Convergence of weighted polynomial multiple ergodic averages

Qing Chu

CONVERGENCE OF WEIGHTED POLYNOMIAL MULTIPLE ERGODIC AVERAGES

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Abstract. We study here weighted polynomial multiple ergodic averages. A sequence of weights is called universally good if any polynomial multiple ergodic average with this sequence of weights converges in L^2. We find a necessary condition and show that for any bounded measurable function ϕ on an ergodic system, the sequence $\phi(T^n x)$ is universally good for almost every x. The linear case was understood by Host and Kra.

1. introduction

In his innovative proof of Szemerédi's Theorem via ergodic theory, Furstenberg introduced certain multiple ergodic averages. There have been many results on these and other nonconventional ergodic averages, including the multiple ergodic theorems of Host and Kra [2], [3], Ziegler [10], Leibman [5]... Recently Host and Kra studied weighted ergodic theorems for multiple averages along arithmetic progressions, and we give a generalization of this result for polynomial averages, showing:

Theorem 1.1. *Let (Y, ν, S) be an ergodic system and $\phi \in L^\infty(\nu)$. Then there exist $Y_0 \subset Y$ with $\nu(Y_0) = 1$ such that, for every $y_0 \in Y_0$, every system (X, μ, T), every $r \geq 1$, all integer polynomials $p_1, \ldots, p_r$ and all functions $f_1, \ldots, f_r \in L^\infty(\mu)$, the averages*

$$\frac{1}{N} \sum_{n=0}^{N-1} \phi(S^n y_0) T^{p_1(n)} f_1 \cdot \ldots \cdot T^{p_r(n)} f_r$$

converge in $L^2(\mu)$.

Throughout this article, by *integer polynomial* we mean a polynomial all of whose coefficients are integers.

The case of $p_i(n) = in$ was proved by Host and Kra [4].

Note that the set Y_0 does not depend on X or on $f_i, i = 1, \ldots, r$. We say that for every $y_0 \in Y_0$, the sequence $\phi(S^n y_0)$ is *universally good for the convergence in mean of polynomial multiple ergodic averages.*

For $r = 1$ and $p(n) = n$, the result follows immediately from the classical Wiener-Winter ergodic Theorem [9] and a corollary of Spectral Theorem. We follow a similar strategy, generalizing the proof in [4] along arithmetic progressions, but need to address some deeper technical issues.

We first recall some definitions, see [1] and [4] for details. Let G be a k-step nilpotent Lie group and $\Gamma \subset G$ be a discrete, cocompact subgroup of G. The

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compact manifold $X = G/\Gamma$ is called a *k-step nilmanifold*. The Haar measure μ of X is the unique probability measure invariant under the left translations $x \mapsto gx$ of G on X. Letting T denote left multiplication by the fixed element $\alpha \in G$, we call (X, μ, T) a *k-step nilsystem*. Let $f\colon X \to C$ be a continuous function, and $x_0 \in X$, the sequence $(f(\alpha^n x_0) : n \in \mathbb{Z})$ is called a *basic k-step nilsequence*. A *k-step nilsequence* is a uniform limit of basic k-step nilsequences.

The proof of Theorem 1.1 is broken down into two pieces. First we give a spectral result for higher sequences generated by polynomial orbits in nilmanifolds:

Theorem 1.2. *(Generalized Corollary of Spectral Theorem) For any $r, b \in \mathbb{N}$, there exists an integer $K \geq 1$ with the following properties: for any bounded sequence $\mathbf{c} = (c_n : n \in \mathbb{Z})$, if the averages*

$$\frac{1}{N}\sum_{n=0}^{N-1} c_n d_n$$

converge as $N \to \infty$ for every K-step nilsequence $\boldsymbol{d} = (d_n : n \in \mathbb{Z})$, then for every system (X, μ, T), all $f_1, \ldots, f_r \in L^\infty(X)$, and all integer polynomials $p_1, \ldots, p_r$ of degree $\leq b$, the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} c_n T^{p_1(n)} f_1 \cdot T^{p_2(n)} f_2 \cdot \ldots \cdot T^{p_r(n)} f_r \tag{1}$$

converge in $L^2(X)$.

The bulk of this paper is devoted to the proof of this theorem. Then our main result follows from the following Generalized Wiener-Wintner Theorem proved by Host and Kra in [4]. The case of a polynomial version of Wiener-Wintner theorem was proved by Lesigne [7] [8].

Theorem 1.3. *(Generalized Wiener-Wintner Theorem [4]) Let (X, μ, T) be an ergodic system and ϕ be a bounded measurable function on X. Then there exists $X_0 \subset X$ with $\mu(X_0) = 1$ such that for every $x \in X_0$, the averages*

$$\frac{1}{N}\sum_{n=0}^{N-1} \phi(T^n x) b_n$$

converge as $N \to \infty$ for every $x \in X_0$ and every nilsequence $b = (b_n : n \in \mathbb{Z})$.

While nilsequences do not appear in the statement of Theorem 1.1, they are used as tools in its proof. Both Theorems 1.2 and 1.3 are of interest on their own, as results on nilsequences.

2. Proof of theorem 1.2

It is sufficient to prove Theorem 1.2 for ergodic systems.

2.1. The next proposition state that in nilsystem, the values along polynomials are in fact the values along ordinary orbits in another nilsystem with different initial value.

Proposition 2.1. *Let $(X = G/\Gamma, T)$ be a nilsystem, $x_0 \in X$, p be an integer polynomial, and $f \in \mathcal{C}(X)$. Then there exists a nilsystem (Y, S), $y_0 \in Y$, $h \in \mathcal{C}(Y)$, such that $f(T^{p(n)} x_0) = h(S^n y_0)$ for every n.*

Proof. Let $(X = G/\Gamma, T)$ be a nilsystem. Suppose $Tx := \alpha x$, for some $\alpha \in G$. Then $T^{p(n)}x = \alpha^{p(n)}x$. Let $g(n) := \alpha^{p(n)}$, then g is a polynomial sequence in G. Let $\pi : G \to X$ be the factorization mapping. We will assume that $x_0 = \pi(\mathbf{1}_G)$; otherwise if $x_0 = \pi(\gamma), \gamma \in G$, we write $g(n)x_0 = g(n)\gamma\pi(\mathbf{1}_G)$, and replace $g(n)$ by $g(n)\gamma$.

Now we have a nilpotent lie group G, a discrete cocompact subgroup Γ and a polynomial sequence g in G. By Proposition 3.14 in Leibman's paper [6], there exist a nilpotent lie group $\widetilde{G}$, a discrete cocompact subgroup $\widetilde{\Gamma}$, an epimorphism $\eta : \widetilde{G} \to G$ with $\eta(\widetilde{\Gamma}) \subseteq \Gamma$, a unipotent automorphism $\widetilde{\tau}$ of $\widetilde{G}$ with $\widetilde{\tau}(\widetilde{\Gamma}) = \widetilde{\Gamma}$, and an element $\widetilde{c} \in \widetilde{G}$ such that

$$g(n) = \eta(\widetilde{\tau}^n(\widetilde{c})), n \in \mathbb{Z}.$$

Let $\widetilde{X} = \widetilde{G}/\widetilde{\Gamma}$ and let $\widetilde{\pi} : \widetilde{G} \to \widetilde{X}$ be the factorization mapping.

The epimorphism $\eta : \widetilde{G} \to G$ factors to a map $\widetilde{X} \to X$ that we write also η, which is onto, that is,

$$\pi \circ \eta = \eta \circ \widetilde{\pi} \ .$$

τ induces an homeomorphism $\widetilde{X} \to \widetilde{X}$, that we write $\widetilde{\tau}$ also:

$$\widetilde{\tau} \circ \widetilde{\pi} = \widetilde{\pi} \circ \widetilde{\tau} \ .$$

Let $\widetilde{x}_0 = \widetilde{\pi}(\mathbf{1}_{\widetilde{G}})$, then

$$\eta(\widetilde{\tau}^n(\widetilde{c}\widetilde{x_0})) = g(n)x_0, \ n \in \mathbb{Z} \ .$$

Let $\widehat{G}$ be the extension of $\widetilde{G}$ by $\widetilde{\tau}$, then $\widehat{G}$ is a nilpotent Lie group. Let $\widehat{\tau}$ be the element in $\widehat{G}$ representing $\widetilde{\tau}$, so that $\widetilde{\tau}(\widetilde{\alpha}) = \widehat{\tau}\widetilde{\alpha}\widehat{\tau}^{-1}$ for any $\widetilde{\alpha} \in \widetilde{G}$.

Let $\widehat{\Gamma}$ be the subgroup of $\widehat{G}$ spanned by Γ and $\widehat{\tau}$, since $\widetilde{\tau}(\widetilde{\Gamma}) = \widetilde{\Gamma}$, one has $\widehat{\Gamma} \cap \widetilde{G} = \widetilde{\Gamma}$. As $\widehat{G}$ is spanned by $\widetilde{G}$ and $\widehat{\tau}$, $\widetilde{G}$ is open in $\widehat{G}$ and by the definition of the relative topology, we have that $\widehat{\Gamma}$ is a discrete subgroup of $\widehat{G}$.

Moreover, $\widetilde{X}$ can be identified with $\widehat{G}/\widehat{\Gamma}$ and we write $\widehat{\pi}\colon \widehat{G} \to \widetilde{X}$ for the quotient map.

Let $\widetilde{x} \in \widetilde{X}$ and $\widetilde{g} \in \widetilde{X}$ with $\widetilde{\pi}(\widetilde{g}) = \widetilde{x}$. We have

$$\widetilde{\tau}(\widetilde{x}) = \widetilde{\pi}(\widetilde{\tau}(\widetilde{g})) = \widehat{\pi}(\widehat{\tau}\widetilde{g}\widetilde{\tau}^{-1}) = \widehat{\pi}(\widehat{\tau}\widetilde{g}) = \widehat{\tau}\widetilde{x}$$

because $\widehat{\tau}^{-1} \in \widehat{\Gamma}$. So for every n,

$$g(n)x_0 = \eta(\widetilde{\tau}^n(\widetilde{c}\widetilde{x_0})) = \eta(\widehat{\tau^n}(\widetilde{c}\widetilde{x_0})) \ .$$

Let $Y = (\widehat{G}/\widehat{\Gamma}, S) = (\widetilde{X}, S)$, $S\widetilde{x} = \widehat{\tau}\widetilde{x}$, and let $h = f \circ \eta$, and $y_0 = \widetilde{cx_0}$. This system and this function satisfy the announced properties. □

2.2. We return to prove theorem 1.2. We may assume that the polynomials $p_1, \ldots, p_r$ are nonconstant and essentially distinct, that is $p_i - p_j \neq$ constant for $i \neq j$.

The following theorem will be proved in the next section.

Theorem 2.2. *For any $r, b \in \mathbb{N}$, there is $k \in \mathbb{N}$, such that for any nonconstant essentially distinct polynomials $p_1, \ldots, p_r : \mathbb{Z} \to \mathbb{Z}$ of degree $\leq b$, for every ergodic system (X, μ, T) every $f_1, \ldots, f_r \in L^\infty(X)$ with $\|f_1\|_k = 0$, and any bounded sequence $\mathbf{c} = (c_n : n \in \mathbb{Z})$, one has*

$$\lim_{N\to\infty} ||\frac{1}{N}\sum_{n=0}^{N-1} c_n T^{p_1(n)} f_1 \cdot \cdots \cdot T^{p_r(n)} f_r||_{L^2(X)} = 0 \tag{2}$$

2.3. We recall a few properties of the seminorms and the factors introduced in [2]. Let (X, μ, T) be an ergodic system. For every $k \geq 1$, $\|\cdot\|k$ is a seminorm on $L^\infty(\mu)$. These seminorms satisfy: for every k and every $f \in L^\infty(\mu)$, $\|f\|_{k+1} \geq \|f\|_k$ and

$$\|f\|_{k+1} = \lim_{N\to\infty} (\sum_{n=0}^{N-1} \|f \cdot T^n \overline{f}\|^{2^k})^{1/2^k}$$

Moreover, for every $k \geq 2$, X admits a factor Z_{k-1} such that, for every $f \in L^\infty(\mu)$, $\|f\|_k = 0$ if and only if $\mathbb{E}(f|Z_{k-1}) = 0$. One of the main results of [2] is that, for every k, Z_k is an inverse limit of k–step nilsystems.

2.4. Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2 from Proposition 2.1 and Theorem 2.2. The proof is exactly the same as the proof of Theorem 2.24 in [2]. For any $r, b \in \mathbb{N}$, let $k \in \mathbb{N}$ be the integer in Theorem 2.2, let Z_{k-1} be the $k-1$–th factor of (X, μ, T) as defined in [4]. By definition, if $\mathbb{E}(f_1|Z_{k-1}) = 0$, then $\|f_1\|_k = 0$, and by Theorem 2.2, the averages (2) converge to zero in $L^2(X)$. We say that the factor Z_{k-1} is the characteristic for the convergence of these averages. Therefore, it suffices to prove the result when the functions are measurable with respect to the factor Z_{k-1}.

Since Z_{k-1} is an inverse limit of $k-1$–step nilsystem. By density, we can assume that (X, μ, T) is a $k-1$–step nilsystem and that the functions $f_1, \ldots, f_r$ are continuous.

But in this case, by Proposition 2.1, for every $x \in X$,and every polynomials $p_1, \ldots, p_r$, there exist nilsystems $(Y_1, S_1), \ldots, (Y_r, S_r)$, $y_i \in Y_i$, and $g_i \in \mathcal{C}(Y_i)$, such that $f_i(T^{p_i(n)}x) = g_i(S_i^n y_i)$, $i = 1, \ldots, r$.

Let K be the maximal order of the nilsystems $(Y_i, S_i), i = 1, \ldots, r$. Then the system $(Y = Y_1 \times \cdots \times Y_r, S = S_1 \times \cdots \times S_r)$ is a K-step nilsystem. Let $g : Y_1 \times \cdots \times Y_r \to \mathbb{R}$ be given by $g(y) = g(y_1, \ldots, y_r) = g_1(y_1) \cdot \ldots \cdot g_r(y_r)$. So the sequence

$$\begin{aligned} &\{f_1(T^{p_1(n)x}) \cdot f_2(T^{p_2(n)}x) \cdot \ldots \cdot f_r(T^{p_r(n)}x)\}_{n\in\mathbb{Z}} \\ &= \{g_1(S_1^n y_1) \cdot g_2(S_2^n y_2) \cdot \ldots \cdot g_r(S_r^n y_r)\}_{n\in\mathbb{Z}} \\ &= \{g(S^n y)\}_{n\in\mathbb{Z}} \end{aligned}$$

is a K-step nilsequence and by hypothesis, the averages (1) converge for every $x \in X$. □

3. PROOF OF THEOREM 2.2

We deduce Theorem 2.2 from a more general result. In this section, we assume that $(X, \mathcal{X}, \mu, T)$ is an ergodic system.

Theorem 3.1. *For any $r, b \in \mathbb{N}$, there exists $k \in \mathbb{N}$, such that for any family of nonconstant essentially distinct polynomials $p_1, \ldots, p_r : \mathbb{Z}^d \to \mathbb{Z}$ of degree $\leq b$, any $f_1, \ldots, f_r \in L^\infty(X)$ with $\|f_1\|_k = 0$, and any$\{c_u\}_{u\in\mathbb{Z}^d}$, with $|c_u| \leq 1$, one has*

$$\lim_{N\to\infty} ||\frac{1}{|\Phi_N|} \sum_{u\in\Phi_N} c_u T^{p_1(u)} f_1 \cdot \ldots \cdot T^{p_r(u)} f_r||_{L^2(X)} = 0$$

for any Følner sequence $\{\Phi_N\}_{N=1}^\infty$ in $\mathbb{Z}^d$.

3.1. We first prove the theorem for polynomials of degree 1 which we call affine functions.

For next lemma, ergodicity is not needed.

Lemma 3.2. *Let $p : \mathbb{Z}^d \to \mathbb{Z}$, $p(u) = a_1u_1 + \cdots + a_du_d$ with $a_1, \ldots, a_u \in \mathbb{Z}$. and not all a_j equal to 0. Let $a = \gcd(a_1, \ldots, a_d)$. Then for any $f \in L^2(X)$ and any Følner sequence $\{\Phi_N\}_{N=1}^{\infty}$ in $\mathbb{Z}^d$, we have*

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{u\in\Phi_N} T^{p(u)} f = \mathbb{E}(f|\mathcal{J}_a)$$

in $L^2(X)$, where $\mathcal{J}_a$ is the factor of X associated with the σ-algebra of T^a-invariant measurable subsets of X.

Proof. Let $E := \{f \in L^2(\mu) \colon f = T^a f\}$, and $F := \{g - T^a g; g \in L^2(\mu)\}$, we know that $E + F$ is dense in $L^2(\mu)$, so it suffices to establish the lemma in the two cases that $f \in E$ and that $f \in F$. When $T^a f = f$, then $T^{p(u)} f = f$ for every u, so the assertion is obvious. Now suppose $f = g - T^a g$ for some $g \in L^2(X)$. Since there exists $h = (h_1, \ldots, h_d) \in \mathbb{Z}^d$ such that $a = gcd(a_1, \ldots, a_d) = h_1a_1 + \cdots + h_da_d$, and by the property of Følner sequence, we have

$$\begin{aligned}\frac{1}{|\Phi_N|} \sum_{u\in\Phi_N} T^{p(u)} f = \frac{1}{|\Phi_N|} \sum_{u\in\Phi_N} (T^{p(u)} g - T^{p(u)+a} g) = \frac{1}{|\Phi_N|} \sum_{u\in\Phi_N} (T^{p(u)} g - T^{p(u+h)} g)\\ = \frac{1}{|\Phi_N|} \sum_{u\in\Phi_N} T^{p(u)} g - \frac{1}{|\Phi_N|} \sum_{u\in h+\Phi_N} T^{p(u)} g\end{aligned}$$

and thus

$$\frac{1}{|\Phi_N|} \sum_{u\in\Phi_N} T^{p(u)} f\|_{L^2(X)} \le \frac{1}{|\Phi_N|} ||g||_{L^2(X)} \cdot |\Phi_N \triangle (\Phi_N + h)| \to 0$$

as $N \to +\infty$. □

Lemma 3.3. *Let $p : \mathbb{Z}^d \to \mathbb{Z}$ be a nonconstant affine function. There exists a constant c such that for any $f \in L^\infty(X)$ and any sequence $\{c_u\}_{u\in\mathbb{Z}^d}$,$|c_u| \le 1$, one has*

$$\limsup_{N\to\infty} ||\frac{1}{|\Phi_N|} \sum_{u\in\Phi_N} c_u T^{p(u)} f||_{L^2(X)} \le c|||f|||_2 \tag{3}$$

for any Følner sequence $\{\Phi_N\}_{N=1}^{\infty}$ in $\mathbb{Z}^d$.

Proof. Let $p(u) = a_0 + a_1u_1 + \cdots + a_du_d$, $u = (u_1, \ldots, u_d) \in \mathbb{Z}^d$, with $a_1, \ldots, a_u \in \mathbb{Z}$. After replacing f by $T^{a_0} f$ we may assume that $a_0 = 0$. Put $a = gcd(a_1, \ldots, a_d)$. Let Q be the left side of (3). Then

$$\begin{aligned}Q^2 &= \limsup_{N\to\infty} ||\frac{1}{|\Phi_N|} \sum_{u\in\Phi_N} c_u T^{p(u)} f||^2_{L^2(X)}\\ &\le \limsup_{N\to\infty} \frac{1}{|\Phi_N|^2} \sum_{u\in\Phi_N} \sum_{v\in\Phi_N} |c_u\overline{c_v} \int T^{p(u-v)} f \cdot \overline{f} d\mu|\end{aligned}$$

By the Cauchy-Schwarz Inequality,

$$Q^4 \le \limsup_{N\to\infty} \frac{1}{|\Phi_N|^2} \sum_{u\in\Phi_N} \sum_{v\in\Phi_N} |\int T^{p(u-v)} f \cdot \overline{f} d\mu|^2$$

Since $\{\Phi_N\}_{N=1}^{\infty}$ is a Følner sequence in $\mathbb{Z}^d$, $\{\Phi_N \times \Phi_N\}_{N=1}^{\infty}$ is a Følner sequence in $\mathbb{Z}^{2d}$, and $p' : (u,v) \longmapsto p(u-v)$ is a nonconstant non zero linear function on $\mathbb{Z}^{2d}$. We have

$$\begin{aligned} Q^4 &\leq \limsup_{N\to\infty} \frac{1}{|\Phi_N|^2} \sum_{(u,v)\in\Phi_N\times\Phi_N} |\int_X T^{p'(u,v)} f \cdot \overline{f} d\mu|^2 \\ &\leq \limsup_{N\to\infty} \frac{1}{|\Phi_N|^2} \sum_{(u,v)\in\Phi_N\times\Phi_N} \int_{X\times X} (T\times T)^{p'(u,v)} f\otimes\overline{f}\cdot f\otimes\overline{f} d\mu\times\mu \end{aligned}$$

Applying Lemma 3.2 to system $(X\times X, \mu\times\mu, T\times T)$ and linear polynomial function p', we get

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|^2} \sum_{(u,v)\in\Phi_N\times\Phi_N} (T\times T)^{p'(u,v)} f\otimes\overline{f} = \mathbb{E}(f\otimes\overline{f}|\mathcal{I}_a),$$

where $\mathcal{I}_a$ is the σ-algebra of $T^a\times T^a$-invariant measurable subsets of $X\times X$. Thus

$$\begin{aligned} Q^4 &\leq \int_{X\times X} |\mathbb{E}(f\otimes\overline{f}|\mathcal{I}_a)|^2 d\mu\times\mu = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_{X\times X}(T\times T)^{an} f\otimes\overline{f}\cdot\overline{f}\otimes f d\mu\times\mu \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|\int_X T^{an} f\cdot\overline{f}d\mu|^2 \leq a\lim\frac{1}{aN}\sum_{n=1}^{aN}|\int_X T^n f\cdot\overline{f}d\mu|^2 = a\|\!|f|\!\|_2^4. \end{aligned}$$

□

Lemma 3.4. *Let $p : \mathbb{Z}^d \to \mathbb{Z}$ be a nonconstant affine function. There exists a constant l such that for any $f \in L^\infty(X)$, any $k \geq 1$ and any Følner sequence $\{\Phi_N\}_{N=1}^{\infty}$ in $\mathbb{Z}^d$, one has*

$$\lim_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{u\in\Phi_N}\|\!|f\cdot T^{p(n)}\overline{f}|\!\|_k^{2^{k+1}} \leq l\|\!|f|\!\|_{k+1}^{2^{k+1}}$$

Proof. See Leibman [5], Lemma 8. □

Proposition 3.5. *Let $p_1, p_2, \ldots, p_r : \mathbb{Z}^d \to \mathbb{Z}$ be nonconstant essentially distinct affine functions. There exists a constant C such that for any $f_1, \ldots, f_r \in L^\infty(X)$ and any sequence $\{c_u\}_{u\in\mathbb{Z}^d}, |c_u| \leq 1$, one has*

$$\limsup_{N\to\infty}||\frac{1}{|\Phi_N|}\sum_{u\in\Phi_N} c_u T^{p_1(u)} f_1\cdot\cdots\cdot T^{p_r(u)} f_r||_{L^2(X)} \leq C\|\!|f_1|\!\|_{r+1}\cdot\prod_{i=2}^{r}||f_i||_{L^\infty(X)}.$$

for any Følner sequence $\{\Phi_N\}_{N=1}^{\infty}$ in $\mathbb{Z}^d$.

Proof. The proof is almost exactly the same as that of Proposition 5 in Leibman's paper [5], the only difference is here we are dealing with the weighted averages. We follow Leibman's proof [5] line by line, giving all the steps but omitting proofs when there are the same.

We will proceed by induction on r. For $r = 1$, the statement is given by Lemma 3.3. Let $r \geq 2$, $f_1, \ldots, f_r \in L^\infty(X)$ and let $\{\Phi_N\}_{N=1}^{\infty}$ be Følner sequence in $\mathbb{Z}^d$. We can assume that $|f_2|, \ldots, |f_r| \leq 1$, and that $p_1(0) = \cdots = p_r(0) = 0$. By Lemma

4(i) in [5], for any finite $F \subset \mathbb{Z}^d$, we get

$$\limsup_{N\to\infty} ||\frac{1}{|\Phi_N|}\sum_{u\in\Phi_N} c_u \prod_{i=1}^{r} T^{p_i(u)} f_i||^2_{L^2(X)}$$

$$\leq \frac{1}{|F|^2}\sum_{v,w\in F}\limsup_{N\to\infty}||\frac{1}{|\Phi_N|}\sum_{u\in\Phi_N} c_{u+v}\overline{c_{u+w}}\prod_{i=1}^{r-1} T^{(p_i-p_r)(u)}(T^{p_i(v)}f_i\cdot T^{p_i(w)}\overline{f_i})||_{L^2(X)}.$$

For fixed v, w, write $d_u^{v,w} = c_{u+v}\overline{c_{u+w}}$, so $|d_u^{v,w}| \leq 1$. We remark that $p_i - p_r$, $1 \leq i \leq r-1$, are non constant essentially distinct affine functions. By the induction hypothesis there exists a constant $C^{'}$, independent of $f_1, \ldots, f_r, \{c_u\}$, v, w, and $\{\Phi_N\}_{N=1}^{\infty}$, such that

$$\limsup_{N\to\infty}||\frac{1}{|\Phi_N|}\sum_{u\in\Phi_N} d_u^{v,w}\prod_{i=1}^{r-1} T^{(p_i-p_r)(u)}(T^{p_i(v)}f_i\cdot T^{p_i(w)}\overline{f_i})||_{L^2(X)}$$
$$\leq C'\|T^{p_1(v)}f_1\cdot T^{p_1(w)}\overline{f_1}\|_r = C'\|T^{p_1(v-w)}f_1\cdot\overline{f_1}\|_r$$

for all $v, w \in \mathbb{Z}^d$. Thus, for any finite set$F \subset \mathbb{Z}^d$, by Hölder inequality,

$$\limsup_{N\to\infty}\|\frac{1}{|\Phi_N|}\sum_{u\in\Phi_N} c_u\prod_{i=1}^{r} T^{p_i(u)}f_i\|_{L^2(X)} \leq {C'}^{1/2}(\frac{1}{|F|^2}\sum_{v,w\in F}\|f_1\cdot T^{p_1(w-v)}\overline{f_1}\|_r^{2^r})^{(1/2)^{r+1}}.$$

Let$\{\Psi_M\}_{M=1}^{\infty}$ be any Følner sequence in $\mathbb{Z}^d$. By Lemma 3.4, we have

$$\frac{1}{|\Psi_M|^2}\sum_{v,w\in\Psi_M}\|f_1\cdot T^{p_1(w-v)}\overline{f_1}\|_r^{2^r} \leq c\|f_1\|_{r+1}^{2^{r+1}},$$

where c is independent on f_1. Substituting the sets Ψ_M, $M \in \mathbb{N}$ for F, we obtain

$$\limsup_{N\to\infty}||\frac{1}{|\Phi_N|}\sum_{u\in\Phi_N} c_u\prod_{i=1}^{r} T^{p_i(u)}f_i||_{L^2(X)} \leq {C^{'}}^{1/2} c^{(1/2)^{r+1}}\|f_1\|_{r+1}.$$

□

3.2. In this section we prove the theorem 2.2 for nonlinear polynomials but in a less general condition.

We call *system* any finite set of integer polynomials $P = \{p_1, \ldots, p_r \colon \mathbb{Z}^d \to \mathbb{Z}\}$.

The *degree* $\deg P$ of the system P is the maximum degree of its elements.

We say two integer polynomials p, q are *equivalent* if $\deg p = \deg q$ and $\deg(p - q) < \deg p$. Let P be a system. We can partition P into equivalence classes for this relation. Let ω_l be the number of classes of degree $0 < l \leq \deg P$. Define the *weight* $\omega(P)$ of system P to be the vector $(\omega_1, \ldots, \omega_{\deg P})$. For two vectors $\omega = (\omega_1, \ldots, \omega_m)$ and $\omega^{'} = (\omega_1^{'}, \ldots, \omega_{m^{'}}^{'})$, we write $\omega < \omega^{'}$, if either $m < m^{'}$ or $m = m^{'}$ and there exists $n \leq m$ such that $\omega_n < \omega_n^{'}$ and $\omega_l = \omega_l^{'}$ for $l = n+1, \ldots, m$. Under this relation, the set of weights of systems becomes well ordered. The *PET-induction* is an induction on the weights of systems.

We call the system $P = \{p_1, \ldots, p_r\}$ *standard* if p_i are nonconstant and essentially distinct and $\deg p_1 = \deg P$. And we say that a certain property holds *for almost all* $\nu \in \mathbb{Z}^d$ if the set of elements of $\mathbb{Z}^d$ for which it does not hold is contained in the set of zeroes of a nontrivial polynomial on $\mathbb{Z}^d$.

Proposition 3.6. *For any $r \in \mathbb{N}$ and any integer vector $w = (w_1, \ldots, w_l)$, there exists $k \in \mathbb{N}$ such that for any standard system $\{p_1, \ldots, p_r : \mathbb{Z}^d \to \mathbb{Z}\}$ of weight ω, any $f_1, \ldots, f_r \in L^\infty(X)$ with $\|f_1\|_k = 0$, and any sequence $\{c_u\}_{u\in\mathbb{Z}^d}$, $|c_u| \leq 1$, one has*

$$\lim_{N\to\infty} || \sum_{u\in\Phi_N} c_u T^{p_1(u)} f_1 \cdot \cdots \cdot T^{p_r(n)} f_r||_{L^2(X)} = 0 \tag{4}$$

for any Følner sequence $\{\Phi_N\}_{N=1}^\infty$ in $\mathbb{Z}^d$.

Proof. The proof of this proposition is almost exactly the same as that of Proposition 9 in Leibman's paper [5],As in the proof of Proposition 3.5, we give all the steps but omit proofs when there are the same as in [5].

We will proceed by PET-induction. For system of degree 1 the proposition is given by Proposition 3.5. Let $P = \{p_1, \ldots, p_r : \mathbb{Z}^d \to \mathbb{Z}\}$ be a standard system of degree ≥ 2 and of weight ω.

Note that for each $s < 2r$, there are only finite many integer vectors $\omega' < \omega$ which are the weights of systems with s elements. We denote the number of vectors here by $\rho(s)$ and these vectors by $\omega_j^s, j = 1, \ldots, \rho(s)$.

By inductive hypothesis, for $s < 2r$, and $1 \leq j \leq \rho(s)$, there exists $k(s, \omega_j^s)$, such that for any standard system $\{p_1, \ldots, p_s : \mathbb{Z}^d \to \mathbb{Z}\}$ of weight ω_j^s , any $f_1, \ldots, f_r \in L^\infty(X)$ with $\|f_1\|_{k(s,\omega_j^s)} = 0$, and any $\{c_u\}_{u\in\mathbb{Z}^d}$, $|c_u| \leq 1$, property(4) holds for any Følner sequence $\{\Phi_N\}_{N=1}^\infty$ in $\mathbb{Z}^d$. Set $k = \max_{1\leq s<2r} \max_{1\leq j\leq\rho(s)} k(s, \omega_j^s)$. We have for any standard system $\{p_1, \ldots, p_s : \mathbb{Z}^d \to \mathbb{Z}\}$ with $s < 2r$ of weight $\omega' < \omega$, any $f_1, \ldots, f_r \in L^\infty(X)$ with $\|f_1\|_k = 0$, and $\{c_u\}_{u\in\mathbb{Z}^d}$, $|c_u| \leq 1$, property (4) holds for any Følner sequence $\{\Phi_N\}_{N=1}^\infty$ in $\mathbb{Z}^d$.

Choose $i_o \in \{2, \ldots, r\}$ such that P_{i_0} has the minimal degree in P; if all polynomials in P have the same degree, choose i_0 such that p_{i_0} is not equivalent to p_1; if all polynomials in P are equivalent, choose i_0 arbitrarily.

Let $I_1 = \{i \in \{1, \ldots, r\} : \deg p_i = 1\}$ and $I_2 = \{i \in \{1, \ldots, r\} : \deg p_i \geq 2\}$. For each $v, w \in \mathbb{Z}^d$, define

$$P_{v,w} = \{p_i(u + v), p_i(u + w) : i \in I_2\} \bigcup \{p_i(u + w) : i \in I_1\},$$

and order the set as $P_{v,w} = \{q_{v,w,1}, \ldots, q_{v,w,s}\}$, where $q_{v,w,1}(u) = p_1(u + v)$, and $q_{v,w,s}(u) = p_{i_0}(u + w)$.

It is easy to check that $P_{v,w}$ is a standard system for almost all $(v, w) \in \mathbb{Z}^{2d}$ and $\omega(P_{v,w}) = \omega(P) = \omega$ for all $v, w \in \mathbb{Z}^d$.

For $v, w \in \mathbb{Z}^d$, define $P'_{v,w} = \{q_{v,w,1} - q_{v,w,s}, \ldots, q_{v,w,s-1} - q_{v,w,s}\}$.

For almost all $(v, w) \in \mathbb{Z}^{2d}$ the polynomials $q_{v,w,j} - q_{v,w,s}, j = 1, \ldots, s - 1$ are nonconstant and essentially distinct, $P'_{v,w}$ is a standard system and $\omega(P'_{v,w}) < \omega(P_{v,w}) = \omega$.

Now let $f_1, \ldots, f_r \in L^\infty(X)$ with $\|f_1\|_k = 0$, and let $\{\Phi_N\}_{N=1}^\infty$ be a Følner sequence in $\mathbb{Z}^d$. We assume that $|f_2|, \ldots, |f_r| \leq 1$. By Lemma 4(i) in [5], for any

finite set $F \subset \mathbb{Z}^d$ we get

$$\limsup_{N\to\infty} ||\frac{1}{|\Phi_N|}\sum_{u\in\Phi_N} c_u \prod_{i=1}^{r} T^{p_i(u)} f_i||^2_{L^2(X)}$$

$$(5) \quad \leq \frac{1}{|F|^2}\sum_{v,w\in F} \limsup_{N\to\infty} ||\frac{1}{|\Phi_N|}\sum_{u\in\Phi_N} c_{u+v}\overline{c_{u+w}} \prod_{j=1}^{s-1} T^{(q_{v,w,j}-q_{v,w,s})(u)} h_{v,w,j}||_{L^2(X)},$$

where $q_{v,w,1},\ldots,q_{v,w,s}$ are the elements of the system $P_{v,w}$ and

$$h_{v,w,j} = \begin{cases} f_i & i \in I_2 \\ \overline{f_i}\cdot T^{p_i(v)-p_i(w)} f_i & i \in I_1. \end{cases}$$

Notice that $\deg p_1 = \deg P \geq 2$, thus $1 \in I_2$, so $h_{v,w,1} = f_1$. For fixed v, w, write $d_u^{v,w} = c_{u+v}\overline{c_{u+w}}$, $|d_u| \leq 1$. By the induction hypothesis applied to system $P'_{v,w}$,

$$\lim_{N\to\infty} ||\frac{1}{|\Phi_N|}\sum_{u\in\Phi_N} d_u^{v,w} \prod_{j=1}^{s-1} T^{(q_{v,w,j}-q_{v,w,s})(u)} h_{v,w,j}||_{L^2(X)} = 0.$$

for almost all $(v,w) \in \mathbb{Z}^{2d}$ such that $P'_{v,w}$ is standard and $\omega(P'_{v,w}) < \omega(P_{v,w}) = \omega$. Since for all other (v,w) this norm is bounded by 1, we have

$$\inf_F \frac{1}{|F|^2}\sum_{v,w\in F} \limsup_{N\to\infty} ||\frac{1}{|\Phi_N|}\sum_{u\in\Phi_N} d_u \prod_{j=1}^{s-1} T^{(q_{v,w,j}-q_{v,w,s})(u)} h_{v,w,j}||_{L^2(X)} = 0.$$

Collecting this formula with the formula (5), we get the announced result. □

3.3. We now turn to the general case.

Proof of Theorem 2.2. The proof of this theorem is almost exactly the same as that of theorem 10 in Leibman's paper [5]. Proposition 3.6 implies the result for standard systems. Let $P = \{p_1,\ldots,p_r\}$ be a (nonstandard) system of nonconstant essentially distinct polynomials $\mathbb{Z}^d \to \mathbb{Z}$ of degree $\leq b$, let $f_1,\ldots,f_r \in L^\infty(X)$ and $\{\Phi_N\}_{N=1}^\infty$ be a Følner sequence in $\mathbb{Z}^d$. By Lemma 4 (ii) in Leibman's paper [5], there exists a Følner sequence $\{\Theta_M\}_{M=1}^\infty$ in $\mathbb{Z}^{3d}$, such that

$$\limsup_{N\to\infty} ||\frac{1}{|\Phi_N|}\sum_{u\in\Phi_N} c_u \prod_{i=1}^{r} T^{p_i(u)} f_i||^2_{L^2(X)}$$

$$\leq \limsup_{N\to\infty} ||\frac{1}{|\Theta_M|}\sum_{(u,v,w)\in\Theta_M} d_{u,v,w} \prod_{i=1}^{r} T^{p_i(u+v)+q(u)} f_i \cdot \prod_{i=1}^{r} T^{p_i(u+w)+q(u)}\overline{f_i}||_{L^2(X)},$$

where $d_{u,v,w} = c_{u+v}\overline{c_{u+w}}, |d_u| \leq 1$, and q is any polynomial $\mathbb{Z}^d \to \mathbb{Z}$ of degree b. The set

$$\{p_1(u+v)+q(u),\ldots,p_r(u+v)+q(u),p_1(u+w)+q(u),\ldots,p_r(u+w)+q(u)\}$$

of polynomials $\mathbb{Z}^{3d} \to \mathbb{Z}$ is a standard system of degree b with $2r$ elements, thus there exists $k \in \mathbb{N}$ (depending on r and b only), such that

$$\limsup_{N\to\infty} ||\frac{1}{|\Theta_M|}\sum_{(u,v,w)\in\Theta_M} d_u \prod_{i=1}^{r} T^{p_i(u+v)+q(u)} f_i \cdot \prod_{i=1}^{r} T^{p_i(u+w)+q(u)}\overline{f_i}||_{L^2(X)} = 0.$$

□

References

1. V. Bergelson, B. Host and B. Kra, with an Appendix by I. Ruzsa. Multiple recurrence and nilsequences. *Inventiones Math.* **160** (2005), 261-303.
2. B. Host and B. Kra. Nonconventional ergodic averages and nilmanifolds. *Ann. Math.* **161** (2005), 397-488.
3. B. Host and B. Kra. Convergence of polynomial ergodic averages. *Isr. J. Math.* **149** (2005) 1-19.
4. B. Host and B. Kra. *Uniformity seminorms on l^∞ and applications.* Available at http://arxiv.org/abs/0711.3637v1.
5. A. Leibman. Convergence of multiple ergodic averages along polynomials of several variables. *Israel Journal of Mathematics.* **146** (2005), 303-315.
6. A. Leibman. Pointwise convergence of ergodic averages for polynomial sequence of translation on a nilmanifold. *Erg. Th. and Dyn. Sys.* **25** (2005), 201-213.
7. E. Lesigne. Un théorème de disjonction de systèmes dynamiques et une généralisation du théorème ergodique de Wiener-Wintner. *Erg. Th. and Dyn. Sys.* **10** (1990), 513521.
8. E. Lesigne. Spectre quasi-discret et théorème ergodique de Wiener-Wintner pour les polynômes. *Erg. Th. and Dyn. Sys.* **13** (1993), 767-784.
9. N. Wiener and A. Wintner. Harmonic analysis and ergodic theory. *Amer. J. Math.* **63** (1941), 415-426.
10. T. Ziegler. A non-conventional ergodic theorem for a nilsystem. *Erg. Th. and Dyn. Sys.* **25** (2005), 1357-1370.

Université Paris-Est, Laboratoire dd'Analyse et de mathématiques appliquées, UMR CNRS 8050, 5 bd Descartes, 77454 Marne la Vallée Cedex 2, France

E-mail address: qing.chu@univ-mlv.fr