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# Deformations of associative submanifolds with boundary

Damien Gayet\*, Frederik Witt†

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## Abstract

Let  $M$  be a compact smooth manifold of holonomy  $G_2$ . We prove that the space of infinitesimal associative deformations of a compact associative submanifold  $Y$  with boundary in a coassociative submanifold  $X$  is the solution space of an elliptic problem. Further, we compute its virtual dimension. For  $\partial Y$  connected it is given by  $\int_{\partial Y} c_1(\nu_X) + 1 - g$ , where  $g$  denotes the genus of  $\partial Y$ ,  $\nu_X$  the orthogonal complement of  $T\partial Y$  in  $TX|_{\partial Y}$  and  $c_1(\nu_X)$  the first Chern class of  $\nu_X$  with respect to its natural complex structure.

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KEYWORDS:  $G_2$  holonomy; calibrated submanifolds; elliptic boundary problems on manifolds

## 1 Introduction

The group  $G_2$  is one of the possible holonomy groups of an irreducible and non-symmetric Riemannian manifold. As such, manifolds of holonomy  $G_2$  were an active area of research in Riemannian geometry, culminating with Joyce’s celebrated construction of compact holonomy  $G_2$  manifolds [18]. As they are necessarily seven dimensional, one refers to  $G_2$  as an *exceptional* holonomy group. In recent years physicists also paid accrued interest to these since the arrival of  $M$  theory.

The deep and rich interplay between geometry and algebra on manifolds with a  $G_2$  structure is reflected in the existence of special submanifolds, namely *associative* ones of dimension 3 and *coassociative* ones of dimension 4. These are particular instances of Harvey’s and Lawson’s *calibrated submanifolds* [13], a notion which also embraces complex submanifolds of a Kähler manifold or special Lagrangian submanifolds of a Calabi–Yau. McLean [25] proved that the infinitesimal coassociative deformations of a coassociative  $X$  is an unobstructed elliptic problem. The dimension of the moduli space is  $b_+^2(X)$ , i.e. the dimension of self-dual harmonic 2-forms on  $X$ . For associative submanifolds, the problem, though still elliptic, is more involved: the virtual dimension vanishes, and as for complex submanifolds, existence of deformations is in general obstructed. The work of Akbulut and Salur [2], [3] studies associative deformations on manifolds with topological  $G_2$  structure, but whose holonomy is not necessarily contained in  $G_2$ , and also addresses smooth- and compactness issues of the deformation spaces.

On the other hand, on symplectic manifolds one is naturally led to study the moduli space of (pseudo-)holomorphic curves with boundary in a lagrangian submanifold [11], [12]. In

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physics, Aganagic and Vafa translated this boundary problem for special lagrangians of a Calabi–Yau into an open string problem [1], following Witten’s use of the moduli space of complex curves in the stringy world [26]. Taking a Calabi–Yau 3–fold  $K$  times a circle yields a natural holonomy  $G_2$  manifold  $M = K \times S^1$ . Moreover, holomorphic curves and special lagrangians times a circle give examples of associative and coassociative submanifolds in  $M$ . In this way, the duality of complex versus special lagrangian submanifolds is matched by the duality of associative versus coassociative submanifolds in a holonomy  $G_2$  manifold. It is therefore natural to study deformations of (co–)associatives with boundary. Inspired by the work of Butscher [6], who investigated deformations of special lagrangians with boundary on a symplectic, codimension 2 submanifold inside some compact Calabi–Yau, Kovalev and Lotay investigated in a recent paper the analogous problem for manifolds with  $G_2$  structure, where a compact coassociative has its boundary in a fixed, codimension 2 submanifold [21]. On the other hand, Leung and Wang studied associatives with boundary conditions. They consider Riemann surfaces in a fixed coassociative  $X$  which they thicken into an associative with boundary in  $X \cup X'$ , where the coassociative  $X'$  is an infinitesimal displacement of  $X$  [23].

In this paper, we consider an associative  $Y$  with boundary in a fixed coassociative  $X$ , and study the space  $\mathcal{M}_{X,Y}$  of infinitesimal associative deformations of  $Y$  such that the boundary stays in  $X$ . Our main result is this: We show that  $\mathcal{M}_{X,Y}$  is the solution space of an elliptic boundary value problem whose index (i.e. the virtual dimension of  $\mathcal{M}_{X,Y}$ ) for connected  $\partial Y$  is given by

$$\text{index} = \int_{\partial Y} c_1(\nu_X) + 1 - g,$$

where  $g$  is the genus of  $\partial Y$ ,  $\nu_X$  the orthogonal complement of  $T\partial Y$  in  $TX|_{\partial Y}$  and  $c_1(\nu_X)$  the first Chern class of  $\nu_X$  with respect to its natural complex structure. If the boundary is not connected, then the index is the sum over all components of  $\partial Y$ . Further, we extend this result to encompass 4–dimensional  $\phi$ –free submanifolds  $X$  (i.e.  $X$  does not contain any associative) and manifolds with topological  $G_2$  structure, but not necessarily of holonomy  $G_2$ . Finally and independently, assuming that  $Y$  is an embedded 3–disk, we associate with  $Y$  an element  $\mu_{G_2}(\partial Y) \in \pi_2(G_2/SO(4)) \cong \mathbb{Z}_2$ , which is best thought of as a  $G_2$  analogon of the Maslov index. Under suitable identifications, we show that

$$\mu_{G_2}(\partial Y) = \int_{\partial Y} c_1(\nu_X) \text{ mod } 2.$$

A further natural issue is to study smooth– and compactness of  $\mathcal{M}_{X,Y}$  in the vein of [3], but we will leave this to another paper. The techniques we use are the standard ones from PDE theory; our reference is [4] whose conventions we shall follow throughout this paper.

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## 2 The group $G_2$

We start by recalling some classical facts about  $G_2$  (cf. for instance [3], [13] and [19]).

## 2.1 The octonions

The octonions define an 8-dimensional, non-associative division algebra  $\mathbb{O} = \mathbb{H} \oplus e\mathbb{H}$  generated by  $\langle \mathbf{1}, i, j, k, e, e \cdot i, e \cdot j, e \cdot k \rangle$ . Taking these generators as an orthonormal basis induces an inner product  $\langle \cdot, \cdot \rangle$  on  $\mathbb{O}$  compatible with the algebra structure. Further, we obtain a *cross product* taking values in the imaginary octonions  $\text{Im } \mathbb{O} = \langle \mathbf{1} \rangle^\perp \cong \mathbb{R}^7$  defined by

$$u \times v = \text{Im}(\bar{v} \cdot u).$$

Here,  $\bar{v}$  is the natural conjugation which sends  $v \in \text{Im } \mathbb{O}$  to  $-v$ . The term cross product is justified by the properties  $u \times v = -v \times u$  and  $|u \times v| = |u \wedge v|$ . Over  $\mathbb{R}^7$ , this yields the 3-form

$$\varphi_0(u, v, w) = \langle u \times v, w \rangle, \quad (1)$$

which expressed in the orthonormal basis  $e_1 = i, e_2 = k, \dots, e_7 = e \cdot k$  can be written explicitly as

$$\varphi_0 = e^{123} + e^1 \wedge (e^{45} + e^{67}) + e^2 \wedge (e^{46} - e^{57}) + e^3 \wedge (-e^{47} - e^{56}). \quad (2)$$

By definition the stabiliser of  $\varphi_0$  inside  $GL(7)$  is  $G_2$ , which is why we refer to any basis  $\{e_j\}$  such that  $\varphi_0$  is of the form (2) as a  $G_2$  *frame*. This is a real algebraic Lie group of dimension 14 defined by the equations

$$G_2 = \{(u_1, u_2, u_3) \in \mathbb{R}^7 \times \mathbb{R}^7 \times \mathbb{R}^7 \mid \langle u_i, u_j \rangle = \delta_{ij}, \varphi_0(u_1, u_2, u_3) = 0\}. \quad (3)$$

Conversely, any  $G_2$  invariant form  $\varphi \in \Lambda^3 \mathbb{R}^{7*}$  induces a positive definite inner product  $\langle \cdot, \cdot \rangle_\varphi$  and a cross product  $\times_\varphi$  as follows. Firstly, with  $\varphi$  we can associate a volume form  $\mu_\varphi$  (which is somehow difficult to write down explicitly, cf. the appendix in [17]). Then we define

$$\langle u, v \rangle_\varphi = ((u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi) / 6\mu_\varphi, \quad \langle u \times_\varphi v, w \rangle_\varphi = \varphi(u, v, w). \quad (4)$$

Further, there exists a *triple cross product* on  $\mathbb{O}$ ,

$$C(u, v, w) = \frac{1}{2}(u \cdot (\bar{v} \cdot w) - z \cdot (\bar{y} \cdot x)). \quad (5)$$

Again, this operation is skew in its arguments and satisfies  $|C(u, v, w)| = |u \wedge v \wedge w|$ . Restricted to  $\mathbb{R}^7$ , we find  $\text{Re } C(u, v, w) = \varphi_0(u, v, w)$  so that on  $G_2$ ,

$$C(u, v, w) = \text{Im } C(u, v, w) = \frac{1}{2}((u \cdot v) \cdot w - u \cdot (v \cdot w)).$$

The last expression is sometimes written as  $[u, v, w]$  and called the *associator*. It is actually a 3-form over  $\mathbb{O}$ , and in analogy with (1), we define a 4-form over  $\mathbb{R}^7$  by

$$\psi_0(u, v, w, x) = \frac{1}{2} \langle u, [v, w, x] \rangle.$$

This form actually coincides with the Hodge dual of  $\varphi_0$ , so that in a  $G_2$  frame  $\{e_j\}$ ,

$$\psi_0 = \star \varphi_0 = -e^{12} \wedge (e^{47} + e^{56}) - e^{13} \wedge (e^{46} - e^{57}) + e^{23} \wedge (e^{45} + e^{67}) + e^{4567}.$$

## 2.2 Associative and coassociative planes

An oriented 3-plane  $Y \subset \mathbb{R}^7$  is *associative* if the 3-form  $\varphi_0$  restricted to  $Y$  coincides with the induced Euclidean volume form on  $Y$ . By  $G^{\varphi_0}(\mathbb{R}^7)$  we denote the subset of associatives inside  $G_3(\mathbb{R}^7)$ , the grassmannian of oriented 3-planes in  $\mathbb{R}^7$ . It is diffeomorphic to  $G_2/SO(4)$ , where the action of  $G_2$  on  $\text{Im } \mathbb{O}$  restricts to the action of  $SO(4)$  on  $\text{Im } \mathbb{H} \oplus \mathbb{H}$  with  $\text{Im } \mathbb{H}$  and  $\mathbb{H}$  isomorphic as  $SO(4)$ -representations to the space of anti-self-dual forms  $\Lambda_-^2 \mathbb{R}^{4*}$  and the standard vector representation  $\mathbb{R}^4$ . As the name suggests, the associator vanishes on associative planes. In fact, associativity is tantamount to saying that the restriction to  $Y$  of the  $\mathbb{R}^7$  valued 3-form  $\chi_0$  defined over  $\mathbb{R}^7$  by

$$\langle \chi_0(u, v, w), x \rangle = \psi_0(u, v, w, x)$$

vanishes.

For the form  $\chi_0$ , we have the important identity

$$\chi_0(u, v, w) = -u \times (v \times w) - \langle u, v \rangle w + \langle u, w \rangle v. \quad (6)$$

In particular, we find  $u \times (u \times a) = -|u|^2 \cdot a$  if  $a$  is orthogonal to  $u$ . Further, as remarked in [3], if  $Y$  is associative, then any  $u \in Y$  of norm 1 induces an hermitian structure  $u \times : Y^\perp \rightarrow Y^\perp$ . It follows that  $Y^\perp$  is the irreducible Clifford module of  $\text{Cliff}(Y, \langle \cdot, \cdot \rangle|_Y)$ . Also note that  $Y^\perp \times Y^\perp \rightarrow Y$ .

Finally, an oriented 4-plane  $X$  is said to be *coassociative* if and only if  $\psi_0$  restricted to  $X$  is equal to the induced Riemannian volume form. This is equivalent to saying that the restriction of  $\varphi_0$  to  $X$  vanishes. As for associatives, we find for the set of coassociatives  $G^{\psi_0}(\mathbb{R}^7) \cong G_2/SO(4)$ .

## 2.3 $G_2$ manifolds

Next, consider a 7-dimensional manifold  $M$ , together with some 3-form  $\varphi$ . If the structure group  $GL(7)$  reduces to  $G_2$ , we say that  $M$  carries a *topological  $G_2$  structure*. The associated  $G_2$  principal frame bundle consists of isomorphisms between  $(T_x M, \varphi_x)$  and  $(\mathbb{R}^7, \varphi_0)$  for  $x \in M$ . In particular, formula (4) gives rise to a globally defined Riemannian metric  $g = g_\varphi$  and a cross product  $\times = \times_\varphi$ , inducing the structure of  $\text{Im } \mathbb{O}$  on any tangent space  $T_x M$ . Similarly, there are global counterparts  $\psi = \star_\varphi \varphi \in \Omega^4(M)$  and  $\chi \in \Gamma(\Lambda^3 T^* M \otimes TM)$  of  $\psi_0$  and  $\chi_0$ .

An oriented 3-dimensional submanifold  $Y$  is called *associative* if the pull-back of  $\varphi$  to  $Y$  is equal to the induced Riemannian volume form. Equivalently, the pull-back of  $\chi$  to  $Y$  is identically zero. An oriented 4-dimensional submanifold  $X$  is called *coassociative* if the pull-back of  $\psi$  to  $X$  is equal to the induced Riemannian volume form. Equivalently, the pull-back of  $\varphi$  to  $X$  is identically zero.

(Co-)Associative manifolds have the important property of being homologically volume minimising if the form  $\varphi$  ( $\psi$ ) is closed [13]. By a result of Fernandez and Gray [10],  $d\varphi = d\psi = 0$  is tantamount to saying that the holonomy group of  $g$  is contained in  $G_2$ . Equivalently, there exist coordinates around each point such that  $\varphi(x) = \varphi_0 + O(|x|^2)$ .

**Example:** A important family of examples is provided by  $K \times S^1$ , where  $(K, \omega, \Omega)$  is a Calabi–Yau 3–fold,  $\omega$  being the Kähler 2–form and  $\Omega$  the holomorphic volume form. In this case,  $\varphi = \operatorname{Re} \Omega + \omega \wedge dt$  and  $\psi = \operatorname{Im} \Omega \wedge dt + \omega^2/2$ . Further, if  $C \subset M$  is a complex curve, then  $Y = C \times S^1$  is associative. If  $L \subset M$  is special lagrangian, then  $Y = L \times \{pt\}$  is associative while  $X = L \times S^1$  is coassociative.

### 3 The geometry of the deformation problem

Let  $M$  be a holonomy  $G_2$  manifold and  $Y \subset M$  a compact associative whose boundary  $\partial Y$  is contained in a fixed 4–submanifold  $X \subset M$ . We wish to study the space  $\mathcal{M}_{X,Y}$  of infinitesimal deformations of  $Y$  in the class of associatives with boundary in  $X$ .

#### 3.1 The closed case

To begin with we rederive McLean’s result for the closed case  $\partial Y = \emptyset$  following Akbulut and Salur [3], who emphasise the systematic use of the cross product  $\times$ . We consider  $\mathcal{M}_Y$ , the space of infinitesimal associative deformations of  $Y$ .

An *associative deformation* of an associative  $Y$  will be a map  $Y \rightarrow Y_t \subset M$ , where  $Y_t$  is a family of associatives for  $t \in (-\epsilon, \epsilon)$ . For convenience, we denote this map also by  $Y_t$ . Since  $Y$  is compact, we may always assume, reparametrising with a time–dependent diffeomorphism if necessary, that  $Y_t$  is a *normal deformation*, i.e. the deformation vector field  $\sigma_t(Y_t(p)) = \frac{\partial Y_s(p)}{\partial s}|_{s=t}$  is normal to  $Y_t$ . Now let  $\nu \rightarrow Y$  denote the normal bundle over  $Y$ , equipped with the natural connection  $\nabla$  coming from the Levi–Civita connection on  $TM$ . If  $\sigma \in \Gamma_Y(\nu)$  is a normal vector field on  $Y$ , then its flow  $\phi_t^\sigma$  gives rise to a deformation  $Y_t = \phi_t^\sigma(Y)$  of  $Y$  for  $t$  small enough. The tangent space at a point  $\phi_t^\sigma(p)$  gives an element in  $G_3(TM)$ , the grassmannian bundle of oriented 3–planes inside  $TM$ . Taking a curve  $\alpha \subset Y$  and abusing notation, we obtain a curve  $\tilde{\alpha}_t(s) = \phi_t^\sigma(\alpha(s))$  in  $G_3(TM)$ . If  $Y_t$  is associative, then  $\tilde{\alpha}_t \subset G^\varphi(TM)$ , the bundle of associative planes in  $TM$ , and therefore, the derivative  $\tilde{\alpha}_t$  has to lie in  $T_{\tilde{\alpha}_t} G^\varphi(TM) \subset T_{\tilde{\alpha}_t} G_3(TM)$ . In the limit where  $t \rightarrow 0$ , we end up with the condition  $\nabla_{\tilde{\alpha}} \sigma \in T_\alpha G^\varphi(TM)$  inside  $T_\alpha G_3(M) \cong T_\alpha^* Y \otimes \nu_\alpha$ . Now Clifford multiplication  $\times$  takes the latter space to  $\nu_\alpha$ . Furthermore, we have the decomposition into  $SO(4)$  irreducibles  $T_\alpha^* Y \otimes \nu_\alpha = \mathfrak{so}(4)^\perp \oplus \mathbb{R}^4$  (the complement of  $\mathfrak{so}(4)$  being taken in  $\mathfrak{g}_2$ ) corresponding to  $T_\alpha G^\varphi(TM)$  and the fibre  $\nu_\alpha$ . Appealing to Schur’s lemma, the kernel of  $\times$  is therefore  $T_\alpha G^\varphi(TM)$ . In other words,  $\sigma$  is required to lie in the kernel of the *Dirac operator*

$$\mathbf{D} : \sigma \in \Gamma_Y(\nu) \xrightarrow{\nabla} \Gamma_Y(T^*Y \otimes \nu) \xrightarrow{g} \Gamma_Y(TY \otimes \nu) \xrightarrow{\times} \Gamma_Y(\nu),$$

which written in a local orthonormal basis  $e_1, e_2, e_3$  of  $Y$  is

$$\mathbf{D}\sigma = e_1 \times \nabla_{e_1} \sigma + e_2 \times \nabla_{e_2} \sigma + e_3 \times \nabla_{e_3} \sigma. \quad (7)$$

Summarising, we derived McLean’s

**Theorem 3.1** [25] *A section  $\sigma \in \Gamma_Y(\nu)$  is the deformation vector field to first order to a family of associatives, i.e. it lies in the Zariski tangent space of the infinitesimal associative deformations of  $Y$ , if and only if  $\mathbf{D}\sigma = 0$ .*

### 3.2 The geometry on the boundary

For  $\partial Y \neq \emptyset$ , we need to understand the geometry on the boundary of  $Y$ .

Fix a collar neighbourhood  $\mathcal{C} \cong \partial Y \times [0, \epsilon)$  of  $\partial Y$  and let  $u$  denote the inward pointing unit vector field defined on  $\mathcal{C}$ . As before,  $\nu \rightarrow Y$  denotes the normal bundle as well as its restriction to  $\partial Y$ . In virtue of Section 2.2,  $\nu|_{\mathcal{C}}$  carries a hermitian structure near the boundary induced by  $u$ , namely

$$G : \nu \rightarrow \nu, \quad G(x) = u \times x.$$

This acts indeed as an isometry with respect to  $g$ , as

$$g(Ga, Gb) = \varphi(u, a, u \times b) = -g(u \times (u \times b), a) = g(a, b)$$

for any  $a, b \in \nu|_{\mathcal{C}}$ . Let  $\nu_X \subset TX|_{\partial Y}$  denote the orthogonal complement of  $T\partial Y$  in  $TX|_{\partial Y}$ .

**Lemma 3.2** *For the bundle  $\nu \rightarrow \partial Y$  holds the following :*

1. *The bundle  $\nu_X$  is contained in  $\nu$  and is stable under  $G$ .*
2. *The orthogonal complement  $\mu_X$  of  $\nu_X$  in  $\nu$  is also stable under  $G$ .*
3. *Viewing  $T\partial Y$ ,  $\nu_X$  and  $\mu_X$  as  $G$ -complex bundles, we have*

$$\bar{\mu}_X \cong \nu_X \otimes_{\mathbb{C}} T\partial Y$$

*as complex bundles, that is  $\mu_X^{0,1} \cong \nu_X^{1,0} \otimes T^{1,0}\partial Y \cong \nu_X^{1,0} \otimes \bar{K}_{\partial Y}$ , where  $K_{\partial Y}$  is the canonical line bundle over  $\partial Y$ .*

**Proof:** Let us fix a local orthonormal basis  $(u, v, w)$  on the boundary by choosing locally a unit vector field  $v \in T\partial Y$ . We then set  $w = u \times v$ , which lies in  $T\partial Y$  in virtue of the associativity of  $Y$ . If  $a \in \nu_X$ , then  $g(a, u) = 0$ , for  $v \times w = u$  and  $\varphi(v, w, a) = 0$ ,  $X$  being coassociative. Clearly, the vectors  $a \times v$  and  $a \times w$  are orthogonal to  $v$  and  $w$  as well as to  $u$ , since

$$g(a \times v, u) = \varphi(a, v, u) = -g(u \times v, a) = -g(w, a) = 0,$$

and similarly for  $a \times w$ . Hence  $a \times v, a \times w \in \nu$ . Further, these vectors are orthogonal to  $TX$ , for  $a, v, w \in TX$  and  $X$  is coassociative, so that for  $n \in \nu_X$  we find  $g(a \times v, n) = \varphi(a, v, n) = 0$  etc. Hence  $a \times v$  and  $a \times w$  span  $\mu_X$ . As a consequence,  $u \times a \in \nu$  is orthogonal to  $\mu_X$  (for  $g(u \times a, a \times v) = \varphi(u, a, a \times v)$  etc.), so that  $\nu_X$  is spanned by  $a$  and  $u \times a = Ga$ . This shows that  $\nu_X$  is stable under  $G$ . On the other hand,  $g(u \times (a \times v), a) = \varphi(u, (a \times v), a) = 0$  and similarly  $g(u \times (a \times v), u \times a) = 0$ , hence  $u \times (a \times v) \in \mu_X$  which shows that  $\mu_X$  is also stable under  $G$ .

The Riemann surface structure on  $\partial Y$  is induced by the hermitian structure  $G = u \times$  (to keep notation tight we abuse notation and also write  $G$  for the endomorphism on  $T\partial Y$  induced by  $u \times$ ), for  $Gv = u \times v = w$  and  $Gw = u \times w = -v$ . The map

$$a \otimes y \in \nu_X \otimes_{\mathbb{C}} T\partial Y \mapsto a \times y \in \bar{\mu}_X,$$

where we now view  $\nu_X$  and  $\bar{\mu}_X$  as complex line bundles via  $G$ , is well-defined and a *real* bundle isomorphism. It remains to see that it is complex-linear, i.e.

$$Ga \times y = a \times Gy = -G(a \times y).$$

This is equivalent to  $(u \times a) \times y$  and  $a \times (u \times y)$  being equal to  $-u \times (a \times y)$ . But this follows from (6) and the skew-symmetry of  $\chi$ . ■

## 4 Ellipticity and index

Let  $\mathbf{D}$  still denote the Dirac operator (7). Since  $\mathbf{D}$  and its extension  $\mathbf{D}^{\mathbb{C}} : \Gamma_Y(\nu^{\mathbb{C}}) \rightarrow \Gamma_Y(\nu^{\mathbb{C}})$  to the complexified bundle  $\nu^{\mathbb{C}} = \nu \otimes \mathbb{C}$  is elliptic, we can consider its index,  $\text{index}(\mathbf{D}^{\mathbb{C}}) = \dim \ker \mathbf{D}^{\mathbb{C}} - \dim \text{coker} \mathbf{D}^{\mathbb{C}}$ . McLean's theorem 3.1 identifies  $\mathcal{M}_Y$ , the space of infinitesimal associative deformations, with  $\ker \mathbf{D}$ . We therefore refer to the index of  $\mathbf{D}^{\mathbb{C}}$  as the *virtual dimension* of  $\mathcal{M}_Y$  which,  $Y$  being odd dimensional, is necessarily 0. In generic situations, one expects the cokernel of  $\mathbf{D}$  to vanish, so that by the Sard–Smale theorem  $\mathcal{M}_Y$  will be a smooth manifold of actual dimension  $\dim \ker \mathbf{D}^{\mathbb{C}} = \text{index} \mathbf{D}^{\mathbb{C}} = 0$ , but in non-generic situations, even for  $\mathcal{M}_Y$  smooth, the dimension might become greater.

Next let  $\mathbf{B} : \Gamma_{\partial Y}(\nu) \rightarrow \Gamma_{\partial Y}(\mu_X)$  be the real operator of order 0 which projects smooth sections of  $\nu = \nu_X \oplus \mu_X$  over  $\partial Y$  to  $\mu_X$ . As a corollary to McLean's theorem 3.1 and Lemma 3.2, the deformation space  $\mathcal{M}_{X,Y}$  can be identified with solutions of the system

$$\mathbf{D}\sigma = 0, \quad \mathbf{B}(\sigma|_{\partial Y}) = 0.$$

In view of applying the standard machinery of index theory to manifolds with boundary, we consider the extended problem over the complexified bundles  $\nu_X^{\mathbb{C}}$  and  $\mu_X^{\mathbb{C}}$ , namely

$$\mathbf{D}^{\mathbb{C}}\sigma = 0, \quad \mathbf{B}^{\mathbb{C}}(\sigma|_{\partial Y}) = 0. \quad (8)$$

In order to compute the virtual dimension for  $\mathcal{M}_{X,Y}$  we need a suitable ellipticity condition on (8).

**Definition 4.1** (cf. [4]) *Let  $\mathcal{D} : \Gamma_Y(S) \rightarrow \Gamma_Y(S)$  be a Dirac operator of some (complex) Clifford bundle  $S \rightarrow Y$  on an odd-dimensional manifold  $Y$  with boundary. Let*

$$\mathcal{Q}_{\mathcal{D}} : \Gamma_{\partial Y}(S) \rightarrow \{\sigma|_{\partial Y} \in \Gamma_{\partial Y}(S) \mid \mathcal{D}\sigma = 0 \text{ in } Y \setminus \partial Y\}$$

*denote the associated Calderón projector [7] which projects smooth sections of  $S|_{\partial Y}$  onto the space of Cauchy data of  $\mathcal{D}$ . An operator  $\mathcal{B} : \Gamma_{\partial Y}(S) \rightarrow \Gamma_{\partial Y}(V)$  of order 0 taking sections of  $S|_{\partial Y}$  to sections of some complex vector bundle  $V$ , is said to define a local elliptic boundary condition if its principal symbol  $\sigma(\mathcal{B})$  satisfies  $\text{Im} \sigma(\mathcal{B}) \cong \pi^*V$  and  $\text{Im} \sigma(\mathcal{B}) = \text{Im}(\sigma(\mathcal{B}) \circ \sigma(\mathcal{Q})) \cong \text{Im} \sigma(\mathcal{Q})$ , where  $\pi : T^*\partial Y \setminus 0 \rightarrow \partial Y$  denotes the natural projection.*

If  $\mathcal{B}$  defines a local elliptic boundary condition, the index

$$\text{index}(\mathcal{D}, \mathcal{B}) = \dim \ker(\mathcal{D} \oplus \mathcal{B}) - \dim \text{coker}(\mathcal{D} \oplus \mathcal{B})$$

is well-defined, finite and depends, as the usual index, only on the homotopy type of the principal symbols involved. So we may always assume, possibly after homotopically deforming the metric, that the Riemannian structure is a product on some collar neighbourhood  $\mathcal{C} \cong \partial Y \times [0, \epsilon)$  in  $Y$ . Then, if  $u$  denotes the inward pointing coordinate vector, the Dirac operator  $\mathcal{D}$  decomposes near the boundary into

$$\mathcal{D} = u \bullet (\nabla_u + \mathbf{R}),$$

where  $u$  induces a unitary automorphism  $G = u \bullet$  of  $S|_{\partial Y}$  squaring to minus the identity. We denote the corresponding  $\pm i$  eigenspaces by  $S^{\pm}$ . The following theorem, whose proof can be also found in [4], is a valuable tool for the explicit computation of an index.

**Theorem 4.1** *Let  $\mathcal{D} : \Gamma_Y(S) \rightarrow \Gamma_Y(S)$  be a Dirac operator over an odd-dimensional manifold  $Y$  with boundary.*

1. *If  $\mathcal{P}^+$  denotes the orthogonal projector<sup>1</sup> onto  $S^+$ , then  $(\mathcal{D}, \mathcal{P}^+)$  is a local elliptic boundary problem with vanishing index.*
2. *Let  $\mathcal{B}_1 : \Gamma_{\partial Y}(S) \rightarrow \Gamma_{\partial Y}(V_1)$  and  $\mathcal{B}_2 : \Gamma_{\partial Y}(S) \rightarrow \Gamma_{\partial Y}(V_2)$  be two orthogonal projectors onto subbundles  $V_{1,2}$  of  $S|_{\partial Y}$  defining a local elliptic boundary problem. Then*

$$\text{index}(\mathcal{D}, \mathcal{B}_2) - \text{index}(\mathcal{D}, \mathcal{B}_1) = \text{index}(\mathcal{B}_2 \mathcal{Q}_{\mathcal{D}} \mathcal{B}_1 : \Gamma_{\partial Y}(V_1) \rightarrow \Gamma_{\partial Y}(V_2)),$$

*where on the right hand side we view  $\mathcal{B}_1$  as an operator  $\Gamma_{\partial Y}(V_1) \rightarrow \Gamma_{\partial Y}(S)$ .*

We are now in a position to prove the central theorem of this paper.

**Theorem 4.2** *The pair  $(\mathbf{D}^{\mathbb{C}}, \mathbf{B}^{\mathbb{C}})$  in (8) defines a local elliptic boundary condition with index*

$$\text{index}(\mathbf{D}^{\mathbb{C}}, \mathbf{B}^{\mathbb{C}}) = \text{index}(\bar{\partial}_{\nu_X}).$$

**Remark:** If the boundary is connected, then Riemann–Roch implies  $\text{index}(\mathbf{D}^{\mathbb{C}}, \mathbf{B}^{\mathbb{C}}) = \int_{\partial Y} c_1(\nu_X) + 1 - g$ , where  $g$  is the genus of  $\partial Y$  and  $c_1(\nu_X)$  is the first Chern class of  $\nu_X$  with respect to the natural complex structure (or equivalently, the natural orientation) induced by  $u$ . In general, the index will therefore be

$$\text{index}(\mathbf{D}^{\mathbb{C}}, \mathbf{B}^{\mathbb{C}}) = \sum_j \int_{\Sigma_{g_j}} c_1(\nu_X|_{\Sigma_{g_j}}) + 1 - g_j$$

where  $\Sigma_{g_j}$  denotes a connected component of  $\partial Y$  of genus  $g_j$ .

**Proof:** As above, consider some collar neighbourhood  $\mathcal{C} \cong \partial Y \times [0, \epsilon)$  of  $\partial Y$  on which we assume the Riemannian structure to be a product. Further, complete the inward pointing coordinate vector  $u$  to a local orthonormal basis  $(v(y, t), w(y, t))$  of  $T_y \partial Y \times \{t\}$  such that  $u \times v = w$ . Near the boundary, we have the decomposition  $\mathbf{D}^{\mathbb{C}} = u \times (\nabla_u + \mathbf{R})$  with

$$\mathbf{R} = w \times \nabla_v - v \times \nabla_w, \tag{9}$$

as follows from  $(a \times b) \times c = -a \times (b \times c)$  valid whenever  $\{a, b, c\}$  is an orthogonal family, cf. (6). The unitary automorphism  $G$  on  $\nu^{\mathbb{C}}$  with eigenspaces  $S^{\pm}$  is just  $G = u \times$ .

Locally, we will work with the following basis of  $\nu^{\mathbb{C}}$ : Choose a nowhere vanishing local section  $a \in \Gamma_{\partial Y}(\nu_X)$  so that  $\nu_X^{\mathbb{C}}$  is spanned by  $\alpha = a - iGa$  and  $\bar{\alpha} = a + iGa$  respectively, cf. Lemma 3.2. Consider then the sections  $\beta = -v \times \bar{\alpha}$  and  $\bar{\beta} = -v \times \alpha$ . Again, the lemma implies that these span  $\mu_X^{\mathbb{C}}$  locally. Further,  $\alpha$  and  $\beta$  span  $S^+$  while  $\bar{\alpha}$  and  $\bar{\beta}$  span  $S^-$ . As an example, take  $G\alpha = Ga + ia = i\alpha$  and  $G\bar{\beta} = v \times (u \times \alpha) = -i\bar{\beta}$  etc. For any subsequent matrix representation over  $\nu^{\mathbb{C}}$ , the ordered basis  $\{\alpha, \beta, \bar{\alpha}, \bar{\beta}\}$  shall be used.

To show that  $\mathbf{B}^{\mathbb{C}}$  defines a local elliptic boundary condition requires the principal symbol  $q = \sigma(\mathbf{Q})$  of the Calderón operator  $\mathbf{Q} = \mathcal{Q}_{\mathbf{D}^{\mathbb{C}}}$  associated with  $\mathbf{D}^{\mathbb{C}}$ . By the Calderón–Seeley theorem (as given in [4]),  $q$  is the projector onto the eigenspace of  $\sigma(\mathbf{R})$  corresponding to

<sup>1</sup>By an *orthogonal projector* we understand an operator  $\mathcal{P}$  of order 0 satisfying  $\mathcal{P} = \mathcal{P}^2 = \mathcal{P}^*$ .

eigenvalues with positive real part. Now with respect to our fixed local ordered basis of  $\nu^{\mathbb{C}}$  around  $x \in \partial Y$ ,  $v$  and  $w$  act via

$$v \times = \begin{pmatrix} \mathbf{0} & 0 & 1 \\ 0 & 1 & \mathbf{0} \\ -1 & 0 & \mathbf{0} \end{pmatrix}, \quad w \times = \begin{pmatrix} \mathbf{0} & 0 & -i \\ 0 & i & \mathbf{0} \\ -i & 0 & \mathbf{0} \end{pmatrix}.$$

This follows from  $v \times \alpha = -\bar{\beta}$ ,  $w \times \alpha = -u \times (v \times \alpha) = -i\bar{\beta}$  etc. For  $(\eta_v, \eta_w) \in T_x^* \partial Y \setminus \{0\}$  of unit norm, we deduce from (9) (with  $\eta = \eta_v + i\eta_w$ ) that

$$\sigma(\mathbf{R})(x, \eta) = i(\eta_v \cdot w \times - \eta_w \cdot v \times) = \begin{pmatrix} \mathbf{0} & 0 & \bar{\eta} \\ 0 & -\eta & \mathbf{0} \\ \eta & 0 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & r_-(x, \eta) \\ r_+(x, \eta) & \mathbf{0} \end{pmatrix}.$$

Now  $r_+(x, \eta)^* = r_-(x, \eta)$  and  $r_+(x, \eta) = r_-(x, \eta)^{-1}$ , so that  $\sigma(\mathbf{R})(x, \eta)^* = \sigma(\mathbf{R})(x, \eta) = \sigma(\mathbf{R})(x, \eta)^{-1}$ . Consequently, the eigenvalues are  $\pm 1$ , and the projector on the eigenspace associated with 1 is given by

$$q(x, \eta) = \frac{1}{2} \begin{pmatrix} \text{Id}_2 & r_-(x, \eta) \\ r_+(x, \eta) & \text{Id}_2 \end{pmatrix}.$$

On the other hand,  $\mathbf{B}^{\mathbb{C}}$  is the orthogonal projector onto  $\mu_X^{\mathbb{C}}$ , so that its principal symbol is the matrix (taken with respect to the fixed basis of  $\nu^{\mathbb{C}}$  and  $\{\beta, \bar{\beta}\}$  of  $\mu_X^{\mathbb{C}}$ )

$$\sigma(\mathbf{B}^{\mathbb{C}})(x, \eta) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which is of full rank. Since

$$\sigma(\mathbf{B}^{\mathbb{C}}) \circ q(x, \eta) = \frac{1}{2} \begin{pmatrix} 0 & 1 & -\bar{\eta} & 0 \\ \eta & 0 & 0 & 1 \end{pmatrix}$$

is also of full rank, the boundary condition defined by  $\mathbf{B}^{\mathbb{C}}$  is local elliptic by Definition 4.1.

It remains to compute the index. Let  $\mathbf{P}^+$  denote the orthogonal projector onto  $S^+$ . In virtue of Theorem 4.1 and the established local ellipticity of  $\mathbf{B}^{\mathbb{C}}$ ,

$$\text{index}(\mathbf{D}^{\mathbb{C}}, \mathbf{B}^{\mathbb{C}}) = \text{index}(\mathbf{B}^{\mathbb{C}} \mathbf{Q} \mathbf{P}^+ : \Gamma_{\partial Y}(S^+) \rightarrow \Gamma_{\partial Y}(\mu_X^{\mathbb{C}})).$$

But the symbol of  $\mathbf{B}^{\mathbb{C}} \mathbf{Q} \mathbf{P}^+$  is just

$$\begin{aligned} \sigma(\mathbf{B}^{\mathbb{C}} \mathbf{Q} \mathbf{P}^+)(x, \eta) &= \sigma(\mathbf{B}^{\mathbb{C}}) \circ q \circ \sigma(\mathbf{P}^+)(x, \eta) \\ &= \frac{1}{2} \begin{pmatrix} 0 & 1 & -\bar{\eta} & 0 \\ \eta & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix} : S_x^+ \rightarrow (\mu_X^{\mathbb{C}})_x, \end{aligned}$$

where the matrix is taken with respect to the basis  $\{\alpha, \beta\}$  of  $S^+$  and  $\{\beta, \bar{\beta}\}$  of  $\mu_X^{\mathbb{C}}$ . In particular, the symbol sends  $\beta$  to  $\beta$  and therefore acts as the identity on  $\mu_X^{1,0} = S^+ \cap \mu_X^{\mathbb{C}}$ .

On the other hand, the induced map  $\nu_X^{1,0} = S^+ \cap \nu_X^{\mathbb{C}} \rightarrow \mu_X^{0,1} = S^- \cap \mu_X^{\mathbb{C}}$  is up to  $-i$  the symbol of the Cauchy–Riemann operator  $\bar{\partial}_{\nu_X}$  on  $\nu_X^{1,0}$  after the identification  $\mu_X^{0,1} \cong \nu_X^{1,0} \otimes \bar{K}_{\partial Y}$  (cf. Lemma 3.2). Indeed, on a trivialisaton of  $\nu_X^{\mathbb{C}}$  it acts as  $\bar{\partial}_{\nu_X} = (\partial_1 + i\partial_2)/2$ , where  $(x_1, x_2)$  are coordinates such that  $\partial_1(x) = v(x)$  and  $\partial_2(x) = w(x)$ . Hence  $\sigma(\bar{\partial}_{\nu_X})(x, \eta) = i(\eta_v + i\eta_w)/2$  which is what we wanted.  $\blacksquare$

## 5 Generalisations of the deformation theorem

**Boundaries in  $\varphi$ -free submanifolds.** For the deformation problem (8) the bundle  $\nu_X$  (together with the bundle  $\mu_X$  it determines) is the only non-intrinsic piece of datum attached to  $Y$ , and its properties were derived using the coassociativity of  $X$ . In fact, following an idea of Harvey and Lawson, the condition on  $X$  may be relaxed as follows:

**Definition 5.1** [14], [15], [16] *A 4-submanifold  $X$  of a topological  $G_2$  manifold  $(M, \varphi)$  is said to be  $\varphi$ -free at  $x$  if  $T_x X$  is  $\varphi$ -free, i.e. if it contains no associative 3-plane:  $G_3(T_x X) \cap G^\varphi(T_x M) = \emptyset$ . We call  $X$   $\varphi$ -free if it is  $\varphi$ -free for all  $x \in X$ .*

**Lemma 5.1** *A submanifold  $X \subset M$  is  $\varphi$ -free at  $x$  if and only if  $\varphi_x|_{(T_x X)^\perp} \not\equiv 0$ . In particular, the set of  $\varphi$ -free planes in  $T_x M$  is open, and a generic 4-submanifold is  $\varphi$ -free.*

**Proof:** If  $E \in G_4(T_x M)$  is not  $\varphi$ -free, it contains an associative 3-plane  $F$  so that  $E^\perp$  is contained in the coassociative plane  $F^\perp$ . Hence  $\varphi_x|_{E^\perp} \equiv 0$  which proves the sufficiency of the condition. For the converse, pick  $E$  such that  $\varphi_x|_{E^\perp} \equiv 0$ . We need to exhibit a coassociative plane  $F^\perp$  containing  $E^\perp$ , so that  $E$  is not  $\varphi$ -free. Writing  $E^\perp = u \wedge v \wedge w$ , choose a vector  $a$  in the orthogonal complement to the linear span of  $u, v, w, u \times v, u \times w, v \times w$  which is at most six dimensional. Then the 4-plane  $F^\perp$  spanned by  $E^\perp$  and  $a$  is coassociative, for  $\varphi|_{F^\perp} \equiv 0$ .  $\blacksquare$

Put differently, for any  $a, b \in T_x X$ , we have  $a \times b \notin T_x X$  and in particular, any coassociative is  $\varphi$ -free. From this point of view the class of  $\varphi$ -free submanifolds in  $G_2$  geometry naturally matches the class of totally real submanifolds (which in particular comprises lagrangians) in Kähler geometry.

Next we wish to investigate  $\mathcal{M}_{X,Y}$  under the assumption that  $X$  is  $\varphi$ -free. Let  $N_X$  be the orthogonal complement of  $T\partial Y$  in  $TX$ , and define  $\nu_X$  to be the image of  $N_X$  under the orthogonal projection  $\pi : TM|_{\partial Y} \rightarrow \nu$ . As before,  $\mu_X$  denotes the orthogonal complement of  $\nu_X$  in  $\nu$ . The geometry on the boundary is specified by following

**Lemma 5.2** *If  $X$  is  $\varphi$ -free, then*

1. *the restriction of  $\pi$  to  $N_X$  defines an isomorphism onto  $\nu_X$ .*
2. *for any non-zero  $b \in \mu_X$ ,  $Gb = u \times b \notin \nu_X$ , that is  $(G\mu_X) \cap \nu_X = \{0\}$ .*

**Proof:** The kernel of  $\pi$  is  $TY|_{\partial Y}$ . Since  $TX$  is  $\varphi$ -free,  $N_X \cap \ker \pi = \{0\}$ , whence the first assertion. Next, suppose there is an  $x \in \partial Y$  and a  $b_0 \in (\mu_X)_x$  of unit norm such that  $Gb_0 \in (\nu_X)_x$ . Let  $b_1 \in (\mu_X)_x$  be a vector orthogonal to  $b_0$ . Then  $Gb_1$  lies in  $(\nu_X)_x$ , for it is orthogonal to  $b_1$  and  $g_x(b_0, Gb_1) = -g_x(Gb_0, b_1) = 0$ . By the first assertion, there exist uniquely determined  $n_{0,1} = m_{0,1} + k_{0,1} \in N_X$  with  $\pi(n_{0,1}) = Gb_{0,1}$ , where  $m_{0,1} \in T_x M|_{\partial Y}$  and  $k_{0,1} \in \ker \pi_x$ . Since  $(N_X)_x \perp T_x \partial Y$ , we have in fact  $n_{0,1} = Gb_{0,1} + \lambda_{0,1} u$

for  $\lambda_{1,2} \in \mathbb{R}$ . Then, it is straightforward to check that the orthogonal complement of  $T_x X|_{\partial Y}$  in  $T_x M|_{\partial Y}$  is spanned by  $A = b_0$ ,  $B = b_1$  and  $C = u - \lambda_0 Gb_0 - \lambda_1 Gb_1$ . However,  $v = A \times B \in T_x Y|_{\partial Y}$  belongs in fact to  $T\partial Y$ , for  $g_x(u, v) = -g_x(Gb_1, b_0) = 0$ . Consequently,  $\varphi_x(A, B, C) = g_x(A \times B, C) = 0$  which contradicts the  $\varphi$ -freeness of  $TX$ , cf. the previous Lemma 5.1.  $\blacksquare$

We can now extend Theorem 4.2 to this more general situation. Let  $Y$  be an associative with boundary in a  $\varphi$ -free  $X$ . If  $X$  intersects  $Y$  orthogonally in  $\partial Y$ , then  $N_X = \nu_X$ , and we can identify the space  $\mathcal{M}_{X,Y}$  of infinitesimal associative deformations with boundary in  $X$  with the space of solutions of (8). For the general case, note that  $N_X \oplus \mu_X$  and  $\nu_X \oplus \mu_X$  are isomorphic subbundles of  $TM|_{\partial Y}$ , and we can homotope  $\nu \rightarrow Y$  into a new bundle  $N \rightarrow Y$  such that  $N|_{\partial Y} = \nu_X \oplus \mu_X$ . Then  $\mathcal{M}_{X,Y}$  can be identified with solutions  $\sigma \in \Gamma_Y(N)$  of

$$\mathbf{D}\pi(\sigma) = 0, \quad \mathbf{B}(\sigma|_{\partial Y}) = 0, \quad (10)$$

where  $\pi : N \rightarrow \nu$  denotes orthogonal projection onto  $\nu$  and  $\mathbf{B} : \Gamma_{\partial Y}(N) \rightarrow \Gamma_{\partial Y}(\mu_X)$  orthogonal projection onto  $\mu_X$  in  $N|_{\partial Y}$ .

**Proposition 5.3** *Let  $Y \subset M$  be an associative with boundary in a  $\varphi$ -free submanifold  $X$ . Then the virtual dimension of  $\mathcal{M}_{X,Y}$  is given by  $\text{index}(\overline{\partial}_{\nu_X})$ .*

**Proof:** We restrict ourselves to the case where  $X$  intersects  $Y$  orthogonally for sake of simplicity; the general case follows from a similar argument. We choose a nowhere vanishing local section  $a$  of  $\nu_X$  and extend  $b = -v \times a$  to a local orthonormal trivialisation  $\{b, \tilde{b}\}$  of  $\mu_X$ . By the previous lemma,  $G\mu_X$  can be graphed over  $\nu_X \oplus \mu_X$ , so that we may take  $\tilde{b}$  to be the orthogonal projection of  $Gb$  to  $\mu_X$ . Let  $(0, 1, s, t)$  be the coordinates of  $\tilde{b}$  with respect to the local basis  $\{b, Gb, v \times b, w \times b\}$  of  $\nu$ , where  $v$  is a nowhere vanishing local section of  $T\partial Y$  and  $w = Gv = u \times v$ . With respect to the basis  $\{\alpha, \beta, \bar{\alpha}, \bar{\beta}\}$  of  $\nu^{\mathbb{C}}$  as given in the proof of Theorem 4.2, the matrix of  $\sigma(\mathbf{B}^{\mathbb{C}})$  can be written as

$$\sigma(\mathbf{B}^{\mathbb{C}})(x, \eta) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ z & -i & \bar{z} & i \end{pmatrix},$$

where  $z = s + it$ . For  $\mathbf{B}^{\mathbb{C}}\mathbf{QP}^+$ , we therefore find

$$\sigma(\mathbf{B}^{\mathbb{C}}\mathbf{QP}^+)(x, \eta) = \frac{1}{2} \begin{pmatrix} \eta & 1 \\ z + i\eta & -i - z\eta \end{pmatrix} : S_x^+ \rightarrow (\mu_X^{\mathbb{C}})_x. \quad (11)$$

The determinant of this matrix is  $(-2i\eta - z - \bar{z}\eta^2)/4$ , and multiplication with  $\bar{\eta}$  shows this to vanish only if  $\text{Re}(\bar{T}\eta) = -i$ . Hence the system (10) is still elliptic. Furthermore,  $F(Gb, t) = \text{tpr}_{\nu_X}(Gb) \oplus \text{pr}_{\mu_X}(Gb)$  is a global homotopy deforming  $G\mu_X$  into  $\mu_X$  and which in particular deforms  $Gb$  into  $\tilde{b}$ . Consequently, the symbol (11) is homotoped into the symbol of  $\overline{\partial}_{\nu_X}$ . By homotopy invariance we recover the same index as before.  $\blacksquare$

**Deformations in topological  $G_2$  manifolds.** Mutatis mutandis, Theorem 4.2 extends to topological  $G_2$  manifolds. First, we recall the following generalisation of McLean's theorem which is due to Akbulut and Salur.

**Theorem 5.4** [3] *On a topological  $G_2$  manifold  $(M, \varphi)$ , the Zariski tangent space of infinitesimal associative deformations of an associative  $Y$  is given by the kernel of some twisted Dirac operator  $\mathbf{D}_a$  for which  $\mathbf{D}_a - \mathbf{D}$  is an operator of order 0 that vanishes if  $\varphi$  is closed.*

Although the kernel will in general depend on  $\mathbf{D}_a$ , the twisting acts only as an operator of order 0 and will therefore have no effect on the symbol of  $\mathbf{D}$ . In particular, we recover the same index. Summarising, we arrive at

**Theorem 5.5** *Let  $(M, \varphi)$  be a topological  $G_2$  manifold and  $Y \subset M$  be an associative with boundary in a  $\varphi$ -free submanifold  $X$ . Then the virtual dimension of  $\mathcal{M}_{X,Y}$  is given by  $\text{index}(\bar{\partial}_{\nu_X})$ .*

## 6 Examples

In this section we consider examples on manifolds of holonomy  $G_2$ .

**A trivial compact example.** Equip the torus  $M = \mathbb{R}^7/\mathbb{Z}^7$  with its standard coordinates  $x^1, \dots, x^7$  and the resulting flat  $G_2$  structure. Then  $Y = \{(x_1, x_2, x_3) \times \{0\} \in T^3 \times T^4 \mid 0 \leq x_1 \leq 1/2\}$  is an associative whose boundary lies in the two coassociatives  $X_1 = \{0\} \times T^4$  and  $X_2 = \{1/2\} \times T^4$ . The induced bundles  $\nu_{X_1}$  and  $\nu_{X_2}$  are both trivial, so that the index vanishes.

**The Calabi-Yau extension.** Take a Riemann surface  $\Sigma$  in a Calabi-Yau 3-fold  $K$  with boundary in a fixed compact special lagrangian  $L$ . Lifting this setup to  $M = K \times S^1$  as in the example of Section 2.3 gives an associative  $Y = \Sigma \times S^1$  whose boundary  $\partial Y = S^1 \times S^1$  is contained in the coassociative  $X = L \times S^1$ . Let  $\mathcal{M}_{L,\Sigma}$  be the space of infinitesimal deformations of  $\Sigma$  through Riemann surfaces inside  $K$  with boundary in  $L$ . In [20], Katz and Liu proved that the virtual dimension of  $\mathcal{M}_{L,\Sigma}$  vanishes. On the other hand, the infinitesimal deformations of  $Y = \Sigma \times S^1$  through associatives boil down to deformations  $\Sigma$  through Riemann surfaces in  $K$  (cf. for instance [22]). Consequently,  $Y$  should be virtually rigid. Indeed, consider the orthogonal complement  $\nu_L = (T\partial\Sigma)^\perp \rightarrow \partial\Sigma = S^1$  inside  $TL|_{\partial\Sigma}$ . By design,  $\nu_{X|\partial Y} \cong p^*\nu_L$ , where  $p: \partial Y = S^1 \times S^1 \subset K \times S^1 \rightarrow S^1 \subset K$  is projection onto the left factor. As the bundle  $\nu_L \rightarrow S^1$  admits a nowhere vanishing section being of rank 2, so does the (oriented) bundle  $\nu_X$ . Consequently,  $\chi(\nu_X) = c_1(\nu_X) = 0$  and we obtain the

**Corollary 6.1** *Let  $K$  be a Calabi-Yau 3-fold and  $\Sigma \subset K$  an embedded compact Riemann surface with connected boundary inside a compact special lagrangian  $L$ . Let  $Y = \Sigma \times S^1$  in  $M = K \times S^1$ , and  $X = L \times S^1$ . Then the virtual dimension of  $\mathcal{M}_{X,Y}$  vanishes.*

**Coassociative germs.** Take in  $M$  an associative submanifold  $Y$  with real analytic boundary  $\partial Y$ , and consider a nowhere vanishing real analytic section  $a \in \Gamma_{\partial Y}(\nu)$ . Since a metric of holonomy  $G_2$  is necessarily Ricci flat, the metric is real analytic in harmonic coordinates [8], and so is the geodesic flow  $\gamma_a: \partial Y \times (-\epsilon, \epsilon) \rightarrow M$  induced by  $a$ , which therefore generates an analytical submanifold  $N$  of dimension 3. Further,  $\varphi(v, w, a) = 0$  for  $v, w \in T\partial Y$ , and since  $\nabla\varphi = 0$ , we conclude that the pull-back of  $\varphi$  to  $N$  vanishes identically. An argument of Cartan-Kähler type invoked by Harvey and Lawson [13] (see also [5]) shows that  $N$  is contained in a coassociative  $X$  whose germ around  $N$  is uniquely determined. Furthermore,  $\nu_X$  is generated by  $a$  and  $u \times a$ , where  $u$  denotes again the inward pointing normal vector field of  $\partial Y$ . For instance, taking  $M = \text{Im } \mathbb{O}$  yields plenty of examples of arbitrary index.

**Associative germs.** In the vein of the previous example, consider a real analytic surface  $\Sigma$  inside a coassociative  $X \subset M$ . As before, Cartan-Kähler theory yields the existence of an associative  $Y$  whose germ is uniquely determined. By using a collar neighbourhood of

$\Sigma$  inside  $Y$  we can construct an associative which we keep on denoting by  $Y$  for simplicity, whose boundary consists of two components  $\Sigma \cup \Sigma'$ . Further, we can translate  $X$  into  $X'$  containing  $\Sigma'$  by a suitable diffeomorphism  $C^1$ -close enough to the identity. Of course, there is no reason for the diffeomorphism to preserve the  $G_2$  structure, so that  $X'$  will not be coassociative. However, as  $\varphi$ -freeness is an open condition,  $X'$  will be still  $\varphi$ -free for suitable  $Y$ , and the generalised deformation theorem applies. Since  $\Sigma'$  is homeomorphic to  $\Sigma$ , but with flipped orientation, we conclude that the virtual dimension of  $\mathcal{M}_{X,Y}$  vanishes.

## 7 A $G_2$ analogon of the Maslov index

Next we wish to introduce a  $G_2$  analogon of the Maslov index, whose construction we briefly recall. Consider an almost complex manifold  $(M^{2m}, J)$  with embedded (not necessarily holomorphic) 2-disk  $D$ , whose boundary lies in a totally real oriented submanifold  $X^m$  (i.e for any  $x \in X$ ,  $T_x X$  does not contain any  $J$ -complex line). Since  $D$  is contractible, the subbundle  $TX|_{\partial D}$  may be regarded as a closed curve in the set of totally real oriented  $m$ -planes in  $\mathbb{C}^m$  after choosing some trivialisation of  $TM|_D$ . On the other hand, this set is parametrised by  $GL_m(\mathbb{C})/GL_m(\mathbb{R})$  and is homotopy equivalent to  $U(m)/O(m)$  – the set of oriented lagrangian submanifolds of  $\mathbb{C}^m$ . By the exact homotopy sequence for fibrations  $\pi_1(U(m)/O(m)) \cong \mathbb{Z}$ , and the Maslov index  $\mu(\partial D)$  of  $\partial D$  is the integer corresponding to the homotopy class induced by  $TX|_D$ .

The natural construction in the  $G_2$  setting should be the following. Let  $Y$  be an embedded (not necessarily associative) 3-disk inside a topological  $G_2$  manifold  $M$  such that  $\partial Y \cong S^2$  lies in some  $\varphi$ -free orientable submanifold  $X$ . Trivialising  $TM|_Y$  and orienting  $X$  suitably yields thus a map from  $S^2$  to the set  $\mathcal{P}_+$  of positively oriented  $\varphi$ -free planes in  $\mathbb{R}^7$ .

**Proposition 7.1** *The set of positively oriented  $\varphi$ -free planes  $\mathcal{P}_+ \subset G_4(\mathbb{R}^7)$  is homotopy equivalent to  $G^{\psi_0}(\mathbb{R}^7) \cong G_2/SO(4)$ , the set of coassociatives. In particular,  $\pi_2(\mathcal{P}_+) \cong \mathbb{Z}_2$ .*

**Definition 7.1** *Let  $Y$  be an embedded associative 3-disk in some topological  $G_2$  manifold. We refer to the integer given by the natural class of  $TX|_{\partial Y}$  in  $\pi_2(\mathcal{P}_+) \cong \mathbb{Z}_2$  as the  $G_2$  Maslov index of  $\partial Y$ , and denote it by  $\mu_{G_2}(\partial Y)$ .*

**Proof:** Instead of  $\mathcal{P}_+$  we shall consider the dual set  $\mathcal{P}_+^\perp \subset G_3(\mathbb{R}^7)$ . Regard  $\varphi : G_3(\mathbb{R}^7) \rightarrow \mathbb{R}$  as a smooth function on the grassmannian of oriented 3-planes. It takes values inside  $[-1, 1]$ , and the fibres are acted on transitively by  $G_2$ . In particular, the two critical values  $\pm 1$  correspond to the set of associatives  $G^\psi(\mathbb{R}^7)$  with  $+$  or  $-$  the natural orientation. Furthermore, any fibre  $\varphi^{-1}(t)$  contains an element of the form

$$F_t = e_1 \wedge e_2 \wedge (te_3 + \sqrt{1-t^2}e_4)$$

with respect to a fixed  $G_2$  frame  $e_1, \dots, e_7$  of Section 2. To prove this, write  $F_t = x \wedge y \wedge z$  for some unit vectors  $x, y, z \in \mathbb{R}^7$ . Now  $G_2$  acts transitively on ordered orthonormal pairs with stabiliser  $SU(2)$  [13], so that we may assume that  $F_t = e_1 \wedge e_2 \wedge z$  upon transformation by a suitable element in  $G_2$ . The  $SU(2)$  action induced by the inclusion into  $G_2$  gives rise to a decomposition  $\mathbb{R}^7 = \text{Im } \mathbb{H} \oplus \mathbb{H}$ , where  $SU(2)$  acts trivially on  $\text{Im } \mathbb{H}$  and  $\mathbb{H} = \mathbb{C}^2$  becomes the standard vector representation. We are still free to modify  $F_t$  without changing  $e_1$  and  $e_2$  by an element in  $SU(2)$ . Since this group acts transitively on the unit sphere in  $\mathbb{C}^2$ , we may transform the unit vector  $z = \sum_{i=3}^7 z^i e_i$  into  $te_3 + \sqrt{1-t^2}e_4$  with  $t = z^3 = \varphi(F_t)$ . From this one easily deduces that (a)  $\pm 1$  are the only critical points and

(b) that these are non-degenerate in the sense that the Hessian of  $\varphi$  is non-degenerate in directions transverse to the orbits  $\varphi^{-1}(\pm 1) \cong G_2/SO(4)$ . Consequently,  $\varphi$  defines a  $G = G_2$  invariant Morse function in the sense of [24]. By a theorem in the same paper, we conclude that  $G_3(\mathbb{R}^7) = \varphi^{-1}([-1, 1])$  is homotopy equivalent to  $\varphi^{-1}([-1, 0])$  with the disk bundle  $G_2 \times_{SO(4)} D^4$  of the normal bundle over  $G_2/SO(4)$  attached. But by Lemma 5.1,  $\mathcal{P}_+^\perp = \varphi^{-1}([-1, 1]) - \varphi^{-1}([-1, 0])$ , so that  $\mathcal{P}_+^\perp$  is homotopy equivalent to the open disk bundle of the normal bundle  $G_2 \times_{SO(4)} \mathbb{R}^4 \rightarrow \varphi^{-1}(1)$ . This, in turn, can be retracted to the base, which is the critical orbit  $\varphi^{-1}(1) \cong G_2/SO(4)$ . In particular,  $\mathcal{P}_+$  is of the same homotopy type as  $G_2/SO(4)$ . Since  $\pi_k(G_2) = 0$  for  $k = 1, 2$  and  $\pi_1(SO(4)) = \mathbb{Z}_2$ , the exact homotopy sequence for fibrations yields the asserted homotopy groups.  $\blacksquare$

A non-trivial representative of  $\pi_2(G_2/SO(4))$  can be constructed as follows. Consider the embedded sphere  $S^2 \hookrightarrow \text{Im } \mathbb{H} \subset \text{Im } \mathbb{H} \oplus \mathbb{H}$  with its natural complex structure  $G = u \times$ , where  $u(x) = x$  now denotes the (outward pointing) position vector field normal to the tangent space. As a complex manifold,  $S^2 \cong \mathbb{C}P^1$ , and we can consider the complex rank 2 bundle  $E = \mathcal{O}(2) \oplus \mathcal{O}(-1)$  (where by abuse of notation  $\mathcal{O}(k)$  denotes the sheaf of sections as well as the corresponding complex line bundle of degree  $k$ ). Here,  $\mathcal{O}(2)$  is the tangent bundle seen as a  $u \times$ -complex line bundle in  $\mathbb{C}P^1 \times \text{Im } \mathbb{H}$  and  $\mathcal{O}(-1)$  is the canonical bundle inside  $\mathbb{C}P^1 \times \mathbb{H}$ . Each fibre  $E_x$  is clearly coassociative, whence the map

$$f : x \in S^2 \mapsto E_x \in G^{\psi_0}(\mathbb{R}^7).$$

It remains to see that the resulting homotopy class  $[f] \in \pi_2(G_2/SO(4))$  is non-trivial. To that end, we show its boundary  $\partial[f]$  to be a generator of  $\pi_1(SO(4)) \cong \mathbb{Z}_2$ . By definition, a representative of  $\partial[f]$  is obtained by compounding  $f$  with the collapsing map  $c : (D^2, S^1) \rightarrow (S^2, N)$  ( $N$  and  $S$  being the north and south pole of  $S^2$ ), lifting the resulting map  $(D^2, S^1) \rightarrow (G^{\psi_0}(\mathbb{R}^7), E_N)$  to a map  $F : (D^2, S^1) \rightarrow (G_2, SO(4))$  with respect to the covering map  $\pi : G_2 \rightarrow G^{\psi_0}(\mathbb{R}^7)$  given by

$$(u_1, u_2, u_3) \mapsto u_1 \wedge u_2 \wedge u_3 \wedge C(u_1, u_2, u_3),$$

cf. (3) and (5), and finally restricting this lift to  $S^1$ . So let  $F : (D^2, S^1) \rightarrow (G_2, SO(4))$  be a lift. Under  $c$  the punctured disk  $D^\times$  gets mapped to  $S^2 \setminus \{S\}$ . There, we can construct a lift  $\tilde{f} = v \wedge u \times v \wedge a : S^2 \setminus \{S\} \rightarrow G_2$  of  $f$  by taking (smooth) unit sections  $v$  and  $a$  of  $\mathcal{O}(2)$  and  $\mathcal{O}(-1)$ . Since  $\pi \circ F|_{D^\times} = \pi \circ \tilde{f} \circ c$ , we have  $F(x) = A_x(\tilde{f} \circ c(x))$  for  $A : D^\times \rightarrow SO(4)$ . In particular,  $F(S^1) = A|_{S^1}(N)$ , and  $A|_{S^1}$  is a transition function of the bundle  $E$ . Therefore, it is of the form

$$A(t) = \begin{pmatrix} e^{2it} & 0 \\ 0 & e^{-it} \end{pmatrix} \in U(2) \quad (12)$$

so that the induced homotopy class clearly generates  $\pi(SO(4))$ . In general, if  $Y$  is an embedded 3-disk inside some topological  $G_2$  manifold  $(M, \varphi)$  with boundary in a coassociative  $X$ , then  $\nu_X$  is a complex bundle of degree  $k$ , that is,  $TX|_{\partial Y} = \mathcal{O}(2) \oplus \mathcal{O}(k)$ . The same reasoning as before yields

$$t \in S^1 \mapsto \begin{pmatrix} e^{2it} & 0 \\ 0 & e^{kit} \end{pmatrix} \in U(2)$$

as a representative of the homotopy class of  $TX|_{\partial Y}$ , which by definition is  $\mu_{G_2}(\partial Y)$ . Now  $k = \int_{S^2} c_1(\nu_X)$  and we thus have proven

**Proposition 7.2** *Let  $Y \subset M$  be an embedded 3-disk with boundary in a coassociative  $X$ . If  $\nu_X$  denotes again the orthogonal complement of  $T\partial Y$  in  $TX$ , then*

$$\mu_{G_2}(\partial Y) = \int_{S^2} c_1(\nu_X) \bmod 2 = \text{index}(\nu_X) \bmod 2 + 1.$$

## References

- [1] M. AGANAGIC, C. VAFA, *Mirror Symmetry, D-Branes and Counting Holomorphic Discs*, hep-th/0012041.
- [2] S. AKBULUT, S. SALUR, *Associative submanifolds of a  $G_2$  manifold*, math/0412032.
- [3] S. AKBULUT, S. SALUR, *Deformations in  $G_2$  manifolds*, Adv. Math. **217** no. 5 (2008), 2130–2140.
- [4] B. BOOSS, K. WOJCIECHOWSKI, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser Verlag, Boston, 1993.
- [5] R. BRYANT, *Calibrated embeddings in the special Lagrangian and coassociative cases*, Ann. Global Anal. Geom. **18** no. 3–4 (2000), 405–435.
- [6] A. BUTSCHER, *Deformations of Minimal Lagrangian Submanifolds with Boundary*, Proc. Am. Math. Soc. **131** no. 6 (2003), 1953–1964.
- [7] A. CALDERÓN, *Lecture notes on pseudo-differential operators and elliptic boundary value problems I.*, Publicaciones del I.A.M., Buenos Aires, 1976.
- [8] D. DETURCK, J. KAZDAN *Some regularity theorems in Riemannian geometry*, Ann. Sci. Éc. Norm. Supér. (4) **14** no. 3 (1981), 249–260.
- [9] S. DONALDSON, R. THOMAS, *Gauge theory in higher dimensions*, in: The geometric universe, pp. 31–47, OUP, Oxford, 1998.
- [10] M. FERNÁNDEZ, A. GRAY, *Riemannian manifolds with structure group  $G_2$* , Ann. Mat. Pura Appl. **132** (1982), 19–45.
- [11] A. FLOER, *Morse theory for Lagrangian intersections*, J. Diff. Geom. **28** no. 3 (1988), 513–547.
- [12] M. GROMOV, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** no. 2 (1985), 307–347.
- [13] R. HARVEY, H. LAWSON, *Calibrated geometries*, Acta Math. **148**, (1982) 47–157.
- [14] R. HARVEY, H. LAWSON, *Plurisubharmonic Functions in Calibrated Geometries*, math/0601484.
- [15] R. HARVEY, H. LAWSON, *Duality of Positive Currents and Plurisubharmonic Functions in Calibrated Geometry*, math/07103921.
- [16] R. HARVEY, H. LAWSON, *Dirichlet Duality and the Nonlinear Dirichlet Problem*, math/07103991.
- [17] N. HITCHIN, *Stable forms and special metrics*, Contemporary Math. **288** (2001), 70–89.

- [18] D. JOYCE, *Compact manifolds with special holonomy*, OUP, Oxford, 2000.
- [19] D. JOYCE, *The exceptional holonomy groups and calibrated geometry*, math/0406011.
- [20] S. KATZ, C. LIU, *Enumerative Geometry of Stable Maps with Lagrangian Boundary Conditions and Multiple Covers of the Disc*, Adv. Theor. Math. Phys. **5** (2002), 1–49.
- [21] A. KOVALEV, J. LOTAY, *Deformations of Compact Coassociative 4-folds with Boundary*, math/07124325.
- [22] J. LEE, N. LEUNG, *Geometric structures on  $G_2$ - and  $Spin(7)$ -manifolds*, math/0202045.
- [23] N. LEUNG, X. WANG, *Intersection theory of coassociative submanifolds in  $G_2$ -manifolds and Seiberg–Witten invariants*, math/0401419.
- [24] K.-H. MAYER,  *$G$ -invariante Morse-Funktionen*, manuscripta math. **63** (1989), 99–114.
- [25] R. MCLEAN, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. **6** no. 4 (1998), 705–747.
- [26] E. WITTEN, *Two-dimensional gravity and intersection theory on moduli space*, in: Surveys in differential geometry, pp. 243–310, Lehigh Univ., Bethlehem, 1991.

D. GAYET: Institut Camille Jordan, CNRS UMR 5208, UFR de mathématiques, Université Lyon I Btiment Braconnier, F-69622 Villeurbanne Cedex, France  
 e-mail: gayet@math.univ-lyon1.fr

F. WITT: FB Mathematik und Informatik, FU Berlin, Arnimallee 3, D-14195 Berlin, F.R.G.  
 e-mail: fwitt@mi.fu-berlin.de