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An alternative expression for the Black-Scholes formula in terms of Brownian first and last passage times

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Abstract: The celebrated Black-Scholes formula which gives the price of a European option, may be expressed as the cumulative function of a last passage time of Brownian motion. A related result involving first passage times is also obtained.

Keywords: Black-Scholes formula, last passage times, time-inversion, Cameron-Martin relationship, reflection principle.
1 Introduction and main results

a) Let \((B_t, t \geq 0)\) denote a one-dimensional Brownian motion starting from 0, and let

\[ E_t = \exp \left( B_t - \frac{t}{2} \right), \quad (t \geq 0) \]

A reduced form of the celebrated Black-Scholes formula is the following.

\[ E[(E_t - K)^+] = N \left( -\frac{\elln K}{\sqrt{t}} + \frac{\sqrt{t}}{2} \right) - K N \left( -\frac{\elln K}{\sqrt{t}} - \frac{\sqrt{t}}{2} \right) \]

where \(K \geq 0\), and as usual:

\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dy \exp \left( -\frac{y^2}{2} \right) \]

In fact, formula \((1_K)^+\) may be split into two parts:

\[ E[\mathcal{E}_t 1_{E_t > K}] = N \left( -\frac{\elln K}{\sqrt{t}} + \frac{\sqrt{t}}{2} \right) \]

\[ K P(E_t > K) = K N \left( -\frac{\elln K}{\sqrt{t}} - \frac{\sqrt{t}}{2} \right) \]

As noted in Section 5 of [1], formula \((1_K)^+\) is obtained in an elementary manner, after performing the change of probability:

\[ P'_{F_t} = \mathcal{E}_t \ast P_{F_t} \]

which transforms \((B_t)\) in \((B_t + t)\) hence \((B_t - \frac{t}{2})\) in \((B_t + \frac{t}{2})\).

b) In this Note, we give and discuss a different representation of \((1_K)^+\).

**Theorem 1.** For any \(K \geq 0\), there are the representations:

\[ E[\mathcal{E}_t 1_{E_t > K}] - K P(E_t > K) = P \left( G^{(1/2)}_{(\elln K)} \leq t \right) \quad (K \geq 0) \]

\[ E[\mathcal{E}_t 1_{E_t > K}] + K P(E_t > K) = P \left( T^{(1/2)}_{(\elln K)} \leq t \right) \quad (K \geq 1) \]

\[ E[\mathcal{E}_t 1_{E_t < K}] + K P(E_t < K) = P \left( T^{(1/2)}_{(\elln K)} \leq t \right) \quad (0 \leq K \leq 1) \]
or, equivalently:

\[(2 \pm_K) \quad E[\mathcal{E}_t 1_{\mathcal{E}_t > K}] + K P(\mathcal{E}_t > K) = K + P\left(T^{(1/2)}_{(\ell n K)} > t\right) \quad (0 \leq K \leq 1)\]

where, for \(\nu \in \mathbb{R}\), and \(B_t^{(\nu)} \equiv B_t + \nu t\), we write:

\[T_a^{(\nu)} = \inf\{t : B_t^{(\nu)} = a\}; \quad G_a^{(\nu)} = \sup\{t : B_t^{(\nu)} = a\}\]

Comment and Complements about Theorem 1:

(i) Our motivation to prove formulae such as \((2 \pm_K)\) was our desire to obtain an expression on the RHS showing in a clear manner that the LHS \((= E((\mathcal{E}_t - K)_+))\) is an increasing function of \(t\). This is not clear from \((1_K)\), although this property of increase is a consequence of the submartingale property of \((\mathcal{E}_t - K)^+\); see Section 4 for a more extended discussion.

(ii) Obviously, an equivalent presentation of the "system" \((2 \pm_K)\) is, for \(K \geq 1:\)

\[(3^+_K \geq 1) \quad E[\mathcal{E}_t 1_{\mathcal{E}_t > K}] = \frac{1}{2} \left\{ P\left(T^{(1/2)}_{(\ell n K)} \leq t\right) + P\left(G^{(1/2)}_{(\ell n K)} \leq t\right) \right\}\]

\[(3^-_{K \geq 1}) \quad K P(\mathcal{E}_t > K) = \frac{1}{2} \left\{ P\left(T^{(1/2)}_{(\ell n K)} \leq t\right) - P\left(G^{(1/2)}_{(\ell n K)} \leq t\right) \right\}
\]

\[= \frac{1}{2} P\left(T^{(1/2)}_{(\ell n K)} \leq t \leq G^{(1/2)}_{(\ell n K)}\right)\]

and, for \(0 \leq K \leq 1:\)

\[(3^+_K \leq 1) \quad E[\mathcal{E}_t 1_{\mathcal{E}_t > K}] = \frac{1}{2} \left\{ 1 + K - P\left(T^{(1/2)}_{(\ell n K)} \leq t \leq G^{(1/2)}_{(\ell n K)}\right) \right\}\]

\[(3^-_{K \leq 1}) \quad K P(\mathcal{E}_t > K) = \frac{1}{2} \left\{ 1 + K - \left[ P\left(T^{(1/2)}_{(\ell n K)} \leq t\right) + P\left(G^{(1/2)}_{(\ell n K)} \leq t\right) \right] \right\}\]

(iii) In order to give formulae \((2 \pm_K)\) an "explicit" character, we now recall the distributions of

\[T_a^{(\nu)} = \inf\{t : B_t^{(\nu)} = a\}; \quad G_a^{(\nu)} = \sup\{t : B_t^{(\nu)} = a\}\]

for \(\nu > 0\), and \(a > 0\) (these formulae will then be used with \(\nu = \frac{1}{2}\) and \(a = \ell n K\)): denoting by \(p_t^{(\nu)}(a)\) the density of \(B_t^{(\nu)}\), we have:

\[P(T_a^{(\nu)} \in dt) = \left(\frac{a}{t}\right) p_t^{(\nu)}(a) dt \equiv \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}(a - \nu t)^2\right) dt\]
whereas

\[ P(G_a^{(\nu)} \in dt) = \nu p_t^{(\nu)}(a) dt \equiv \frac{\nu}{\sqrt{2\pi t}} \exp \left( -\frac{1}{2t} (a - \nu t)^2 \right) dt \]

Formula (4) may be obtained from the combination of (4) for \( \nu = 0 \), which is very well known, followed by an application of the Cameron-Martin relationship between the laws of \( B^{(\nu)} \) and \( B \). Formula (5) is a particular case of a more general formula for last passage times of transient diffusions, obtained in Pitman-Yor [3].

(iv) Although this is not strictly necessary at this point (but will be useful in our proof of Theorem 1), we also present the distributions of \( T_a^{(-\nu)} \) and \( G_a^{(-\nu)} \), for \( a \geq 0 \).

In fact, they may be obtained easily from those of \( T_a^{(\nu)} \) and \( G_a^{(\nu)} \) thanks to the Cameron-Martin absolute continuity relationships:

\[ W_t^{(-\nu)} |_{\mathcal{F}_{T_a} \cap (T_a < \infty)} = \exp(-2\nu a) \cdot W_t^{(\nu)} |_{\mathcal{F}_{T_a}} \]

\[ W_t^{(-\nu)} |_{\mathcal{F}_{G_a} \cap (G_a > 0)} = \exp(-2\nu a) \cdot W_t^{(\nu)} |_{\mathcal{F}_{G_a}} \]

where, here, \( W^{(\mu)} \) denotes the law of \((B_t + \mu t, t \geq 0)\) on canonical space, and \( T_a \), resp. \( G_a \), is the first, resp. last, hitting time of \( a \) by the coordinate process.

Thus, we deduce from formulae (6), (4) and (5) that:

\[ P(G_a^{(-\nu)} > 0) = P(T_a^{(-\nu)} < \infty) = \exp(-2\nu a) \]

whereas:

\[ P(T_a^{(-\nu)} \in dt) = \left( \frac{2}{t} \right) p_t^{(-\nu)}(a) dt = \frac{a}{\sqrt{2\pi t^3}} \exp \left( -\frac{1}{2t^2} (a + \nu t)^2 \right) dt \]

\[ P(G_a^{(-\nu)} \in dt) = \nu p_t^{(-\nu)}(a) dt = \frac{\nu}{\sqrt{2\pi t}} \exp \left( -\frac{1}{2t} (a + \nu t)^2 \right) dt \]

\[ \text{Organisation of the remainder of the paper:} \]

- In Section 2, we prove Theorem 1, independently from formulae (1⁺) and (1⁻)
- In Section 3, we give an elementary proof of the agreement between formulae (1⁺) and (3⁺)
- Section 4 concludes, by setting the matter in a broader context.
2 Proof of Theorem 1

Clearly, in order to prove Theorem 1, it now suffices to prove \((2^-_K), (3^-_{K\geq 1})\)
and \((2^+_K)\).

▶ a) Proof of \((2^-_K)\) (for any \(K \geq 0\))

We shall show that

\[
E[|\mathcal{E}_t - K|] = K P\left(0 < G^{(-1/2)}_{(\ell n K)} \leq t\right) = P\left(G^{(+1/2)}_{(\ell n K)} \leq t\right)
\]

The proof of \((ii)\) follows from the relationship between the laws of \(G^{(-\nu)}_a\) and \(G^{(\nu)}_a\) as discussed in Section 1; namely:

\[
P(G^{(-\nu)}_a > 0) = \exp(-2\nu a)\) and \(P(G^{(-\nu)}_a \in dt|G^{(-\nu)}_a > 0) = P(G^{(\nu)}_a \in dt)
\]

For the proof of \((i)\), we rely upon the following formula

\[
P(G^{(\mu)}_a \geq t|\mathcal{F}_t) = \left(\frac{\exp(2\mu a)}{\exp(2\mu B_t^{(\mu)})}\right) \wedge 1\)

which is valid for all \(\mu \in \mathbb{R}\); this is a particular case of the results for last passage times of a transient real-valued diffusion, as discussed in Pitman-Yor [3].

In particular, for \(\mu = -\nu, \nu > 0\), we get:

\[
\exp(2\nu a)P(0 < G^{(-\nu)}_a \leq t|\mathcal{F}_t) = (\exp(2\nu B_t^{(-\nu)}) - \exp(2\nu a))^+
\]

This obviously proves \((i)\), by taking \(a = (\ell n K), \nu = 1/2\).

▶ b) Proof of \((3^-_{K\geq 1})\) (for \(K \geq 1\))

Using again formula \((8)\), we see that \((3^-_{K\geq 1})\) is equivalent to:

\[
 KP\left(B_t - \frac{t}{2} > (\ell n K)\right) = \frac{1}{2} E\left(1_{(G^{(1/2)}_{(\ell n K)} \leq t)} \cdot \left(\frac{K}{\exp\left(B_t + \frac{1}{2}\right) \wedge 1}\right)\right)
\]

We now use the Cameron-Martin relationship on both sides to reduce the statement of \((9_K)\) to a statement about standard Brownian motion \((B_t)\), for which we denote: \(M_t = \sup_{s \leq t} B_s\). Thus, we find that \((9_K)\) is equivalent to:

\[
 K E\left(1_{(B_t > (\ell n K))} \cdot \exp\left(-\frac{B_t}{2}\right)\right)
\]
\[
\frac{1}{2} E \left( 1_{\{M_t > (\ln K)\}} \cdot \left( \frac{K}{\exp(B_t)} \wedge 1 \right) \cdot \exp \left( \frac{B_t}{2} \right) \right)
\]

We now decompose the RHS of \((10_K)\) in a sum of two quantities:

\[
\frac{1}{2} \left\{ \begin{array}{l}
E \left[ 1_{\{M_t > (\ln K)\}} \cdot 1_{\{B_t > \ln K\}} \cdot K \cdot \exp \left( -\frac{B_t}{2} \right) \right] \\
+ E \left[ 1_{\{M_t > \ln K\}} \cdot 1_{\{B_t < \ln K\}} \cdot \exp \left( \frac{B_t}{2} \right) \right]
\end{array} \right. 
\]

Thus, \((10_K)\) now gets simplified to the equivalent form:

\[
\frac{K}{2} E \left( 1_{\{B_t > (\ln K)\}} \cdot \exp \left( -\frac{B_t}{2} \right) \right)
\]

which, taking \(x = (\ln K)\), may be written as:

\[
(12_x) \quad E \left( 1_{\{B_t > x\}} \cdot \exp \left( x - \frac{B_t}{2} \right) \right) = E \left( 1_{\{M_t > x > B_t\}} \cdot \exp \left( \frac{B_t}{2} \right) \right)
\]

We now show \((12_x)\), from the right to the left, as a consequence of the reflection principle:

conditionally on \(\mathcal{F}_{T_x}\), and \(T_x < t\), we have:

\[B_t - x = \hat{B}_{(t-T_x)}, \text{ with } \hat{B} \text{ independent from } \mathcal{F}_{T_x};\]

hence, under this condition, the reflection principle boils down to:

\[
(\mathcal{R}) \quad B_t - x \overset{(\text{law})}{=} -(B_t - x)
\]

Thus, the RHS of \((12_x)\) is:

\[
\begin{align*}
&= E \left( 1_{\{T_x < t\}} \cdot 1_{\{B_t - x < 0\}} \cdot \exp \left( \frac{1}{2} \{x + (B_t - x)\} \right) \right) \\
&= E \left( 1_{\{T_x < t\}} \cdot 1_{\{B_t - x < 0\}} \cdot \exp \left( \frac{1}{2} \{x - (B_t - x)\} \right) \right) \\
&= E \left( 1_{\{B_t > x\}} \cdot \exp \left( x - \frac{B_t}{2} \right) \right), \text{ which is the LHS of } (12_x)
\end{align*}
\]

This proves \((3_{K \geq 1})\), and, with \((2^-_K), (2^+_{K \geq 1})\).
c) We now prove that $(2_{K \geq 1}^+) \implies (2_{K \leq 1}^-)$ (for $0 \leq K \leq 1$):

We introduce the probability $P'$ such that:

$$P'_x = \mathcal{E}_t \cdot P_x$$

We note that, under $P'$, $\frac{1}{\mathcal{E}_t} := \hat{\mathcal{E}}_t = \exp \left( \hat{B}_t - \frac{t}{2} \right)$, for a new Brownian motion $(\hat{B}_t, t \geq 0)$. Thus, the LHS of $(2_{K \leq 1}^-)$ writes:

$$P' \left( \mathcal{E}_t < K \right) + K P' \left( \hat{\mathcal{E}}_t - 1_{(\mathcal{E}_t < K)} \right)$$

$$= P' \left( \hat{\mathcal{E}}_t > \frac{1}{K} \right) + K E' \left( 1_{(\hat{\mathcal{E}}_t > \frac{1}{K})} \right)$$

$$= K E' \left( 1_{(\hat{\mathcal{E}}_t > \frac{1}{K})} \right) + \frac{1}{K} P' \left( \hat{\mathcal{E}}_t > \frac{1}{K} \right)$$

$$= K P \left( T_{(\ln K)}^{(1/2)} \leq t \right) \quad \text{(from $(2_{K \leq 1}^-)$)}$$

$$= K P \left( T_{(\ln K)}^{(-1/2)} \leq t \right) \quad \text{(by symmetry)}$$

$$= P \left( T_{(\ln K)}^{(1/2)} \leq t \right) \quad \text{(from (6))}$$

d) Finally, we observe that $(2_{K \leq 1}^-)$ is equivalent to $(2_{K \leq 1}^{++})$; since:

$$E(\mathcal{E}_t 1_{(\mathcal{E}_t < K)}) + K P(\mathcal{E}_t < K)$$

$$= 1 - E(\mathcal{E}_t 1_{(\mathcal{E}_t > K)}) + K (1 - P(\mathcal{E}_t > K)) \quad \text{(since } E(\mathcal{E}_t) = 1)$$

$$= 1 + K - \{E(\mathcal{E}_t 1_{(\mathcal{E}_t > K)}) + K P(\mathcal{E}_t > K)\}$$

3 On the agreement between the classical Black-Scholes formula $(1_K^\pm)$ and our main result

a) We now check in an elementary manner formulae $(2_K^\pm)$ by comparing their LHS, as given from the "traditional" Black-Scholes formulae $(1_K^\pm)$, with their RHS, as given by (4) and (5).

b) The case $K \geq 1$. Since both sides of $(2_K^-)$ and $(2_{K \geq 1}^+)$ are equal to 0 for $t = 0$, we need only check that the derivatives in $t$ are equal; thus, our task is to show:

$$\frac{\partial}{\partial t} \left\{ N \left( -\frac{\ell n K}{\sqrt{t}} + \frac{\sqrt{t}}{2} \right) - K N \left( -\frac{\ell n K}{\sqrt{t}} - \frac{\sqrt{t}}{2} \right) \right\}$$

7
\[
\left( \frac{K}{2} \right) \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{1}{2t} \left( (\elln K) + t \right)^2 \right)
\]

\[\text{(13)}_{K\geq1}\]

\[
\frac{\partial}{\partial t} \left\{ N \left( -\frac{\elln K}{\sqrt{t}} + \frac{\sqrt{t}}{2} \right) + K N \left( -\frac{\elln K}{\sqrt{t}} - \frac{\sqrt{t}}{2} \right) \right\}
\]

\[= K (\elln K) \frac{1}{\sqrt{2\pi t^3}} \exp \left( -\frac{1}{2t} \left( (\elln K) + t \right)^2 \right)
\]

(We also see on these expressions the relationships between \(p_t(-1/2)(x)\) and \(p_t(+1/2)(x)\).)

To prove \((13)_K\) and \((13)_{K>1}\), we compute:

- \[
\frac{\partial}{\partial t} \left( N \left( -\frac{\elln K}{\sqrt{t}} + \frac{\sqrt{t}}{2} \right) \right)
\]

\[
= \frac{\sqrt{K}}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{(\elln K)^2}{t} + \frac{t}{4} \right) \right) \left( \frac{\partial}{\partial t} \left\{ -\frac{(\elln K)}{\sqrt{t}} + \frac{\sqrt{t}}{2} \right\} \right)
\]

- \[
K \frac{\partial}{\partial t} \left( N \left( -\frac{\elln K}{\sqrt{t}} - \frac{\sqrt{t}}{2} \right) \right)
\]

\[
= \frac{\sqrt{K}}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{(\elln K)^2}{t} + \frac{t}{4} \right) \right) \left( \frac{\partial}{\partial t} \left\{ -\frac{(\elln K)}{\sqrt{t}} - \frac{\sqrt{t}}{2} \right\} \right)
\]

and \((13)_K\) and \((13)_{K>1}\) are then obtained by elementary algebraic manipulations. (In fact, it is these very manipulations which led us to believe in the truth of Theorem 1!!!).

\[\text{c)}\) The case \(0 \leq K \leq 1\). Since both sides of \((2)_{K\leq1}^{++}\) are equal to \(1 + K\) for \(t = 0\), it suffices to prove, for \(K \leq 1\):

\[
\frac{\partial}{\partial t} \left\{ N \left( -\frac{\elln K}{\sqrt{t}} + \frac{\sqrt{t}}{2} \right) + K N \left( -\frac{\elln K}{\sqrt{t}} - \frac{\sqrt{t}}{2} \right) \right\}
\]

\[= \frac{\partial}{\partial t} P \left( T_{(\elln K)}>t \right)
\]

and this latter relation follows immediately from the computations done in point \(b)\) above and from \((7)\).

Note also that \((2)_{K>1}^-\) and \((2)_{K\leq1}^+\) coincide for \(K = 1\).
4 Further remarks and conclusion

▶ a) Easy variants of formulae \((2_{K}^{\pm})\) may be written, e.g. by using the scaling property of Brownian motion, so that \(T_{a}^{(\nu)}\) and \(G_{a}^{(\nu)}\) appear on the RHS of \((2_{K}^{\pm})\); however, writing down these variants would only complicate unnecessarily these formulae.

▶ b) We recall that, as a consequence of the time inversion property of Brownian motion, there are the relations:

\[
(T_{a}^{(\nu)}, G_{a}^{(\nu)}) \overset{\text{(law)}}{=} \left(\frac{1}{G_{\nu}^{(a)}}, \frac{1}{T_{\nu}^{(a)}}\right)
\]

(see e.g., Pitman-Yor [3] for a more general discussion).

In particular, for \(a = 0\), one gets:

\[
G_{0}^{(\nu)} \overset{\text{(law)}}{=} \frac{1}{T_{0}^{(0)}} \overset{\text{(law)}}{=} \frac{B_{1}^{2}}{\nu^{2}}
\]

In particular, formula \((2_{K=1}^{-})\) becomes:

\[
\int_{0}^{\infty} \theta(dt)E[(E_{t} - 1)^{\pm}] = E[(E_{t} - 1)^{-}] = P(4B_{1}^{2} \leq t)
\]

which allowed us to answer M. Qian’s question [4]: is there a simple formula for:

\[
\int_{0}^{\infty} \theta(dt)E[(E_{t} - 1)^{\pm}]
\]

where \(\theta(dt)\) is a probability on \(\mathbb{R}_{+}\)?

From \((14_{1})\), we easily obtain:

\[
\int_{0}^{\infty} \theta(dt)E[(E_{t} - 1)^{\pm}] = E[\bar{\theta}(4B_{1}^{2})]
\]

where \(\bar{\theta}(x) = \theta([x, \infty))\) is the tail of \(\theta\).

To particularise even more, we give the explicit form of the Laplace transform:

\[
\int_{0}^{\infty} dt e^{-\lambda t} E[(E_{t} - 1)^{\pm}] = \frac{1}{\lambda} E[\exp(-\lambda(4B_{1}^{2}))]
\]

\[
= \frac{1}{\lambda} \frac{1}{\sqrt{1 + 8\lambda}}
\]
It is this question which set us on the general quest for a representation of
\[ E[(\mathcal{E}_t - K)^+] \]
as a cumulative distribution function in \( t \).

\( \blacktriangleright \) c) We come back to the time inversion property of BM, in order to throw another light upon our main result \((2^\gamma_\nu)\), which relates the European call price with the cumulative function of last Brownian passage times. (This paragraph has been partly inspired by unpublished notes by Peter Carr [2].) Indeed, a variant of \((2^\gamma_\nu)\) is the following:

for every \( t \geq 0, \ K \geq 0 \), and \( \phi : C([0, t]) \to \mathbb{R}_+ \), measurable,

\[ E\left[ \phi(B_u, u \leq t)(K - \mathcal{E}_t)^+ \right] = K \ E\left[ \phi(B_u, u \leq t)1_{\mathcal{G}_K \leq t} \right] \]

where \( \mathcal{G}_K = \sup \{ u : E_u = K \} \).

Writing \((17)\) in terms of the Brownian motion \((\hat{B}_v, v \geq 0)\) such that: \( B_u = u\hat{B}_{1/u} \), and setting \( s = 1/t \), it is clearly seen that \((17)\) is equivalent to:

\[ K \ P(\hat{T}_{1/2}^{(-\ell nK)} \geq s|\hat{B}_s) = \left( K - \exp \left( \frac{1}{s} \hat{B}_s - \frac{1}{2s} \right) \right)^+ \]

where: \( \hat{T}_a^{(\nu)} = \inf \{ u : \hat{B}_u + \nu u = a \} \).

Since hats are no longer necessary for our purpose, we drop them, and we now look for an independent proof of:

\[ P(\hat{T}_{1/2}^{(-\ell nK)} \geq s|B_s = x) = \left( 1 - \exp \left( \frac{x}{s} - \frac{1}{2s} \right) \right)^+ \]

On the LHS of \((18)\), we may replace \((B_s = x)\) by \((B_s - s(\ell nK) = x - s(\ell nK))\). Now, as a consequence of the Cameron-Martin relationship, the conditional expectation:

\[ E[F(B_u - \nu u, u \leq s)|B_s - \nu s = y] \]
does not depend on \( \nu \); hence, \((18)\) is equivalent to:

\[ P(\hat{T}_{1/2}^{(-\ell nK)} \geq s|B_s = x - s(\ell nK)) = \left( 1 - \exp \left( \frac{x - \frac{y}{s}}{s} \right) \right)^+ \]

which simplifies to:

\[ P(\sup_{u \leq s} B_u < \frac{1}{2}|B_s = y) = \left( 1 - \exp \left( \frac{y - \frac{1}{s}}{s} \right) \right)^+ \]
or, by scaling:

\[ P(\sup_{u \leq 1} B_u < \frac{1}{2\sqrt{s}} | B_1 = \frac{y}{\sqrt{s}}) = \left(1 - \exp \left(\frac{1}{\sqrt{s}} \left(\frac{y}{\sqrt{s}} - \frac{1}{2\sqrt{s}}\right)\right)\right)^+ \]

This is equivalent to:

\[ P(\sup_{u \leq 1} B_u < \sigma | B_1 = y) = (1 - \exp(2\sigma(y - \sigma))^+ \]

for \( \sigma \geq 0 \), and \( y \in \mathbb{R} \).

This formula is trivial for \( \sigma < y \), and, for \( \sigma \geq y \), it follows from the classical formula:

\[ P(\sup_{u \leq 1} B_u \in d\sigma, B_1 \in da) = \frac{da \, d\sigma}{\sqrt{2\pi}} \frac{2(2\sigma - a)e^{-(2\sigma-a)^2/2}}{1_{\{a<\sigma, \sigma \geq 0\}}} \]

\[ \textbf{d)} \quad \text{In a future work, we plan to study more generally how quantities such as the calls and puts:} \]

\[ E[(S_t - K)^+] \text{ and } E[(S_t - K)^-] \]

associated with a general \( \mathbb{R}^+ \)-valued continuous local martingale \((S_t, t \geq 0)\) may be written in terms of cumulative functions.

\[ \textbf{e)} \quad \text{In [1], the authors present eight different approaches to the Black-Scholes formula, among which the change of numéraire approach (Section 5 of [1]), and the local time approach (Section 6 of [1]). This local time approach, together with (2K) yields the relationship:} \]

\[ P\left(G^{(1/2)}_{(t\in K)} \leq t\right) = \left(\frac{1}{2}\right)^+ \quad E[\mathcal{L}_t^K] \]

where \((\mathcal{L}_t^K, t \geq 0)\) denotes the local time at level \( K \) of \((\mathcal{E}_t, t \geq 0)\). It is this kind of relationship (21) which is central in the obtention in [3] of a general expression for the law of a last passage time of a transient diffusion. However, to our knowledge, despite the remarkable survey [1] of methods leading to the Black-Scholes formula, no interpretation of this formula seems to have been made in terms of last passage times distributions.

\section*{References}


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