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# Local limit theorems for Brownian additive functionals and penalisation of Brownian paths, IX

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**Abstract** We obtain a local limit theorem for the laws of a class of Brownian additive functionals and we apply this result to a penalisation problem. We study precisely the case of the additive functional :  $\left(A_t^- := \int_0^t 1_{X_s < 0} ds, t \geq 0\right)$ . On the other hand, we describe Feynman-Kac type penalisation results for long Brownian bridges thus completing some similar previous study for standard Brownian motion (see [RVY,I]).

**Key words** Limit theorems for additive functionals, Feynman-Kac functionals, long Brownian bridges.

**2000 Mathematics Subject Classification** 60 F 17, 60 G 44, 60 J 25, 60 J 35, 60 J 55, 60 J 57, 60 J 60, 60 J 65.

## 0 Notations and introduction

### 0.1 Notations

- $(\Omega, (X_t, \mathcal{F}_t)_{t \geq 0}, \mathcal{F}_\infty = \vee_{t \geq 0} \mathcal{F}_t, P_x (x \in \mathbb{R}))$  denotes the canonical realisation of the one-dimensional Wiener process.  $\Omega = \mathcal{C}([0, \infty[ \rightarrow \mathbb{R})$  is the space of continuous functions on  $[0, \infty[$ ,  $(X_t, t \geq 0)$  the coordinates process on this space,  $(\mathcal{F}_t, t \geq 0)$  its natural filtration and  $(P_x, x \in \mathbb{R})$  the family of Wiener measures on  $(\Omega, \mathcal{F}_\infty)$ , with  $P_x(X_0 = x) = 1$ . When  $x = 0$ , we write simply  $P$  for  $P_0$ .
- We denote by  $(L_t^x, t \geq 0, x \in \mathbb{R})$  the jointly continuous family of the local times of  $(X_t, t \geq 0)$ . We denote  $(L_t, t \geq 0)$  for  $(L_t^0, t \geq 0)$ , the (continuous) local time process at level 0 and by  $(\tau_l, l \geq 0)$  its right-continuous inverse :

$$\tau_l := \inf\{s > 0; L_s > l\} \quad (l \geq 0) \quad (0.1)$$

- To  $q$  a positive Radon measure on  $\mathbb{R}$ ,  $q \neq 0$ , we associate the continuous additive functional :

$$A_t^q := \int_{\mathbb{R}} L_t^x q(dx) \quad (0.2)$$

When  $q$  admits a density with respect to Lebesgue measure, we keep the former notation by still writing  $q$  for the density; we have :

$$A_t^q = \int_0^t q(X_s) ds \quad (0.3)$$

from the occupation density formula for Brownian motion.

Throughout the following, we shall assume that  $q$  satisfies one of the three following hypotheses :

H1. (The integrable case)  $\int_{\mathbb{R}} (1 + |x|) q(dx) < \infty$ .

H2. (The left unilateral case)  $\int_{-\infty}^0 (1 + |x|) q(dx) < \infty$  and there exists  $\alpha < 1$  such that  $\liminf_{x \rightarrow \infty} x^{2\alpha} q^{(a)}(x) \geq b > 0$  where  $q^{(a)}$  denotes the absolutely continuous part of  $q$ .

H3. (The right unilateral case)  $\int_0^{\infty} (1 + |x|) q(dx) < \infty$  and there exists  $\alpha < 1$  such that  $\liminf_{x \rightarrow -\infty} |x|^{2\alpha} q^{(a)}(x) \geq b > 0$ .

Of course, if the pair  $((X_t, A_t^q), t \geq 0)$  satisfies H2 (resp. H3), then the pair  $((-X_t, A_t^q), t \geq 0)$  satisfies H3 (resp. H2).

## 0.2 Introduction

### 0.2.1

In [RVY, I], we obtained the following results :

- i) Under H1, H2 or H3, for any  $\lambda > 0$  :

$$\lim_{t \rightarrow \infty} \sqrt{t} E_x \left[ e^{\left(-\frac{\lambda}{2} A_t^q\right)} \right] = \varphi_{\lambda q}(x) \quad (0.4)$$

(Later, we shall give other presentations of  $\varphi_{\lambda q}$ ).

- ii) For any  $s \geq 0$  and  $\Lambda_s \in \mathcal{F}_s$  :

$$\lim_{t \rightarrow \infty} \frac{E \left[ 1_{\Lambda_s} e^{\left(-\frac{\lambda}{2} A_t^q\right)} \right]}{E \left[ e^{\left(-\frac{\lambda}{2} A_t^q\right)} \right]} := Q^{(\lambda q)}(\Lambda_s) \quad (0.5)$$

where formula (0.5) induces a probability  $Q^{(\lambda q)}$  on  $(\Omega, \mathcal{F}_\infty)$  (see [RVY,I] for more details about this penalisation result).

The first part of this work consists in :

- Using the result i) to obtain a limit theorem relative to the law of the additive functional  $(A_t^q, t \geq 0)$ . This is the content of Theorem 1.1.
- Obtaining a penalisation result, which is more general than (0.5) i.e, by replacing the exponential function  $x \longrightarrow e^{-\frac{\lambda x}{2}}$  by a more general function. This is the content of Theorem 1.5.

### 0.2.2

In Section 2 of this work, we study in detail the situation where  $q = 1_{]-\infty, 0]}$  i.e,

$A_t^q := A_t^- := \int_0^t 1_{X_s < 0} ds$ . In particular, we prove a penalisation theorem for long Brownian bridges : this is the content of Theorem 2.1

### 0.2.3

Section 3 of this work is devoted to the study of Feynman-Kac penalisation for long Brownian bridges, which generalizes what we have done in [RVY,I] for standard Brownian motion. This is the content of Theorem 3.1.

To summarize, this work extends, in the above directions, our preceding work [RVY,I].

# 1 A local limit theorem for the laws of some Brownian additive functionals and a penalisation result

## 1.1 A local limit theorem

**Theorem 1.1.** *Let  $q$  satisfy one of the hypotheses H1, H2 or H3, and let  $(A_t^q, t \geq 0)$  be defined by (0.2) (or (0.3)). Then, for every  $x \in \mathbb{R}$ , there exists a positive,  $\sigma$ -finite measure  $\nu_x$ , carried by  $\mathbb{R}_+$ , such that :*

$$\sqrt{t} P_x(A_t^q \in dz) \xrightarrow[t \rightarrow \infty]{} \nu_x(dz) \tag{1.1}$$

*The convergence in (1.1) is understood in the following sense : for any function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  Borel, and sub-exponential i.e. : there exist two positive constants  $C_1$  and  $C_2$  such that :*

$$0 \leq f(x) \leq C_1 e^{-C_2 x}$$

*then*

$$\sqrt{t} E_x[f(A_t^q)] \xrightarrow[t \rightarrow \infty]{} \int_{\mathbb{R}_+} f(z) \nu_x(dz)$$

*The measure  $\nu_x$  is characterized by :*

$$\int_0^\infty e^{-\frac{\lambda}{2}y} \nu_x(dy) = \varphi_{\lambda q}(x) \tag{1.2}$$

## 1.2 Proof of Theorem 1.1

We first begin with some precisions, taken from [RVY, I], (see also S. Kotani, [K]) about  $\varphi_{\lambda q}$ , which was defined from (0.3) but admits at least another characterization, namely :  $\varphi_{\lambda q}$  is the unique solution of the Sturm-Liouville equation :

$$\varphi''(dx) = \lambda \varphi(x) q(dx) \tag{1.3}$$

This equation is taken in the sense of Schwartz distributions, and subject to the following boundary conditions :

$$\text{Under H1. : } \quad \varphi'(+\infty) = -\varphi'(-\infty) = \sqrt{\frac{2}{\pi}} \tag{1.4}$$

$$\text{Under H2. : } \quad \varphi'(-\infty) = -\sqrt{\frac{2}{\pi}} \text{ and } \varphi(+\infty) = 0 \tag{1.5}$$

$$\text{Under H3. : } \quad \varphi'(+\infty) = \sqrt{\frac{2}{\pi}} \text{ and } \varphi(-\infty) = 0 \tag{1.6}$$

Theorem 1.1. is now an immediate consequence of the next lemma.

**Lemma 1.2.** *Under either of the hypotheses H1, H2, or H3, the function :  $\lambda \longrightarrow \varphi_{\lambda q}(x)$  ( $\lambda > 0$ ) is, for any real  $x$ , completely monotone, i.e., it satisfies :*

$$(-1)^n \frac{\partial^n}{\partial \lambda^n} \varphi_{\lambda q}(x) \geq 0 \quad (1.7)$$

Consequently, there exists a positive,  $\sigma$ -finite measure  $\nu_x$ , carried by  $\mathbb{R}_+$ , such that :

$$\varphi_{\lambda q}(x) = \int_0^\infty e^{-\frac{\lambda}{2}z} \nu_x(dz) \quad (1.8)$$

We shall give two proofs for Lemma 1.2.

### 1.3 A first proof of Lemma 1.2

We define, for every  $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$  and every real  $h \neq 0$  :

$$D_h f(\lambda) := \frac{f(\lambda + h) - f(\lambda)}{h} \quad (1.9)$$

For  $f(\lambda) := \exp - \frac{\lambda}{2} A_t^q$ , we get :

$$(D_h)^n(f)(\lambda) = e^{-\frac{\lambda A_t^q}{2}} \left( \frac{e^{-\frac{A_t^q h}{2}} - 1}{h} \right)^n$$

and, hence for all  $h \neq 0$  :

$$(-1)^n (D_h)^n(f)(\lambda) \geq 0 \quad (1.10)$$

Consequently, taking the expectation of the LHS in (1.10), we obtain :

$$\sqrt{t} (-1)^n E_x \left[ (D_h)^n \left( \exp - \frac{\bullet}{2} A_t^q \right) \right] \geq 0 \quad (1.11)$$

Hence, from (0.4) :

$$\sqrt{t} (-1)^n E_x \left[ (D_h)^n \left( \exp - \frac{\bullet}{2} A_t^q \right) \right] \xrightarrow{t \rightarrow \infty} (-1)^n (D_h)^n (\varphi_{\bullet q}(x))$$

Thus :

$$(-1)^n (D_h)^n (\varphi_{\bullet q}(x))(\lambda) \geq 0 \quad (1.12)$$

Letting  $h \rightarrow 0$  in (1.12), and using the fact that :  $D_h f \xrightarrow{h \rightarrow 0} f'$ , we get :

$$(-1)^n \frac{\partial^n}{\partial \lambda^n} (\varphi_{\lambda q}(x)) \geq 0 \quad (1.13)$$

## 1.4 A second proof of Lemma 1.2

We shall only give this second proof under the hypothesis H1 and for  $x = 0$ . In [RVY, I], Proposition 4.13, formula (4.43), we have obtained the following explicit formula for  $\varphi_{\lambda q}(0)$  :

$$\begin{aligned} \varphi_{\lambda q}(0) = \frac{1}{\sqrt{2\pi}} \int_0^\infty & \left[ Q_l^{(0)}(\exp - \lambda \langle Y, q^- \rangle) \cdot Q_l^{(2)}(\exp - \lambda \langle Y, q^+ \rangle) \right. \\ & \left. + Q_l^{(2)}(\exp - \lambda \langle Y, q^- \rangle) Q_l^{(0)}(\exp - \lambda \langle Y, q^+ \rangle) \right] dl \end{aligned} \quad (1.14)$$

where, in this formula (1.14), the process  $(Y_x, x \geq 0)$  is, under  $Q_l^{(0)}$ , (resp. under  $Q_l^{(2)}$ ), a squared Bessel process with dimension 0, (resp. 2), starting from  $l$ , and we denote :

$$\langle Y, q^+ \rangle = \int_0^\infty Y_x q(dx) ; \quad \langle Y, q^- \rangle = \int_{-\infty}^0 Y_{-x} q(dx) \quad (1.15)$$

It is then clear from (1.14) that :  $\lambda \longrightarrow \varphi_{\lambda q}(0)$  is the Laplace transform of a positive measure, as an integral, with respect to the parameter  $l$  of the product of two Laplace transforms of positive measures (indexed by  $l$ ).

We shall now give some examples for which the measure  $\nu_x$  may be computed explicitly. We recall that  $\nu_x$  is characterized by :  $\int_0^\infty e^{-\frac{\lambda}{2}z} \nu_x(dz) = \varphi_{\lambda q}(x)$  where  $\varphi_{\lambda q}(x)$  is given by (1.3) ... (1.6).

## 1.5 Computation of $\nu_x$ for $q(dy) = \delta_0(dy)$

In this case, the hypothesis H1 is verified and  $A_t^q = L_t$ , is the local time at level 0

$$\begin{aligned} \varphi_{\lambda q}(x) &= \sqrt{\frac{2}{\pi}} \left( \frac{2}{\lambda} + |x| \right) \quad (\text{cf [RVY, I], Ex. 4.8, p. 199-200}) \\ &= \int_0^\infty e^{-\frac{\lambda}{2}z} \left( \sqrt{\frac{2}{\pi}} 1_{z \geq 0} dz + \sqrt{\frac{2}{\pi}} |x| \delta_0(dz) \right) \end{aligned} \quad (1.16)$$

Thus :

$$\nu_x(dz) = \sqrt{\frac{2}{\pi}} 1_{[0, \infty[}(z) dz + \sqrt{\frac{2}{\pi}} |x| \delta_0(dz) \quad (1.17)$$

## 1.6 Computation of $\nu_x$ for $q(dy) = \delta_a(dy) + \delta_b(dy)$ with $(a < b)$

In this case, the hypothesis H1 is satisfied and  $A_t^q = L_t^a + L_t^b$  where  $(L_t^a, t \geq 0)$  resp.  $(L_t^b, t \geq 0)$  denotes the local time at level  $a$ , resp. at level  $b$ . We know (see [RVY, I], Ex.

4.8, p. 199-200) that

$$\begin{aligned} \varphi_{\lambda q}(x) &= \begin{cases} \sqrt{\frac{2}{\pi}} \left( \frac{1}{\lambda} + x - b \right) & \text{if } x > b \\ \sqrt{\frac{2}{\pi}} \frac{1}{\lambda} & \text{if } x \in [a, b] \\ \sqrt{\frac{2}{\pi}} \left( \frac{1}{\lambda} + a - x \right) & \text{if } x < a \end{cases} \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{\lambda}{2}z} \left\{ \frac{1}{2} dz + (x-b)1_{x>b}\delta_0(dz) + (a-x)1_{x<a}\delta_0(dz) \right\} \end{aligned} \quad (1.18)$$

Hence :

$$\nu_x(dz) = \sqrt{\frac{2}{\pi}} \left\{ \frac{1}{2} 1_{[0, \infty[}(z) dz + (x-b)^+ \delta_0(dz) + (a-x)^+ \delta_0(dz) \right\} \quad (1.19)$$

### 1.7 Computation of $\nu_x$ , for $q(y) = e^{2y}$

In this case, the hypothesis H2 is satisfied and  $A_t^q = \int_0^t e^{2X_s} ds$ .

To begin with, we show :

$$\varphi_{\lambda q}(x) = \sqrt{\frac{2}{\pi}} K_0(\sqrt{\lambda} e^x) \quad (1.20)$$

where  $K_0$  denotes the Bessel-Mc Donald function with index 0 (see [Leb], p. 108).

Let  $\psi(x) := \sqrt{\frac{2}{\pi}} K_0(\sqrt{\lambda} e^x)$ . To check (1.20), it suffices to see that :

$$\psi''(x) = \lambda e^{2x} \psi(x), \quad \psi(x) \xrightarrow{x \rightarrow \infty} 0, \quad \psi'(x) \xrightarrow{x \rightarrow -\infty} -\sqrt{\frac{2}{\pi}} \quad (1.21)$$

Now (1.21) follows from (see [Leb], p. 110) :

$$K_0' = -K_1, \quad -K_1'(z) = \frac{1}{z} K_1(z) + K_0(z)$$

and

$$\begin{aligned} \psi(x) &\underset{x \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi}} \left( \frac{\pi}{2\sqrt{\lambda} e^x} \right) e^{-\sqrt{\lambda} e^x} \xrightarrow{x \rightarrow \infty} 0 \quad ([\text{Leb}], p.123) \\ \psi'(x) &= -\sqrt{\frac{2}{\pi}} \sqrt{\lambda} e^x K_1(\sqrt{\lambda} e^x) \underset{x \rightarrow -\infty}{\sim} -\sqrt{\frac{2}{\pi}} \sqrt{\lambda} e^x \frac{1}{2} \frac{2}{\sqrt{\lambda} e^x} \xrightarrow{x \rightarrow -\infty} -\sqrt{\frac{2}{\pi}} \quad ([\text{Leb}], p.111) \end{aligned}$$

This proves (1.20). But, we also have :

$$\begin{aligned} K_0(\sqrt{\lambda} e^x) &= \frac{1}{2} \int_0^\infty e^{-t - \frac{\lambda e^{2x}}{4t}} \frac{dt}{t} \quad (\text{cf } [\text{Leb}], p. 119) \\ &= \frac{1}{2} \int_0^\infty e^{-\frac{\lambda u}{2} - \frac{e^{2x}}{2u}} \frac{du}{u} \end{aligned}$$



Hence :

$$\nu_x(dz) = \frac{1}{\sqrt{2\pi}} e^{-\frac{e^{2x}}{2z}} 1_{[0,\infty[}(z) \frac{dz}{z} \quad (1.22)$$

### 1.8 Computation of $\nu_x$ for $q_0(dx) = 1_{]-\infty,0]}(x)dx$

Here, it is the hypothesis H3 which is satisfied, and

$$A_t^{q_0} = \int_0^t 1_{]-\infty,0]}(X_s) ds$$

By scaling, one has, under  $P_0 : A_t^{q_0} \stackrel{(law)}{=} t A_1^{q_0}$ , and it is well known that under  $P_0$ ,  $A_1^{q_0}$  follows the arc sine law, i.e., the beta  $\left(\frac{1}{2}, \frac{1}{2}\right)$  law. We shall recall the law of  $A_t^{q_0}$  under  $P_x$  for any  $x \in \mathbb{R}$ , (see Subsection 2.1 below), which will allow to obtain the following result :

$$\nu_x(dz) = x_+ \sqrt{\frac{2}{\pi}} \delta_0(dz) + \frac{1}{\pi} e^{-\frac{x^2}{2z}} 1_{[0,\infty[}(z) \frac{dz}{\sqrt{z}} \quad (1.23)$$

For the moment, we shall prove (1.23) without using the explicit law of  $A_t^{q_0}$ . For this purpose, we already observe that :

$$\varphi_{\lambda q_0}(x) = \sqrt{\frac{2}{\pi}} \left\{ e^{x\sqrt{\lambda}} \frac{1}{\sqrt{\lambda}} 1_{x \leq 0} + \left(x + \frac{1}{\sqrt{\lambda}}\right) 1_{x > 0} \right\} \quad (1.24)$$

Indeed we have :

$$\varphi_{\lambda q_0}''(x) = \lambda 1_{]-\infty,0]}(x) \varphi_{\lambda q_0}(x), \quad \varphi_{\lambda q_0}'(+\infty) = \sqrt{\frac{2}{\pi}}, \quad \varphi_{\lambda q_0}(-\infty) = 0.$$

Then, it remains to see that :

$$\int_0^\infty e^{-\frac{\lambda}{2}z} \nu_x(dz) = \varphi_{\lambda q_0}(x) \quad (1.25)$$

where  $\nu_x$  is defined via (1.23) and  $\varphi_{\lambda q_0}(x)$  by (1.24). Now, for  $x > 0$ , one has :

$$\begin{aligned} \int_0^\infty e^{-\frac{\lambda}{2}z} \nu_x(dz) &= x_+ \sqrt{\frac{2}{\pi}} + \frac{1}{\pi} \int_0^\infty e^{-\frac{\lambda z}{2}} \frac{dz}{\sqrt{z}} \\ &= x_+ \sqrt{\frac{2}{\pi}} + \frac{1}{\pi} \sqrt{\frac{2}{\lambda}} \Gamma(1/2) = x_+ \sqrt{\frac{2}{\pi}} + \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\lambda}} = \varphi_{\lambda q_0}(x) \end{aligned}$$

whereas for  $x < 0$  :

$$\int_0^\infty e^{-\frac{\lambda}{2}z} \nu_x(dz) = \frac{1}{\pi} \int_0^\infty e^{-\frac{\lambda}{2}z - \frac{x^2}{2z}} \frac{dz}{\sqrt{z}} = \frac{2}{\pi} K_{1/2}(|x|\sqrt{\lambda}) \left(\frac{x^2}{\lambda}\right)^{1/4} \quad (\text{see [Leb], p. 119})$$

However, one has :  $K_{1/2}(|x|\sqrt{\lambda}) = \left(\frac{\pi}{2|x|\sqrt{\lambda}}\right)^{1/2} e^{-|x|\sqrt{\lambda}}$ . Hence :

$$\int_0^\infty e^{-\frac{\lambda}{2}z} \nu_x(dz) = \frac{2}{\pi} \left(\frac{x^2}{\lambda}\right)^{1/4} \left(\frac{\pi}{2|x|\sqrt{\lambda}}\right)^{1/2} e^{-|x|\sqrt{\lambda}} = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\lambda}} e^{-|x|\sqrt{\lambda}} = \varphi_{\lambda q_0}(x)$$

### 1.9 Computation of $\nu_x$ when $q(y) = 1_{[a,b]}(y)$ ( $a < b$ )

The hypothesis H1 is satisfied and  $A_t^q = \int_0^t 1_{[a,b]}(X_s) ds$ . We shall prove that :

$$\nu_x^{(a,b)}(dz) = \begin{cases} \sqrt{\frac{2}{\pi}}(x-b)\delta_0(dz) + \frac{1}{\pi\sqrt{z}} 1_{[0,\infty[}(z)dz \left(1 + 2 \sum_{n=1}^{\infty} e^{-\frac{n^2(b-a)^2}{2z}}\right) & \text{if } x > b \\ \sqrt{\frac{2}{\pi}}(a-x)\delta_0(dz) + \frac{1}{\pi\sqrt{z}} 1_{[0,\infty[}(z)dz \left(1 + 2 \sum_{n=1}^{\infty} e^{-\frac{n^2(b-a)^2}{2z}}\right) & \text{if } x < a \\ \frac{1}{\pi\sqrt{z}} \sum_{n=0}^{\infty} \left( e^{-\frac{(n(b-a)+b-x)^2}{2z}} + e^{-\frac{(n(b-a)+(x-a))^2}{2z}} \right) 1_{[0,\infty[}(z)dz & \text{if } x \in [a, b] \end{cases} \quad (1.26)$$

Here, the explicit form of  $\varphi_{\lambda q}^{(a,b)}(x)$  is (see [RVY, I], Ex. 4.7, p. 199) :

$$\varphi_{\lambda q}^{(a,b)}(x) = \begin{cases} \sqrt{\frac{2}{\pi}} \left( \frac{1}{\sqrt{\lambda} \tanh(\sqrt{\lambda} \frac{b-a}{2})} + x - b \right) & \text{if } x > b \\ \sqrt{\frac{2}{\pi}} \left( \frac{1}{\sqrt{\lambda} \tanh(\sqrt{\lambda} \frac{b-a}{2})} + a - x \right) & \text{if } x < a \\ \sqrt{\frac{2}{\pi}} \left( \frac{\cosh(\sqrt{\lambda}(x - \frac{a+b}{2}))}{\sqrt{\lambda}(\sinh \sqrt{\lambda} \frac{b-a}{2})} \right) & \text{if } x \in [a, b] \end{cases} \quad (1.27)$$

It now remains to prove that :

$$\int_0^{\infty} e^{-\frac{\lambda}{2}z} \nu_x^{(a,b)}(dz) = \varphi_{\lambda q}^{(a,b)}(x) \quad (1.28)$$

where  $\nu_x^{(a,b)}$  is defined via (1.26) and  $\varphi_{\lambda q}^{(a,b)}$  via (1.27). But, (1.28) follows, after some elementary computations from the identities, for every real  $u$  and  $v > 0$  :

$$\begin{aligned} \frac{\cosh(\sqrt{\lambda}u)}{\sqrt{\lambda} \sinh(\sqrt{\lambda}v)} &= \sum_{n=0}^{\infty} \int_0^{\infty} dh \left( e^{-\sqrt{\lambda}(h+(2n+1)v-u)} + e^{-\sqrt{\lambda}(h+(2n+1)v+u)} \right) \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} dh \int_0^{\infty} ds \left( H_{h+(2n+1)v-u}(s) + H_{h+(2n+1)v+u}(s) \right) e^{-\lambda s} \end{aligned} \quad (1.29)$$

with

$$H_a(u) := \frac{a}{2\sqrt{\pi}u^3} e^{-a^2/4u} = \frac{-1}{\sqrt{\pi}u} \frac{\partial}{\partial a} \left( e^{-\frac{a^2}{4u}} \right) \quad (a > 0)$$

Passing from (1.29) to (1.30) is obtained by using the elementary formula :

$$e^{-\sqrt{\lambda}a} = \int_0^{\infty} e^{-\lambda u} H_a(u) du = \int_0^{\infty} e^{-\lambda u} \frac{a}{2\sqrt{\pi}u^3} e^{-\frac{a^2}{4u}} du \quad (1.31)$$

(Note that (1.31) is nothing else but a translation of :  $E(e^{-\frac{\lambda^2}{2}T_a}) = \exp(-\lambda a)$ , where  $T_a$  denotes the hitting time of level  $a > 0$  by Brownian motion starting from 0, and  $H_a$  is the density of  $T_{\frac{a}{\sqrt{2}}}$ )

We now show (1.29).

$$\begin{aligned}
\frac{\cosh(\sqrt{\lambda}u)}{\sqrt{\lambda} \sinh(\sqrt{\lambda}v)} &= \frac{1}{\sqrt{\lambda}} e^{-\sqrt{\lambda}(v-u)} \frac{1 + e^{-2\sqrt{\lambda}u}}{1 - e^{-2\sqrt{\lambda}v}} \\
&= \frac{1}{\sqrt{\lambda}} e^{-\sqrt{\lambda}(v-u)} (1 + e^{-2\sqrt{\lambda}u}) \left( \sum_{n=0}^{\infty} e^{-2n\sqrt{\lambda}v} \right) \\
&= \frac{1}{\sqrt{\lambda}} \left\{ \sum_{n=0}^{\infty} e^{-\sqrt{\lambda}\{v-u+2nv\}} + \sum_{n=0}^{\infty} e^{-\sqrt{\lambda}\{2(u+nv)+(v-u)\}} \right\} \\
&= \frac{1}{\sqrt{\lambda}} \left\{ \sum_{n=0}^{\infty} \left( e^{-\sqrt{\lambda}((2n+1)v-u)} + e^{-\sqrt{\lambda}(u+(2n+1)v)} \right) \right\} \\
&= \int_0^{\infty} e^{-\sqrt{\lambda}h} \left\{ \sum_{n=0}^{\infty} e^{-\sqrt{\lambda}((2n+1)v-u)} + e^{-\sqrt{\lambda}(u+(2n+1)v)} \right\} dh \\
&= \sum_{n=0}^{\infty} \int_0^{\infty} \left( e^{-\sqrt{\lambda}(h+(2n+1)v-u)} + e^{-\sqrt{\lambda}(h+(2n+1)v+u)} \right) dh.
\end{aligned}$$

**Remark 1.9**

(i) If in formula (1.26), we take :  $b = 0$ , and we let  $a$  tend to  $-\infty$ , we obtain :

$$\lim_{a \rightarrow -\infty} \nu^{a,0}(dz) = \begin{cases} \sqrt{\frac{2}{\pi}} x_+ \delta_0(dz) + \frac{1}{\pi\sqrt{z}} 1_{[0,\infty[}(z) dz & \text{if } x > 0 \\ \frac{1}{\pi\sqrt{z}} e^{-\frac{x^2}{2z}} 1_{[0,\infty[}(z) dz & \text{if } x \leq 0 \end{cases} \quad (1.32)$$

We note that the RHS of (1.32) is nothing else but the measure  $\nu_x$  associated with  $q_0(y) = 1_{]-\infty,0]}$  (see (1.23)). This may be interpreted as "a continuity property" of  $\varphi^{a,b}$ , as  $a \rightarrow -\infty$ .

(ii) In the same spirit, but taking up now the computation from subsection 1.9, where we choose for  $q$  the function :  $q^{(c)}(y) = \frac{1}{2c} 1_{[-c,+c]}(y)$ , we have :

$$\int_0^{\infty} e^{-\frac{\lambda z}{2}} \nu_x^{(c)}(dz) \xrightarrow{c \rightarrow 0} \int_0^{\infty} e^{-\frac{\lambda z}{2}} \nu_x(dz) = \sqrt{\frac{2}{\pi}} \left( \frac{2}{\lambda} + |x| \right) \quad (1.33)$$

where  $\nu_x$  is the measure associated to  $q(dz) = \delta_0(dz)$  (see (1.16)). In other terms, since :  $\frac{1}{2c} \int_0^t 1_{[-c,c]}(X_s) ds \xrightarrow{c \rightarrow 0} L_t$  a.s., we witness there also a "continuity property" of  $\nu_x^{(c)}$  as  $c \rightarrow 0$ . Let us show (1.33) for  $x = 0$  ; from (1.27) :

$$\int_0^{\infty} e^{-\frac{\lambda z}{2}} \nu_0^{(c)}(dz) = \sqrt{\frac{2}{\pi}} \left\{ \frac{\cosh\left(\sqrt{\frac{\lambda}{2c}} c\right)}{\sqrt{\frac{\lambda}{2c}} \sinh\left(\sqrt{\frac{\lambda}{2c}} \cdot c\right)} \right\} \xrightarrow{c \rightarrow 0} \sqrt{\frac{2}{\pi}} \cdot \frac{2}{\lambda}$$

and for  $x \neq 0$ , and  $c$  small enough, we obtain from (1.27) that :

$$\int_0^\infty e^{-\frac{\lambda z}{2}} \nu_x^{(c)}(dz) = \sqrt{\frac{2}{\pi}} \left\{ \frac{1}{\sqrt{\frac{\lambda}{2c}} \tanh\left(\sqrt{\frac{\lambda}{2c}} \cdot c\right)} + |x - c| \right\} \xrightarrow{c \rightarrow 0} \sqrt{\frac{2}{\pi}} \left( \frac{2}{\lambda} + |x| \right)$$

### 1.10 Computation of $\nu_x$ when $q(y) = 1_{[0, \infty[}(y)y^\alpha$ , $\alpha > 0$

The hypothesis H2 is satisfied, and we have :  $A_t^q = \int_0^t \mathbf{1}_{(X_s > 0)} X_s^\alpha ds$ .

We now show the existence of a constant  $C_\alpha > 0$  such that :

$$\nu_0(dz) = \frac{C_\alpha}{z^{\frac{1+\alpha}{2+\alpha}}} 1_{[0, \infty[}(z) dz \quad (1.34)$$

Indeed, thanks to the scaling property, we have :

$$\begin{aligned} E_0\left(e^{-\frac{\lambda}{2} \int_0^t \mathbf{1}_{X_s > 0} X_s^\alpha ds}\right) &= E_0\left(e^{-\frac{\lambda}{2} t^{1+\alpha/2} A_1^q}\right) \\ &= E_0\left(\exp\left(-\frac{1}{2} A_{\lambda^{\frac{2}{2+\alpha}} t}^q\right)\right) \end{aligned} \quad (1.35)$$

Thus, multiplying (1.35) by  $\sqrt{t}$  and letting  $t$  tend to  $+\infty$ , we obtain :

$$\varphi_{\lambda q}(0) = \frac{1}{\lambda^{\frac{1}{2+\alpha}}} \varphi_{1 \cdot q}(0) = \frac{1}{\lambda^{\frac{1}{2+\alpha}}} c'_\alpha = c_\alpha \int_0^\infty e^{-\frac{\lambda}{2} z} \frac{dz}{z^{\frac{1+\alpha}{2+\alpha}}}$$

The same computations, performed this time with  $x \neq 0$ , lead to :

$$\varphi_{\lambda q}(x) = \frac{1}{\lambda^{\frac{1}{2+\alpha}}} \varphi_{1 \cdot q}(x \lambda^{\frac{1}{2+\alpha}}), \text{ i.e. } \nu_x = \frac{1}{\lambda^{\frac{1}{2+\alpha}}} \nu_{x \lambda^{\frac{1}{2+\alpha}}}^{(\lambda)},$$

where  $\nu^{(\lambda)}$  is the image of  $\nu$  by the application  $z \rightarrow \lambda z$ .

#### Question 1.4

We know (see [R,Y], chap. X) that, if  $q$  is an integrable function, then :

$$\frac{1}{\sqrt{t}} \int_0^t q(x + X_s) ds \xrightarrow[t \rightarrow \infty]{\text{law}} \left( \int q(x + y) dy \right) |N| = \left( \int q(y) dy \right) |N| \quad (1.36)$$

where  $N$  is a standard Gaussian variable, and on the LHS of (1.36),  $(X_s, s \geq 0)$  is a Brownian motion starting from 0. Let  $g$  denote the density of the r.v.  $\bar{q}|N|$  with  $\bar{q} = \int q(y) dy$  that is :

$$g(z) = \frac{1}{\bar{q}} \sqrt{\frac{2}{\pi}} e^{-\frac{z^2}{2(\bar{q})^2}} 1_{[0, \infty[}(z)$$

Let us now consider the supplementary hypothesis  $\tilde{H}$ , which seems reasonable enough in view of (1.36) that the density  $g_t(x, \cdot)$  of the r.v.  $\frac{1}{\sqrt{t}} \int_0^t q(x + X_s) ds$  converges, as  $t \rightarrow \infty$ , uniformly on every compact, towards  $g$ .

However, this would imply that, for every function  $h$ , which is continuous with compact support, one would have :

$$\begin{aligned} \sqrt{t} E_x [h(A_t^q)] &= \sqrt{t} E_x \left[ h \left( \frac{A_t^q}{\sqrt{t}} \cdot \sqrt{t} \right) \right] \\ &= \sqrt{t} \int_0^\infty h(z\sqrt{t}) g_t(x, z) dz \\ &= \int_0^\infty h(y) g_t \left( x, \frac{y}{\sqrt{t}} \right) dy \xrightarrow{t \rightarrow \infty} \int_0^\infty h(y) g(0) dy \end{aligned}$$

But, from Theorem 1.1., we know that :

$$\sqrt{t} E_x [h(A_t^q)] \xrightarrow{t \rightarrow \infty} \int_{\mathbb{R}_+} h(x) \nu_x(dz)$$

Thus, this would imply that the measure  $\nu_x(dz)$  would be equal to :

$$\frac{1}{q} \sqrt{\frac{2}{\pi}} 1_{[0, \infty[} dz \tag{1.37}$$

so that, the measure  $\nu_x$  would not depend on  $x$ , and would be proportional to Lebesgue measure on  $\mathbb{R}_+$ . But clearly, this is not the case for either of the examples in subsections 1.4 to 1.9. Consequently, the hypothesis  $\tilde{H}$  is not satisfied for the corresponding  $q$ 's. It would be of interest to know for which  $q$ 's, if any, it is satisfied.

### 1.11 Penalisation by $h(A_t^q)$

Let  $q$  satisfy one of the previous hypotheses H1, H2 or H3, and denote, as before :

$A_t^q = \int_{\mathbb{R}} L_t^x q(dx) \left( = \int_0^t q(X_s) ds \quad \text{if } q \text{ admits a density} \right)$ . Let now  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that :

$$\sqrt{t} E_x [h(A_t^q)] \xrightarrow{t \rightarrow \infty} \int_0^\infty h(z) \nu_x(dz) \tag{1.38}$$

Then, (1.38) is satisfied, from Theorem 1.1., as soon as  $h$  is sub-exponential (for example if  $h$  is continuous, with compact support). We shall now study the penalisation of Wiener measure by the functional  $h(A_t^q)$ , i.e. : we shall study the limit, as  $t \rightarrow \infty$ , of :

$$\frac{E_x(1_{\Lambda_s} h(A_t^q))}{E_x(h(A_t^q))} \quad (s \geq 0, \Lambda_s \in \mathcal{F}_s) \tag{1.39}$$

We have already made this study in two situations :

- 1)  $q(dy) = \delta_0(dy)$  then  $A_t^q = L_t$  (cf [RVY, II]) ;
- 2)  $A_t^q = \int_{\mathbb{R}} L_t^y q(dy)$  and  $h(u) = \exp\left(-\frac{\lambda}{2} u\right)$  (cf [RVY, I]).

This time, Theorem 1.1. allows us to obtain :

**Theorem 1.5.** : *Let  $q, A^q$  and  $h$  as above. Then :*

1) *For every  $s \geq 0$ , and every  $\Lambda_s \in \mathcal{F}_s$  :*

$$\lim_{t \rightarrow \infty} \frac{E_x(\mathbf{1}_{\Lambda_s} h(A_t^q))}{E_x(h(A_t^q))} \text{ exists} \quad (1.40)$$

2) *This limit equals  $E_x(\mathbf{1}_{\Lambda_s} M_s^{h,q}) := Q^{h,q}(\Lambda_s)$ , where*

$$M_s^{h,q} := \frac{\int_{\mathbb{R}_+} \nu_{X_s}(dz) h(z + A_s^q)}{\int_{\mathbb{R}_+} \nu_x(dz) h(z)} \quad (1.41)$$

*Furthermore,  $(M_s^{h,q}, s \geq 0)$  is a positive martingale. In the case when  $h(u) := e^{-\frac{\lambda}{2}u}$  ( $u, \lambda \geq 0$ ), we then obtain :*

$$M_s^{h,q} = \frac{\varphi_{\lambda q}(X_s)}{\varphi_{\lambda q}(x)} \exp\left(-\frac{\lambda}{2} A_s^q\right) \quad (1.42)$$

### **Proof of Theorem 1.5.**

We have :

$$\frac{E_x(\mathbf{1}_{\Lambda_s} h(A_t^q))}{E_x(h(A_t^q))} = \frac{E_x(\mathbf{1}_{\Lambda_s} E_b(h(a + A_{t-s}^q)))}{E_x(h(A_t^q))}$$

from the Markov property, where  $b = X_s$  and  $a = A_s^q$ . Thus, from Theorem 1.1. :

$$E_x(h(A_t^q)) \underset{t \rightarrow \infty}{\sim} \frac{1}{\sqrt{t}} \int_0^\infty \nu_x(dz) h(z)$$

and  $E_b(h(a + A_{t-s}^q)) \underset{t \rightarrow \infty}{\sim} \frac{1}{\sqrt{t-s}} \int_0^\infty \nu_b(dz) h(a + z)$

Hence :

$$\begin{aligned} \frac{E_x(\mathbf{1}_{\Lambda_s} h(A_t^q))}{E_x(h(A_t^q))} &\underset{t \rightarrow \infty}{\sim} \frac{\sqrt{t}}{\sqrt{t-s}} \frac{E_x(\mathbf{1}_{\Lambda_s} \int_{\mathbb{R}_+} \nu_{X_s}(dz) h(z + A_s^q))}{\int_{\mathbb{R}_+} h(z) \nu_x(dz)} \\ &\xrightarrow[t \rightarrow \infty]{} E_x(\mathbf{1}_{\Lambda_s} M_s^{h,q}) \end{aligned}$$

In the preceding lines, we have been a little careless concerning the exchange of limit and expectation. Likewise, although it is easy to see that  $(M_s^{h,q}, s \geq 0)$  is a local martingale, some care is needed in order to show that it is a true martingale. However, all this is correct as soon as  $h$  is sub-exponential. We leave details to the reader.

## 2 A detailed study for $q_0 = 1_{]-\infty, 0]}$ , $A_t^- := \int_0^t 1_{(X_s < 0)} ds$

Throughout this section, we choose  $q_0 = 1_{]-\infty, 0]}$ . Thus, the hypothesis H3 is now satisfied. We shall study this situation in detail, which we are able to do as we know (see [Y]) the law of  $A_t^{q_0} = \int_0^t q_0(X_s) ds$  under  $P_x$ , for every real  $x$  (see (2.5) and (2.7) below). We shall, successively :

- compute explicitly the measure  $\nu_x$  starting from the knowledge of the law of  $A_t^{q_0}$  and we shall recover the result of subsection 1.8 above ;
- study the penalisation, not only of the process  $(X_t, t \geq 0)$  by  $h(A_t^{q_0})$ , but also the penalisation of the "long bridges" by this functional ;
- describe precisely the behavior of the canonical process under the probability  $Q^{h, q_0}$ , where  $Q^{h, q_0}$  is defined via :

$$Q^{h, q_0}(\Lambda_s) = E(1_{\Lambda_s} M_s^{h, q_0}) \quad (s \geq 0, \Lambda_s \in \mathcal{F}_s) \quad (2.1)$$

### 2.1 The law of $A_t^-$ and the computation of $\nu_x$

To simplify notation, we denote :

$$A_t^- = \int_0^t 1_{(X_s < 0)} ds = \int_0^t q_0(X_s) ds \quad (2.2)$$

We recall the following result, which is found in [Y]. For any  $f : [0, 1] \rightarrow \mathbb{R}_+$ , Borel, sub-exponential (see Theorem 1.1) and any  $y > 0$  :

$$E_0 \left[ f \left( \int_0^1 1_{(X_s < y)} ds \right) \right] = \int_0^1 \frac{du}{\pi \sqrt{u(1-u)}} e^{-\frac{y^2}{2u}} f(u) + f(1) \sqrt{\frac{2}{\pi}} \int_0^y e^{-\frac{\alpha^2}{2}} d\alpha \quad (2.3)$$

whereas, for any  $y < 0$ , we use :

$$\int_0^1 1_{(X_s < y)} ds \stackrel{\text{law}}{=} \int_0^1 1_{(X_s > -y)} ds \stackrel{\text{law}}{=} 1 - \int_0^1 1_{(X_s < -y)} ds \quad (2.4)$$

and by the scaling property :

$$\begin{aligned} E_x \left[ f \left( \int_0^t 1_{(X_s < 0)} ds \right) \right] \\ = E_x(f(A_t^-)) = E_0 \left( f \left( \int_0^t 1_{(X_s < -x)} ds \right) \right) = E_0 \left( f \left( t \int_0^1 1_{(X_s < -\frac{x}{\sqrt{t}})} ds \right) \right) \end{aligned}$$

hence, from (2.3) and (2.4), if  $x \leq 0$  :

$$E_x[f(A_t^-)] = \int_0^t \frac{dv}{\pi \sqrt{v(t-v)}} e^{-\frac{x^2}{2v}} f(v) + f(t) \sqrt{\frac{2}{\pi}} \int_0^{\frac{|x|}{\sqrt{t}}} e^{-\frac{\alpha^2}{2}} d\alpha \quad (2.5)$$

$$\underset{t \rightarrow \infty}{\sim} \frac{1}{\pi \sqrt{t}} \int_0^\infty \frac{dv}{\sqrt{v}} e^{-\frac{x^2}{2v}} f(v) \quad (2.6)$$

whereas, if  $x > 0$  :

$$E_x[f(A_t^-)] = f(0)\sqrt{\frac{2}{\pi}} \int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{\alpha^2}{2}} d\alpha + \int_0^t \frac{dv}{\pi\sqrt{v(t-v)}} e^{-\frac{x^2}{2(t-v)}} f(v) \quad (2.7)$$

$$\underset{t \rightarrow \infty}{\sim} f(0)\sqrt{\frac{2}{\pi}} \int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{\alpha^2}{2}} d\alpha + \frac{1}{\pi\sqrt{t}} \int_0^\infty \frac{dv}{\sqrt{v}} f(v) \quad (2.8)$$

Thus, we obtain, for  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , Borel, sub-exponential (and then  $\int_0^\infty \frac{dv}{\sqrt{v}} f(v) < \infty$ )

$$\sqrt{t} E_x[f(A_t^-)] \xrightarrow{t \rightarrow \infty} \int_0^\infty f(z) \nu_x(dz)$$

with

$$\nu_x(dz) = x_+ \sqrt{\frac{2}{\pi}} \delta_0(dz) + \frac{1}{\pi} e^{-\frac{x^2}{2z}} \mathbf{1}_{[0, \infty[}(z) \frac{dz}{\sqrt{z}}$$

which is precisely (1.23).

## 2.2 Penalisation by $h(A_t^-)$ . A study of "long bridges" and of the $Q^h$ -process

We recall that, from (2.5), the density of  $A_t^-$  under  $P_0$ , which we denote by  $p_{A_t^-}$ , equals :

$$p_{A_t^-}(y) = \frac{1}{\pi} \frac{1}{\sqrt{y(t-y)}} \mathbf{1}_{[0, t]}(y) \quad (: \text{ the arc sine law}). \quad (2.9)$$

Throughout the following,  $h$  denotes a function from  $\mathbb{R}_+$  to  $\mathbb{R}_+$  such that :

$$\int_0^\infty \frac{dy}{\sqrt{y}} h(y) < \infty$$

and we assume, without loss of generality, that :

$$\int_0^\infty \frac{dy}{\sqrt{y}} h(y) = 1 \quad (2.10)$$

### Theorem 2.1

1) For every  $s \geq 0$  and every  $\Lambda_s \in \mathcal{F}_s$  :

$$\lim_{t \rightarrow \infty} E_0(\mathbf{1}_{\Lambda_s} | A_t^- = a) = Q^{(a)}(\Lambda_s) \quad (2.11)$$

with

$$\begin{aligned} Q^{(a)}(\Lambda_s) &:= \sqrt{\frac{2}{\pi}} \frac{\mathbf{1}_{a < s}}{\sqrt{s-a}} E_0(\mathbf{1}_{\Lambda_s} X_s^+ | A_s^- = a) \\ &+ E\left[\mathbf{1}_{\Lambda_s} \sqrt{\frac{a}{a-A_s^-}} \mathbf{1}_{(A_s^- < a)} e^{-\frac{(X_s^-)^2}{2(a-A_s^-)}}\right] \end{aligned} \quad (2.12)$$

(recall that  $X_s^+ = 0 \vee X_s$ ,  $X_s^- = -(X_s \wedge 0)$  and  $A_s^- = \int_0^s \mathbf{1}_{X_u < 0} du$ )



2) For every function  $h$  which satisfies (2.10), for every  $s \geq 0$  and any  $\Lambda_s \in \mathcal{F}_s$  :

$$\lim_{t \rightarrow \infty} \frac{E_0(1_{\Lambda_s} h(A_t^-))}{E_0(h(A_t^-))} = E(1_{\Lambda_s} M_s^h) \quad (2.13)$$

where  $(M_s^h, s \geq 0)$  is the positive martingale given by :

$$M_s^h := \sqrt{2\pi} X_s^+ h(A_s^-) + \int_0^\infty \frac{dy}{\sqrt{y}} e^{-\frac{(X_s^-)^2}{2y}} h(A_s^- + y) \quad (2.14)$$

(note that  $M_0^h = 1$ ).

3) Formula (2.13) induces a probability  $Q^h$  on  $(\Omega, \mathcal{F}_\infty)$ , which admits the following disintegration :

$$Q^h(\Lambda) = \int_0^\infty Q^{(a)}(\Lambda) \frac{h(a)}{\sqrt{a}} da \quad (\Lambda \in \mathcal{F}_\infty) \quad (2.15)$$

where  $Q^{(a)}$  is given by (2.12).

4) Under  $Q^h$ , the canonical process  $(X_t, t \geq 0)$  satisfies :

$$i) \quad A_\infty^- \text{ is finite a.s., and admits as density } \frac{h(y)}{\sqrt{y}} 1_{y>0} \quad (2.16)$$

$$ii) \quad \text{let } g = \inf\{t; A_t^- = A_\infty^-\} = \sup\{t; X_t \leq 0\} \quad (2.17)$$

Then  $Q^h(g < \infty) = 1$

iii) the processes  $(X_t, t \leq g)$  and  $(X_{g+t}, t \geq 0)$  are independent ;

iv) the process  $(X_{g+t}, t \geq 0)$  is a 3-dimensional Bessel process starting from 0.

Moreover, while proving Theorem 2.1, we shall give a precise description of the process  $(X_t; t \leq g)$ .

## 2.3 Proof of Theorem 2.1

### 2.3.1 Proof of point 1) in Theorem 2.1

For this purpose, we choose a function  $h$ , which is Borel, positive, and satisfies (2.10).

We first write :

$$E_0(1_{\Lambda_s} h(A_t^-)) = \int_0^t E_0(1_{\Lambda_s} | A_t^- = a) p_{A_t^-}(a) h(a) da \quad (2.18)$$

Then, conditioning with respect to  $\mathcal{F}_s$ , we obtain :

$$E_0(1_{\Lambda_s} h(A_t^-)) = E_0\left(1_{\Lambda_s} E_0\left(h\left(a + \int_0^{t-s} 1_{(X_u < -x)} du\right)\right)\right) \quad (2.19)$$

with  $a = A_s^-$  and  $x = X_s$ . Using now (2.5) and (2.6), we obtain :

$$\begin{aligned} & E_0(1_{\Lambda_s} h(A_t^-)) \\ &= E_0\left(1_{\Lambda_s} 1_{x < 0} \left( \int_0^{t-s} \frac{dv}{\pi \sqrt{v(t-s-v)}} e^{-\frac{x^2}{2v}} h(a+v) \right) + h(a+t-s) \psi\left(\frac{|x|}{\sqrt{t-s}}\right) \right) \\ &+ E_0\left(1_{\Lambda_s} 1_{x > 0} \left[ h(a) \psi\left(\frac{x}{\sqrt{t-s}}\right) + \int_0^{t-s} \frac{dv}{\pi \sqrt{v(t-s-v)}} e^{-\frac{x^2}{2(t-s-v)}} h(a+v) \right] \right) \end{aligned} \quad (2.20)$$

$$:= (1)_t + (2)_t \quad (2.21)$$

where  $\psi\left(\frac{x}{\sqrt{t}}\right) := P\left(|N| \leq \frac{x}{\sqrt{t}}\right) = \sqrt{\frac{2}{\pi}} \int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{\alpha^2}{2}} d\alpha \underset{t \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi}} \frac{x}{\sqrt{t}}$ .

We now study successively  $(1)_t$  and  $(2)_t$ . We rewrite  $(1)_t$  in the form :

$$\begin{aligned} & (1)_t \\ &= \int_0^s p_{A_s^-}(a) da E_0\left(1_{\Lambda_s} 1_{X_s < 0} \left( \int_0^{t-s} \frac{dv}{\pi \sqrt{v(t-s-v)}} e^{-\frac{X_s^2}{2v}} h(a+v) \right) \right. \\ &\quad \left. + h(a+t-s) \psi\left(\frac{|X_s|}{\sqrt{t-s}}\right) \middle| A_s^- = a \right) \\ &= \int_0^{t-s} \frac{dv}{\pi \sqrt{v(t-s-v)}} \int_v^{v+s} da p_{A_s^-}(a-v) E_0\left(1_{\Lambda_s} 1_{X_s < 0} e^{-\frac{X_s^2}{2v}} \middle| A_s^- = a-v \right) h(a) \\ &\quad + \int_{t-s}^t p_{A_s^-}(a+s-t) E_0\left(1_{\Lambda_s} 1_{X_s < 0} \psi\left(\frac{|X_s|}{\sqrt{t-s}}\right) \middle| A_s^- = a+s-t \right) h(a) da \end{aligned} \quad (2.22)$$

Similarly :

$$(2)_t = \int_0^s p_{A_s^-}(a) E_0\left(1_{\Lambda_s} 1_{X_s > 0} \psi\left(\frac{|X_s|}{\sqrt{t-s}}\right) \middle| A_s^- = a \right) h(a) da \quad (2.23)$$

$$\begin{aligned} & + \int_0^s da p_{A_s^-}(a) \int_0^{t-s} \frac{dv}{\pi \sqrt{v(t-s-v)}} E_0\left(1_{\Lambda_s} 1_{X_s > 0} e^{-\frac{X_s^2}{2(t-s-v)}} \middle| A_s^- = a \right) h(a+v) \\ &= \int_0^s p_{A_s^-}(a) E_0\left(1_{\Lambda_s} 1_{X_s > 0} \psi\left(\frac{|X_s|}{\sqrt{t-s}}\right) \middle| A_s^- = a \right) h(a) da \quad (2.24) \\ & + \int_0^{t-s} \frac{dv}{\pi \sqrt{v(t-s-v)}} \int_v^{v+s} p_{A_s^-}(a-v) E_0\left(1_{\Lambda_s} 1_{X_s > 0} e^{-\frac{X_s^2}{2(t-s-v)}} \middle| A_s^- = a-v \right) h(a) da \end{aligned}$$

Then, comparing (2.18), (2.22), (2.24) and identifying the "coefficient of  $h(a)$ ", it follows that :

$$E_0(1_{\Lambda_s} | A_t^- = a) = (\tilde{1})_t + (\tilde{2})_t$$

with :

$$\begin{aligned}
& (\tilde{1})_t \\
&= \frac{1}{p_{A_t^-}(a)} \int_0^{t-s} \frac{dv}{\pi \sqrt{v(t-s-v)}} 1_{v < a < v+s} p_{A_s^-}(a-v) E_0 \left( 1_{\Lambda_s} 1_{X_s < 0} e^{-\frac{X_s^2}{2v}} \mid A_s^- = a-v \right) \\
&\quad + \frac{1}{p_{A_t^-}(a)} 1_{t-s < a < t} p_{A_s^-}(a+s-t) E_0 \left( 1_{\Lambda_s} 1_{X_s < 0} \psi \left( \frac{|X_s|}{\sqrt{t-s}} \right) \mid A_s^- = a+s-t \right) \\
&\xrightarrow{t \rightarrow \infty} \int_0^s \frac{\sqrt{a}}{\sqrt{a-w}} p_{A_s^-}(w) E_0 \left( 1_{\Lambda_s} 1_{X_s < 0} e^{-\frac{X_s^2}{2(a-w)}} \mid A_s^- = w \right) dw \\
&= E_0 \left( 1_{\Lambda_s} 1_{X_s < 0} e^{-\frac{X_s^2}{2(a-A_s^-)}} \frac{\sqrt{a}}{\sqrt{a-A_s^-}} 1_{A_s^- < a} \right) \tag{2.25}
\end{aligned}$$

since  $p_{A_t^-}(a) = \frac{1}{\pi \sqrt{a(t-a)}} 1_{[0,t]}(a)$ . Similarly, one has :

$$\begin{aligned}
& (\tilde{2})_t \\
&= \frac{p_{A_s^-}(a)}{p_{A_t^-}(a)} 1_{a < s} E_0 \left( 1_{\Lambda_s} 1_{X_s > 0} \psi \left( \frac{|X_s|}{\sqrt{t-s}} \right) \mid A_s^- = a \right) \\
&\quad + \frac{1}{p_{A_t^-}(a)} \int_0^{t-s} 1_{v < a < v+s} \frac{dv}{\pi \sqrt{v(t-s-v)}} p_{A_s^-}(a-v) E_0 \left( 1_{\Lambda_s} 1_{X_s > 0} e^{-\frac{X_s^2}{2(t-s-v)}} \mid A_s^- = a-v \right) \\
&\xrightarrow{t \rightarrow \infty} \frac{1_{a < s}}{\sqrt{s-a}} \sqrt{\frac{2}{\pi}} E_0 \left( 1_{\Lambda_s} X_s^+ \mid A_s^- = a \right) + E_0 \left( 1_{\Lambda_s} 1_{X_s > 0} \sqrt{\frac{a}{a-A_s^-}} 1_{A_s^- < a} \right)
\end{aligned}$$

Hence, point 1 of Theorem 2.1 follows.

### 2.3.2 Proof of point 2 in Theorem 2.1

In fact, this point has already been shown while proving Theorem 1.5. With the help of the form (1.41) of  $M^h$  and the explicit computation of  $\nu_x$  (see formula (1.23)), we obtain :

$$\begin{aligned}
M_s^h &= \frac{\int_0^\infty \nu_{X_s}(dy) h(A_s^- + y)}{\int_0^\infty \nu_0(dy) h(y)} = \frac{\int_0^\infty h(A_s^- + y) \left[ X_s^+ \sqrt{\frac{2}{\pi}} \delta_0(dy) + \frac{1}{\pi} e^{-\frac{(X_s^-)^2}{2y}} \frac{dy}{\sqrt{y}} \right]}{\frac{1}{\pi} \int_0^\infty h(y) \frac{dy}{\sqrt{y}}} \\
&= \sqrt{2\pi} X_s^+ h(A_s^-) + \int_0^\infty \frac{dy}{\sqrt{y}} e^{-\frac{(X_s^-)^2}{2y}} h(A_s^- + y) \tag{2.26}
\end{aligned}$$

Now, clearly, this point 1 of Theorem 2.1 which we just proved implies also point 2 of the same Theorem 2.1. Indeed, we have :

$$\frac{E_0(1_{\Lambda_s} h(A_t^-))}{E_0(h(A_t^-))} = \frac{\int_0^t E_0(1_{\Lambda_s} \mid A_t^- = a) h(a) p_{A_t^-}(a) da}{\int_0^t h(a) p_{A_t^-}(a) da}$$

From the above point 1, and with the help of the explicit form of  $p_{A_t^-}(a)$  as given by (2.9) the above quantity converges, as  $t \rightarrow \infty$ , towards :

$$\frac{\int_0^\infty \frac{da}{\sqrt{a}} Q^{(a)}(\Lambda_s) h(a)}{\int_0^\infty \frac{h(a) da}{\sqrt{a}}} = \int_0^\infty \frac{h(a)}{\sqrt{a}} Q^{(a)}(\Lambda_s) da \quad (2.27)$$

since we assumed :  $\int_0^\infty \frac{h(a) da}{\sqrt{a}} = 1$ .

It now remains to compute :

$$\begin{aligned} \int_0^\infty \frac{h(a)}{\sqrt{a}} Q^{(a)}(\Lambda_s) da &= \sqrt{\frac{2}{\pi}} \int_0^s \frac{1}{\sqrt{a(s-a)}} E_0(1_{\Lambda_s} X_s^+ | A_s^- = a) h(a) da \quad (2.28) \\ &+ \int_0^\infty \frac{h(a)}{\sqrt{a}} E_0\left(1_{\Lambda_s} \sqrt{\frac{a}{a-A_s^-}} 1_{A_s^- < a} e^{-\frac{(X_s^-)^2}{2(a-A_s^-)}}\right) da \quad (\text{from (2.12)}) \\ &= \sqrt{2\pi} \int_0^s p_{A_s^-}(a) E_0(1_{\Lambda_s} X_s^+ | A_s^- = a) h(a) da \\ &+ \int_0^\infty \frac{dy}{\sqrt{y}} E_0\left(1_{\Lambda_s} e^{-\frac{(X_s^-)^2}{2y}} h(A_s^- + y)\right) \\ &\quad (\text{after the change of variable } a - A_s^- = y) \\ &= \sqrt{2\pi} E_0\left(1_{\Lambda_s} X_s^+ h(A_s^-)\right) + \int_0^\infty E_0\left(1_{\Lambda_s} e^{-\frac{(X_s^-)^2}{2y}} h(A_s^- + y)\right) \frac{dy}{\sqrt{y}} \\ &= E_0(1_{\Lambda_s} M_s^h) \quad (\text{from (2.14)}) \quad (2.29) \end{aligned}$$

We now remark that point 3 in Theorem 2.1 states precisely formula (2.29) we just established.

### 2.3.3 Proofs of points 4i) and 4ii) in Theorem 2.1

a) From formula (2.15) and from Doob's optional sampling theorem, we deduce :

$$Q^h(A_\infty^- > a) = E[M_{\sigma_a}^h], \quad \text{with } \sigma_a := \inf\{t; A_t^- > a\} \quad (2.30)$$

But :

$$\begin{aligned} M_{\sigma_a}^h &= \sqrt{2\pi} h(a) X_{\sigma_a}^+ + \int_0^\infty \frac{dy}{\sqrt{y}} e^{-\frac{(X_{\sigma_a}^-)^2}{2y}} h(a+y) \\ &= \int_0^\infty \frac{dy}{\sqrt{y}} e^{-\frac{(X_{\sigma_a}^-)^2}{2y}} h(a+y) \quad \text{since } X_{\sigma_a}^+ = 0 \end{aligned}$$

We recall that the process  $(X_{\sigma_a}^-; a \geq 0)$  is distributed as the reflecting Brownian motion  $(|X_a|, a \geq 0)$ , where  $(X_a, a > 0)$  is a standard Brownian motion starting from 0 (see, e.g., [K.S], Th. 3.1, p. 419). Hence, we obtain :

$$\begin{aligned} E[M_{\sigma_a}^h] &= \int_0^\infty \frac{dy}{\sqrt{y}} h(a+y) E(e^{-\frac{x_a^2}{2y}}) \\ &= \int_0^\infty \frac{dy}{\sqrt{y}} h(a+y) \sqrt{\frac{y}{y+a}} = \int_0^\infty \frac{dy}{\sqrt{y+a}} h(a+y) = \int_a^\infty \frac{dy}{\sqrt{y}} h(y) \end{aligned}$$

b) We now remark that it is easy to recover the law of  $A_\infty^-$  under  $Q^h$  from points 1 and 2 in Theorem 2.1. We may already prove that, under  $Q^{(a)}$ , one has  $A_\infty^- = a$  a.s. Indeed, this follows from :

$$\begin{aligned} \text{if } b > a, \quad Q^{(a)}(A_s^- > b) &= \sqrt{\frac{2}{\pi}} \frac{1_{a < s}}{\sqrt{s-a}} E_0(X_s^+ 1_{A_s^- > b} | A_s^- = a) \\ &+ E_0\left(\sqrt{\frac{a}{a-A_s^-}} 1_{b < A_s^- < a} e^{-\frac{(X_s^-)^2}{2(a-A_s^-)}}\right) = 0 \end{aligned}$$

Hence, passing to the limit as  $s \rightarrow \infty$ , if  $b > a$  :  $Q^a(A_\infty^- > b) = 0$

On the other hand, it is clear that  $E_0(1_{A_s^- \leq a} | A_t^- = a) = 1$  ( $t > s$ ), hence, passing to the limit as  $t \rightarrow \infty$ , and then, letting  $s \rightarrow \infty$  we obtain :

$$Q^{(a)}(A_\infty^- \leq a) = 1$$

Finally, from (2.15), we get :

$$Q^h(A_\infty^- \leq b) = \int_0^\infty \frac{h(a)}{\sqrt{a}} Q^{(a)}(A_\infty^- \leq b) da = \int_0^b \frac{h(a)}{\sqrt{a}} da.$$

### 2.3.4 Computation of Azéma's supermartingale $Z_t := Q^h(g > t | \mathcal{F}_t)$

Our proof of points 4iii) and 4iv) in Theorem 2.1 is based on the theory of enlargements of filtration (cf [J] or [JY]). In order to apply this theory, we need to calculate Azéma's supermartingale  $Q^h(g > t | \mathcal{F}_t)$ . We start with this computation.

$$\text{Let } g = \inf\{t \geq 0 ; A_t^- = A_\infty^-\} = \sup\{t \geq 0 ; X_t \leq 0\} \quad (2.31)$$

**Lemma 2.2** *The following explicit formula holds :*

$$Z_t := Q^{(h)}(g > t | \mathcal{F}_t) = 1_{(X_t < 0)} + 1_{(X_t > 0)} \frac{\int_0^\infty \frac{dv}{\sqrt{v}} h(A_t^- + v)}{M_t^h} \quad (2.32)$$

**Proof of Lemma 2.2** We note that, for  $\Lambda_t \in \mathcal{F}_t$  :

$$Q^h(1_{g>t} 1_{\Lambda_t}) = Q^h(1_{\Lambda_t} 1_{X_t < 0}) + Q^h(1_{\Lambda_t} 1_{X_t > 0} 1_{d_t < \infty})$$

(where  $d_t$  denotes the first return time to 0 after time  $t$ )

$$= Q^h(1_{\Lambda_t} 1_{X_t < 0}) + E(1_{\Lambda_t} 1_{X_t > 0} M_{d_t}^h)$$

We have :

$$M_{d_t}^h = \sqrt{2\pi} h(A_{d_t}^-) X_{d_t}^+ + \int_0^\infty \frac{dv}{\sqrt{v}} h(A_{d_t}^- + v) = \int_0^\infty \frac{dv}{\sqrt{v}} h(A_t^- + v) \quad (\text{from (2.14)})$$

since  $X_{d_t} = 0$  and  $A_{d_t}^- = A_t^-$  on the set  $(X_t > 0)$ .

Hence :

$$Q^h(1_{g>t} 1_{\Lambda_t}) = Q^h\left(1_{\Lambda_t} \left(1_{X_t < 0} + 1_{X_t > 0} \frac{\int_0^\infty \frac{dw}{\sqrt{w}} h(A_t^- + w)}{M_t^h}\right)\right)$$

This proves (2.32) and Lemma 2.2.

### 2.3.5 Proof of $Q^h(g < \infty) = 1$

We deduce from (2.32) that :

$$\begin{aligned} Q[g < t] &= 1 - Q[g > t] \\ &= 1 - E\left[1_{X_t < 0} M_t^h + 1_{X_t > 0} \int_0^\infty \frac{dv}{\sqrt{v}} h(A_t^- + v)\right] \\ &= \sqrt{2\pi} E[X_t^+ h(A_t^-)] \end{aligned}$$

(from (2.26) and since  $(M_t^h, t \geq 0)$  is a martingale s.t.  $E(M_t^h) = 1$ )

$$\begin{aligned} &= \frac{\sqrt{2\pi}}{2} E\left(\int_0^t h(A_s^-) dL_s\right) \quad \text{from It\^o-Tanaka formula} \\ &\xrightarrow{t \rightarrow \infty} \sqrt{\frac{\pi}{2}} E\left(\int_0^\infty h(A_s^-) dL_s\right) = \sqrt{\frac{\pi}{2}} E\left(\int_0^\infty h(a) dL_{\sigma_a}\right) \end{aligned}$$

(where  $(\sigma_a, a \geq 0)$  denotes the right continuous inverse of  $(A_t^-, t \geq 0)$ )

$$= 2\sqrt{\frac{\pi}{2}} E\left(\int_0^\infty h(a) dL_a\right)$$

(since  $(X_{\sigma_a}^-, a \geq 0)$  is distributed as  $(|X_a|, a \geq 0)$ ; cf point  $a$ ) of Subsection 2.3.3)

$$= 2\sqrt{\frac{\pi}{2}} \int_0^\infty h(a) E(dL_a) = \int_0^\infty \frac{h(a)}{\sqrt{a}} da = 1$$

since  $E(L_a) = \sqrt{a} \cdot \sqrt{\frac{2}{\pi}}$ .

### 2.3.6 Description of the canonical process $(X_t, t \geq 0)$ under $Q^h$

For this purpose, we shall use the technique of enlargement of filtrations. Thus, let  $(\mathcal{G}_t, t \geq 0)$  denote the smallest filtration which makes  $g$  a  $(\mathcal{G}_t, t \geq 0)$  stopping time, and which contains  $(\mathcal{F}_t, t \geq 0)$ .

The application of Girsanov's Theorem and (2.14) imply the existence of a  $(\mathcal{F}_t, Q^h)$  Brownian motion  $(\beta_t, t \geq 0)$  such that, under  $Q^h$  :

$$X_t = \beta_t + \int_0^t \frac{1}{M_s^h} \left\{ \sqrt{2\pi} h(A_s^-) 1_{X_s > 0} - \left( \int_0^\infty \frac{dw}{w^{3/2}} e^{-\frac{(X_s^-)^2}{2w}} h(A_s^- + w) \right) X_s^- \right\} ds \quad (2.33)$$

We now apply the enlargement formulae (cf [J], [JY], [MY]). We first observe that :

$$dZ_t = -\frac{\int_0^\infty \frac{dw}{\sqrt{w}} h(A_t^- + w)}{(M_t^h)^2} (\sqrt{2\pi} h(A_t^-) 1_{X_t > 0} dX_t) + dV_t \quad (2.34)$$

where  $(V_t, t \geq 0)$  has bounded variations and therefore :

$$d\langle Z, X \rangle_t = -\frac{\int_0^\infty \frac{dw}{\sqrt{w}} h(A_t^- + w)}{(M_t^h)^2} \sqrt{2\pi} h(A_t^-) 1_{(X_t > 0)} dt \quad (2.35)$$

Thus, there exists a  $((\mathcal{G}_t, t \geq 0), Q^h)$  Brownian motion  $(\tilde{\beta}_t, t \geq 0)$  such that :

$$\begin{aligned} dX_t &= d\tilde{\beta}_t + \frac{1}{M_t^h} \left\{ \sqrt{2\pi} h(A_t^-) 1_{X_t > 0} - \left( \int_0^\infty \frac{dw}{w^{3/2}} e^{-\frac{(X_t^-)^2}{2w}} h(A_t^- + w) \right) X_t^- \right\} dt \\ &+ 1_{t < g} \left[ - \frac{\int_0^\infty \frac{dw}{\sqrt{w}} h(A_t^- + w)}{(M_t^h)^2} (\sqrt{2\pi} h(A_t^-) 1_{X_t > 0}) \cdot \frac{M_t^h}{\int_0^\infty \frac{dw}{\sqrt{w}} h(A_t^- + w)} \right] dt \\ &- 1_{t > g} \left[ - \frac{\int_0^\infty \frac{dw}{\sqrt{w}} h(A_t^- + w)}{(M_t^h)^2} (\sqrt{2\pi} h(A_t^-) 1_{X_t > 0}) \cdot \frac{M_t^h}{\sqrt{2\pi} h(A_t^-) X_t^+} \right] dt \end{aligned}$$

This yields, after some simplifications :

$$X_t = \tilde{\beta}_t - \int_0^{t \wedge g} \frac{1}{M_s^h} \left( \int_0^\infty \frac{dw}{w^{3/2}} e^{-\frac{(X_s^-)^2}{2w}} h(A_s^- + w) \right) X_s^- ds + \int_{t \wedge g}^t \frac{ds}{X_s} \quad (2.36)$$

since, after  $g$ ,  $X_t^- = 0$ , hence  $X_t^+ = X_t$ .

Points 4 iii) and iv) now follow immediately from (2.36).

**Remark 2.3** When  $h(x) = e^{-\frac{\lambda x}{2}}$  ( $\lambda > 0, x \geq 0$ ), the equation (2.36) simplifies as :

$$X_t = \tilde{\beta}_t - \int_0^{t \wedge g} \frac{\sqrt{X_s^-} \lambda^{\frac{1}{2}}}{\lambda^{\frac{1}{4}} \sqrt{2\pi} X_s^+ - \sqrt{X_s^-}} ds + \int_{t \wedge g}^t \frac{ds}{X_s}. \quad (2.37)$$

This formula (2.37) follows from :

$$\begin{aligned} \int_0^\infty \frac{dw}{w^{3/2}} e^{-\frac{(X_s^-)^2}{2w} - \frac{\lambda}{2} w} dw &= \left( \frac{(X_s^-)^2}{\lambda} \right)^{-1/4} 2 K_{-1/2}(\sqrt{\lambda} X_s^-) \\ \int_0^\infty \frac{dw}{w^{1/2}} e^{-\frac{(X_s^-)^2}{2w} - \frac{\lambda}{2} w} dw &= \left( \frac{(X_s^-)^2}{\lambda} \right)^{+1/4} 2 K_{1/2}(\sqrt{\lambda} X_s^-) \end{aligned}$$

and from :  $K_{-\frac{1}{2}}(z) = K_{\frac{1}{2}}(z) = \left( \frac{\pi}{2z} \right)^{\frac{1}{2}} e^{-z}$  ([Leb], p. 112 and p. 119).

### 2.3.7 Markovian limit process

Theorem 2.1 shows that the process  $(X_t, t \geq 0)$  is not Markovian under  $Q^h$ , whereas the 2-dimensional process  $((X_t, A_t^-), t \geq 0)$  is Markovian.

Indeed,  $g$  is not a  $(\mathcal{F}_t, t \geq 0)$  stopping time and the dynamics of  $(X_t)$  is not the same before and after  $g$ . On the other hand, we know (see [RVY, I]) that if  $h(x) := e^{-\frac{\lambda}{2}x}$  ( $\lambda, x \geq 0$ ), then the  $Q^h$ -process is Markovian. It is the diffusion with infinitesimal generator  $L^h$  :

$$L^h f(x) = \frac{1}{2} f''(x) + \frac{\varphi'}{\varphi}(x) f'(x), \quad f \in C_b^2$$

where  $\varphi$  denotes the unique solution of  $\varphi'' = \lambda\varphi, \varphi(-\infty) = 0; \varphi'(+\infty) = \sqrt{\frac{2}{\pi}}$ . In this case, the solution of this equation (see (1.24)) takes the explicit form :

$$\varphi_\lambda(x) = \sqrt{\frac{2}{\pi}} \left\{ e^{x\sqrt{\lambda}} \frac{1}{\sqrt{\lambda}} 1_{x \leq 0} + \left( x + \frac{1}{\sqrt{\lambda}} \right) 1_{x > 0} \right\} \quad (2.38)$$

Under  $Q^h$ , we obtain :

$$X_t = B_t + \int_0^t \frac{du}{X_u^+ + \frac{1}{\sqrt{\lambda}}} \quad (\text{compare with (2.37)}) \quad (2.39)$$

where  $(B_t, t \geq 0)$  is a  $((\mathcal{F}_t, t \geq 0), Q^h)$  Brownian motion. The martingale  $(M_s^h, s \geq 0)$  is equal to :

$$M_s^h = \varphi_\lambda(X_s) \exp\left(-\frac{\lambda}{2} \int_0^s 1_{]-\infty, 0]}(X_u) du\right) \quad (2.40)$$

This example motivated us to raise the question : which are the functions  $h$  such that the  $Q^h$  process is Markovian ? The answer is given by the following :

**Proposition 2.4** : *Let  $h$  be regular, bounded, satisfying equation (2.10) i.e,  $\int_0^\infty \frac{dy}{\sqrt{y}} h(y) = 1$  and such that the process  $(X_t, t \geq 0)$  is Markov under  $Q^h$ . Then, there exists  $\lambda \geq 0$  such that  $h(x) = \sqrt{\frac{\lambda}{2\pi}} e^{-\frac{\lambda}{2}x}$  ( $x \geq 0$ ).*

#### Proof of Proposition 2.4

To answer this question, we come back to equation (2.33). The problem is to find under which conditions the drift term :

$$\frac{\sqrt{2\pi} h(A_t^-) 1_{X_t > 0} - \left( \int_0^\infty \frac{dw}{w^{3/2}} e^{-\frac{(X_t^-)^2}{2w}} h(A_t^- + w) \right) X_t^-}{\sqrt{2\pi} h(A_t^-) X_t^+ + \int_0^\infty \frac{dw}{w^{1/2}} e^{-\frac{(X_t^-)^2}{2w}} h(A_t^- + w)} \quad (2.41)$$

does not depend on  $A_t^-$ . Considering this expression when  $X_t < 0$ , the problem amounts to study the functions  $h$  for which :

$$\frac{x \int_0^\infty \frac{dw}{w^{3/2}} e^{-\frac{x^2}{2w}} h(a+w)}{\int_0^\infty \frac{dw}{w^{1/2}} e^{-\frac{x^2}{2w}} h(a+w)} := k(x) \quad (2.42)$$

does not depend on  $a$ . (2.42) may be written :  $\frac{\partial}{\partial x} \log(\theta(x, a)) = -k(x)$  where we have

denoted :  $\theta(x, a) := \int_0^\infty \frac{dw}{\sqrt{w}} e^{-\frac{x^2}{2w}} h(a+w)$ .

Hence, by integration we obtain the existence of two functions  $\varphi_1$  and  $\varphi_2$  such that :

$$\int_0^\infty \frac{x}{\sqrt{2\pi w^3}} e^{-\frac{x^2}{2w}} h(a+w) dw = \varphi_1(a) \varphi_2(x) \quad (2.43)$$

Letting  $x \rightarrow 0$  in (2.43), we obtain  $h(a) = \varphi_1(a) \varphi_2(0)$ . Note that the LHS in (2.43) writes  $E(h(a+T_x))$ , where  $(T_x, x \geq 0)$  is the  $\frac{1}{2}$ -stable subordinator of Brownian first hitting times. Hence we have :

$$E[h(a+T_x)] = P_x(h)(a) = E[\varphi_1(a+T_x) \varphi_2(0)] = \varphi_1(a) \varphi_2(0) \quad (2.44)$$



where  $(P_x, x \geq 0)$  denotes the semi-group associated with the subordinator  $(T_x, x \geq 0)$ , whose infinitesimal generator is  $\left(\frac{\partial^2}{\partial x^2}\right)^{\frac{1}{2}}$ . In other terms, from (2.43), we get :

$$P_x \varphi_1(a) = \frac{\varphi_2(x)}{\varphi_2(0)} \varphi_1(a) \quad (2.45)$$

$\varphi_1$  is an eigenfunction of  $P_x$ , and consequently an eigenfunction of  $\frac{\partial^2}{\partial x^2}$ .  $\varphi_1$  being positive and bounded :  $\varphi_1(a) = c e^{-\frac{\lambda}{2}a}$  ( $a, \lambda \geq 0$ ) and  $h(a) = c e^{-\frac{\lambda}{2}a} \varphi_2(0) = c' e^{-\frac{\lambda}{2}a}$ .

### **3 A local limit theorem for a class of additive functionals of the "long Brownian bridges"**

#### **3.1 Statement of Theorem 3.1**

In this section, our aim is to obtain results similar to those in Section 1, but, now, Brownian motion  $(X_s, s \geq 0)$  is being replaced by the Brownian bridge with length  $t$ , with  $t \rightarrow \infty$ .  $q$  denotes a function from  $\mathbb{R}$  to  $\mathbb{R}_+$ , which is Borel, and such that :

$$0 < \int_{-\infty}^{\infty} (1 + x^2) q(x) dx < \infty. \quad (3.1)$$

We let :

$$A_t^q := \int_0^t q(X_s) ds \quad (3.2)$$

#### **Theorem 3.1**

1) For every  $x$  and  $y \in \mathbb{R}$ , and  $\mu > 0$  :

$$E_x \left( \exp \left( -\frac{\mu}{2} \int_0^t q(X_s) ds \right) \middle| X_t = y \right) \underset{t \rightarrow \infty}{\sim} \frac{\pi}{2} \frac{\varphi_{\mu q}(x) \varphi_{\mu q}(y)}{t} \quad (3.3)$$

where  $\varphi_{\mu q}$  denotes the unique solution of :

$$\varphi'' = (\mu q) \cdot \varphi, \quad \lim_{x \rightarrow +\infty} \varphi'(x) = - \lim_{x \rightarrow -\infty} \varphi'(x) = \sqrt{\frac{2}{\pi}} \quad (3.4)$$

$$2) \lim_{t \rightarrow \infty} t P_x (A_t^q \in dz | X_t = y) = \nu_x * \nu_y (dz) \quad (3.5)$$

where  $\nu_x$  and  $\nu_y$  have been defined in Theorem 1.1. The convergence in (3.5) has the same meaning as in Theorem 1.1.

#### **3.2 Proof of Theorem 3.1**

Without loss of generality, we shall assume that  $\mu = 1$ .

### 3.2.1 An auxiliary lemma

**Lemma 3.2** *There exists a constant  $C > 0$ , depending only on  $q$ , such that :*

$$E_x \left( \exp \left( - \frac{1}{2} \int_0^t q(X_s) ds \right) \middle| X_t = y \right) \leq C e^{\frac{(x-y)^2}{2t}} \frac{(1+|x|)(1+|y|)}{1+t} \quad (3.6)$$

### 3.2.2 Proof of lemma 3.2

1) As an intermediary result, we already show that :

$$E_x \left( \exp \left( - \frac{1}{2} \int_0^t q(X_s) ds \right) \middle| X_t = y \right) \leq C e^{\frac{(x-y)^2}{2t}} \frac{1+|x|}{\sqrt{1+t}} \quad (3.7)$$

for a constant  $C$  which does not depend on  $x, y, t$ .

To prove (3.7), we condition with respect to  $X_{t/2}$ , and we get :

$$\begin{aligned} E_x \left( \exp \left( - \frac{1}{2} \int_0^t q(X_s) ds \right) \middle| X_t = y \right) e^{-\frac{(x-y)^2}{2t}} &= \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} E_x \left( \exp - \frac{1}{2} A_{t/2}^q \middle| X_{t/2} = c \right) \\ E_c \left( \exp - \frac{1}{2} A_{t/2}^q \middle| X_{t/2} = y \right) &\cdot e^{-\frac{(x-c)^2}{t} - \frac{(y-c)^2}{t}} dc \end{aligned} \quad (3.8)$$

In (3.8), we majorize  $E_c \left( \exp - \frac{1}{2} A_{t/2}^q \middle| X_{t/2} = y \right)$  by 1, and we get :

$$\begin{aligned} E_x \left( \exp - \frac{1}{2} A_t^q \middle| X_t = y \right) e^{-\frac{(x-y)^2}{2t}} &\leq \frac{1}{\sqrt{\pi t}} \cdot \int_{-\infty}^{\infty} E_x \left( \exp - \frac{1}{2} A_{t/2}^q \middle| X_{t/2} = c \right) e^{-\frac{(x-c)^2}{2 \cdot t/2}} dc \\ &\leq E_x \left( \exp - \frac{1}{2} A_{t/2}^q \right) \leq C \frac{1+|x|}{\sqrt{1+t}} \end{aligned}$$

from Lemma 4.3 in [RVY, I]. Thus, we have obtained (3.7).

2) Then, plugging the estimate (3.7) in (3.8), we obtain :

$$\begin{aligned} E_x \left( \exp \left( - \frac{1}{2} A_t^q \right) \middle| X_t = y \right) &\leq e^{\frac{(x-y)^2}{2t}} \frac{C(1+|y|)}{\sqrt{1+t}} \int_{-\infty}^{\infty} E_x \left( \exp - \frac{1}{2} A_{t/2}^q \middle| X_{t/2} = c \right) \frac{e^{-\frac{(x-c)^2}{2 \cdot t/2}}}{\sqrt{2\pi t/2}} dc \\ &\leq \frac{C(1+|x|)(1+|y|)}{1+t} \cdot e^{\frac{(x-y)^2}{2t}} \end{aligned}$$

since :

$$E_c \left( \exp - \frac{1}{2} A_{t/2}^q \middle| X_{t/2} = y \right) = E_y \left( \exp - \frac{1}{2} A_{t/2}^q \middle| X_{t/2} = c \right)$$

and

$$e^{\frac{(x-y)^2}{2t}} \cdot \frac{C(1+|y|)}{\sqrt{1+t}} E_x \left( \exp - \frac{1}{2} A_{t/2}^q \middle| X_{t/2} = c \right) \leq e^{\frac{(x-y)^2}{2t}} C \frac{(1+|y|)(1+|x|)}{1+t}$$

by applying once again Lemma 4.3 in [RVY, I].

### 3.2.3 Another auxiliary lemma

**Lemma 3.3** Let  $Z(t, x, y) := E_x \left( \exp - \frac{1}{2} A_t^q | X_t = y \right)$ . We also denote by  $U(t, x, y)$  the solution of :

$$\begin{cases} \frac{\partial U}{\partial t}(t, x, y) - \frac{1}{2} \frac{\partial^2 U}{\partial x^2}(t, x, y) + \frac{1}{2} U(t, x, y) q(x) = 0 \\ U(0, \bullet, y) = \delta_y \end{cases} \quad (3.9)$$

Then :  $Z(t, x, y) = \sqrt{2\pi t} e^{-\frac{(x-y)^2}{2t}} U(t, x, y)$ .  
In particular, it follows from Lemma 3.2, that :

$$U(t, x, y) \leq C \frac{(1 + |x|)(1 + |y|)}{(1 + t)^{3/2}} \quad (t \geq 1) \quad (3.10)$$

### 3.2.4 Proof of Lemma 3.3

We know that, for every regular function  $f$  :

$$Z^f(t, x) := E_x \left[ \exp \left( - \frac{1}{2} A_t^q \right) f(X_t) \right]$$

is solution of :

$$\frac{\partial Z^f}{\partial t} - \frac{1}{2} \frac{\partial^2 Z^f}{\partial x^2} + \frac{1}{2} Z^f \cdot q = 0, \quad Z^f(0, x) = f(x) \quad (3.11)$$

It suffices, in order to obtain Lemma 3.3, to write :

$$Z(t, x, y) = \lim_{\varepsilon \downarrow 0} \frac{E_x \left[ \left( \exp \left( - \frac{1}{2} A_t^q \right) \right) f_\varepsilon(X_t) \right]}{E_x(f_\varepsilon(X_t))}$$

where  $f_\varepsilon$  is a family of functions which converges weakly towards  $\delta_y$ , and to use (3.11).

### 3.2.5 We define, for every $\lambda > 0$

$$A(\lambda, x, y) = \int_0^\infty e^{-\lambda t} U(t, x, y) dt \quad (3.12)$$

Since  $Z(t, x, y)$  is a decreasing function of  $t$ , we deduce the following equivalences from the Tauberian theorem :

$$\begin{aligned} i) \quad Z(t, x, y) &\underset{t \rightarrow \infty}{\sim} \frac{\pi}{2} \frac{\varphi_q(x) \varphi_q(y)}{t} \\ ii) \quad U(t, x, y) &\underset{t \rightarrow \infty}{\sim} \frac{\sqrt{\pi}}{2\sqrt{2}} \frac{\varphi_q(x) \varphi_q(y)}{t^{3/2}} \\ iii) \quad \left| \frac{\partial}{\partial \lambda} A(\lambda, x, y) \right| &= - \frac{\partial}{\partial \lambda} A(\lambda, x, y) \underset{\lambda \rightarrow 0}{\sim} \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{\lambda}} \varphi_q(x) \varphi_q(y) \end{aligned} \quad (3.13)$$

We shall now show (3.13). We already deduce from Lemmas 3.2 and 3.3 that :

$$\lim_{\lambda \rightarrow 0} A(\lambda, x, y) = \int_0^\infty U(t, x, y) dt < \infty \quad (3.14)$$

$$A(\lambda, x, y) \leq C(1 + |x|)(1 + |y|) \quad (3.15)$$

$$\left| \frac{\partial}{\partial \lambda} A(\lambda, x, y) \right| \leq \frac{C}{\sqrt{\lambda}} (1 + |x|)(1 + |y|) \quad (3.16)$$

To prove (3.13) we shall show that :  $\psi(x, y) := \lim_{\lambda \rightarrow 0} \sqrt{\lambda} \frac{\partial}{\partial \lambda} A(\lambda, x, y)$  satisfies the Sturm-Liouville equation (for any fixed  $y$ ) :

$$\frac{\partial^2}{\partial x^2} \psi = \psi q, \quad \text{with adequate limit conditions in } x = \pm\infty \quad (3.17)$$

### 3.2.6 We get, from (3.9)

$$U(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} - \frac{1}{2} \int_0^t ds \int_{-\infty}^\infty \frac{e^{-\frac{(x-z)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} U(s, z, y) q(z) dz \quad (3.18)$$

Thus, after taking the Laplace transform in the variable  $t$  of the two sides of (3.18), we obtain :

$$A(\lambda, x, y) = g_\lambda(x, y) - \frac{1}{2} \int_{-\infty}^\infty g_\lambda(x, z) A(\lambda, z, y) q(z) dz \quad (3.19)$$

where  $g_\lambda$  denotes the density of the resolvent kernel of Brownian motion :

$$g_\lambda(x, z) = \frac{1}{\sqrt{2\lambda}} e^{-|x-z|\sqrt{2\lambda}}$$

We write (3.19) in the form :

$$A(\lambda, x, y) = G_\lambda \left[ \delta_y - \frac{1}{2} (A(\lambda, \bullet, y) q(\bullet)) \right] (x) \quad (3.20)$$

with for any Radon measure  $\mu(dz)$  :

$$G_\lambda \mu(x) := \int_{-\infty}^\infty g_\lambda(x, z) \mu(dz) \quad (3.21)$$

and we use the resolvent equation :  $\frac{\partial^2}{\partial x^2} G_\lambda \mu = -2\mu + 2\lambda G_\lambda \mu$ , to obtain :

$$\frac{\partial^2}{\partial x^2} A(\lambda, x, y) = 2\lambda A(\lambda, x, y) - [2\delta_y - A(\lambda, x, y) q(x)] \quad (3.22)$$

As a consequence, differentiating with respect to  $\lambda$ , then multiplying by  $\sqrt{\lambda}$ , we obtain :

$$\frac{\partial^2}{\partial x^2} \left( \sqrt{\lambda} \frac{\partial A}{\partial \lambda}(\lambda, x, y) \right) - \sqrt{\lambda} \frac{\partial A}{\partial \lambda}(\lambda, x, y) q(x) = 2\sqrt{\lambda} A(\lambda, x, y) + 2\lambda^{3/2} \frac{\partial A}{\partial \lambda}(\lambda, x, y) \quad (3.23)$$

Hence, from (3.16) and (3.15), and denoting  $\tilde{A}(\lambda, x, y) := \sqrt{\lambda} \frac{\partial A}{\partial \lambda}(\lambda, x, y)$ , it follows that :

$$\left| \frac{\partial^2}{\partial x^2} (\tilde{A}(\lambda, x, y)) - \tilde{A}(\lambda, x, y) q(x) \right| \leq C \sqrt{\lambda} (1 + |x|)(1 + |y|) \quad (\lambda \rightarrow 0) \quad (3.24)$$

(3.24) is the first step to prove that  $\tilde{A}(\lambda, x, y)$  converges, as  $\lambda \rightarrow 0$ , to a solution of the Sturm-Liouville equation (3.17).

### 3.2.7 We now examine the limit conditions in $x = \pm\infty$

We come back to equation (3.19) which we differentiate with respect to  $\lambda$ , then we multiply by  $\lambda$  :

$$\begin{aligned}\sqrt{\lambda}\tilde{A}(\lambda, x, y) &= -\frac{1}{2}A(\lambda, x, y) - \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|\sqrt{2\lambda}}|x-z|(\delta_y(dz) - A(\lambda, z, y)q(z)dz) \\ &\quad - \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-|x-z|\sqrt{2\lambda}}\tilde{A}(\lambda, x, z)q(z)dz\end{aligned}\quad (3.25)$$

From (3.16) and (3.14), respectively we deduce that :

$$\sqrt{\lambda}\tilde{A}(\lambda, x, y) \xrightarrow{\lambda \rightarrow 0} 0 \quad \text{and} \quad A(\lambda, x, y) \text{ converges as } \lambda \rightarrow 0.$$

hence, from (3.25), since  $\int_{-\infty}^{\infty} (1+x^2)q(dz) < \infty$  :

$$\lim_{\lambda \rightarrow 0} \int_{-\infty}^{\infty} e^{-|x-z|\sqrt{2\lambda}}\tilde{A}(\lambda, x, z)q(z)dz \text{ exists.} \quad (3.26)$$

On the other hand, differentiating (3.25) with respect to  $x$ , we obtain :

$$\frac{\partial \tilde{A}}{\partial x}(\lambda, x, y) = \frac{\partial B}{\partial x} - \frac{1}{2} \left\{ \int_{-\infty}^x e^{-|x-z|\sqrt{2\lambda}}\tilde{A}(\lambda, z, y)q(z)dz + \int_x^{\infty} e^{-|x-z|\sqrt{2\lambda}}\tilde{A}(\lambda, z, y)q(z)dz \right\} \quad (3.27)$$

with

$$B := -\frac{1}{2\sqrt{\lambda}} \left\{ A + \int_{-\infty}^{\infty} e^{-|x-z|\sqrt{2\lambda}}(A(\lambda, z, y)q(z)dz - \delta_y(dz)) \right\} \quad (3.28)$$

We deduce from (3.27), (3.26) and (3.28) that :

$$\lim_{\substack{\lambda \rightarrow 0 \\ x \rightarrow \infty}} \frac{\partial}{\partial x} \tilde{A}(\lambda, x, y) = - \lim_{\substack{\lambda \rightarrow 0 \\ x \rightarrow -\infty}} \frac{\partial}{\partial x} \tilde{A}(\lambda, x, y) = C(y) \quad (3.29)$$

(cf [RVY, I], p. 194-197 for similar computations). Thus, from the equivalence between i), ii) and iii) which we recalled in (3.13), we get :

$$E_x \left[ \exp - \frac{1}{2} \int_0^t q(X_s)ds \mid X_t = y \right] \underset{t \rightarrow \infty}{\sim} \frac{\psi(x, y)}{t} \quad (3.30)$$

where  $\psi$  is solution to :

$$\frac{\partial^2 \psi}{\partial x^2}(x, y) = \psi(x, y)q(x), \quad \lim_{x \rightarrow +\infty} \frac{\partial \psi}{\partial x}(x, y) = - \lim_{x \rightarrow -\infty} \frac{\partial \psi}{\partial x}(x, y) = C(y) \quad (3.31)$$

Thus, from the definition of  $\varphi_q$  (see(3.4)), we get :

$$\psi(x, y) = C(y) \sqrt{\frac{\pi}{2}} \varphi_q(x)$$

Now, since  $Z(t, x, y)$  is symmetric in  $x$  and  $y$  :

$$\psi(x, y) = K \varphi_q(x) \varphi_q(y) \quad (3.32)$$

**It remains to determine the value of  $K$**  For this purpose, we write :

$$\varphi_q(x) = E_x \left( \left( \exp \left( -\frac{1}{2} \int_0^t ds q(X_s) \right) \right) \varphi_q(X_t) \right)$$

(since  $\varphi_q(X_t) \exp \left( -\frac{1}{2} A_t^q \right)$ ,  $t \geq 0$  is a martingale)

$$\begin{aligned} &= \int_{-\infty}^{\infty} E_x \left( \exp \left( -\frac{1}{2} A_t^q \right) | X_t = y \right) \frac{e^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}} \varphi_q(y) dy \\ &\underset{t \rightarrow \infty}{\sim} K \int_{-\infty}^{\infty} \frac{\varphi_q(x) \varphi_q(y)}{t} \varphi_q(y) \frac{e^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}} dy \\ &= \frac{K \varphi_q(x)}{t} E_x \left( \varphi_q^2(X_t) \right) \underset{t \rightarrow \infty}{\sim} \frac{K \varphi_q(x)}{t} \frac{2}{\pi} t \end{aligned}$$

since  $\varphi_q(z) \underset{|z| \rightarrow \infty}{\sim} \sqrt{\frac{2}{\pi}} |z|$ . Hence  $K \frac{2}{\pi} = 1$ , that is :  $K = \frac{\pi}{2}$ .

Thus, we have obtained point 1 of Theorem 3.1.

### 3.2.8 Point 2 of Theorem 3.1 may be proven

with the help of (3.3), exactly as Theorem 1.1.

**Remark 3.2** Under our hypothesis H1 on  $q$ , there is the equivalence :

$$Z(t, x, y) \equiv E_x \left( \exp -\frac{1}{2} \int_0^t q(X_s) ds | X_t = y \right) \underset{t \rightarrow \infty}{\sim} \frac{\pi}{2t} \varphi_q(x) \varphi_q(y) \quad (3.33)$$

Intuitively, we may think of the bridge of duration  $t$  going from  $x$  to  $y$  as "resembling", as  $t \rightarrow \infty$ , to the concatenation of two brownian motions each being defined on a time interval  $\left[0, \frac{t}{2}\right]$ , with the first one starting from  $x$  and the second one, after time reversal, starting from  $y$ , these two parts being independent. If this were true, then :

$$\begin{aligned} Z(t, x, y) &= E_x \left( \exp -\frac{1}{2} A_t^q | X_t = y \right) = E_x \left( \exp -\frac{1}{2} A_{t/2}^q \right) \cdot E_y \left( \exp -\frac{1}{2} A_{t/2}^q \right) \\ &\underset{t \rightarrow \infty}{\sim} \frac{\varphi_q(x)}{\sqrt{t/2}} \cdot \frac{\varphi_q(y)}{\sqrt{t/2}} = \frac{4}{\pi} \left( \frac{\pi}{2} \frac{\varphi_q(x) \varphi_q(y)}{t} \right) \end{aligned}$$

Thus, comparing with (3.33) the factor  $\frac{4}{\pi}$  which we just obtained measures, in some sense, the default of independence of these two Brownian components.

**Remark 3.3** Theorem 3.1 allows to "penalize long Brownian Bridges". More precisely, for every  $s \geq 0$  and  $\Lambda_s \in \mathcal{F}_s$  :

$$\frac{E_x \left( 1_{\Lambda_s} \exp \left( -\frac{1}{2} A_t^q \right) | X_t = y \right)}{E_x \left( \exp \left( -\frac{1}{2} A_t^q \right) | X_t = y \right)} \xrightarrow{t \rightarrow \infty} E_x (1_{\Lambda_s} M_s^\varphi) \quad (3.34)$$

with  $M_s^\varphi := \frac{\varphi_q(X_s)}{\varphi_q(x)} \exp\left(-\frac{1}{2} A_s^q\right)$ , et  $(M_s^\varphi, s \geq 0)$  is a positive martingale. In other terms, comparing with Theorem 5.1 in [RVY, I], the penalisation is the same for "long bridges" as for Brownian motion itself. Once more (see [RVY, III]), we obtain that a long bridge of duration  $t$ , as  $t \rightarrow \infty$ , behaves as a standard Brownian motion.

### 3.2.9 Finally, we show (3.34)

$$\begin{aligned} \frac{E_x\left(1_{\Lambda_s} \exp\left(-\frac{1}{2} A_t^q\right) | X_t = y\right)}{E_x\left(\exp\left(-\frac{1}{2} A_t^q\right) | X_t = y\right)} &= \frac{E_x\left(1_{\Lambda_s} \left(\exp\left(-\frac{1}{2} A_s^q\right)\right) E_{X_s, s}\left(\exp\left(-\frac{1}{2} \int_s^t q(X_u) du\right) | X_t = y\right)\right)}{E_x\left(\exp\left(-\frac{1}{2} A_t^q\right) | X_t = y\right)} \\ &\underset{t \rightarrow \infty}{\sim} \frac{E_x\left(1_{\Lambda_s} \exp\left(-\frac{1}{2} A_s^q\right) \cdot \frac{\varphi_q(X_s) \varphi_q(y)}{t-s}\right)}{\frac{\varphi_q(x) \varphi_q(y)}{t}} \underset{t \rightarrow \infty}{\longrightarrow} E_x(1_{\Lambda_s} M_s^\varphi). \end{aligned}$$

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