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► **To cite this version:**

D. Madan, Bernard Roynette, Marc Yor. An alternative expression for the Black-Scholes formula in terms of Brownian first and last passage times. 2008. hal-00257403

HAL Id: hal-00257403

<https://hal.science/hal-00257403>

Preprint submitted on 19 Feb 2008

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An alternative expression for the Black-Scholes formula in terms of Brownian first and last passage times

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February 11, 2008

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Abstract: The celebrated Black-Scholes formula which gives the price of a European option, may be expressed as the cumulative function of a last passage time of Brownian motion. A related result involving first passage times is also obtained.

Keywords: Black-Scholes formula, last passage times, time-inversion, Cameron-Martin relationship, reflection principle.

1 Introduction and main results

► **a)** Let $(B_t, t \geq 0)$ denote a one-dimensional Brownian motion starting from 0, and let

$$\mathcal{E}_t = \exp\left(B_t - \frac{t}{2}\right), \quad (t \geq 0)$$

A reduced form of the celebrated Black-Scholes formula is the following.

$$(1_{\mathbf{K}}) \quad E[(\mathcal{E}_t - K)^+] = \mathcal{N}\left(-\frac{\ln K}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right) - K \mathcal{N}\left(-\frac{\ln K}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right)$$

where $K \geq 0$, and as usual:

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x dy \exp\left(-\frac{y^2}{2}\right)$$

In fact, formula $(1_{\mathbf{K}})$ may be split into two parts:

$$(1_{\mathbf{K}}^+) \quad E[\mathcal{E}_t 1_{\mathcal{E}_t > K}] = \mathcal{N}\left(-\frac{\ln K}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right)$$

$$(1_{\mathbf{K}}^-) \quad K P(\mathcal{E}_t > K) = K \mathcal{N}\left(-\frac{\ln K}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right)$$

As noted in Section 5 of [1], formula $(1_{\mathbf{K}}^\pm)$ is obtained in an elementary manner, after performing the change of probability:

$$P'_{|\mathcal{F}_t} = \mathcal{E}_t \bullet P_{|\mathcal{F}_t}$$

which transforms (B_t) in $(B_t + t)$ hence $(B_t - \frac{t}{2})$ in $(B_t + \frac{t}{2})$.

► **b)** In this Note, we give and discuss a different representation of $(1_{\mathbf{K}}^\pm)$.

Theorem 1. *For any $K \geq 0$, there are the representations:*

$$(2_{\mathbf{K}}^-) \quad E[\mathcal{E}_t 1_{(\mathcal{E}_t > K)}] - K P(\mathcal{E}_t > K) = P\left(G_{(\ln K)}^{(1/2)} \leq t\right) \quad (K \geq 0)$$

$$(2_{\mathbf{K} \geq 1}^+) \quad E[\mathcal{E}_t 1_{\mathcal{E}_t > K}] + K P(\mathcal{E}_t > K) = P\left(T_{(\ln K)}^{(1/2)} \leq t\right) \quad (K \geq 1)$$

$$(2_{\mathbf{K} \leq 1}^+) \quad E[\mathcal{E}_t 1_{\mathcal{E}_t < K}] + K P(\mathcal{E}_t < K) = P\left(T_{(\ln K)}^{(1/2)} \leq t\right) \quad (0 \leq K \leq 1)$$

or, equivalently:

$$(\mathbf{2}_{\mathbf{K} \leq 1}^{++}) \quad E[\mathcal{E}_t 1_{\mathcal{E}_t > K}] + K P(\mathcal{E}_t > K) = K + P\left(T_{(\ell n K)}^{(1/2)} > t\right) \quad (0 \leq K \leq 1)$$

where, for $\nu \in \mathbb{R}$, and $B_t^{(\nu)} \equiv B_t + \nu t$, we write:

$$T_a^{(\nu)} = \inf\{t : B_t^{(\nu)} = a\}; \quad G_a^{(\nu)} = \sup\{t : B_t^{(\nu)} = a\}$$

Comment and Complements about Theorem 1:

(i) Our motivation to prove formulae such as $(\mathbf{2}_{\mathbf{K}}^-)$ was our desire to obtain an expression on the RHS showing in a clear manner that the LHS ($= E((\mathcal{E}_t - K)_+)$) is an increasing function of t .

This is not clear from $(\mathbf{1}_{\mathbf{K}})$, although this property of increase is a consequence of the submartingale property of $(\mathcal{E}_t - K)^+$; see Section 4 for a more extended discussion.

(ii) Obviously, an equivalent presentation of the "system" $(\mathbf{2}_{\mathbf{K}}^\pm)$ is, for $K \geq 1$:

$$(\mathbf{3}_{\mathbf{K} \geq 1}^+) \quad E[\mathcal{E}_t 1_{(\mathcal{E}_t > K)}] = \frac{1}{2} \left\{ P\left(T_{(\ell n K)}^{(1/2)} \leq t\right) + P\left(G_{(\ell n K)}^{(1/2)} \leq t\right) \right\}$$

$$\begin{aligned} (\mathbf{3}_{\mathbf{K} \geq 1}^-) \quad K P(\mathcal{E}_t > K) &= \frac{1}{2} \left\{ P\left(T_{(\ell n K)}^{(1/2)} \leq t\right) - P\left(G_{(\ell n K)}^{(1/2)} \leq t\right) \right\} \\ &= \frac{1}{2} P\left(T_{(\ell n K)}^{(1/2)} \leq t \leq G_{(\ell n K)}^{(1/2)}\right) \end{aligned}$$

and, for $0 \leq K \leq 1$:

$$(\mathbf{3}_{\mathbf{K} \leq 1}^+) \quad E[\mathcal{E}_t 1_{(\mathcal{E}_t > K)}] = \frac{1}{2} \left\{ 1 + K - P\left(T_{(\ell n K)}^{(1/2)} \leq t \leq G_{(\ell n K)}^{(1/2)}\right) \right\}$$

$$(\mathbf{3}_{\mathbf{K} \leq 1}^-) \quad K P(\mathcal{E}_t > K) = \frac{1}{2} \left\{ 1 + K - \left[P\left(T_{(\ell n K)}^{(1/2)} \leq t\right) + P\left(G_{(\ell n K)}^{(1/2)} \leq t\right) \right] \right\}$$

(iii) In order to give formulae $(\mathbf{2}_{\mathbf{K}}^\pm)$ an "explicit" character, we now recall the distributions of

$$T_a^{(\nu)} = \inf\{t : B_t^{(\nu)} = a\} \text{ and } G_a^{(\nu)} = \sup\{t : B_t^{(\nu)} = a\},$$

for $\nu > 0$, and $a > 0$ (these formulae will then be used with $\nu = \frac{1}{2}$ and $a = \ell n K$): denoting by $p_t^{(\nu)}(a)$ the density of $B_t^{(\nu)}$, we have:

$$(4) \quad P(T_a^{(\nu)} \in dt) = \left(\frac{a}{t}\right) p_t^{(\nu)}(a) dt \equiv \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}(a - \nu t)^2\right) dt$$

whereas

$$(5) \quad P(G_a^{(\nu)} \in dt) = \nu p_t^{(\nu)}(a) dt \equiv \frac{\nu}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(a - \nu t)^2\right) dt$$

Formula (4) may be obtained from the combination of (4) for $\nu = 0$, which is very well known, followed by an application of the Cameron-Martin relationship between the laws of $B^{(\nu)}$ and B . Formula (5) is a particular case of a more general formula for last passage times of transient diffusions, obtained in Pitman-Yor [3].

(iv) Although this is not strictly necessary at this point (but will be useful in our proof of Theorem 1), we also present the distributions of $T_a^{(-\nu)}$ and $G_a^{(-\nu)}$, for $a \geq 0$.

In fact, they may be obtained easily from those of $T_a^{(\nu)}$ and $G_a^{(\nu)}$ thanks to the Cameron-Martin absolute continuity relationships:

$$(6) \quad \begin{aligned} W_{|\mathcal{F}_{T_a} \cap (T_a < \infty)}^{(-\nu)} &= \exp(-2\nu a) \cdot W_{|\mathcal{F}_{T_a}}^{(\nu)} \\ W_{|\mathcal{F}_{G_a} \cap (G_a > 0)}^{(-\nu)} &= \exp(-2\nu a) \cdot W_{|\mathcal{F}_{G_a}}^{(\nu)} \end{aligned}$$

where, here, $W^{(\mu)}$ denotes the law of $(B_t + \mu t, t \geq 0)$ on canonical space, and T_a , resp. G_a , is the first, resp. last, hitting time of a by the coordinate process.

Thus, we deduce from formulae (6), (4) and (5) that:

$$P(G_a^{(-\nu)} > 0) = P(T_a^{(-\nu)} < \infty) = \exp(-2\nu a)$$

whereas:

$$(7) \quad \begin{aligned} P(T_a^{(-\nu)} \in dt) &= \left(\frac{a}{t}\right) p_t^{(-\nu)}(a) dt = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}(a + \nu t)^2\right) dt \\ P(G_a^{(-\nu)} \in dt) &= \nu p_t^{(-\nu)}(a) dt = \frac{\nu}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(a + \nu t)^2\right) dt \end{aligned}$$

► c) Organisation of the remainder of the paper:

- In Section 2, we prove Theorem 1, independently from formulae $(\mathbf{1}_K^\pm)$
- In Section 3, we give an elementary proof of the agreement between formulae $(\mathbf{1}_K^\pm)$ and $(\mathbf{3}_K^\pm)$
- Section 4 concludes, by setting the matter in a broader context.

2 Proof of Theorem 1

Clearly, in order to prove Theorem 1, it now suffices to prove $(\mathbf{2}_{\mathbf{K}}^-)$, $(\mathbf{3}_{\mathbf{K} \geq 1}^-)$ and $(\mathbf{2}_{\mathbf{K} \leq 1}^+)$.

► **a) Proof of $(\mathbf{2}_{\mathbf{K}}^-)$** (for any $K \geq 0$)

We shall show that

$$E[(\mathcal{E}_t - K)^+] \stackrel{(i)}{=} K P\left(0 < G_{(\ell n K)}^{(-1/2)} \leq t\right) \stackrel{(ii)}{=} P\left(G_{(\ell n K)}^{(+1/2)} \leq t\right)$$

The proof of (ii) follows from the relationship between the laws of $G_a^{(-\nu)}$ and $G_a^{(\nu)}$ as discussed in Section 1; namely:

$$P(G_a^{(-\nu)} > 0) = \exp(-2\nu a) \text{ and } P(G_a^{(-\nu)} \in dt | G_a^{(-\nu)} > 0) = P(G_a^{(\nu)} \in dt)$$

For the proof of (i), we rely upon the following formula

$$(8) \quad P(G_a^{(\mu)} \geq t | \mathcal{F}_t) = \left(\frac{\exp(2\mu a)}{\exp(2\mu B_t^{(\mu)})} \right) \wedge 1$$

which is valid for all $\mu \in \mathbb{R}$; this is a particular case of the results for last passage times of a transient real-valued diffusion, as discussed in Pitman-Yor [3].

In particular, for $\mu = -\nu$, $\nu > 0$, we get:

$$\exp(2\nu a) P(0 < G_a^{(-\nu)} \leq t | \mathcal{F}_t) = (\exp(2\nu B_t^{(-\nu)}) - \exp(2\nu a))^+$$

This obviously proves (i), by taking $a = (\ell n K)$, $\nu = 1/2$.

► **b) Proof of $(\mathbf{3}_{\mathbf{K} \geq 1}^-)$** (for $K \geq 1$)

Using again formula (8), we see that $(\mathbf{3}_{\mathbf{K} \geq 1}^-)$ is equivalent to:

$$(9_{\mathbf{K}}) \quad K P\left(B_t - \frac{t}{2} > (\ell n K)\right) = \frac{1}{2} E\left(1_{(T_{(\ell n K)}^{(1/2)} \leq t)} \cdot \left(\frac{K}{\exp(B_t + \frac{1}{2})} \wedge 1\right)\right)$$

We now use the Cameron-Martin relationship on both sides to reduce the statement of $(9_{\mathbf{K}})$ to a statement about standard Brownian motion (B_t) , for which we denote: $M_t = \sup_{s \leq t} B_s$. Thus, we find that $(9_{\mathbf{K}})$ is equivalent to:

$$(10_{\mathbf{K}}) \quad K E\left(1_{(B_t > (\ell n K))} \cdot \exp\left(-\frac{B_t}{2}\right)\right)$$

$$= \frac{1}{2} E \left(\mathbf{1}_{(M_t > (\ell n K))} \cdot \left(\frac{K}{\exp(B_t)} \wedge 1 \right) \cdot \exp \left(\frac{B_t}{2} \right) \right)$$

We now decompose the RHS of $(\mathbf{10}_K)$ in a sum of two quantities:

$$\frac{1}{2} \left\{ \begin{array}{l} E \left[\mathbf{1}_{(M_t > (\ell n K))} \cdot \mathbf{1}_{(B_t > \ell n K)} \cdot K \exp \left(-\frac{B_t}{2} \right) \right] \\ + E \left[\mathbf{1}_{(M_t > \ell n K)} \cdot \mathbf{1}_{(B_t < \ell n K)} \cdot \exp \left(+\frac{B_t}{2} \right) \right] \end{array} \right\}$$

Thus, $(\mathbf{10}_K)$ now gets simplified to the equivalent form:

$$\begin{aligned} (\mathbf{11}_K) \quad & \frac{K}{2} E \left(\mathbf{1}_{(B_t > (\ell n K))} \cdot \exp \left(-\frac{B_t}{2} \right) \right) \\ & = \frac{1}{2} E \left(\mathbf{1}_{(M_t > (\ell n K))} \cdot \mathbf{1}_{(B_t < \ell n K)} \cdot \exp \left(\frac{B_t}{2} \right) \right) \end{aligned}$$

which, taking $x = (\ell n K)$, may be written as:

$$(\mathbf{12}_x) \quad E \left(\mathbf{1}_{(B_t > x)} \cdot \exp \left(x - \frac{B_t}{2} \right) \right) = E \left(\mathbf{1}_{(M_t > x > B_t)} \cdot \exp \left(\frac{B_t}{2} \right) \right)$$

We now show $(\mathbf{12}_x)$, from the right to the left, as a consequence of the reflection principle:

conditionally on \mathcal{F}_{T_x} , and $T_x < t$, we have:

$$B_t - x = \widehat{B}_{(t-T_x)}, \text{ with } \widehat{B} \text{ independent from } \mathcal{F}_{T_x};$$

hence, under this condition, the reflection principle boils down to:

$$(\mathbf{13}) \quad B_t - x \stackrel{(\text{law})}{=} -(B_t - x)$$

Thus, the RHS of $(\mathbf{12}_x)$ is:

$$\begin{aligned} & E \left(\mathbf{1}_{(T_x < t)} \cdot \mathbf{1}_{(B_t - x < 0)} \cdot \exp \left(\frac{1}{2} \{x + (B_t - x)\} \right) \right) \\ & \stackrel{(\text{from } (\mathbf{13}))}{=} E \left(\mathbf{1}_{(T_x < t)} \cdot \mathbf{1}_{(B_t - x < 0)} \cdot \exp \left(\frac{1}{2} \{x - (B_t - x)\} \right) \right) \\ & = E \left(\mathbf{1}_{(B_t > x)} \cdot \exp \left(x - \frac{B_t}{2} \right) \right), \text{ which is the LHS of } (\mathbf{12}_x) \end{aligned}$$

This proves $(\mathbf{3}_{K \geq 1})$, and, with $(\mathbf{2}_K^-)$, $(\mathbf{2}_{K \geq 1}^+)$.

► **c)** We now prove that $(\mathbf{2}_{K \geq 1}^+)$ implies $(\mathbf{2}_{K \leq 1}^+)$ (for $0 \leq K \leq 1$):
We introduce the probability P' such that:

$$P'_{\mathcal{F}_t} = \mathcal{E}_t \bullet P_{\mathcal{F}_t}$$

We note that, under P' , $\frac{1}{\mathcal{E}_t} := \widehat{\mathcal{E}}_t = \exp\left(\widehat{B}_t - \frac{t}{2}\right)$, for a new Brownian motion $(\widehat{B}_t, t \geq 0)$. Thus, the LHS of $(\mathbf{2}_{K \leq 1}^+)$ writes:

$$\begin{aligned} & P'(\mathcal{E}_t < K) + K P'(\widehat{\mathcal{E}}_t - 1_{(\mathcal{E}_t < K)}) \\ = & P'\left(\widehat{\mathcal{E}}_t > \frac{1}{K}\right) + K E'\left(1_{(\widehat{\mathcal{E}}_t > \frac{1}{K})}\right) \\ = & K E'\left(1_{(\widehat{\mathcal{E}}_t > \frac{1}{K})}\right) + \frac{1}{K} P'\left(\widehat{\mathcal{E}}_t > \frac{1}{K}\right) \\ = & K P\left(T_{(\ln K)}^{(1/2)} \leq t\right) \quad (\text{from } (\mathbf{2}_{\frac{1}{K}}^+)) \\ = & K P\left(T_{(\ln K)}^{(-1/2)} \leq t\right) \quad (\text{by symmetry}) \\ = & P\left(T_{(\ln K)}^{(1/2)} \leq t\right) \quad (\text{from (6)}) \end{aligned}$$

► **d)** Finally, we observe that $(\mathbf{2}_{K \leq 1}^+)$ is equivalent to $(\mathbf{2}_{K \leq 1}^{++})$, since:

$$\begin{aligned} & E(\mathcal{E}_t 1_{(\mathcal{E}_t < K)}) + K P(\mathcal{E}_t < K) \\ = & 1 - E(\mathcal{E}_t 1_{(\mathcal{E}_t > K)}) + K(1 - P(\mathcal{E}_t > K)) \quad (\text{since } E(\mathcal{E}_t) = 1) \\ = & 1 + K - \{E(\mathcal{E}_t 1_{(\mathcal{E}_t > K)}) + K P(\mathcal{E}_t > K)\} \end{aligned}$$

3 On the agreement between the classical Black-Scholes formula $(\mathbf{1}_{\mathbf{K}}^\pm)$ and our main result

► **a)** We now check in an elementary manner formulae $(\mathbf{2}_{\mathbf{K}}^\pm)$ by comparing their LHS, as given from the "traditional" Black-Scholes formulae $(\mathbf{1}_{\mathbf{K}}^\pm)$, with their RHS, as given by (4) and (5).

► **b)** The case $K \geq 1$. Since both sides of $(\mathbf{2}_{\mathbf{K}}^-)$ and $(\mathbf{2}_{K \geq 1}^+)$ are equal to 0 for $t = 0$, we need only check that the derivatives in t are equal; thus, our task is to show:

$$(13_{\mathbf{K}}^-) \quad \frac{\partial}{\partial t} \left\{ \mathcal{N}\left(-\frac{\ln K}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right) - K \mathcal{N}\left(-\frac{\ln K}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right) \right\}$$

$$\begin{aligned}
&= \left(\frac{K}{2}\right) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t} \left((\ell n K) + \frac{t}{2}\right)^2\right) \\
(13_{\mathbf{K} \geq 1}^+) \quad &\frac{\partial}{\partial t} \left\{ \mathcal{N}\left(-\frac{\ell n K}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right) + K \mathcal{N}\left(-\frac{\ell n K}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right) \right\} \\
&= K(\ell n K) \frac{1}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t} \left((\ell n K) + \frac{t}{2}\right)^2\right)
\end{aligned}$$

(We also see on these expressions the relationships between $p_t^{(-1/2)}(x)$ and $p_t^{(+1/2)}(x)$).

To prove $(13_{\mathbf{K}}^-)$ and $(13_{\mathbf{K} \geq 1}^+)$, we compute:

$$\begin{aligned}
&\bullet \quad \frac{\partial}{\partial t} \left(\mathcal{N}\left(-\frac{\ell n K}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right) \right) \\
&= \frac{\sqrt{K}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{(\ell n K)^2}{t} + \frac{t}{4}\right)\right) \left(\frac{\partial}{\partial t} \left\{ -\frac{(\ell n K)}{\sqrt{t}} + \frac{\sqrt{t}}{2} \right\} \right) \\
&\bullet \quad K \frac{\partial}{\partial t} \left(\mathcal{N}\left(-\frac{\ell n K}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right) \right) \\
&= \frac{\sqrt{K}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{(\ell n K)^2}{t} + \frac{t}{4}\right)\right) \left(\frac{\partial}{\partial t} \left\{ -\frac{(\ell n K)}{\sqrt{t}} - \frac{\sqrt{t}}{2} \right\} \right)
\end{aligned}$$

and $(13_{\mathbf{K}}^-)$ and $(13_{\mathbf{K} \geq 1}^+)$ are then obtained by elementary algebraic manipulations. (In fact, it is these very manipulations which led us to believe in the truth of Theorem 1!!).

► **c)** The case $0 \leq K \leq 1$. Since both sides of $(2_{K \leq 1}^{++})$ are equal to $1 + K$ for $t = 0$, it suffices to prove, for $K \leq 1$:

$$\begin{aligned}
&\frac{\partial}{\partial t} \left\{ \mathcal{N}\left(-\frac{\log K}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right) + K \mathcal{N}\left(-\frac{\log K}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right) \right\} \\
&= \frac{\partial}{\partial t} P\left(T_{(\ell n K)}^{(1/2)} > t\right)
\end{aligned}$$

and this latter relation follows immediately from the computations done in point **b)** above and from (7).

Note also that $(2_{K \geq 1}^+)$ and $(2_{K \leq 1}^+)$ coincide for $K = 1$.

4 Further remarks and conclusion

► **a)** Easy variants of formulae $(\mathbf{2}_{\mathbf{K}}^{\pm})$ may be written, e.g. by using the scaling property of Brownian motion, so that $T_a^{(\nu)}$ and $G_a^{(\nu)}$ appear on the RHS of $(\mathbf{2}_{\mathbf{K}}^{\pm})$; however, writing down these variants would only complicate unnecessarily these formulae.

► **b)** We recall that, as a consequence of the time inversion property of Brownian motion, there are the relations:

$$(T_a^{(\nu)}, G_a^{(\nu)}) \stackrel{(\text{law})}{=} \left(\frac{1}{G_\nu^{(a)}}, \frac{1}{T_\nu^{(a)}} \right)$$

(see e.g., Pitman-Yor [3] for a more general discussion).

In particular, for $a = 0$, one gets:

$$G_0^{(\nu)} \stackrel{(\text{law})}{=} \frac{1}{T_\nu^{(0)}} \stackrel{(\text{law})}{=} \frac{B_1^2}{\nu^2}$$

In particular, formula $(\mathbf{2}_{\mathbf{K}=1}^-)$ becomes:

$$(14_1) \quad E[(\mathcal{E}_t - 1)^+] = E[(\mathcal{E}_t - 1)^-] = P(4B_1^2 \leq t)$$

which allowed us to answer M. Qian's question [4]:

is there a simple formula for:

$$\int_0^\infty \theta(dt) E[(\mathcal{E}_t - 1)^\pm]$$

where $\theta(dt)$ is a probability on \mathbb{R}_+ ?

From (14_1) , we easily obtain:

$$(15) \quad \int_0^\infty \theta(dt) E[(\mathcal{E}_t - 1)^\pm] = E[\bar{\theta}(4B_1^2)]$$

where $\bar{\theta}(x) = \theta([x, \infty))$ is the tail of θ .

To particularise even more, we give the explicit form of the Laplace transform:

$$(16) \quad \int_0^\infty dt e^{-\lambda t} E[(\mathcal{E}_t - 1)^\pm] = \frac{1}{\lambda} E[\exp(-\lambda(4B_1^2))] \\ = \frac{1}{\lambda} \frac{1}{\sqrt{1 + 8\lambda}}$$

It is this question which set us on the general quest for a representation of

$$E[(\mathcal{E}_t - K)^+]$$

as a cumulative distribution function in t .

► **c)** We come back to the time inversion property of BM, in order to throw another light upon our main result $(\mathbf{2}_K^-)$, which relates the European call price with the cumulative function of last Brownian passage times. (This paragraph has been partly inspired by unpublished notes by Peter Carr [2].) Indeed, a variant of $(\mathbf{2}_K^-)$ is the following:

for every $t \geq 0$, $K \geq 0$, and $\phi : C([0, t]) \rightarrow \mathbb{R}_+$, measurable,

$$(17) \quad E[\phi(B_u, u \leq t)(K - \mathcal{E}_t)^+] = K E[\phi(B_u, u \leq t)1_{(\mathcal{G}_K \leq t)}]$$

where $\mathcal{G}_K = \sup\{u : \mathcal{E}_u = K\}$.

Writing (17) in terms of the Brownian motion $(\widehat{B}_v, v \geq 0)$ such that: $B_u = u\widehat{B}_{(1/u)}$, and setting $s = 1/t$, it is clearly seen that (17) is equivalent to:

$$K P(\widehat{T}_{1/2}^{(-\ell n K)} \geq s | \widehat{B}_s) = \left(K - \exp\left(\frac{1}{s}\widehat{B}_s - \frac{1}{2s}\right) \right)^+$$

where: $\widehat{T}_a^{(\nu)} = \inf\{u : \widehat{B}_u + \nu u = a\}$.

Since hats are no longer necessary for our purpose, we drop them, and we now look for an independent proof of:

$$(18) \quad P(T_{1/2}^{(-\ell n K)} \geq s | B_s = x) = \left(1 - \frac{1}{K} \exp\left(\frac{x}{s} - \frac{1}{2s}\right) \right)^+$$

On the LHS of (18), we may replace $(B_s = x)$ by $(B_s - s(\ell n K) = x - s(\ell n K))$. Now, as a consequence of the Cameron-Martin relationship, the conditional expectation:

$$E[F(B_u - \nu u, u \leq s) | B_s - \nu s = y]$$

does not depend on ν ; hence, (18) is equivalent to:

$$P(T_{1/2}^{(-\ell n K)} \geq s | B_s = x - s(\ell n K)) = \left(1 - \frac{1}{K} \exp\left(\frac{x - \frac{1}{2}}{s}\right) \right)^+$$

which simplifies to:

$$P(\sup_{u \leq s} B_u < \frac{1}{2} | B_s = y) = \left(1 - \exp\left(\frac{y - \frac{1}{2}}{s}\right) \right)^+$$

or, by scaling:

$$P(\sup_{u \leq 1} B_u < \frac{1}{2\sqrt{s}} | B_1 = \frac{y}{\sqrt{s}}) = \left(1 - \exp\left(\frac{1}{\sqrt{s}} \left(\frac{y}{\sqrt{s}} - \frac{1}{2\sqrt{s}}\right)\right)\right)^+$$

This is equivalent to:

$$(19) \quad P(\sup_{u \leq 1} B_u < \sigma | B_1 = y) = (1 - \exp(2\sigma(y - \sigma)))^+$$

for $\sigma \geq 0$, and $y \in \mathbb{R}$.

This formula is trivial for $\sigma < y$, and, for $\sigma \geq y$, it follows from the classical formula:

$$(20) \quad P(\sup_{u \leq 1} B_u \in d\sigma, B_1 \in da) = \frac{da d\sigma}{\sqrt{2\pi}} 2(2\sigma - a) e^{-\frac{(2\sigma - a)^2}{2}} 1_{\{a < \sigma; \sigma \geq 0\}}$$

► **d)** In a future work, we plan to study more generally how quantities such as the calls and puts:

$$E[(S_t - K)^+] \text{ and } E[(S_t - K)^-]$$

associated with a general \mathbb{R}_+ -valued continuous local martingale $(S_t, t \geq 0)$ may be written in terms of cumulative functions.

► **e)** In [1], the authors present eight different approaches to the Black-Scholes formula, among which the change of numéraire approach (Section 5 of [1]), and the local time approach (Section 6 of [1]). This local time approach, together with $(\mathbf{2}_{\mathbf{K}}^-)$ yields the relationship:

$$(21) \quad P\left(G_{(\ell_n K)}^{(1/2)} \leq t\right) = \left(\frac{1}{2}\right) E[\mathcal{L}_t^K]$$

where $(\mathcal{L}_t^K, t \geq 0)$ denotes the local time at level K of $(\mathcal{E}_t, t \geq 0)$. It is this kind of relationship $(\mathbf{21})$ which is central in the obtention in [3] of a general expression for the law of a last passage time of a transient diffusion. However, to our knowledge, despite the remarkable survey [1] of methods leading to the Black-Scholes formula, no interpretation of this formula seems to have been made in terms of last passage times distributions.

References

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Acknowledgments: We are grateful to M. Qian for raising the question mentioned in Section 4, and to T. Fujita for pointing out reference [1].