle N_k, \bar{u}_L^{int} \rangle N_k.$$

Hence, in the rest of the section, we take

$$(5.15) \quad c_k(t) = \hat{\gamma}_k \exp \left( - \left( |k_h|^2 + \sqrt{\frac{\nu}{\epsilon}} A_k \right) t \right).$$

By doing so, we have neglected the vertical viscosity term  $\nu \partial_z^2$ .

**Remark 5.2.** (i) *Notice that with the scaling we have chosen for the wind-stress, there is no Ekman pumping due to the wind. Indeed, the Ekman pumping term is of order  $\sqrt{\epsilon\nu}\beta$ , which vanishes as  $\epsilon, \nu \rightarrow 0$  according to hypotheses (2.3).*

(ii) *We emphasize that the operator  $S_{Ekman}$  constructed above depends on  $\nu$  and  $\epsilon$  through the matrix  $P$ , the vectors  $w_{\lambda^\pm}$  and the eigenvalues  $\lambda_k^\pm$ . However, it is useful, for later purposes, to compute the leading order terms in  $A_k$ , which amounts to deriving an equation for the limit of the term  $\bar{u}_L^{int}$  as  $\epsilon, \nu$  vanish. Hence we now compute the limit of  $A_k$  as  $\epsilon, \nu \rightarrow 0$ .*

Recall that  $n_1(k)$  and  $n_2(k)$  are given by (1.8). Thus, at first order,

$$\begin{aligned} A_k &= \frac{|k_h|}{2\pi|k|^2} \sum_{\sigma \in \{-1, 1\}} \frac{n_1(k) - i\sigma n_2(k)}{2\lambda_k^\sigma} (ik_1 - \sigma k_2) + o(1) \\ &= \frac{|k_h|^2}{8\sqrt{2}\pi^2|k|^2} \left[ \frac{1 - \lambda_k}{\sqrt{1 + \lambda_k}} (1 - i) + \frac{1 + \lambda_k}{\sqrt{1 - \lambda_k}} (1 + i) \right] + o(1) \\ &= R_k + iI_k + o(1) \end{aligned}$$

where  $R_k$  and  $I_k$  are real numbers given by

$$(5.16) \quad R_k := \frac{1 - \lambda_k^2}{8\sqrt{2}\pi^2} \left( \frac{1 + \lambda_k}{\sqrt{1 - \lambda_k}} + \frac{1 - \lambda_k}{\sqrt{1 + \lambda_k}} \right) > 0$$

$$(5.17) \quad I_k := \frac{1 - \lambda_k^2}{8\sqrt{2}\pi^2} \left( \frac{1 + \lambda_k}{\sqrt{1 - \lambda_k}} - \frac{1 - \lambda_k}{\sqrt{1 + \lambda_k}} \right).$$

Recalling the definition of  $\lambda_k$ , we deduce that

$$R_k \geq C \frac{|k_h|^2}{|k|^2},$$

and thus

$$\Re(A_k) \geq C \frac{|k_h|^2}{|k|^2}$$

for  $\epsilon, \nu$  small enough.

To conclude this paragraph, we now give estimates on the boundary conditions  $\delta_{0,h}, \delta_{0,3}$  in the norm  $\|\cdot\|_s$ .

**Lemma 5.3.** *Assume that  $\delta_{0,h}, \delta_{0,3}$  are given by (5.10)-(5.11). Then the following estimates hold*

$$\|\delta_{0,h}(t)\|_s \leq \left( \sum_k |k|^{2(s+1)} |c_k(t)|^2 \right)^{1/2} \leq C \|\gamma\|_{H^{s+1}(\omega)},$$

and

$$\|\delta_{0,3}(t)\|_s \leq \|\delta_{0,h}(t)\|_{s+1} \leq C \|\gamma\|_{H^{s+2}(\omega)}.$$

*Proof.* The bound on  $\delta_{0,h}$  is easily deduced from inequality (5.13) together with formula (5.10) and the Cauchy-Schwarz inequality. Concerning the other bound, let us recall that if  $\mu = -\lambda_k$ , for  $k \in \mathbf{Z}^3$ , then the decay rates  $\lambda^\pm(-\lambda_k, k_h)$  satisfy

$$|\lambda^\pm| \leq C \left( \frac{|k_3|}{|k_h|} + 1 \right).$$

Plugging this estimate into (5.11) yields the desired inequality.  $\square$

**5.4. Estimates on the boundary layer and corrector terms.** Now that  $\bar{u}^{int}$  is rigorously defined by Lemma 5.1, we may define the other terms  $v^{int,0}$ ,  $\delta u^{int,0}$  and  $u^{BL,0}$ . We have gathered in this paragraph some estimates which are needed in the proof of Theorem 2.1. Before deriving these estimates, let us emphasize that equation (2.1) supplemented with homogeneous boundary conditions at  $z = 0$  and  $z = 1$  is a contraction in  $L^2$ . As a consequence, it is sufficient to prove the Theorem for arbitrarily smooth initial data. Thus, without any loss of generality, we assume from now on that the initial data  $\gamma$  only has a finite number of Fourier modes, that is

$$\gamma = \sum_{|k_h| \leq N} \sum_{|k_3| \leq K} \hat{\gamma}_k N_k,$$

and we do not keep track of the regularity of  $\gamma$ , since all  $H^r$  norms are controlled by the  $L^2$  norm (with constants depending on  $N$  and  $K$ ).

- The boundary layer term of order zero, denoted by  $u^{BL}$ , is defined by

$$u^{BL,0} = \mathcal{B}(\delta_{0,h}, 0),$$

where  $\delta_{0,h}$  is given by (5.10). Thus we deduce that the decay rates  $\lambda^\pm(\mu, k_h)$  in the boundary layer term  $u^{BL,0}$  are all of order one. Consequently, according to (3.11), the boundary layer term  $\bar{u}^{BL,0}$  satisfies

$$\begin{aligned} & \|\bar{u}_h^{BL,0}(t)\|_{L^2(\omega)} + (\epsilon\nu)^{-1/2} \|\bar{u}_3^{BL,0}(t)\|_{L^2(\omega)} \\ (5.18) \quad & \leq C \|\delta_{0,h}(t)\|_{H^1(\omega_h)} (\epsilon\nu)^{1/4} \end{aligned}$$

$$(5.19) \quad \leq C \|\gamma\|_{L^2(\omega)} (\epsilon\nu)^{1/4}.$$

Moreover, the definition of  $c_k$  entails that  $\bar{u}^{BL,0}$  is an approximate solution of (2.1), with an error term (due to the fact that  $\partial_t c_k$  does not vanish), which is bounded in  $L^2(\omega)$  by

$$\begin{aligned} & C(\epsilon\nu)^{1/4} \left\| \partial_t \bar{u}_L^{int} \right\|_{L^2([0,T] \times \omega)} \\ & \leq C(\epsilon\nu)^{1/4} \left( \sum_k \int_0^T |\hat{\gamma}_k|^2 \left( |k_h|^4 + \frac{\nu}{\epsilon} |A_k|^2 \right) \exp\left(-c\sqrt{\frac{\nu}{\epsilon}}t\right) dt \right)^{1/2} \\ & \leq C(\epsilon\nu)^{1/4} \left[ 1 + \sqrt{\frac{\nu}{\epsilon}} \right]^{1/2} \|\gamma\|_{L^2(\omega)} \\ & \leq C((\epsilon\nu)^{1/4} + \nu^{1/2}). \end{aligned}$$

Thus we may include the corresponding error term in the function  $\eta$  of Proposition 4.1.

Notice that the Dirichlet boundary condition at  $z = 0$  also generates a bounded vertical profile, namely

$$\begin{aligned} v^{sing,0}(t, x) & := -\frac{1}{2\pi} \sum_{k_3 \in \mathbf{Z}^*} c_{(0,0,k_3)}(t) e^{i \operatorname{sign}(k_3) \frac{t}{\epsilon}} \begin{pmatrix} \operatorname{sign}(k_3) \\ i \\ 0 \end{pmatrix} \\ & = -\frac{1}{2\pi} \sum_{k_3 \in \mathbf{Z}^*} \hat{\gamma}_{(0,0,k_3)} e^{i \operatorname{sign}(k_3) \frac{t}{\epsilon}} \begin{pmatrix} \operatorname{sign}(k_3) \\ i \\ 0 \end{pmatrix} \end{aligned}$$

Exactly as in paragraph 4.3, we then define a corrector  $\bar{u}^{sing,0}$  and a function

$$(5.20) \quad u^{sing,0} = \bar{u}^{sing,0} + v^{sing,0}$$

such that

$$\begin{aligned} \partial_t \bar{u}^{sing,0} + \frac{1}{\epsilon} e_3 \wedge \bar{u}^{sing,0} - \nu \partial_z^2 \bar{u}^{sing,0} &= 0 \\ \bar{u}|_{t=0} &= -v|_{t=0}, \\ \bar{u}|_{z=0} &= 0, \quad \partial_z \bar{u}|_{z=1} = 0. \end{aligned}$$

We have

$$\|u^{sing,0}\|_{L^\infty([0,\infty), L^2(\omega))} \leq C \|\gamma\|_{L^2(\omega)}$$

and

$$\|u^{sing,0}(t)\|_{L^2(\omega)} \leq C \|\gamma\|_{L^2(\omega)} (\nu t)^{1/4}.$$

• The term  $v^{int,0}$  is given by (5.6), in which  $\delta_{1,3} = 0$  and  $\delta_{0,3}$  is defined in (5.11). As a consequence,  $v^{int,0}$  satisfies the estimate

$$(5.21) \quad \begin{aligned} \|v^{int,0}(t)\|_{L^2(\omega)} &\leq C \|\delta_{0,3}(t)\|_{L^2(\omega_h)} (\epsilon\nu)^{1/2} \\ &\leq C \|\gamma\|_{L^2(\omega)} (\epsilon\nu)^{1/2}. \end{aligned}$$



• At last, the term  $\delta u^{int,0}$  is given by equation (5.9). As stated earlier, we choose a special solution of (5.9) in order to keep track of the exponential decay of  $\delta_{0,3}$ . Indeed, we have, for all  $k \in \mathbf{Z}^3 \setminus \{0\}$ ,

$$\hat{\delta}_{0,3}(-\lambda_k, k_h, t) = i\hat{\gamma}_k \exp\left(-\left(|k_h|^2 + \sqrt{\frac{\nu}{\epsilon}} A_k\right)t\right) \sum_{\sigma \in \{-1,1\}} \frac{n_\sigma(k)}{\lambda_k^\sigma} k_h \cdot w^{\lambda_k^\sigma}.$$

Thus we choose for  $\delta_{cl}$ ,  $|l| \leq A$ , the special solution constructed in Remark 6.1 in the Appendix. With this choice, we obtain

$$\|\delta u_A^{int,0}(t)\|_{H^2} \leq C\epsilon^{1/4}\nu^{1/2} \left( \sum_{k \in \mathbf{Z}^3} \frac{(1 + |k_3|)^4}{\left|\frac{1}{|k_3|^3} - \sqrt{\epsilon\nu}|\Im(A_k)|\right|^2} |\gamma_k|^2 \exp\left(-2\sqrt{\frac{\nu}{\epsilon}}\Re(A_k)t\right) \right)^{1/2},$$

where  $\Im(x)$  denotes the imaginary part of a complex number  $x$ . According to Remark 5.2,  $\Im(A_k) = I_k + o(1)$ , and thus, using the explicit formula for  $I_k$  derived in Remark 5.2,  $\Im(A_k) = O(1)$ . Moreover, since  $\hat{\gamma}_k = 0$  for  $k$  large enough, we deduce that

$$\frac{1}{|k_3|^3} - \sqrt{\epsilon\nu}|\Im(A_k)| \geq \frac{1}{2|k_3|^3}$$

for  $\epsilon, \nu$  small enough and for all  $k$  such that  $\hat{\gamma}_k \neq 0$ . The above estimate then becomes

$$(5.22) \quad \|\delta u_A^{int,0}(t)\|_{H^2} \leq C\epsilon^{1/4}\nu^{1/2} \|\gamma\|_{L^2(\omega)} \exp\left(-c\sqrt{\frac{\nu}{\epsilon}}t\right).$$

**5.5. Conclusion: proof of Theorem 2.1 when  $\sigma = 0$ .** The idea is to use the construction of the previous paragraphs in order to compute an approximate solution of the evolution equation (2.1), which satisfies the boundary conditions up to sufficiently small error terms. The order of approximation required on the boundary condition is quantified by Lemma 1 in the Appendix, and thus we build interior and boundary layer terms until the conditions of the Lemma 1 are met.

Let us now explain the construction in detail.

• First, we set

$$u^0 := u^{int} + u^{sing,0} + u^{BL,0},$$

where  $u^{int}$ ,  $u^{sing,0}$  and  $u^{BL,0}$  have been defined in the previous paragraphs. We have seen that  $u^0$  is an approximate solution of the evolution equation (2.1), with error terms which are all  $o(1)$  in  $L^2$ . We now evaluate the error on the boundary conditions. Indeed, setting  $u' := u - u^0$ , we have proved that  $u'$  is an approximate solution of (2.1), with some boundary conditions  $\tilde{\delta}_0, \delta'_1$ , namely

$$\begin{aligned} u_h|_{z=0} &= \delta'_{0,h}, & \partial_z u'_h|_{z=1} &= \delta'_{1,h}, \\ u'_3|_{z=0} &= \delta'_{0,3}, & u'_3|_{z=1} &= \delta'_{0,1}. \end{aligned}$$

Thus we have to estimate  $\gamma' := u'|_{t=0}$ , together with the terms  $\delta'_0, \delta'_1$ .

First, since  $\bar{u}|_{t=0}^{int} = \gamma$  and  $u|_{t=0}^{sing,0} = 0$ , we obtain

$$(5.23) \quad \gamma' = -u|_{t=0}^{BL,0} - v|_{t=0}^{int,0} - \delta u|_{t=0}^{int,0},$$

where  $u|_{t=0}^{BL,0}$ ,  $v|_{t=0}^{int,0}$  and  $\delta u|_{t=0}^{int,0}$  satisfy the estimates (5.18), (5.21), and (5.22) respectively. Thus, remembering that  $\gamma$  has a finite number of Fourier modes, we have

$$\|\gamma'\|_{L^2} \leq C \left( \|\gamma\|_{L^2}(\epsilon\nu)^{1/4} + \|\gamma\|_{L^2}(\epsilon\nu)^{1/2} + \|\gamma\|_{L^2}\epsilon^{1/4}\nu^{1/2} \right).$$

Then, by construction of the operators  $\mathcal{U}$  and  $\mathcal{B}$ , the horizontal remainder boundary term at  $z = 1$  is exponentially small: indeed, we have  $\partial_z u|_{z=1}^{int} = 0$ , and consequently,

$$(5.24) \quad \delta'_{1,h} = - \sum_{\mu, k_h} \sum_{\sigma \in \{-1,1\}} \alpha_\sigma^0 \frac{\lambda^\sigma}{\sqrt{\epsilon\nu}} e^{-\frac{\lambda^\sigma}{\sqrt{\epsilon\nu}}} w_{\lambda^\sigma} e^{ik_h \cdot x_h} e^{i\mu \frac{t}{\epsilon}}.$$

We infer that

$$(5.25) \quad \begin{aligned} \|\delta'_{1,h}\|_0^2 &\leq C \exp\left(-\frac{C}{K\sqrt{\epsilon\nu}}\right) \sum_{|k_h| \leq N} \sum_{|k_3| \leq K} |\hat{\delta}_{0,h}(-\lambda_k, k_h, t)|^2 \\ &\leq C \|\delta_{0,h}\|_0^2 \exp\left(-\frac{C}{K\sqrt{\epsilon\nu}}\right). \end{aligned}$$

Similarly,

$$(5.26) \quad \delta'_{1,3} = \sum_{\mu, k_h} \sum_{\sigma \in \{-1,1\}} \alpha_\sigma^0 \frac{\sqrt{\epsilon\nu}}{\lambda^\sigma} ik_h \cdot w_{\lambda^\sigma} e^{-\frac{\lambda^\sigma}{\sqrt{\epsilon\nu}}} e^{ik_h \cdot x_h} e^{i\mu \frac{t}{\epsilon}},$$

and thus

$$(5.27) \quad \|\delta'_{1,3}\|_0 \leq CK\sqrt{\epsilon\nu} \exp\left(-\frac{C}{K\sqrt{\epsilon\nu}}\right) \|\delta_{0,h}\|_2.$$

The treatment of the vertical boundary condition at  $z = 0$  is easier. Indeed, since  $\delta_1 = 0$ , we have  $\delta'_{0,3} = 0$ , because

$$(5.28) \quad \delta'_{0,3} = - \sum_{\mu, k_h} \sum_{\sigma \in \{-1,1\}} \alpha_\sigma^1 \frac{\epsilon\nu}{(\lambda^\sigma)^2} ik_h \cdot w_{\lambda^\sigma} e^{-\frac{\lambda^\sigma}{\sqrt{\epsilon\nu}}} e^{ik_h \cdot x_h} e^{i\mu \frac{t}{\epsilon}} = 0.$$

There remains to compute  $\delta'_{0,h}$ ; because of the terms  $\delta u_A^{int,0}$  and  $v^{int,0}$ ,  $\delta'_{0,h}$  is the largest term of all. Precisely, we have

$$(5.29) \quad \begin{aligned} \delta'_{0,h}(t) &= - \left[ v|_{z=0}^{int,0}(t) + \delta u_{A,h}|_{z=0}^{int,0}(t) \right] \\ &= -\sqrt{\epsilon\nu} \sum_{\mu, k_h \neq 0} ik_h \cdot \frac{\hat{\delta}_{0,3}(\mu, k_h, t)}{|k_h|^2} e^{ik_h \cdot x_h} e^{i\mu \frac{t}{\epsilon}} \\ (5.30) \quad &- \sum_{k_h} \sum_{|k_3| \leq A} \delta c_k(t) e^{ik_h \cdot x_h} e^{-i\lambda_k \frac{t}{\epsilon}} n_h(k), \end{aligned}$$

and thus there exists a constant  $c > 0$  such that for all  $t \geq 0$

$$\begin{aligned} \|\delta'_{0,h}(t)\|_{L^2} &\leq C \left( \sqrt{\epsilon\nu} \|\delta_{0,3}(t)\|_0 + \|\delta u_A^{int,0}(t)\|_{H^1} \right) \\ &\leq C \epsilon^{1/4} \nu^{1/2} \|\gamma\|_{L^2} \exp \left( -c \sqrt{\frac{\nu}{\epsilon}} t \right). \end{aligned}$$

Now, the remaining boundary terms  $\delta'_{1,h}, \delta'_{1,3}, \delta'_{0,3}$  are all of order  $o(\epsilon)$  according to (5.25)-(5.28). Notice furthermore that by construction,

$$\int_{\omega_h} \delta'_{j,3} = 0 \quad \text{for } j = 0, 1.$$

Consequently,  $\delta'_{1,h}, \delta'_{1,3}, \delta'_{0,3}$  all match the conditions of the stopping Lemma 1.

• We now have to continue the construction with the “bad” part of the remaining boundary conditions, i.e.  $\delta'_{0,h}$ . Let us define the boundary layer term

$$\delta u^{BL,0} := \mathcal{B}(\delta'_{0,h}, 0).$$

According to (3.11),

$$\|\delta u^{BL,0}\|_{L^\infty([0,\infty), L^2(\omega))} \leq C(\epsilon\nu)^{1/4} \|\delta'_{0,h}\|_0 \leq C \epsilon^{1/2} \nu^{3/4} \|\gamma\|_{L^2},$$

and  $\delta u^{BL,0}$  is an approximate solution of equation (2.1) with a  $o(1)$  error term. Moreover, notice that for all  $t \geq 0$ , for all  $s \geq 0$ ,

$$\|\delta u_{3|z=0}^{BL,0}(t)\|_{H^s(\omega_h)} \leq C \epsilon^{3/4} \nu \|\gamma\|_{L^2} \exp \left( -c \sqrt{\frac{\nu}{\epsilon}} t \right).$$

We deduce that for all  $T > 0$ ,

$$\|\delta u_{3|z=0}^{BL,0}\|_{L^2((0,T), H^s(\omega_h))} \leq C \epsilon^{3/4} \nu \|\gamma\|_{L^2} \left( \frac{\epsilon}{\nu} \right)^{1/4} = o(\epsilon).$$

Thus  $\delta u_{3|z=0}^{BL,0}$  satisfies the hypotheses of Lemma 1. Additionally,  $\delta u_{|z=1}^{BL,0}$  is exponentially small, and thus also satisfies the conditions of Lemma 1.

• We now define the approximate solution  $u_{app}$  by

$$u_{app} := u^{int} + \bar{u}^{BL,0} + u^{sing,0} + \delta u^{BL,0} + w^{stop},$$

where  $w^{stop}$  is defined by Lemma 1 with the remaining boundary conditions. By construction,  $u_{app}$  is an approximate solution of the evolution equation (2.1), with

$$u_{app}|_{t=0} = u|_{t=0} + o(1),$$

and  $u_{app}$  satisfies homogeneous boundary conditions at  $z = 0$  and  $z = 1$ . By a simple energy estimate analogous to that of Proposition 4.1, we deduce that

$$\|u - u_{app}\|_{L^\infty((0,T), L^2(\omega))} \rightarrow 0 \quad \forall T > 0.$$

Since all the terms in  $u_{app}$  except  $\bar{u}_L^{int}$  and  $u^{sing,0}$  are  $o(1)$  in  $L^2$  norm, Theorem 2.1 is proved.

**Remark 5.4.** *The proof of Theorem 2.1 for  $\sigma = 0$  is valid for all ranges of  $\epsilon, \nu$  such that  $\epsilon, \nu \rightarrow 0$ . In particular, we do not assume that  $\nu = O(\epsilon)$ . However, in the case  $\nu \gg \epsilon$ , all the modes such that  $k_h \neq 0$  in  $\bar{u}_L^{int}$  are of order  $\exp(-c\sqrt{\nu/\epsilon}t)$ , and vanish exponentially for all  $t > 0$ . Thus the effect of the heterogeneous horizontal modes of the initial data vanishes outside an initial layer of size  $\sqrt{\epsilon/\nu}$ . On the other hand, the modes corresponding to  $k_h = 0$  are not damped, and give rise to the singular profile  $u^{sing,0}$ . Eventually, for  $t \gg \sqrt{\epsilon/\nu}$ , we have*

$$u(t) \approx \sum_{k_3 \in \mathbf{Z}^*} \hat{\gamma}_{(0,0,k_3)} N_{(0,0,k_3)} + u^{sing,0}.$$

## 6. CONCLUDING REMARKS

The previous study allows to characterize the linear response of the ocean to some wind forcing, which admits fast time oscillations and may be resonant with the Coriolis force. In addition to the usual Ekman layer, we have exhibited another - much larger - boundary layer, and some global vertical profile. Note that these effects do not modify the mean motion (i.e. the  $L^2$  asymptotics) when considering moderate times, say for instance  $t \ll \frac{1}{\nu}$ .

The destabilization will appear for longer times. If the forcing is resonant, the penalization by the Coriolis force has no averaging effect on  $\sigma_{res}$  and we are brought back to the study of a heat equation with small conductivity  $\nu$ . The point is therefore to study the long time behaviour of this equation, for which the vertical profile is a stationary solution, and a global attractor.

**6.1. Nonlinear stability of the singular profile  $v^{sing}$ .** In order to take into account more physics in our model, the first point is to understand the nonlinear response of the ocean to the same wind forcing. In other words, we are interested in the asymptotic behaviour of the full Navier-Stokes-Coriolis equation (1.5)(2.2) including in particular the nonlinear contribution of the convection.

In the case of a non-resonant forcing, the asymptotic motion of the fluid is obtained by some filtering method : there is indeed two time scales, a rapid time scale at which the fluid oscillates according to the modes of the linear penalization, and a slow one which characterizes the nonlinear evolution of the wave envelopes. The boundary effects do not play any role in the nonlinear process since they are localized in the vicinity of the surface. They contribute to the envelope equations only by the Ekman pumping. In the case of a resonant forcing, the boundary effects - which are not expected to be localized in the same way - could play a different role.

In a forthcoming paper [5], we investigate in particular the role of the stationary profile

$$v^{sing} = \beta\sigma_+(1,0)z \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} e^{i\frac{z}{\epsilon}} + \beta\sigma_-(-1,0)z \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} e^{-i\frac{z}{\epsilon}}$$

The first observation is that it is also a stationary solution to the nonlinear equation (1.5). We indeed check that the convection term

$$(v^{sing} \cdot \nabla)v^{sing} = (v_h^{sing} \cdot \nabla_h)v^{sing} + v_3^{sing}\partial_z v^{sing} = 0.$$

We are then able to prove that  $v^{sing}$  is nonlinearly stable in the sense that solutions to (1.5)-(2.2) with

$$\sigma = \sigma_+(1,0) \begin{pmatrix} 1 \\ i \end{pmatrix} e^{i\frac{z}{\epsilon}} + \sigma_-(-1,0) \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\frac{z}{\epsilon}}$$

remain close to  $v^{sing}$  in a suitable norm and on a time interval independent of  $\epsilon, \nu$ .

The difficulty here comes from the fact that we have **no uniform a priori bound** on  $u$ . The energy estimate on  $u$  has indeed a source term - coming from the surface boundary condition - which is of order  $O(\beta\nu^{1/2}|\sigma|)$ . The  $L^2$  bound on  $u$  is therefore exponentially increasing, with a rate depending on  $\epsilon, \nu$  and  $\beta$ . Instead of a priori estimates, we have thus to use **loop estimates**, and to conclude by some Gronwall's argument.

**6.2. Towards a more physically relevant model.** The present theory of the wind-driven circulation of a fluid of uniform density is actually inadequate to capture the velocity structure of the oceans. We indeed expect the wind forcing to modify in depth the circulation. The profile arising from the resonant part of the forcing and the Ekman pumping are not enough to get a relevant description of that vertical structure.

We will mention here many phenomena that have been neglected in our study and which seem to be crucial to obtain realistic models.

- we first need to consider the **variations of the Coriolis parameter**, keeping at least the  $\beta$ -plane approximation :

$$\Omega = f + \beta y$$

where  $y$  is the coordinate measuring the latitude. Such a spatial dependence of  $\Omega$  is necessary to derive Sverdrup's theory of horizontal transport, which is still one of the foundations of all theories of the ocean circulation (see [17] for instance).

From a mathematical point of view, we refer to [4][7] and references therein for some preliminary studies on inhomogeneous rotating fluids.

- the vertical structure of the ocean circulation is also related to the variations of the density  $\rho$ , the so-called **stratification** of the oceans. The theoretical works of Rhines and Young [18] have brought some

understanding about geostrophic contours, potential vorticity homogenization and their role in shaping the pattern of circulation. Luyten, Pedlosky and Stommel [13] have then developed a theory for the full density and velocity structure of the wind-driven circulation by going beyond the quasi-geostrophic approximation to consider the important effect of the ventilation of the thermocline which occurs as oceanic density surfaces rise to intersect the oceanic mixed layer.

However, to our knowledge, there is no mathematical contribution on that topics, the first difficulty being to determine some suitable functional framework to deal with the inhomogeneous incompressible Navier-Stokes equations. We refer to [3] and [12] for more precisions on the inhomogeneous incompressible Navier-Stokes equations.

- we finally have to take into account the bottom topography which may have an important contribution to the mean circulation as proved for instance in [4] or [8].

The crucial point to understand these features from a mathematical point of view is to get a description of the boundary layer operator which is not based on the Fourier transform, but on the spectral decomposition of the Coriolis operator. The Coriolis penalization becomes indeed in the two first cases a skew-symmetric operator with non-constant coefficients (depending on  $\Omega$  and  $\rho$ ). We therefore have to develop new tools to obtain the asymptotic expansions in a more abstract and systematic way.

#### APPENDIX A : THE STOPPING CONDITION

We have postponed here the statement and the proof of the stopping condition since it is just a technical result (based on straightforward computations) which is used in several places (Sections 4 and 5).

**Lemma 1** (Stopping condition). *Let  $\delta_0, \delta_1 \in L^\infty(\mathbf{R}^+, H^2(\omega_h))$  be two families such that*

$$\int (\delta_{1,3} - \delta_{0,3}) dx_h = 0$$

and

$$\frac{1}{\epsilon} \|\delta_i\|_{H^1(\omega_h)} \rightarrow 0, \quad \|\delta_i\|_{H^3(\omega_h)} \rightarrow 0 \text{ and } \|\partial_t \delta_i\|_{H^1(\omega_h)} \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$

Then there exists a family  $w \in L^\infty(\mathbf{R}^+, L^2(\Omega))$  with  $\nabla \cdot w = 0$  such that

$$w|_{z=0} = \delta_0, \quad w_3|_{z=1} = \delta_{1,3} \text{ and } \partial_z w_h|_{z=1} = \delta_{1,h}$$

and satisfying the following estimates

$$\|w\|_{L^2(\Omega)} \rightarrow 0 \text{ and } \left\| \partial_t w + \frac{1}{\epsilon} Lw - \nu \partial_{zz} w - \Delta_h w \right\|_{L^2(\Omega)} \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

*Proof.* Here we have to build a family  $w \in L^\infty(\mathbf{R}^+, L^2(\Omega))$  with  $\nabla \cdot w = 0$  such that

$$w|_{z=0} = \delta_0, \quad w_3|_{z=1} = \delta_{1,3} \text{ and } \partial_z w_h|_{z=1} = \delta_{1,h}.$$

Of course it is not uniquely defined. We just want to obtain one such family satisfying further suitable continuity estimates.

Given any two-dimensional vector field  $w_h$ , we get a divergence-free vector field by setting

$$w_3(x_h, z) = w_3(x_h, 0) - \int_0^z (\partial_1 w_1 + \partial_2 w_2)(x_h, z') dz'.$$

In order that the boundary conditions on  $w_3$  are satisfied, the only condition on  $w_h$  is therefore

$$\int_0^1 (\partial_1 w_1 + \partial_2 w_2)(x_h, z') dz' + \delta_{1,3}(x_h) - \delta_{0,3}(x_h) = 0.$$

We therefore choose

$$\begin{aligned} w_1(x_h, z) &= \delta_{0,1}(x_h) + \delta_{1,1}(x_h)z + \partial_1 \phi(x_h)z(1-z)^2, \\ w_2(x_h, z) &= \delta_{0,2}(x_h) + \delta_{1,2}(x_h)z + \partial_2 \phi(x_h)z(1-z)^2, \end{aligned}$$

with

$$\nabla_h \cdot \delta_{0,h} + \frac{1}{2} \nabla_h \cdot \delta_{1,h} + \frac{1}{12} \Delta_h \phi + \delta_{1,3} - \delta_{0,3} = 0.$$

Standard elliptic estimates give for any  $s \geq 0$

$$\|\phi\|_{H^{s+1}(\omega_h)} \leq C(\|\delta_0\|_{H^s(\omega_h)} + \|\delta_1\|_{H^s(\omega_h)}).$$

Therefore

$$\|w\|_{H^2(\Omega)} \leq C(\|\delta_0\|_{H^3(\omega_h)} + \|\delta_1\|_{H^3(\omega_h)})$$

so that, using the assumptions on  $\delta_0, \delta_1$ ,

$$\|w\|_{H^2(\Omega)} \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Furthermore, since  $w$  is given in terms of  $\delta_0, \delta_1$  by linear relations with constant coefficients,

$$\|\partial_t w\|_{L^2(\Omega)} \leq C(\|\partial_t \delta_0\|_{H^1(\omega_h)} + \|\partial_t \delta_1\|_{H^1(\omega_h)}).$$

We conclude, using again the assumptions on  $\delta_0, \delta_1$  that

$$\left\| \partial_t w + \frac{1}{\epsilon} Lw - \nu \partial_{zz} w - \Delta_h w \right\|_{L^2(\Omega)} \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

□

## APPENDIX B : THE SMALL DIVISOR ESTIMATE

We recall here the by-now standard arguments used to obtain some estimate for the solution to fast-oscillating linear equation with non-resonant source terms :

$$(6.1) \quad \partial_t w + \frac{1}{\epsilon} \mathbb{P}(w) - \nu \Delta_h w - \nu \partial_{zz}^2 w = \Sigma$$

where the horizontal Fourier mode  $l_h$  is fixed and

$$\Sigma(t) = e^{il_h \cdot x_h} \sum_{\mu} \sum_{\substack{k_3 \in \mathbf{Z} \\ \mu \neq -\lambda_k}} s(\mu, k, t) e^{i\mu \frac{t}{\epsilon}} N_k.$$

We further assume that the frequencies  $\mu$  belong either to  $\{-\lambda_l, l_3 \in \mathbf{Z}^3\}$ , or to some finite set  $M$ . The integer  $k$  in the right-hand side above stands for  $(l_h, k_3)$ .

The small divisor estimate is the following:

**Lemma 2.** *Let  $w$  be the solution of (6.1), i.e. for all  $l = (l_h, l_3)$  with  $l_3 \in \mathbf{Z}$ ,*

$$\partial_t w_l + (|l_h|^2 + \nu' |l_3|^2) w_l = \sum_{\mu \neq -\lambda_l} s(\mu, l, t) e^{i(\mu + \lambda_l) \frac{t}{\epsilon}}.$$

*Then there exists a constant  $C$  such that for all  $t > 0$ ,  $r > 0$ , for all  $K > 0$ , we have*

$$\begin{aligned} \|\mathbb{P}_K w(t)\|_{H^r(\omega)} &\leq C\epsilon \left\{ \|s|_{t=0}\|_{r,K} \exp(-(|l_h|^2 + \nu' l_3^2)t) + \|s(t)\|_{r,K} \right\} \\ &+ C\epsilon \int_0^t \|\partial_t s(u)\|_{r,K} \exp(-(|l_h|^2 + \nu' l_3^2)(t-u)) du \\ &+ C\epsilon \sup_{u \in [0,t]} \|s(u)\|_{r,K} \\ &+ \|\mathbb{P}_K w|_{t=0}\|_{H^r(\omega)}, \end{aligned}$$

where the norm  $\|\cdot\|_{r,K}$  is defined by

$$\begin{aligned} \|s(t)\|_{r,K}^2 &:= \sum_{|l| \leq K} \sum_{\substack{k_3 \in \mathbf{Z} \\ k_3 \neq l_3}} |k_3|^8 |l|^{2r} |s(-\lambda_k, l, t)|^2 \\ &+ \sum_{|l| \leq K} \sum_{\substack{\mu \in M \\ \mu \neq -\lambda_l}} (1 + \mathbf{1}_{|\mu|=1} |l|^4) |l|^{2r} |s(\mu, l, t)|^2. \end{aligned}$$

We recall that the notation  $\mathbb{P}_K$  stands for the projection onto the vector space generated by  $N_k$  for  $|k| \leq K$ .

*Proof.* For all  $K > 0$ , define

$$w_K := \mathbb{P}_K w = \sum_{|k| \leq K} w_l N_l.$$



We deduce from Duhamel's formula that

$$(6.2) \quad |w_l(t)| \leq |w_l(0)| \exp(-(|l_h|^2 + \nu' l_3^2)t) + \left| \int_0^t \sum_{\mu \neq -\lambda_l} s(\mu, l, u) e^{i(\mu + \lambda_l) \frac{u}{\epsilon}} \exp(-(|l_h|^2 + \nu' l_3^2)(t - u)) du \right|.$$

Integrating by parts, we get

$$\begin{aligned} & \left| \int_0^t s(\mu, l, u) e^{i(\lambda_l + \mu) \frac{u}{\epsilon}} \exp(-(|l_h|^2 + \nu' l_3^2)(t - u)) du \right| \\ & \leq \frac{\epsilon}{|\lambda_l + \mu|} |s(\mu, l, t)| + \frac{\epsilon}{|\lambda_l + \mu|} |s(\mu, l_h, 0)| e^{-(|l_h|^2 + \nu' l_3^2)t} \\ & + \frac{\epsilon}{|\lambda_l + \mu|} \int_0^t (|l_h|^2 + \nu' |l_3|^2) |s(\mu, l, u)| \exp(-(|l_h|^2 + \nu' l_3^2)(t - u)) du \\ & + \frac{\epsilon}{|\lambda_l + \mu|} \int_0^t |\partial_u s(\mu, l, u)| \exp(-(|l_h|^2 + \nu' l_3^2)(t - u)) du. \end{aligned}$$

Plugging this inequality back into (6.2), we deduce that

$$\begin{aligned} |w_l(t)| & \leq |w_l(0)| \exp(-(|l_h|^2 + \nu' l_3^2)t) \\ & + C\epsilon \sum_{\mu \neq -\lambda_l} \frac{|s(\mu, l, t)|}{|\lambda_l + \mu|} \\ & + C\epsilon \sum_{\mu \neq -\lambda_l} \frac{|s(\mu, l_h, 0)|}{|\lambda_l + \mu|} \exp(-(|l_h|^2 + \nu' l_3^2)t) \\ & + C\epsilon \int_0^t F_l(u) \exp(-(|l_h|^2 + \nu' l_3^2)(t - u)) du, \end{aligned}$$

where

$$\begin{aligned} F_l(u) & := \sum_{\mu \neq -\lambda_l} \frac{1}{|\lambda_l + \mu|} |\partial_t s(\mu, l, u)| \\ & + (|l_h|^2 + \nu' |l_3|^2) \sum_{\mu \neq -\lambda_l} \frac{1}{|\lambda_l + \mu|} |s(\mu, l, u)|. \end{aligned}$$

There remains to derive bounds for quantities of the type

$$\sum_{\mu \neq -\lambda_l} \frac{1}{|\mu + \lambda_l|} |s(\mu, l, u)|.$$

Remember that either  $\mu = -\lambda_k$  for some  $k = (l_h, k_3) \in \mathbf{Z}^3$  with  $k_3 \neq -l_3$ , or  $\mu \in M$ , where  $M$  is a finite set. Thus

$$\begin{aligned}
 & \left( \sum_{\mu \neq -\lambda_l} \frac{1}{|\mu + \lambda_l|} |s(\mu, l, u)| \right)^2 \\
 \leq & 2 \left( \sum_{k_3 \neq l_3} \frac{1}{|\lambda_l - \lambda_k|} |s(-\lambda_k, l, u)| \right)^2 \\
 & + 2 \left( \sum_{\substack{\mu \in M, \\ \mu \neq -\lambda_l}} \frac{1}{|\mu + \lambda_l|} |s(\mu, l, u)| \right)^2 \\
 \leq & C \sum_{k_3 \neq l_3} |k_3|^2 \frac{1}{|\lambda_l - \lambda_k|^2} |s(-\lambda_k, l, u)|^2 \\
 & + C \sum_{\substack{\mu \in M, \\ \mu \neq -\lambda_l}} \frac{1}{|\mu + \lambda_l|^2} |s(\mu, l, u)|^2.
 \end{aligned}$$

Notice that the function  $l_3 \mapsto \lambda_l$  is monotonous for  $l_h$  fixed. Hence  $|\lambda_l - \lambda_k|$  is minimal for  $k_3 = l_3 \pm 1$ . Consequently, it is easily checked that for all  $l_3 \in \mathbf{Z}$ ,

$$|\lambda_l - \lambda_k|^{-1} \leq C \frac{|k|^3}{|l_h|^2}.$$

Moreover, if  $\mu \in M$ , then either  $\mu \notin \{0, 1, -1\}$ , and then  $\mu$  is not an accumulation point of  $\text{sp}(iL)$ ; in this case

$$|\lambda_l - \mu|^{-1} \leq C,$$

or  $\mu = 0$ , and then

$$|\lambda_l|^{-1} \leq C \frac{|l|}{|l_3|},$$

or  $\mu \in \{1, -1\}$ , and then

$$(6.3) \quad |\lambda_l - \mu|^{-1} \leq C \frac{|l|^2}{|l_h|^2}.$$

Gathering all these results we get

$$\begin{aligned}
 |w_l(t)| & \leq |w_l(0)| + C\epsilon D_l^0(t) \\
 (6.4) \quad & + C\epsilon \int_0^t D_l^1(u) \exp\left(-\left(\frac{|l_h|^2}{2} + \nu l_3^2\right)(t-u)\right) du,
 \end{aligned}$$

where the constant  $C$  depends on  $k_h$  ( $C = O(|k_h|^2)$ ) and

$$\begin{aligned}
D_l^0(t) &:= \left[ \sum_{k_3} |k_3|^8 |s(-\lambda_k, l, 0)|^2 \right]^{1/2} \exp(-(|l_h|^2 + \nu' l_3^2)t) \\
&+ \sum_{\substack{\mu \in M, \\ \mu \neq -\lambda_l}} (1 + \mathbf{1}_{|\mu|=1} |l_3|^2) |s(\mu, l, 0)| \exp(-(|l_h|^2 + \nu' l_3^2)t) \\
&+ \left[ \sum_{k_3} |k_3|^8 |s(-\lambda_k, l, t)|^2 \right]^{1/2} \\
&+ \sum_{\substack{\mu \in M, \\ \mu \neq -\lambda_l}} (1 + \mathbf{1}_{|\mu|=1} |l_3|^2) |s(\mu, l, t)|
\end{aligned}$$

and

$$\begin{aligned}
D_l^1(u) &:= \left[ \sum_{k_3} |k_3|^8 |\partial_t s(-\lambda_k, l, u)|^2 \right]^{1/2} \\
&+ \sum_j \sum_{\substack{\mu \in M, \\ \mu \neq -\lambda_l}} (1 + \mathbf{1}_{|\mu|=1} |l_3|^2) |\partial_t s(\mu, l, u)| \\
&+ (|l_h|^2 + \nu' |l_3|^2) \left[ \sum_{k_3} |k_3|^8 |s(-\lambda_k, l, u)|^2 \right]^{1/2} \\
&+ (|l_h|^2 + \nu' |l_3|^2) \sum_{\substack{\mu \in M, \\ \mu \neq -\lambda_l}} (1 + \mathbf{1}_{|\mu|=1} |l_3|^2) |s(\mu, l_h, u)|.
\end{aligned}$$

The estimate of Lemma 2 follows.  $\square$

**Remark 6.1.** Assume that the Fourier coefficients of  $s$  have exponential decay with respect to time, meaning that for all  $(\mu, l)$ , there exists  $r_0(\mu, l) \in \mathbf{C}$ , and  $c(\mu, l) \in \mathbf{C}$  with nonnegative real part such that

$$s(\mu, l, t) = r_0(\mu, l) \exp(-c(\mu, l)t).$$

Then provided the sequence  $r_0(\mu, l)$  is sufficiently convergent, a special solution of (6.1) can be built, which preserves the exponential decay property. Indeed, for all  $l \in \mathbf{Z}^3$ , set

$$w_l(t) := \sum_{\mu \neq -\lambda_l} r_0(\mu, l) \frac{\exp(i(\lambda_l + \mu)\frac{t}{\epsilon} - c(\mu, l)t)}{i\frac{\lambda_l + \mu}{\epsilon} - c(\mu, l) + |l_h|^2 + \nu |l_3|^3}.$$

Then it can be readily checked that  $w$  is a solution of (6.1), and moreover

$$|w_l(t)| \leq \epsilon \sum_{\mu \neq -\lambda_l} \frac{1}{|\lambda_l + \mu - \epsilon \Im(c(\mu, l))|} |r_0(\mu, l)| \exp(-\Re(c(\mu, l))t).$$

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