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► **To cite this version:**

Aline Lefebvre, Bertrand Maury, Aline Lefebvre-Lepot. Micro-Macro Modelling of an Array of Spheres Interacting Through Lubrication Forces. 2005. hal-00257250v1

**HAL Id: hal-00257250**

**<https://hal.science/hal-00257250v1>**

Preprint submitted on 18 Feb 2008 (v1), last revised 5 Sep 2012 (v2)

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# MICRO-MACRO MODELLING OF AN ARRAY OF SPHERES INTERACTING THROUGH LUBRICATION FORCES

A. LEFEBVRE, B. MAURY

ABSTRACT. We consider here a discrete system of spheres interacting through a lubrication force. This force is dissipative, and singular near contact: it behaves like the reciprocal of interparticle distance. We propose a macroscopic constitutive equation which is built as the natural continuous counterpart of this microscopic lubrication model. This model, which is of the newtonian type, relies on an elongational viscosity, which is proportional to the reciprocal of the local fluid fraction. We then establish the convergence in a weak sense of solutions to the discrete problem towards the solution to the partial differential equation which we identified as the macroscopic constitutive equation.

RÉSUMÉ. Nous considérons ici un système discret de sphères en interaction à travers une force de lubrification. Cette force est dissipative et singulière près du contact : elle se comporte comme l'inverse de la distance interparticulaire. Nous proposons une équation constitutive macroscopique qui est construite comme le pendant continu de ce modèle discret de lubrification. Ce modèle, de type Newtonien, repose sur une viscosité élongationnelle proportionnelle à l'inverse de la fraction locale de fluide. Nous établissons ensuite la convergence dans un sens faible des solutions du problème discret vers les solutions de l'équation aux dérivées partielles que nous avons identifiée comme l'équation macroscopique constitutive.

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*Key words and phrases.* Lubrication, fluid-solid interaction, homogeneization, macroscopic model.  
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## 1. INTRODUCTION

Description of the macroscopic behaviour of diluted suspensions of rigid particles in a Newtonian fluid has given rise to a large amount of papers over the last century. The first models (see [6], [8, §22] where Einstein's approach is detailed, or [5]) were based on the assumption that particles do not interact, which restricts their domain of validity to highly diluted suspension. This approach was extended to semi-diluted suspension ([1]), which leads to second order asymptotic expansions of the apparent viscosity with respect to the solid fraction.

More recently, some authors have investigated the other end of the dilution scale: the case of highly packed suspensions of rigid spheres. Direct or semi-direct numerical simulations have been carried out to investigate the behaviour of highly concentrated fluid-particle mixtures ([2], [9]).

In [10], a first attempt was proposed to investigate the behaviour of the apparent shear viscosity of a suspension in the neighbourhood of the maximal packing solid fraction  $\Phi_{\max}$ . A model is proposed, which gives a shear viscosity which is of the order  $(1 - \Phi/\Phi_{\max})^{-2}$  where  $\Phi$  is the solid fraction. In [4], the authors investigate the asymptotic behaviour, as  $\varepsilon$  goes to 0, of a set of particles under the assumption that distances between neighbouring particles is subject to behave like  $\varepsilon$ . In this framework, the authors establish that the apparent shear viscosity behaves like  $1/\varepsilon^{3/2}$ . This approach extends a previous work ([7]) where periodic arrays of spheres are considered. In this context, the elongational viscosity can be shown to behave like  $1/d$  where  $d$  is the constant distance between neighbouring spheres.

The approach we propose here is based on a simpler model from the geometric standpoint, as the spheres are supposed to be aligned. On the other hand it generalizes these works in the sense that no assumption is made on the distances: the macroscopic behaviour depends on the solid fraction only. Contacts between neighbouring particles are even allowed, and a special attention has been paid to the way we express the continuous model so that macroscopic clusters can be taken into account (the local viscosity within a cluster is infinite).

We prove, as expected, that the limit elongational viscosity behaves singularly with respects to the vanishing fluid fraction. This approach leads to an equation of the elliptic type

$$-\partial_x \left( \frac{1}{1-\rho} \partial_x v \right) = \rho f,$$

where  $v$  is the velocity field,  $\rho$  the solid fraction (which is 1 when all particles are in contact), and  $f$  an external body force.

## 2. DISCRETE MODEL

Consider two rigid spheres imbedded in a viscous fluid, subject to move horizontally. Denoting by  $q_1$  and  $q_2$  the abscisses of their centers, by  $u_1$  and  $u_2$  their instantaneous



with  $d_i = q_i - q_{i-1} - 2\varepsilon$ . Consider now a set of forces  $f_1, f_2, \dots, f_{N-1}$ , and the corresponding vector  $\mathbf{f}$ . From (1), the balance of forces reads

$$(3) \quad -A(\mathbf{q})\mathbf{u} + \mathbf{f} = 0.$$

**Proposition 2.2.** *Given a strictly  $\varepsilon$ -feasible vector  $\mathbf{q} \in \mathbb{R}^{N-1}$ , a force field  $\mathbf{f} \in \mathbb{R}^{N-1}$ , problem (3) has a unique solution  $\mathbf{u}$ , and we shall write*

$$\mathbf{u} = (u_i)_{1 \leq i \leq N-1} = \mathcal{P}(\mathbf{q}, \mathbf{f}, \varepsilon).$$

This solution can be expressed

$$(4) \quad u_i = \frac{1}{D_N} \left\{ (D_N - D_i) \sum_{k=1}^i D_k f_k + D_i \sum_{k=i+1}^{N-1} (D_N - D_k) f_k \right\} \quad \forall i = 1 \dots N-1$$

$$\text{with } D_i = \sum_{j=1}^i d_j.$$

*Proof.* Matrix  $A$ , which is similar to the matrix obtained by discretizing the Laplace operator with Dirichlet boundary condition by finite differences, is symmetric positive definite, and the vector  $\mathbf{u}$  is immediately checked to solve the system.  $\square$

This approach can be extended to  $\varepsilon$ -feasible situations in a large sense (particles are allowed to get into contact). As the interaction force (which tends to penalize the relative velocity) blows up when particles tend to get into contact, we simply consider that two particles in contact have the same velocity. This situation can be formalized the following way: The  $N+1$  particules form  $N_c$  clusters, and the  $k$ -th cluster contains the  $N_k+1$  particles  $i_k, i_k+1, \dots, i_k+N_k$  (see Fig. 3). The balance of forces now reads

$$(5) \quad \forall i \notin \cup_k [i_k, i_k + N_k], \quad \frac{u_{i+1} - u_i}{d_{i+1}} - \frac{u_i - u_{i-1}}{d_i} = -f_i,$$

$$(6) \quad \forall k \in [1, N_c], \quad \begin{cases} u_{i_k} = u_{i_k+1} = \dots = u_{i_k+N_k}, \\ \frac{u_{i_k+N_k+1} - u_{i_k+N_k}}{d_{i_k+N_k+1}} - \frac{u_{i_k} - u_{i_k-1}}{d_{i_k}} = - \sum_{i=i_k}^{i_k+N_k} f_i. \end{cases}$$

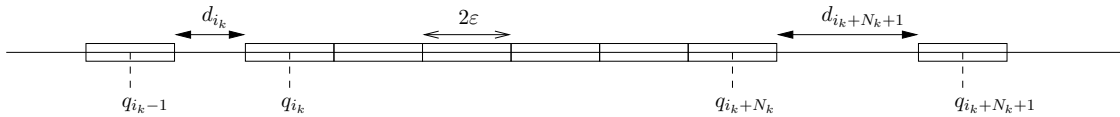


FIGURE 3. Non-strictly  $\varepsilon$ -feasible configuration

**Proposition 2.3.** *Given an  $\varepsilon$ -feasible vector  $\mathbf{q} \in \mathbb{R}^{N-1}$ , a force field  $\mathbf{f} \in \mathbb{R}^{N-1}$ , problem (5)(6) has a unique solution, and we shall write as before*

$$\mathbf{u} = (u_i)_{1 \leq i \leq N-1} = \mathcal{P}(\mathbf{q}, \mathbf{f}, \varepsilon).$$

An explicit expression of this solution is given by (4).

**Remark 2.4.** It is possible to set extremal velocities  $u_0$  and  $u_N$  to non-zero values : in that case, the balance of forces is given by

$$A(\mathbf{q})\mathbf{u} = \mathbf{f} + \mathbf{b},$$

where  $\mathbf{b}$  contains the non homogeneous dirichlet conditions :  $\mathbf{b} = (u_0/d_1, 0, \dots, 0, u_N/d_N)^t$ . The extension of Proposition 2.3 to that case is straightforward.

### 3. MICRO TO MACRO MODELLING: A HEURISTIC APPROACH

The purpose of this section is to derive heuristically a macroscopic constitutive equation from the discrete balance of forces.

We consider an  $\varepsilon$ -feasible configuration of  $N + 1$  particles (see definition 2.1)  $\mathbf{q}$ . We suppose that  $u_0$  and  $u_N$  are given, and the external force is taken 0. The balance of forces reads

$$(7) \quad A(\mathbf{q})\mathbf{u} = \mathbf{b},$$

where  $A$  is given by (2) and  $\mathbf{b}$  contains the non homogeneous dirichlet conditions  $\mathbf{b} = (u_0/d_1, 0, \dots, 0, u_N/d_N)^t$ .

The force exerted on the Nth particle by the others is

$$f_N^{sys} = \frac{u_{N-1} - u_N}{d_N}.$$

As the solution to (7) can be expressed

$$u_i = u_0 + \frac{D_i}{D_N}(u_N - u_0),$$

it comes

$$(8) \quad f_N^{sys} = \frac{u_0 - u_N}{D_N},$$

where  $D_N = \sum_{j=1}^N d_j$  is the quantity of vacuum between  $q_0$  and  $q_N$ .

Given this discrete stress tensor, we shall conjecture the continuous one. Suppose the density of particles is high and note the vacuum fraction in the neighbourhood of a point  $x$  by  $D(x)$ . The force exerted on point  $x$  by the part of the system on its right side is given by

$$F_{\{\text{right side of } x\} \rightarrow \{x\}} = \lim_{\eta \rightarrow 0} F_{[x, x+\eta] \rightarrow \{x\}}.$$

Using (8), it can be expressed by

$$F_{\{\text{right side of } x\} \rightarrow \{x\}} \approx \lim_{\eta \rightarrow 0} \frac{v(x + \eta) - v(x)}{\int_x^{x+\eta} D(x) dx},$$

which gives the following continuous stress tensor:

$$F_{\{\text{right side of } x\} \rightarrow \{x\}} \approx \frac{1}{D(x)} \partial_x v(x).$$

Following a very standard approach, we establish now the resulting equation which expresses the local balances of forces overall a one-dimensional domain. We denote the solid fraction by  $\rho$  and suppose that an external force with density  $f$  is exerted on the system. We express the balance of forces on a part  $\Omega = [a, b]$  of the system. The external force exerted on  $\Omega$  is given by

$$F_{ext} = \int_{\Omega} \rho f$$

and the force exerted by the rest of the system on  $\Omega$  is

$$F_{sys} = \frac{\partial_x v(b)}{D(b)} - \frac{\partial_x v(a)}{D(a)} = \int_{\Omega} \partial_x \left( \frac{1}{D(x)} \partial_x v(x) \right),$$

so that the balance of forces on  $\Omega$  reads

$$\int_{\Omega} \rho f + \int_{\Omega} \partial_x \left( \frac{1}{D(x)} \partial_x v(x) \right) = 0.$$

This last result being true for all  $\Omega$  we have

$$-\partial_x \left( \frac{1}{D} \partial_x v \right) = \rho f$$

and, since  $D = 1 - \rho$ , we finally obtain

$$(9) \quad -\partial_x \left( \frac{1}{1 - \rho} \partial_x v \right) = \rho f.$$

#### 4. ASYMPTOTIC BEHAVIOUR OF THE DISCRETE SOLUTIONS

Let  $I$  denote  $]0, 1[$ . Firstly, we build a new operator  $\tilde{\mathcal{P}}$ , which is our key tool to connect the microscopic level to the macroscopic one. This operator is defined the following way: Given  $\varepsilon > 0$ , an  $\varepsilon$ -feasible position vector  $\mathbf{q}$  (as stated by Definition 2.1, it represents a distribution of particles whose centers are located at  $q_i$ ,  $i = 1, \dots, N$ , with common radius  $\varepsilon$ ), a force density  $f \in L^1(I)$ , we define vector  $\mathbf{u}$  as  $\mathcal{P}(\mathbf{q}, \mathbf{f}^\varepsilon, \varepsilon)$  (see Proposition 2.2 or 2.3, depending on whether  $\mathbf{q}$  is strictly feasible or feasible), where  $\mathbf{f}^\varepsilon$  is defined by

$$f_i^\varepsilon = \frac{1}{2\varepsilon} \int_{q_i - \varepsilon}^{q_i + \varepsilon} f(s) ds \text{ for } 1 \leq i \leq N - 1.$$

Now, to this vector  $\mathbf{u} = \mathcal{P}(\mathbf{q}, \mathbf{f}^\varepsilon, \varepsilon) \in \mathbb{R}^{N-1}$ , we associate a piecewise affine function  $u$  defined by

$$(10) \quad u \in C^0(\bar{I}), \quad u \text{ affine on } [q_i, q_{i+1}] \quad \forall i = 0, \dots, N - 1, \quad u(q_i) = \mathbf{u}_i \quad \forall i = 0, \dots, N.$$

We shall write  $u = \tilde{\mathcal{P}}(\mathbf{q}, f, \varepsilon)$ .

In the same spirit, for any  $\varepsilon > 0$ , and any  $\varepsilon$ -feasible position vector  $\mathbf{q} \in \mathbb{R}^{N-1}$ , we define  $\chi(\mathbf{q}, \varepsilon)$  as the characteristic function of the solid phase associated to the  $(\mathbf{q}, \varepsilon)$  distribution:

$$(11) \quad \chi(\mathbf{q}, \varepsilon) = \sum_{i=1}^{N-1} \mathbb{1}_{]q_i - \varepsilon, q_i + \varepsilon[} + \mathbb{1}_{]0, \varepsilon[} + \mathbb{1}_{]1 - \varepsilon, 1[}.$$

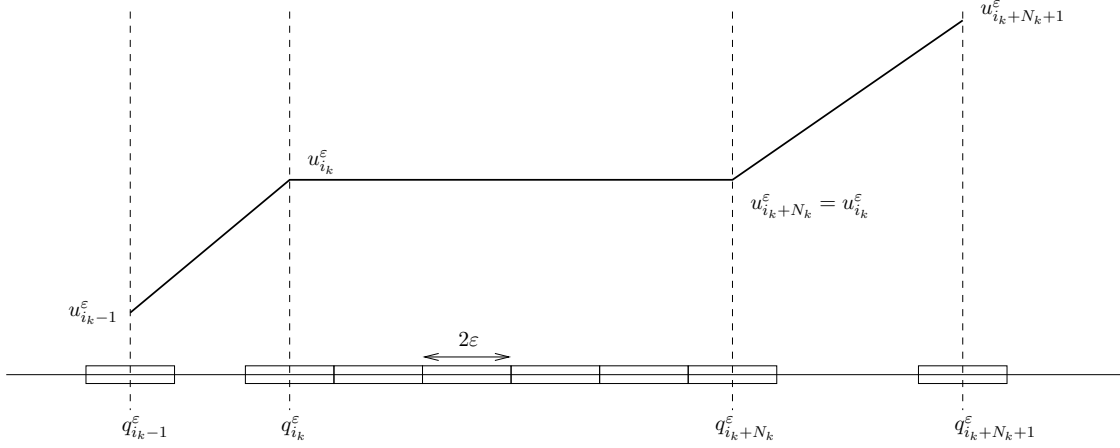


FIGURE 4. Definition of  $u^\varepsilon$

Before we state the main convergence theorem, we still have to give a sense to (9) when the density  $\rho \in [0, 1]$  is allowed to take value 1, even on a set of non-zero measure.

**Proposition 4.1.** *Let  $K : I \mapsto \mathbb{R}^+ \cup \{+\infty\}$  be measurable,  $K(x) \geq \alpha > 0$  for almost every  $x \in I$ ,  $\varphi \in H^{-1}(I)$ , and let  $J$  be defined as*

$$v \in H_0^1(I) \mapsto J(v) = \int_I K(x) |\partial_x v|^2 - \langle \varphi, v \rangle \in \mathbb{R}^+ \cup \{+\infty\}.$$

*There exists a unique  $u \in H_0^1(I)$  which realizes the minimum of  $J$  over  $H_0^1(I)$ . If there exists  $f \in L^1$  such that  $\langle \varphi, v \rangle = \int f v$ , we shall say that  $u$  is a generalized solution to*

$$-\partial_x (K(x) \partial_x u) = f.$$

*Proof.* The functional  $J$  is convex (strictly convex over its domain), coercive, and it can be written

$$J(v) = \sup_{n \in \mathbb{N}} \left( \int_I \min(K(x), n) |\partial_x v|^2 - \int_I f v \right),$$

thus it is l.s.c. as a supremum of a family of l.s.c functions. Therefore it admits a unique minimizer. Note that the minimization problem is equivalent to the problem which consists in minimizing the same functional  $J$  over the set

$$(12) \quad H_K = \left\{ v \in H_0^1(I), \partial_x v = 0 \text{ a.e. in } D(K)^c \text{ and } \int_{D(K)} K(x) |\partial_x v|^2 < \infty \right\}.$$



where  $D(K) = \{x \in I, K(x) < +\infty\}$  is the domain of  $K$ . Consequently,  $u$  is characterized by the variational formulation

$$(13) \quad u \in H_K, \quad \int_{D(K)} K(x) \partial_x u \partial_x v = \int_I f v \quad \forall v \in H_K. \quad \square$$

We may now state the convergence result.

**Theorem 4.2.** *Let  $f \in L^1(I)$  be a force density, and  $\rho \in L^\infty(I)$  a solid fraction, with  $\rho(x) \in [0, 1]$  a.e. in  $I$ . Let  $(\mathbf{q}^\varepsilon)_\varepsilon$  be a sequence of  $\varepsilon$ -feasible position vectors (see Definition 2.1),  $\mathbf{q}^\varepsilon \in \mathbb{R}^{N^\varepsilon}$  with  $N_\varepsilon = 1/\varepsilon$ . We introduce  $\chi^\varepsilon = \chi(\mathbf{q}^\varepsilon, \varepsilon)$  (see (11)), and we assume that  $\chi^\varepsilon$  converges toward  $\rho$  in  $L^\infty(I)$  weak- $\star$ .*

*Then  $u^\varepsilon = \tilde{\mathcal{P}}(\mathbf{q}^\varepsilon, f, \varepsilon)$  solution to the discrete model (see (10)) converges weakly toward  $u$  in  $H_0^1(I)$  as  $\varepsilon$  converges to 0, where  $u$  is the solution to*

$$(14) \quad -\partial_x \left( \frac{1}{1-\rho} \partial_x u \right) = \rho f,$$

*in the sense of Proposition 4.1 (i.e. characterized by (13)).*

*Proof.* The proof is based on some technical lemmas. For readability reasons, we postpone the proofs of the lemmas to the end of the section.

As a first step, we define  $\rho^\varepsilon$ , piecewise constant, as the proportion of solid on each subinterval  $[q_{i-1}^\varepsilon, q_i^\varepsilon]$  in the following way : let  $d_i$  be the distance between particles  $i-1$  and  $i$ , then

$$\forall i = 0, \dots, N^\varepsilon - 1, \quad \rho^\varepsilon = \rho_i^\varepsilon = 1 - \frac{d_{i+1}^\varepsilon}{q_{i+1}^\varepsilon - q_i^\varepsilon} \text{ on } [q_{i-1}^\varepsilon, q_i^\varepsilon].$$

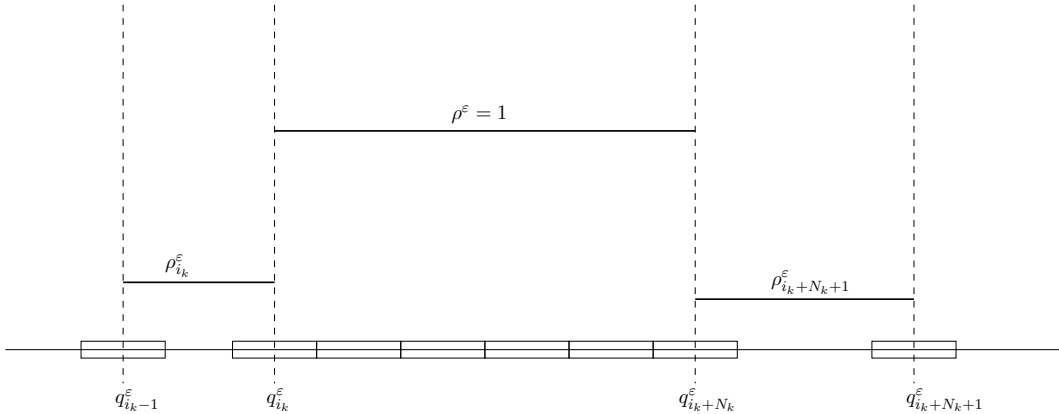


FIGURE 5.  $\rho^\varepsilon$

Define now  $w^\varepsilon$ , constant by part, as the discrete counterpart of  $\partial_x u/(1 - \rho)$  given on  $[q_{i-1}^\varepsilon, q_i^\varepsilon]$  by

$$(15) \quad \begin{cases} w^\varepsilon = w_i^\varepsilon = \frac{1}{1 - \rho_i^\varepsilon} \partial_x u^\varepsilon & , \text{ if } \rho_i^\varepsilon < 1 \\ w^\varepsilon = w_i^\varepsilon = \beta_i^\varepsilon & , \text{ if } \rho_i^\varepsilon = 1 \end{cases}$$

where  $(\beta_i^\varepsilon)_{i_k \leq i \leq i_k + N_k}$  corresponding to the  $k$ th cluster, is the solution to the following system:

$$(16) \quad \begin{cases} \beta_{i_k+1}^\varepsilon & - \frac{u_{i_k}^\varepsilon - u_{i_k-1}^\varepsilon}{d_{i_k}^\varepsilon} & = & -2\varepsilon f_{i_k}^\varepsilon, \\ \beta_{i_k+2}^\varepsilon & - \beta_{i_k+1}^\varepsilon & = & -2\varepsilon f_{i_k+1}^\varepsilon, \\ & \vdots & & \\ \beta_{i_k+N_k}^\varepsilon & - \beta_{i_k+N_k-1}^\varepsilon & = & -2\varepsilon f_{i_k+N_k-1}^\varepsilon, \\ \frac{u_{i_k+N_k+1}^\varepsilon - u_{i_k+N_k}^\varepsilon}{d_{i_k+N_k+1}^\varepsilon} & - \beta_{i_k+N_k}^\varepsilon & = & -2\varepsilon f_{i_k+N_k}^\varepsilon. \end{cases}$$

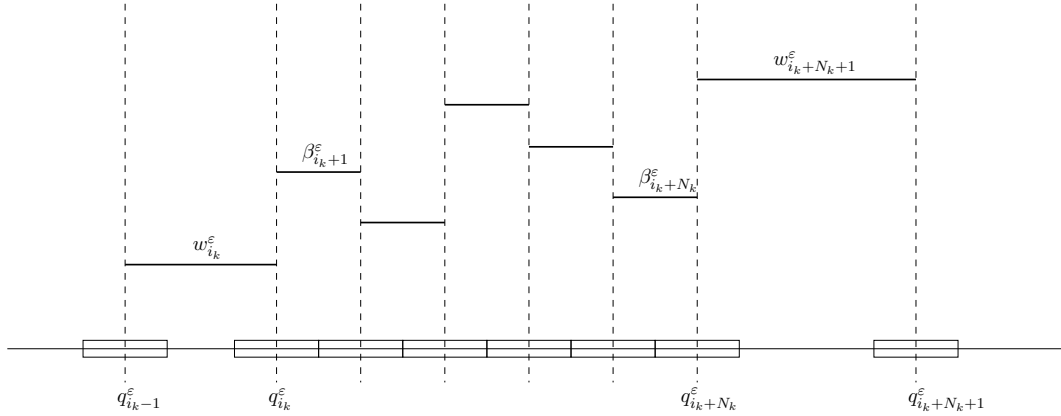


FIGURE 6.  $w^\varepsilon$

Note that, summing up all equations of (16) we recognize the balance of forces on the  $k$ th cluster given by (6).

**Remark 4.3.** The idea behind the above construction is that  $\beta_i$  can be seen as the cohesion force between particles  $i - 1$  and  $i$ . A first way to notice it is to note that  $\mathbf{u}^\varepsilon$  is the limit of  $\mathbf{u}^{\varepsilon, \eta}$  where  $\mathbf{u}^{\varepsilon, \eta}$  is the solution to system (3) with  $d_i = \eta > 0$  for  $i$  between  $i_k + 1$  and  $i_k + N_k$  and that we have

$$\forall k, \forall j \in [1, N_k], \beta_{i_k+j}^\varepsilon = \lim_{\eta \rightarrow 0} \frac{u_{i_k+j}^{\varepsilon, \eta} - u_{i_k+j-1}^{\varepsilon, \eta}}{\eta}.$$

Another way to define these cohesion forces is to consider the following minimization problem : minimize

$$J(\mathbf{v}) = \sum_{i \notin \cup_k [i_k+1, i_k+N_k]} \frac{1}{2} \frac{(v_i - v_{i-1})^2}{d_i^\varepsilon} + \sum_{i=1}^{N-1} f_i^\varepsilon v_i$$

over  $K = \{\mathbf{v} , \forall i \in \cup_k [i_k + 1, i_k + N_k], v_i = v_{i-1}\}$ . This problem is equivalent to problem (5)(6) and  $\beta_i^\varepsilon$  turns up as the Lagrange multiplier associated to the constraint  $v_i = v_{i-1}$ .

The next lemma shows that (14) is true at the discrete level :

**Lemma 4.4.** *For any  $\varepsilon > 0$ ,*

$$(17) \quad -\partial_x (w^\varepsilon) = 2\varepsilon f^\varepsilon$$

in  $H^{-1}(I)$ , where  $f^\varepsilon = \sum_{i=1}^{N_\varepsilon-1} f_i^\varepsilon \delta_{q_i^\varepsilon}$  ( $\delta_{q_i^\varepsilon}$  is the Dirac measure at point  $q_i^\varepsilon$ ).

The idea of the proof is now to let  $\varepsilon$  go to 0 in (17). To that purpose, we first study  $\rho^\varepsilon$  and  $2\varepsilon f^\varepsilon$  when  $\varepsilon$  tends to zero:

**Lemma 4.5.**  $\rho^\varepsilon \xrightarrow{*} \rho$  in  $L^\infty(I)$ .

**Lemma 4.6.**  $2\varepsilon f^\varepsilon \xrightarrow{*} \rho f$  in  $H^{-1}(I)$ .

Let us now establish that  $u^\varepsilon$  is bounded in  $H_0^1(I)$ . Applying Lemma 4.4 to the test function  $u^\varepsilon$  gives, using that  $\mathbf{q}^\varepsilon$  is  $\varepsilon$ -feasible,

$$\|u^\varepsilon\|_{H_0^1(I)}^2 \leq \|u^\varepsilon\|_{L^\infty(I)} \|f\|_{L^1(I)} \leq C \|u^\varepsilon\|_{H_0^1(I)} \|f\|_{L^1(I)}.$$

Hence,  $(u^\varepsilon)_\varepsilon$  is bounded in  $H_0^1(I)$  and we can extract a subsequence of  $(u^\varepsilon)_\varepsilon$  (still denoted by  $(u^\varepsilon)_\varepsilon$ ) such that  $u^\varepsilon \rightharpoonup u$  in  $H_0^1(I)$ . In order to pass to the limit in the left-hand side of (17), we are going to prove that there exists a subsequence of  $(w^\varepsilon)_\varepsilon$  converging to a  $w$  in  $L^1(I)$ . This will follow from the next lemmas

**Lemma 4.7.**  $(w^\varepsilon)_\varepsilon$  is bounded in  $L^\infty(I)$ .

**Lemma 4.8.**  $(w^\varepsilon)_\varepsilon$  has uniform bounded variation.

From these two lemmas it follows that  $(w^\varepsilon)_\varepsilon$  is bounded in the space of functions of bounded variation  $BV(I)$  and by compact injection of  $BV(I)$  in  $L^1(I)$  we can find  $w \in L^1(I) \cap BV(I)$  and a subsequence of  $(w^\varepsilon)_\varepsilon$  (still denoted by  $(w^\varepsilon)_\varepsilon$ ) such that

$$(18) \quad w^\varepsilon \rightarrow w \text{ in } L^1(I) \text{ and a.e.}$$

Then, convergence of  $w^\varepsilon$  and  $\rho^\varepsilon$  make it possible to establish the following lemma:

**Lemma 4.9.**

$$\partial_x u^\varepsilon \rightharpoonup (1 - \rho)w \text{ in } L^2(I).$$

We now come to the last step of the proof : let  $\varepsilon$  tend to zero in (17) and obtain asymptotically

$$-\partial_x \left( \frac{1}{1 - \rho} \partial_x u \right) = \rho f,$$

in the sense of Proposition 4.1, characterized by (13)

First, Lemma 4.9 implies  $\partial_x u = (1 - \rho)w$ , so  $u \in H_{1/(1-\rho)}$  (defined by (12)). Then, by Lemma 4.4,

$$\int_I w^\varepsilon v' = \langle 2\varepsilon f^\varepsilon, v \rangle \quad \forall v \in H_0^1(I).$$

Passing to the limit on  $\varepsilon$  gives, using (18), Lemma 4.7 and Lemma 4.6,

$$\int_I wv' = \int_I \rho f v \quad \forall v \in H_0^1(I).$$

Finally, for any  $v \in H_{1/(1-\rho)}$ ,

$$\begin{aligned} \int_I \rho f v &= \int_I wv' = \int_{\rho \neq 1} wv' \\ &= \int_{\rho \neq 1} \frac{(1-\rho)w}{1-\rho} v' = \int_{\rho \neq 1} \frac{\partial_x u}{1-\rho} v' \end{aligned}$$

and we conclude that  $u$  is the solution to (14).

So, we proved that there exists a subsequence of  $(u^\varepsilon)_\varepsilon$  converging to  $u$  as  $\varepsilon$  tends to zero. Since the same work can be done for each subsequence of  $(u^\varepsilon)_\varepsilon$ , we conclude that  $(u^\varepsilon)_\varepsilon$  itself converges to  $u$ , which completes the proof of the theorem.  $\square$

**Proof of Lemma 4.4:**  $-\partial_x(w^\varepsilon) = 2\varepsilon f^\varepsilon$  in  $H^{-1}(I)$ .

First, a simple computation shows that

$$\begin{aligned} \partial_x w^\varepsilon &= \sum_{\substack{i \in 1..N^\varepsilon - 1 \\ d_i > 0, d_{i+1} > 0}} \left( \frac{u_{i+1}^\varepsilon - u_i^\varepsilon}{d_{i+1}^\varepsilon} - \frac{u_i^\varepsilon - u_{i-1}^\varepsilon}{d_i^\varepsilon} \right) \delta_{q_i^\varepsilon} + \sum_{k=1}^{N_c} \sum_{i=i_k+1}^{i_k+N_k-1} (\beta_i^\varepsilon - \beta_{i-1}^\varepsilon) \delta_{q_i^\varepsilon} \\ &+ \sum_{k=1}^{N_c} \left( \beta_{i_k+1}^\varepsilon - \frac{u_{i_k}^\varepsilon - u_{i_k-1}^\varepsilon}{d_{i_k}^\varepsilon} \right) \delta_{q_{i_k}^\varepsilon} + \sum_{k=1}^{N_c} \left( \frac{u_{i_k+N_k+1}^\varepsilon - u_{i_k+N_k}^\varepsilon}{d_{i_k+N_k+1}^\varepsilon} - \beta_{i_k+N_k}^\varepsilon \right) \delta_{q_{i_k+N_k}^\varepsilon} \end{aligned}$$

in the sense of distributions. Then, combining this with system (5)(6) and with the definition of  $\beta_i$  (16), we get

$$\partial_x w^\varepsilon = - \sum_{i=1}^{N^\varepsilon-1} 2\varepsilon f_i^\varepsilon \delta_{q_i^\varepsilon}.$$

By density of  $C_0^\infty(I)$  in  $H_0^1(I)$  and continuous injection of  $H_0^1(I)$  in  $C^0(I)$ , this result holds in  $H^{-1}(I)$  as required.

**Proof of Lemma 4.5 :**  $\rho^\varepsilon \xrightarrow{*} \rho$  in  $L^\infty(I)$ .

Since  $\rho^\varepsilon - \rho = (\rho^\varepsilon - \chi^\varepsilon) + (\chi^\varepsilon - \rho)$  and  $\chi^\varepsilon \xrightarrow{*} \rho$  in  $L^\infty(I)$ , the result will follow provided we show that

$$\forall \varphi \in L^1(I), \quad \lim_{\varepsilon \rightarrow 0} \left( \int_I \chi^\varepsilon \varphi - \int_I \rho^\varepsilon \varphi \right) = 0.$$

By density of the set of stairs functions in  $L^1(I)$ , and by using the fact that  $(\rho^\varepsilon)_\varepsilon$  and  $(\chi^\varepsilon)_\varepsilon$  are bounded in  $L^\infty(I)$ , this in turn will follow from

$$\forall \varphi \text{ piecewise constant on } I, \quad \lim_{\varepsilon \rightarrow 0} \left( \int_I \chi^\varepsilon \varphi - \int_I \rho^\varepsilon \varphi \right) = 0.$$

To show this, it suffices to prove that

$$\forall \alpha, \beta \quad 0 < \alpha < \beta < 1 \quad \lim_{\varepsilon \rightarrow 0} \left( \int_\alpha^\beta \chi^\varepsilon - \int_\alpha^\beta \rho^\varepsilon \right) = 0.$$

In order to do so take  $\alpha$  and  $\beta$  such that  $0 < \alpha < \beta < 1$  and denote the particules whose center is in  $[\alpha, \beta]$  by  $q_{i_0}^\varepsilon, q_{i_0+1}^\varepsilon, \dots, q_{j_0}^\varepsilon$ . Since  $\int_{q_i^\varepsilon}^{q_{i+1}^\varepsilon} (\chi^\varepsilon - \rho^\varepsilon) = 2\varepsilon - (q_{i+1}^\varepsilon - q_i^\varepsilon - d_{i+1}^\varepsilon) = 0$  for  $1 \leq i \leq N_\varepsilon - 1$ , we have

$$\int_\alpha^\beta \chi^\varepsilon - \int_\alpha^\beta \rho^\varepsilon = \int_\alpha^{q_{i_0}^\varepsilon} (\chi^\varepsilon - \rho^\varepsilon) - \int_{q_{j_0}^\varepsilon}^\beta (\chi^\varepsilon - \rho^\varepsilon).$$

Then, a simple computation shows that the left-hand side converges to zero as  $\varepsilon$  tends to zero wich completes the proof of the lemma.

**Proof of Lemma 4.6 :**  $2\varepsilon f^\varepsilon \xrightarrow{*} \rho f$  in  $H^{-1}(I)$ .

By injection of  $H_0^1(I)$  in  $\mathcal{C}^0(I)$  it suffices to show that

$$\forall \varphi \in \mathcal{C}^0(I), \quad \lim_{\varepsilon \rightarrow 0} \left\langle 2\varepsilon \sum_{i=1}^{N_\varepsilon-1} f_i^\varepsilon \delta_{q_i^\varepsilon}, \varphi \right\rangle - \int_I \rho f \varphi = 0.$$

Moreover, we can write

$$2\varepsilon \sum_{i=1}^{N_\varepsilon-1} f_i^\varepsilon \delta_{q_i^\varepsilon} - \rho f = \left( \sum_{i=1}^{N_\varepsilon-1} f_i^\varepsilon \delta_{q_i^\varepsilon} - \chi^\varepsilon f \right) + (\chi^\varepsilon f - \rho f).$$

Then, using the fact that  $\chi^\varepsilon \xrightarrow{*} \rho$ , the required result will follow as soon as we prove that

$$\forall \varphi \in \mathcal{C}^0(I), \quad \lim_{\varepsilon \rightarrow 0} \Delta_\varepsilon = 0,$$

where

$$\Delta_\varepsilon = \left\langle 2\varepsilon \sum_{i=1}^{N_\varepsilon-1} f_i^\varepsilon \delta_{q_i^\varepsilon}, \varphi \right\rangle - \int_I \chi^\varepsilon f \varphi.$$

To obtain this, merely compute

$$\Delta_\varepsilon = \sum_{i=1}^{N_\varepsilon-1} \int_{q_i^\varepsilon - \varepsilon}^{q_i^\varepsilon + \varepsilon} [\varphi(q_i^\varepsilon) - \varphi] f - \int_0^\varepsilon \varphi f - \int_{1-\varepsilon}^1 \varphi f$$

and use uniform continuity of  $\varphi$  together with the fact that  $\mathbf{q}^\varepsilon$  is  $\varepsilon$ -feasible to define  $e(\delta)$  by

$$e(\delta) = \sup_{x, y \in I, |x-y| < \delta} |\varphi(x) - \varphi(y)|$$

and show

$$|\Delta_\varepsilon| \leq e(\varepsilon) \sum_{i=1}^{N_\varepsilon-1} \int_{q_i^\varepsilon-\varepsilon}^{q_i^\varepsilon+\varepsilon} |f| + \int_0^\varepsilon |\varphi||f| + \int_{1-\varepsilon}^1 |\varphi||f| \leq e(\varepsilon) \|f\|_{L^1(I)} + \int_0^\varepsilon |\varphi||f| + \int_{1-\varepsilon}^1 |\varphi||f|.$$

The result follows from this by letting  $\varepsilon \rightarrow 0$  and using the fact that  $\varphi f \in L^1(I)$ .  $\square$

**Proof of Lemma 4.7:**  $(w^\varepsilon)_\varepsilon$  is bounded in  $L^\infty(I)$ .

By (15),  $w^\varepsilon$  is piecewise constant, equal to  $w_i^\varepsilon$  on  $[q_{i-1}^\varepsilon, q_i^\varepsilon]$ , where  $w_i^\varepsilon = \frac{1}{1-\rho_i^\varepsilon} \partial_x u^\varepsilon = \frac{u_i^\varepsilon - u_{i-1}^\varepsilon}{d_i^\varepsilon}$  if  $\rho_i^\varepsilon < 1$ , and  $w_i^\varepsilon = \beta_i^\varepsilon$  otherwise. Note that, by (16) and the fact that  $\mathbf{q}^\varepsilon$  is  $\varepsilon$ -feasible

$$\forall k = 1 \dots N_c, \quad \forall j = 1 \dots N_k, \quad |\beta_{i_k+j}^\varepsilon| = \left| w_{i_k-1}^\varepsilon - 2\varepsilon \sum_{m=i_k}^{i_k+j-1} f_m \right| \leq |w_{i_k-1}^\varepsilon| + \|f\|_{L^1(I)}.$$

Therefore, to show that  $\|w^\varepsilon\|_{L^\infty(I)}$  is bounded it suffices to obtain an upper-bound for  $(w_i^\varepsilon)_i$  s.t.  $\rho_i < 1$ . In order to do so, a simple computation gives using (4)

$$\forall i \text{ s.t. } \rho_i < 1, \quad w_i^\varepsilon = 2\varepsilon \sum_{k=i+1}^{N_\varepsilon-1} \frac{D_{N_\varepsilon} - D_k}{D_{N_\varepsilon}} f_k^\varepsilon - 2\varepsilon \sum_{k=1}^i \frac{D_k}{D_{N_\varepsilon}} f_k^\varepsilon.$$

Then, from  $\frac{D_{N_\varepsilon} - D_k}{D_{N_\varepsilon}} \leq 1$ ,  $\frac{D_k}{D_{N_\varepsilon}} \leq 1$  and the fact that  $\mathbf{q}^\varepsilon$  is  $\varepsilon$ -feasible it follows that

$$|w_i^\varepsilon| \leq \sum_{k=1}^{N_\varepsilon-1} |2\varepsilon f_k^\varepsilon| \leq \|f\|_{L^1(I)}$$

and we conclude that

$$\|w^\varepsilon\|_{L^\infty(I)} \leq 2\|f\|_{L^1(I)},$$

which completes the proof of the lemma.  $\square$

**Proof of Lemma 4.8:**  $(w^\varepsilon)_\varepsilon$  has uniform bounded variation.

We recall that the variation of a function is defined by

$$\text{Var}(f) = \sup \left\{ \int_I f \varphi' : \varphi \in \mathcal{C}_c^1(I), \|\varphi\|_{L^\infty(I)} \leq 1 \right\}.$$

By Lemma 4.4 applied to any test function  $\varphi \in \mathcal{C}_c^1(I)$  and the fact that  $\mathbf{q}^\varepsilon$  is  $\varepsilon$ -feasible, we see that

$$(19) \quad \text{Var}(w^\varepsilon) \leq \|f\|_{L^1(I)},$$

so  $w^\varepsilon$  has uniform bounded variation.  $\square$

**Proof of Lemma 4.9 :**  $\partial_x u^\varepsilon \rightharpoonup (1 - \rho)w$  in  $L^2(I)$ .

Writing

$$\partial_x u^\varepsilon - (1 - \rho)w = (1 - \rho^\varepsilon)w^\varepsilon - (1 - \rho)w = \{((1 - \rho^\varepsilon) - (1 - \rho))w\} + \{(1 - \rho^\varepsilon)(w^\varepsilon - w)\},$$

we shall prove the weak convergence to zero in  $L^2(I)$  of both term of the left-hand side. For the first term, take  $\varphi$  in  $L^2(I)$ . Using that  $w \in BV(I) \subset L^\infty(I)$ , it follows that  $w\varphi \in L^1(I)$  and by Lemma 4.5

$$\lim_{\varepsilon \rightarrow 0} \int_I ((1 - \rho^\varepsilon) - (1 - \rho)) w\varphi = 0,$$

as required.

We shall now prove the convergence of the second term to zero. By density of  $\mathcal{C}_0^\infty(I)$  in  $L^2(I)$  and Lemma 4.7 it suffices to take  $\varphi$  in  $\mathcal{C}_0^\infty(I)$ . Therefore, the result follows immediately from

$$\left| \int_I (1 - \rho^\varepsilon)(w^\varepsilon - w)\varphi \right| \leq \|\varphi\|_\infty \int_I |w^\varepsilon - w|$$

together with (18).

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