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# VARIATIONS ON LOG SARKISOV PROGRAM FOR SURFACES

ADRIEN DUBOULOZ AND STÉPHANE LAMY

ABSTRACT. Let  $(S, B_S)$  be the log-pair associated with a compactification of a given smooth quasi-projective surface  $V$ . Under the assumption that the boundary  $B_S$  is irreducible, we propose an algorithm, in the spirit of the (log) Sarkisov program, to factorize any automorphism of  $V$  into a sequence of elementary links in the framework of the logarithmic Mori theory. The new noteworthy feature of our algorithm is that all the blow-ups and contractions involved in the process occur on the boundary.

## INTRODUCTION

Let  $V$  be a smooth quasi-projective surface. We plan to describe the automorphisms of  $V$  when there exists a compactification  $V \subset S$  where  $S$  is a (possibly singular) projective surface with  $S \setminus V$  equal to an irreducible curve. More precisely, we look for a decomposition in the framework of logarithmic Mori theory for automorphisms of  $V$  that do not extend as biregular automorphisms on  $S$ . In this introduction we suppose the reader has some familiarity with the basics of Mori theory ([15] is an agreeable introductory book); let us recall simply that a Mori fibration  $X$  is a  $\mathbb{Q}$  factorial normal variety with at most terminal singularities endowed with a fibration  $g : X \rightarrow Y$  with connected fibers above a normal variety  $Y$  of a dimension strictly less than  $X$ , such that all the curves contracted by  $g$  are numerically proportional and of negative intersection with the canonical divisor  $K_X$ : a Mori fibration should be thought as a “simplest possible” representative in its birational class.

The Sarkisov program, written out in dimension 3 by Corti in 1995 [3], is an algorithm to decompose a birational map  $f : Y \dashrightarrow Y'$  between Mori fibrations into so-called elementary links. The algorithm works in principle in arbitrary dimension (as soon as the MMP = “minimal model program” is proved); the general idea is that one decomposes  $f$  with the help of a sequence of intermediate varieties between  $Y$  and  $Y'$ , and that we have control of the complexity of these varieties in the sense that, modulo isomorphism in codimension 1, at most one divisorial contraction is sufficient to come back to a Mori fibration. Here is a brief description of the algorithm. We start by taking a resolution  $Y \xleftarrow{\pi} X \xrightarrow{\pi'} Y'$  of the base points of  $f$ , where  $X$  is a smooth projective variety, and we choose an ample divisor  $H'$  on  $Y'$ . We note  $H_Y \subset Y$  (or  $H_X \subset X$ , etc...) the strict transform of a general member of the linear system  $|H'|$ , and  $C_i \subset X$  the irreducible components of the exceptional locus of  $\pi$ . We write down the ramification formulas

$$K_X = \pi^*K_Y + \sum c_i C_i \quad \text{and} \quad H_X = \pi^*H_Y - \sum m_i C_i$$

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*Date:* february 2008.

and we define the maximal multiplicity  $\lambda$  as the maximum of the  $\lambda_i = \frac{m_i}{c_i}$ . On the other hand we define the degree  $\mu$  of  $f$  as the rational number  $\frac{H_Y \cdot C}{-K_Y \cdot C}$  where  $C$  is any curve contained in a fiber of the Mori fibration on  $Y$ . In the case  $\lambda > \mu$ , that we feel is the general case, the algorithm predicts the existence of a maximal extraction  $Z \rightarrow Y$  (we take the terminology from [4], in [15] the same operation is called a maximal divisorial blow-up), which by definition is an extremal divisorial contraction whose exceptional divisor realizes the maximal multiplicity  $\lambda$ . Then either  $Z$  is itself a Mori fibration, or there exists another extremal divisorial contraction on  $Z$  (possibly preceded by a sequence of log flips, that are isomorphisms in codimension 1) that brings us back to a Mori fibration. These operations done, one shows that we have simplified  $f$  in the sense that : either  $\mu$  went down; or  $\mu$  remained constant but  $\lambda$  went down; or  $\mu$  and  $\lambda$  remained constant but the number of exceptional divisors in  $X$  realizing the multiplicity  $\lambda$  went down. As we can see the algorithm is quite complex, not to mention the case  $\lambda \leq \mu$  which is also intricate, and that we do not detail further.

In 1997 Bruno and Matsuki [2] published a logarithmic version of this algorithm : the log-Sarkisov program. In this new situation there exist some distinguished divisors  $B_Y$  and  $B_{Y'}$  on the varieties  $Y$  and  $Y'$  : this arises naturally when  $Y$  and  $Y'$  are compactifications of a fixed quasi-projective variety  $V$ ; by analogy with this case we say in general that  $B_Y$  is the boundary divisor of  $Y$ . The idea is that the algorithm remains formally the same, where  $K_Y + B_Y$  now plays the role of the canonical divisor  $K_Y$ . The degree  $\mu$  in this context is defined as  $\mu = \frac{H_Y \cdot C}{-(K_Y + B_Y) \cdot C}$ , where  $C$  is in some fiber of the log Mori fibration on  $Y$ . In addition to the ramification formulas for  $K_X$  and  $H_X$  we now have a similar formula for the boundary :

$$B_X = \pi^* B_Y - \sum b_i C_i$$

and the maximal multiplicity is defined as the maximum of the  $\lambda_i = \frac{m_i}{c_i - b_i}$ . Bruno and Matsuki worked out a log-Sarkisov algorithm in two cases :

- (1) In dimension 3, for boundary divisors whose all coefficients are strictly less than 1 (the precise technical condition is  $(Y, B_Y)$  klt, for kawamata log terminal);
- (2) In dimension 2, for boundary divisors whose coefficients are less or equal to 1 (the technical condition is dlt, for divisorially log terminal).

The expressed hope is that a refinement of such an algorithm could allow us to understand the structure of polynomial automorphisms of  $\mathbb{C}^3$ . We have in mind to compactify  $\mathbb{C}^3$  by the projective space  $\mathbb{P}^3$ , and to apply the algorithm to the birational map from  $\mathbb{P}^3$  to  $\mathbb{P}^3$  induced by an automorphism of  $\mathbb{C}^3$ . A technical problem is that the boundary in this situation is the plane at infinity, with coefficient  $+1$ , and therefore we are not in the klt framework. Nevertheless in dimension 2 this obstacle disappears, and we might feel free to think that everything is done in the case of surfaces.

Now here is the example that initially gave us the motivation to write on the log-Sarkisov program in dimension 2 in spite of the existence of the results by Bruno-Matsuki. Let us consider an affine quadric surface  $V$ , for instance we can take  $V = \{w^2 + uv = 1\} \subset \mathbb{C}^3$ . Such a surface is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  minus a diagonal  $D$ . Let  $f$  be the rational map

$$f : (x, y) \in \mathbb{C}^2 \dashrightarrow \left( x + \frac{1}{x+y}, y - \frac{1}{x+y} \right) \in \mathbb{C}^2.$$

This map preserves the levels  $x + y = cte$ , extends as a birational map from  $S = \mathbb{P}^1 \times \mathbb{P}^1$  to  $S' = \mathbb{P}^1 \times \mathbb{P}^1$ , and induces an isomorphism on  $V = \mathbb{P}^1 \times \mathbb{P}^1 \setminus D$  where  $D$  is the diagonal  $x + y = 0$ .

The unique base point is the point  $p = [1 : 0], [1 : 0]$ , and the unique contracted curve is the diagonal  $D$ . We can resolve  $f$  by performing 4 blow-ups that give rise to divisors  $C_1, \dots, C_4$  arranged as on figure 1 (all these claims are not difficult to check by straightforward calculations in local charts; the reader may also look in [14]).

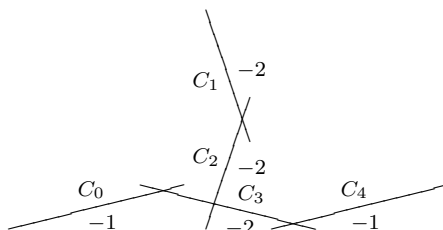


FIGURE 1. Resolution of  $f$ .

The divisor  $C_0$  is (the strict transform of) the diagonal on  $S$ , and  $C_4$  is the diagonal on  $S'$ . Let us choose  $H' = D$  as an ample divisor on  $S'$ , then we compute the coefficients in the ramification formulas :

$$K_X = \pi^*K_S + \sum c_i C_i, B_X = \pi^*B_S - \sum b_i C_i \text{ and } H_X = \pi^*H_S - \sum m_i C_i,$$

in order to deduce the  $\lambda_i = \frac{m_i}{c_i - b_i}$ . The  $c_i$  and  $b_i$  are easy to compute; for the  $m_i$  it is sufficient to check that in this particular example the strict transform  $H_S$  of a general member of  $|D|$  is a smooth curve. We obtain the following results :

	$c_i$	$b_i$	$m_i$	$\lambda_i$
$C_1$	1	0	1	1
$C_2$	2	1	2	2
$C_3$	3	2	3	3
$C_4$	4	2	4	2

Thus the maximal multiplicity is realized by the divisor  $C_3$ . We can construct the maximal extraction of the maximal singularity in the following way : blow-up three times to produce  $C_1, C_2$  and  $C_3$ , then contract  $C_1$  and  $C_2$  creating a singular point (this is a so-called Hirzebruch-Jung singularity, noted  $A_{3,2}$ ). We obtain a surface  $Z$  that compactifies the affine quadric  $V$  by two curves :  $C_0$  and  $C_3$  (the latter supporting the unique singular point on the surface). After this maximal extraction is made we notice that there exists 4 curves on  $Z$  that correspond to  $K + B$  negative extremal rays :

- The strict transforms of the 2 rules  $D_+$  and  $D_-$  crossing at  $p$  : it is one of these two curves that the Bruno-Matsuki's algorithm imposes to contract (precisely : the one that was a fiber for the chosen structure of Mori fibration on  $\mathbb{P}^1 \times \mathbb{P}^1$ );
- $C_3$ , which is the exceptional divisor associated with the maximal multiplicity (that we have just constructed);
- $C_0$ , which is the strict transform of the diagonal on  $S$  : it is this curve that our algorithm will impose to contract.

This elementary example shows that the log-Sarkisov algorithm proposed by Bruno-Matsuki is not fully satisfying in the sense that there is no reason why it should respect the surface  $V$  (the two authors were well aware of this fact, see [2, problem 4.4]). It would be natural to hope for

an algorithm where all the blow-ups and contractions occur on the boundary divisor. This is such an algorithm, “a variation on the log-Sarkisov theme”, that we propose in this paper.

Our main result reads as follows, where the notion of “admissible compactification” will be defined and discussed on paragraph 1.1 below.

**Theorem 1.** *Let  $f : V \xrightarrow{\sim} V'$  be an isomorphism of smooth quasi-projective surfaces, and let  $S, S'$  be admissible compactifications of  $V$  (or equivalently of  $V'$ ) such that the boundary divisors  $B_S, B_{S'}$  have irreducible support. Then if the induced birational map  $f : S \dashrightarrow S'$  is not an isomorphism, we can decompose  $f$  into a finite sequence of  $n$  links of the following form*

$$\begin{array}{ccc} & Z_i & \\ & \swarrow \quad \searrow & \\ S_{i-1} & & S_i \end{array}$$

where  $S_0 = S, S_1, \dots, S_n = S'$  are admissible compactifications of  $V$  with an irreducible boundary,  $Z_i$  is for all  $i = 1, \dots, n$  an admissible compactification of  $V$  with two boundary components, and  $Z_i \rightarrow S_{i-1}, Z_i \rightarrow S_i$  are the contractions associated with one of the two  $K + B$  negative extremal rays contained in the boundary  $B_{Z_i}$ .

## 1. THE FACTORIZATION ALGORITHM

**1.1. Admissible surfaces.** Here we discuss the class of admissible compactifications, and show that the hypothesis made on the singularities and the geometry of the boundary are, in a sense, optimal. Let us mention that it is relevant to consider quasi-projective surfaces  $V$  and not only affine ones; for instance  $\mathbb{P}^1 \times \mathbb{P}^1$  minus a fiber is a non affine surface with a rich group of automorphisms.

*Singularities.* First of all an automorphism of a quasi-projective normal surface  $V$  extends as an automorphism of the minimal desingularization of  $V$ ; this remark allows us to restrict without loss of generality to the case of a smooth surface  $V$ . On the other hand it is natural to allow some kind of singularities on the compactifications  $S$  of  $V$ , indeed the log-MMP can produce a singular variety after an extremal contraction even if the variety we started with was smooth (this is true even for surfaces). The widest framework where the Mori Program is (essentially) established in arbitrary dimension is the one of pairs  $(Y, B_Y)$  with dlt (divisorially log terminal) singularities. For the general definition of dlt singularities we refer the reader to [12, def. 2.37]; in the case of a pair  $(S, B_S)$  with  $S$  a projective surface,  $B_S = \sum E_i$  a non empty reduced divisor (i.e. all the coefficients of the  $E_i$  are equal to 1) and  $S \setminus B_S$  smooth this is equivalent to ask for the following properties :

- Any singular point  $p$  of  $S$  is a point of  $B_S$  which is not a crossing  $E_i \cap E_j$ ;
- The  $E_i$  are smooth irreducible curves with normal crossings;
- A singular point  $p$  is locally isomorphic to a quotient of  $\mathbb{C}^2$  by a cyclic group, that is to say the Hirzebruch-Jung singularities  $A_{n,q}$  are the only ones allowed (the reader may find a discussion of these classical singularities for instance in [1, p.99]). Furthermore if  $C_1, \dots, C_s$  is the minimal chain of rational curves (each with self-intersection  $\leq -2$ ) that desingularizes  $p$ , this is the first curve  $C_1$  that meets transversally the strict transform of  $B_S$ .

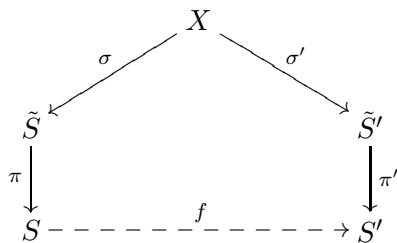
With the same notation, in the ramification formula

$$K_{\tilde{S}} + B_{\tilde{S}} = \pi^*(K_S + B_S) + \sum a_i C_i$$

we have  $a_i > 0$  for all  $i$  (this is in fact the definition of  $p$  being a log terminal singularity). The characterization above comes from [12, prop. 2.42] and from the local description of log terminal singularities when the boundary is reduced and non empty, that can be found in [11, see in particular p.57, case(3)]. The interested reader may consult chapter 3 of [5] by Fujino for a complete discussion on the diverse existing definitions of log terminal singularities.

*Geometry of the boundary.* A first observation is that it is unreasonable to try to extend the statement of the theorem to the case of surfaces with reducible boundary. Let us suppose indeed that  $f : S \dashrightarrow S'$  is a birational map with  $B_S$  reducible. Let  $g : Z \rightarrow S$  be a birational morphism whose exceptional locus  $E$  is irreducible. Then either the image  $g(E)$  is located at the intersection point of two components of  $B_S$ , and the remark 4 (see further, at the end of paragraph 1.2) implies that  $g$  can not be a  $K + B$  negative contraction ; either  $g(E)$  is a general point of a component  $E_i$  of  $B_S$ , and this time the same argument forbids the contraction of  $E_i$  to be  $K + B$  negative. A very simple explicit example is given by the identity map  $\mathbb{C}^2 \rightarrow \mathbb{C}^2$  viewed as a map from  $\mathbb{P}^1 \times \mathbb{P}^1$  to  $\mathbb{P}^2$  : it admits an unique base point located at the intersection of the two lines at infinity, and the blow-up of this point is not a  $K + B$  negative contraction (it is only a  $K$  negative contraction).

A second observation is that the existence of a birational map that is not a morphism  $f : S \dashrightarrow S'$  imposes strong constraints on the irreducible boundary  $E_0 = B_S$ . Let us introduce some notations that will also serve in the proof of the theorem. Let  $\pi : \tilde{S} \rightarrow S$  and  $\pi' : \tilde{S}' \rightarrow S'$  be minimal resolutions of the singularities of  $S$  and  $S'$  respectively; from the characterization of dlt singularities we deduce that the total transforms of  $B_S$  and  $B_{S'}$  are simple normal crossing divisors of  $\tilde{S}$  and  $\tilde{S}'$  respectively. Let  $\tilde{S} \xleftarrow{\sigma} X \xrightarrow{\sigma'} \tilde{S}'$  be a minimal resolution of the base points of  $f$  view as a birational map between  $\tilde{S}$  and  $\tilde{S}'$ .



We still denote by  $E_0$  the strict transform of  $B_S$  in  $X$  or in  $\tilde{S}$ . Remember (see [10, th. 5.2 p.410]) that in general if  $h : M \rightarrow M'$  is a birational map between normal surfaces, and  $p \in M$  is a base point of  $h$ , then there exists a curve  $C \subset M'$  such that  $h^{-1}(C) = p$ . This implies that at every step of the resolution  $\sigma$  of  $f$  there is only one base point, which is the preimage of  $B_{S'}$ . Thus the last blow-up of the sequence  $\sigma$  produces a divisor which is the strict transform of  $B_{S'}$ , and the last blow-up of the sequence  $\sigma'$  produces  $E_0 = B_S$ . That is to say the curve  $E_0$  in  $X$  can be contracted to a smooth point. Therefore  $E_0$  is rational (as are all the other components of  $B_X$ , by construction), and in  $\tilde{S}$  we have  $E_0^2 \geq 0$  because the self-intersection of  $E_0$  should become  $-1$  in  $X$  after a (non-empty) sequence of blow-ups whose at least the first is located on  $E_0$ . Furthermore, after the contraction of  $E_0$  from  $X$  the boundary is still a simple normal

crossing divisor. In consequence,  $E_0$  admits at most two neighboring components in  $B_X$ . This implies that  $B_S$  supports at most two singularities, and if  $B_S$  contains exactly two singularities, then the base point of  $f$  must coincide with one of these singularities.

*Admissible class.* We suppose given a smooth quasi-projective surface  $V$ . In view of the observations above, it is natural to define the class of admissible surfaces as the set of pairs  $(S, B_S)$  with the following properties :

- $S$  is a projective compactification of  $V$ , that is we have a fixed isomorphism  $S \setminus B_S \xrightarrow{\sim} V$ ;
- $B_S = \sum E_i$  is a reduced divisor with each  $E_i$  isomorphic to  $\mathbb{P}^1$ ;
- $(S, B_S)$  admits only dlt singularities;
- If  $B_S$  is irreducible then  $S$  admits at most two singularities.

We allow the possibility of a reducible boundary mainly in order to include the surfaces  $Z_i$  with two boundary components that appear in the theorem. In this case, we will observe in the course of the demonstration that each boundary component supports at most one singularity.

*Remark 2.* The class of admissible surfaces we just defined contains in particular the class of affine surfaces that admit a compactification by a chain of smooth rational curves, which has been studied by Danilov-Gizatullin [8, 9]. Our theorem thus applies to these surfaces. Indeed, each surface of this kind admits at least one compactification by a chain of rational curves  $C_0, C_1, \dots, C_r$ ,  $r \geq 1$ , whose self-intersections are respectively  $0, a_1, \dots, a_r$ , where  $a_1 \leq -1$  and  $a_i \leq -2$  for all  $i = 2, \dots, r$ . After contracting the curves  $C_1, \dots, C_r$ , we obtain an admissible compactification  $S$  with an irreducible boundary  $B_S = C_0$ . These surfaces always admit a very rich automorphism group. In particular, it acts on the surface with an open orbit of finite complement (see [7]).

**1.2. Proof of the theorem.** As above, let  $\pi : \tilde{S} \rightarrow S$  and  $\pi' : \tilde{S}' \rightarrow S'$  denote the minimal resolutions of the singularities and let  $\tilde{S} \xleftarrow{\sigma} X \xrightarrow{\sigma'} \tilde{S}'$  be a minimal resolution of the base points of  $f$ . The divisor  $B_X$  is then a tree of rational curves, whose irreducible components are exceptional for at least one of the two morphisms  $\pi \circ \sigma$  or  $\pi' \circ \sigma'$ , thus they have all a strictly negative self-intersection. Since  $B_X$  is a tree, there exists a unique sub-chain  $E_0, E_1, \dots, E_n$  of  $B_X$  such that  $E_0$  and  $E_n$  are the strict transforms of  $B_S$  and  $B_{S'}$  respectively. The minimality hypotheses imply that  $E_0$  and  $E_n$  are the only irreducible components with self-intersection  $-1$  in  $B_X$ . The demonstration proceeds by induction on the number  $n$  of components in the chain joining the strict transforms of  $B_S$  and  $B_{S'}$ , which will also be the number of links necessary to factorize  $f$ .

We use the same notation for the curves  $E_i$ ,  $i = 0, \dots, n$  and their images or strict transforms in the different surfaces that will come into play. The self-intersection of  $E_0$  is positive in  $\tilde{S}$  by hypothesis. By definition of the resolution  $X$ , the divisor  $E_1$  is produced by blowing-up successively the base points of  $f$  as long as they lie on  $E_0$ ,  $E_1$  being the last divisor produced by this process. Let  $Y \rightarrow \tilde{S}$  be the intermediate surface thus obtained. By construction, the image of the curves contracted by the induced birational morphism  $X \rightarrow Y$  are all located outside  $E_0$ . The self-intersections of  $E_0$  in  $X$  or in  $Y$  must in particular be equal (to  $-1$ ). The divisor  $B_Y$  is then a chain that looks as in figure 2. The wavy curves labelled “Sing” correspond to the (possible) chains of rational curves obtained by desingularization of  $S$ , and the wavy curve labelled “Aux” corresponds to the (possible) chain of auxiliary rational curves, each one with self-intersection  $-2$ , obtained by resolving the base points of  $f$  before getting  $E_1$ .

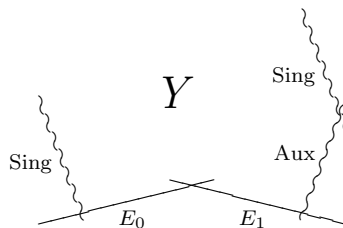
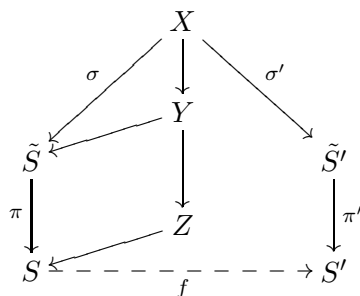
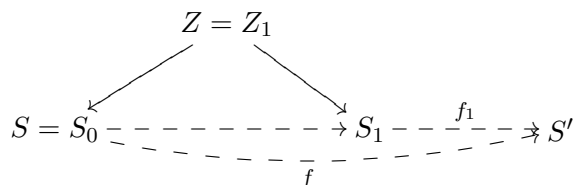


FIGURE 2. The boundary divisor of  $Y$ .

The lemma 3 ensures that running the  $K + B$  MMP we can successively contract all the components of the boundary  $B_Y$  with the exceptions of  $E_0$  and  $E_1$ . Indeed at each step, the extremities of the boundary chain support at most one singularity and thus are  $K + B$  negative, with negative self-intersection. This implies that they give rise to divisorial extremal contractions. We note  $(Z, E_0 + E_1)$  the dlt pair obtained from the pair  $(Y, B_Y)$  by this process.



By construction,  $Z$  dominates  $S$  via the divisorial contraction of the  $K + B$  negative curve  $E_1$ . Again by the lemma 3,  $E_0$  is  $K + B$  negative, with self-intersection strictly negative in  $Z$ . So there exists a  $K + B$  divisorial extremal contraction  $Z \rightarrow S_1$  contracting exactly  $E_0$ . We obtain the first expected link and the map  $f : S \dashrightarrow S'$  factorizes via a birational map  $f_1 : S_1 \dashrightarrow S'$  for which it is straightforward to check that the length of the chain defined at the beginning of the proof is equal to  $n - 1$ .



We conclude by induction that we can factorize  $f$  by exactly  $n$  links. □

**Lemma 3.** *Let  $(S, B_S)$  be an admissible surface.*

- (1) *A curve  $C \subset B_S$  with only one neighboring component in  $B_S$  and supporting at most one singularity of  $S$  is  $K_S + B_S$  negative.*
- (2) *For a curve  $C \subset B_S$  supporting exactly one singularity  $p$  of  $S$  to satisfy  $C^2 < 0$ , it is sufficient that its strict transform  $\bar{C}$  in the minimal resolution of  $p$  satisfies  $\bar{C}^2 < 0$ .*



*Proof.* (1) If  $C$  does not support any singularity of  $S$ , we have

$$(K_S + B_S) \cdot C = (K_S + C) \cdot C + 1 = -2 + 1 = -1$$

by adjunction. Otherwise, let  $\pi : \bar{S} \rightarrow S$  be a minimal resolution of the unique singularity supported by  $C$ . Let  $E$  be the unique exceptional curve of  $\pi$  that meets the strict transform  $\bar{C}$  of  $C$ . We write  $K_{\bar{S}} + B_{\bar{S}} = \pi^*(K_S + B_S) + aE + R$ , where  $R$  (here and further in the proof) denotes an exceptional divisor for  $\pi$ , whose support does not meet  $\bar{C}$ , and all of whose coefficients may vary. We have then

$$\begin{aligned} (K_S + B_S) \cdot C &= (K_S + C) \cdot C + 1 = \pi^*(K_S + C) \cdot \pi^*C + 1 \\ &= (K_{\bar{S}} + \bar{C} + (1 - a)E + R) \cdot \bar{C} + 1 = \pi^*(K_S + C) \cdot \bar{C} + 1 \\ &= (K_{\bar{S}} + \bar{C}) \cdot \bar{C} + 1 - a + 1 = -2 + 2 - a \\ &= -a < 0 \end{aligned}$$

because  $(S, B_S)$  dlt pair implies  $a > 0$ .

(2) Let  $\pi : \bar{S} \rightarrow S$  be as above. We write  $\bar{C} = \pi^*C - bE - R$  and  $K_{\bar{S}} = \pi^*K_S - cE - R$  where  $c \geq 0$  (otherwise  $p$  would be a smooth point) and  $b > 0$ . We have

$$\begin{aligned} C^2 = (\pi^*C)^2 &= (\bar{C} + bE + R)^2 = \bar{C}^2 + 2b + (bE + R)^2 \\ &= \bar{C}^2 + 2b + (bE + R) \cdot (\pi^*C - \bar{C}) = \bar{C}^2 + 2b - bE \cdot \bar{C} \\ &= \bar{C}^2 + b \end{aligned}$$

On the other hand, in the logarithmic ramification formula above, we have  $a = 1 - b - c > 0$  because  $(S, B_S)$  is a dlt pair. So  $1 > b$ , and therefore  $C^2 < \bar{C}^2 + 1$ : this gives the assertion of the lemma.  $\square$

*Remark 4.* The lemma contains what is strictly necessary for the demonstration of the theorem. Nevertheless an easy refinement of the first assertion of the lemma leads to a more precise characterisation of a  $K_S + B_S$  negative boundary component  $C$  of a dlt pair  $(S, B_S)$ . In particular :

*A curve  $C$  in  $B_S$  with at least two neighboring components in  $B_S$  can never be  $K_S + B_S$  negative.*

Indeed, if we note  $n$  the number of neighbors of  $C$  in  $B_S$  and  $p_1, \dots, p_r$  the singular points of  $S$  supported along  $C$ , the same argument as in the proof of the lemma shows that

$$(K_S + B_S) \cdot C = -2 + \sum_{i=1}^r (1 - a_{i,1}) + n,$$

where for all  $i = 1, \dots, r$ ,  $a_{i,1} > 0$  is the log discrepancy of the unique exceptional divisor  $E_{i,1}$  in the minimal resolution of the singular point  $p_i$  that meets the strict transform of  $C$ . As above we can show that the log discrepancies  $a_{i,j} > 0$  of the irreducible components  $E_{i,1}, \dots, E_{i,r_i}$  of the chain of exceptional divisors in the minimal resolution of  $p_i$  are all strictly less than 1 (write  $a_{i,j}$  as  $a_{i,j} = 1 - b_{i,j} - c_{i,j}$  as in the end of the proof of the lemma). If we note  $\pi : \bar{S} \rightarrow S$  the

surface obtained by taking the minimal resolution of all the singular points  $p_i$ , we have

$$\begin{aligned} a_{i,1}E_{i,1}^2 + a_{i,2} &= (\pi^*(K_S + B_S) + \sum_{i,j} a_{i,j}E_{i,j}) \cdot E_{i,1} \\ &= (K_{\bar{S}} + B_{\bar{S}}) \cdot E_{i,1} = (K_{\bar{S}} + \bar{C} + E_{i,1} + E_{i,2}) \cdot E_{i,1} \\ &= -1 + E_{i,2} \cdot E_{i,1} \end{aligned}$$

where, by convention,  $a_{i,2} = E_{i,2} \cdot E_{i,1} = 0$  if  $p_i$  admits a resolution by a blow-up with a unique exceptional divisor  $E_{i,1}$ . In all cases we have  $a_{i,1} \leq 1/2$  because  $E_{i,1}^2 \leq -2$  and  $a_{i,2} < 1$ . Finally we get

$$n - 2 + r > (K_S + B_S) \cdot C \geq n - 2 + \frac{r}{2}.$$

For  $n = 2$ , this gives the assertion above. For  $n = 1$  and  $r = 1$  we get again the first assertion of the lemma. For  $n = 0$  and  $r = 2$  we obtain : if  $B_S = C$  is irreducible and supports two singularities then  $C$  is  $K + B$  negative (possibly without being extremal).

The inequalities above do not *a priori* exclude the possibility for a curve  $C$  without a neighbor and supporting three singularities to be  $K_S + B_S$  negative. But in this case the singularities could no longer be of an arbitrary type. For instance, the computations above shows that an isolated component  $C$  supporting exactly three  $A_{3,1}$  singularities (i.e., each one admits a resolution by a unique exceptional rational curve with self-intersection  $-3$ ) satisfies  $(K_S + B_S) \cdot C = 0$ . On the other hand, let us remember from the paragraph 1.1 that if  $S$  admits more than two singularities then any birational map  $S \dashrightarrow S'$  induced by an automorphism of  $V$  is in fact a morphism.

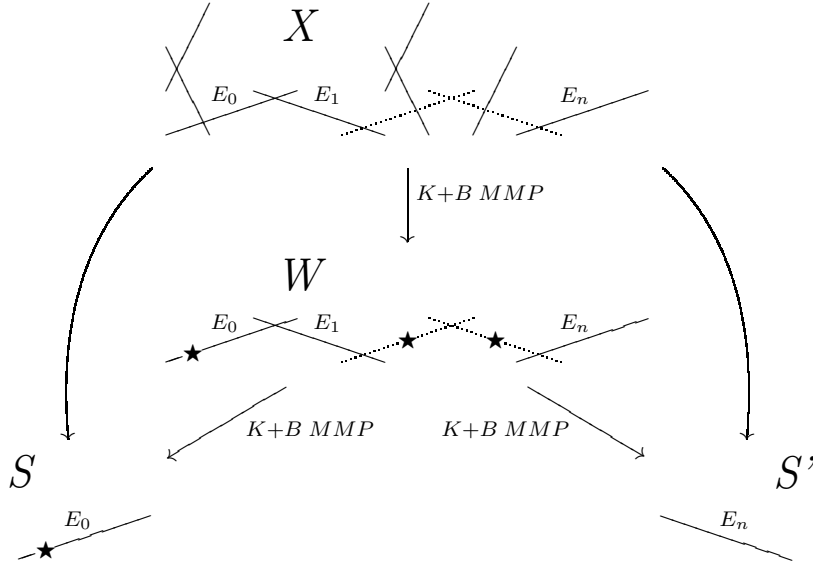
We should finally remark that neither the lemma nor the above argument tell something about the possible  $K + B$  negative curves that do not belong to the boundary : in the example given in the introduction, we had four  $K + B$  negative extremal rays, only two of which were within the boundary.

**1.3. Comments and complements.** Here we discuss additional properties of our algorithm which may lead to a better understanding of the construction and give an insight of the possible generalizations in higher dimensions.

Let us first consider again the subchain  $E_0, \dots, E_n$  of rational curves in the boundary  $B_X$  of  $X$  defined in the proof. Lemma 3 guarantees that all the irreducible components of  $B_X$  except the ones contained in that chain can be successively contracted by a process of the  $K + B$  MMP. The surface  $W$  obtained by this procedure has boundary  $B_W = \sum_{i=0}^n E_i$  and dominates both  $S$  and  $S'$  by a sequence of  $K + B$  negative divisorial contractions : see figure 3 (note that  $W$  is in general singular). It follows that the pairs  $(S, B_S)$  and  $(S', B_{S'})$  considered in the theorem are always log-MMP related in the sense of Matsuki [15, p.128]. This leads to the following result.

**Proposition 1.** *The birational morphism  $Z \rightarrow S$  with exceptional divisor  $E_1$  constructed in the proof of the theorem is a maximal extraction .*

*Proof.* A maximal extraction (see [15, prop. 13-1-8] and [2, p.485] for the logarithmic case) is obtained from a surface which dominates  $S$  and  $S'$  by a process of the  $K + B$ -MMP. So we may start with the surface  $W$  just constructed. The precise procedure consists in running a  $K + B + \frac{1}{\lambda}H$ -MMP over  $S$ , where  $\lambda$  and  $H$  have been defined in the introduction. The crucial observation is that each extremal divisorial contraction of this log-MMP is also a one of the genuine  $K + B$ -MMP (this is obvious for surfaces as  $H$  is nef, but this property actually holds

FIGURE 3. Log-MMP relation between  $S$  and  $S'$ .

in any dimension). The fact that we are running a MMP over  $S$  guarantees that the only curves affected by the procedure are contained in the boundary. By virtue of lemma 3 and remark 4, the only  $K + B$  negative extremal rays contained in a chain are its terminal components. So there exists a unique sequence of  $K + B$  negative divisorial contractions from  $W$  to  $S$ . It follows that the last one  $Z \rightarrow S$ , which has for exceptional divisor  $E_1$ , is a maximal extraction.  $\square$

The Sarkisov program has been initially designed as an algorithm to factorize birational maps between a class of varieties as simple as possible in the context of the Minimal Model Program, namely, Mori fiber spaces. Here we replaced Mori fiber spaces by another class of very simple objects : dlt pairs  $(S, B_S)$  with an irreducible boundary  $B_S$ . It may happen that certain pairs  $(S, B_S)$  also admits a structure of a log Mori fiber space. This holds for instance for admissible compactifications of the affine plane  $\mathbb{C}^2$  by a smooth rational curve. Indeed, the latter admits a trivial Mori fibration  $S \rightarrow pt$ , due to the fact that their Picard group is of rank one. Using Proposition 1 above, it is not difficult to check that for such surfaces our algorithm coincides with the log-Sarkisov program of Bruno-Matsuki. Furthermore, the factorization enjoys the following property.

**Proposition 2.** *Let  $S$  and  $S'$  be admissible surfaces equipped with a structure of trivial Mori fibration and let  $f : S \dashrightarrow S'$  be a birational map extending an automorphism of  $V$ . Then each link of our algorithm strictly decreases the log-Sarkisov degree  $\mu$ .*

*Proof.* Letting  $\mu_S$  and  $\mu_{S_1}$  be the degrees of  $f : S \dashrightarrow S'$  and  $f_1 : S_1 \dashrightarrow S'$  respectively (see the proof of the theorem for the notation), the ramification formulas read

$$\begin{aligned} K_Z + B_Z + \frac{1}{\mu_S} H_Z &= g^*(K_S + B_S + \frac{1}{\mu_S} H_S) + (c - b - \frac{m}{\mu_S}) E_1 \\ &= g^*(K_{S_1} + B_{S_1} + \frac{1}{\mu_S} H_{S_1}) + \star E_0, \end{aligned}$$

where  $\lambda = m/(c - b)$  is the maximal multiplicity and where  $\star$  is a coefficient which plays no role in the sequel. By definition,  $K_S + B_S + \frac{1}{\mu_S} H_S \equiv 0$  and due to the fact that all the curves in  $S_1$  are numerically proportional, we have  $\mu_{S_1} < \mu_S$  provided that  $(K_{S_1} + B_{S_1} + \frac{1}{\mu_S} H_{S_1}) \cdot C < 0$  for an arbitrary curve in  $S_1$ . Since  $f$  is not an isomorphism the logarithmic version of the Noether-Fano criterion [15, prop. 13-1-3] guarantees that  $\lambda > \mu_S$ . Thus, given a curve  $C \subset Z$  intersecting  $E_1$  but not  $E_0$ , one checks that its image in  $S_1$  satisfies

$$\begin{aligned} (K_{S_1} + B_{S_1} + \frac{1}{\mu_S} H_{S_1}) \cdot C &= (g^*(K_{S_1} + B_{S_1} + \frac{1}{\mu_S} H_{S_1}) + \star E_0) \cdot C \\ &= (c - b) \underbrace{(1 - \lambda/\mu_S)}_{<0} \underbrace{E_1 \cdot C}_{>0} < 0 \end{aligned}$$

as desired.  $\square$

## 2. EXAMPLES

**2.1. Automorphisms of  $\mathbb{C}^2$ .** Since our hope is that a possible generalization of our results in higher dimension could help to understand the structure of the automorphism groups of the affine space  $\mathbb{C}^3$ , at least we would expect that the well-known structure of the automorphism group of  $\mathbb{C}^2$  can be recovered using our algorithm. It turns out that this property holds true as expected. Here we just describe without proof the links which appear in the decomposition of an automorphisms of  $\mathbb{C}^2$ . For the details we refer the reader to [13] and [15], which contain proofs of Jung's Theorem derived from the philosophy of (log)-Sarkisov program (but not formulated in the language of Mori Theory in the former).

Let  $f$  be a polynomial automorphism of  $\mathbb{C}^2$  and let  $f = a_0 e_1 \cdots e_n a_n$  be a decomposition of  $f$  into a product of affine automorphisms  $a_i$  and triangular automorphisms  $e_j$ . Up to the composition by the affine automorphism  $a_n^{-1}$ , we may write

$$f = j_1 \cdots j_n = \underbrace{a_0 e_1 a_0^{-1}}_{j_1} \underbrace{a_0 a_1 e_2 (a_0 a_1)^{-1}}_{j_2} \cdots$$

where each automorphism  $j_i$  preserves a foliation of  $\mathbb{C}^2$  by parallel lines. These automorphisms are called de Jonquières automorphisms. Applying our theorem to each  $j_i$ , which is assumed to be of degree  $d_i \geq 2$ , leads to a factorization in  $2d_i - 2$  links of the form

$$\mathbb{P}^2 \longleftrightarrow \mathbb{P}^2(2) \longleftrightarrow \mathbb{P}^2(3) \longleftrightarrow \cdots \longleftrightarrow \mathbb{P}^2(d_i) \longleftrightarrow \cdots \longleftrightarrow \mathbb{P}^2(2) \longleftrightarrow \mathbb{P}^2$$

where each  $\longleftrightarrow$  denotes an elementary link, and where  $\mathbb{P}^2(d)$  denotes the weighted projective plane  $\mathbb{P}^2(d, 1, 1)$ , obtained from the Hirzebruch surface  $\mathbb{F}_d \rightarrow \mathbb{P}^1$  by contracting the section with self-intersection  $-d$ . The automorphism  $j_i$  extends to an automorphism  $\sigma_i$  of  $\mathbb{P}^2(d_i)$ , and the above decomposition can be thought as a conjugation  $j_i = \varphi^{-1} \sigma_i \varphi$ , where  $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2(d_i)$  denotes the birational map induced by the identity on  $\mathbb{C}^2$ .

If the decomposition of  $f$  in  $\text{Aut}(\mathbb{C}^2)$  is reduced, then two successive automorphisms  $j_i$  and  $j_{i+1}$  do not preserve a common foliation. Therefore, the base points of  $j_{i+1}$  et  $j_i^{-1}$  considered as birational endomorphisms of  $\mathbb{P}^2$  are distinct. Under this hypothesis, one verifies that our algorithm produces a factorization of  $f$  which coincides with the one obtained by concatenating the respective decompositions of the  $j_i$ 's.

$$\underbrace{\underbrace{\mathbb{P}^2 \dots \mathbb{P}^2(d_1) \dots \mathbb{P}^2}_{j_1} \dots \underbrace{\mathbb{P}^2 \dots \mathbb{P}^2(d_2) \dots \mathbb{P}^2}_{j_2} \dots \dots \dots \mathbb{P}^2 \dots \mathbb{P}^2(d_n) \dots \mathbb{P}^2}_{j_n}}_f$$

**2.2. Automorphism of the affine quadric surface  $\mathbb{P}^1 \times \mathbb{P}^1 \setminus D$ .** We again consider the birational map  $f : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  given in the introduction. With the notation of the proof of the theorem, the union of the boundary of the resolution  $X$  constructed in the introduction and of the strict transforms  $D_+$  and  $D_-$  of the members of the standard rulings on  $\mathbb{P}^1 \times \mathbb{P}^1$  passing through the base point  $p = ([1 : 0], [1 : 0])$  of  $f$  is described by the figure 4.

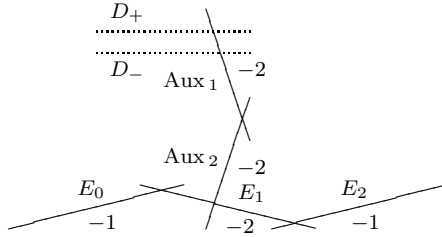
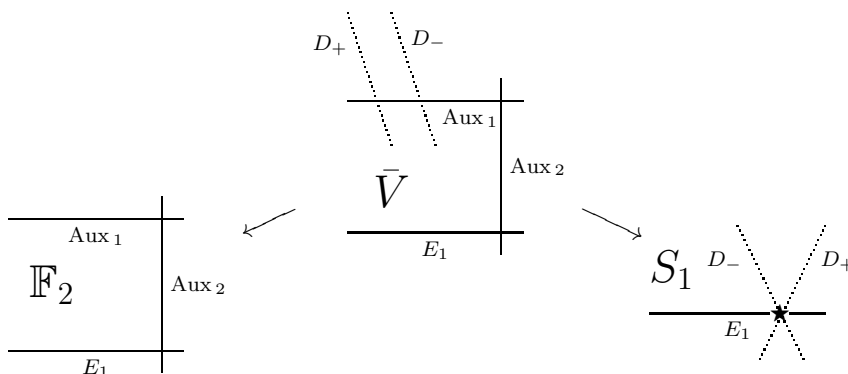


FIGURE 4. Resolution of  $f$ .

Our algorithm gives a factorization  $f : \mathbb{P}^1 \times \mathbb{P}^1 \leftrightarrow S_1 \leftrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ , where the surface  $S_1$  is obtained from  $X$  by contracting the curves  $E_0, E_2$  onto smooth points and the two auxiliary curves onto a singularity of type  $A_{3,2}$  supported on  $E_1$ . The Picard group of  $S_1$  is isomorphic to  $\mathbb{Z}^2$ , generated by the strict transforms of  $D_+$  and  $D_-$ , and the latter also generate the only  $K + B$ -negative extremal rays on  $S_1$ . One checks further that  $S_1$  dominates  $\mathbb{P}^2$  via the divisorial contraction of any of these two curves. So, in contrast with the situation in the log-Sarkisov program of Bruno-Matsuki,  $S_1$  does not admit a Mori fiber space structure.

We may identify the affine quadric  $V = \{w^2 + uv = 1\} \subset \mathbb{C}^3$  with  $\mathbb{P}^1 \times \mathbb{P}^1 \setminus D$  via the open immersion  $(u, v, w) \mapsto ([u : w + 1], [u : 1 - w])$ . Then, the automorphism of  $\mathbb{P}^1 \times \mathbb{P}^1 \setminus D$  induced by  $f$  coincides with the unique automorphism of  $V$  lifting the triangular automorphism  $(u, w) \mapsto (u, w + u^2/2)$  of  $\mathbb{C}^2$  via the birational morphism  $V \rightarrow \mathbb{C}^2$ ,  $(u, v, w) \mapsto (u, w)$ . The latter triangular automorphism uniquely extends to a biregular automorphism  $\phi$  of the Hirzebruch surface  $\mathbb{F}_2 \rightarrow \mathbb{P}^1$  via the open immersion of  $\mathbb{C}^2$  in  $\mathbb{F}_2$  as the complement of the union of a fiber  $E_1$  and of the section  $\text{Aux}_2$  with self-intersection  $-2$ . In turn, the birational morphism  $V \rightarrow \mathbb{C}^2$  lifts to an open immersion of  $V$  into the projective surface  $\bar{V}$  obtained from  $\mathbb{F}_2$  by blowing-up the two points  $q_{\pm} = (0, \pm 1) \subset \mathbb{C}^2 \subset \mathbb{F}_2$  with exceptional divisors  $D_{\pm}$  respectively. The boundary  $B_{\bar{V}}$  consists in the union of the strict transforms of  $\text{Aux}_2, E_1$  and of the fiber  $\text{Aux}_1$  of  $\mathbb{F}_2 \rightarrow \mathbb{P}^1$  containing the points  $q_{\pm}$ .


 FIGURE 5. Sequence of blow-ups and contractions from  $\mathbb{F}^2$  to  $S_1$ .

The automorphism  $\phi$  of  $\mathbb{F}_2$  lifts to an automorphism of  $\bar{V}$  which restricts on  $V$  to the automorphism induced by  $f$ . The latter descends to a biregular automorphism with the same property on the surface isomorphic to  $S_1$  obtained from  $\bar{V}$  by contracting the curves  $\text{Aux}_1$  et  $\text{Aux}_2$ .

More generally, similar arguments show that every automorphism  $f$  of  $V$  induced by a birational transformation

$$(x, y) \mapsto \left( x + P\left(\frac{1}{x+y}\right), y - P\left(\frac{1}{x+y}\right) \right),$$

of  $\mathbb{C}^2$ , where  $P$  is a polynomial of degree  $d \geq 1$ , extends to a biregular automorphism of the projective surface  $\hat{\mathbb{P}}^2(d)$  obtained from the Hirzebruch surface  $\mathbb{F}_d$ , with negative section  $C$ , by first blowing-up two distinct points in a fiber  $F \setminus C$  of  $\mathbb{F}_d \rightarrow \mathbb{P}^1$  and then contracting successively the strict transforms of  $F$  and  $C$ . By construction,  $\hat{\mathbb{P}}^2(d)$  dominates the weighted projective plane  $\mathbb{P}^2(d)$  via the the divisorial contraction of any of the strict transforms of the exceptional divisors of the first blow-up. One checks further that the factorization of  $f : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  given by our algorithm consists of  $2d$  links

$$\mathbb{P}^1 \times \mathbb{P}^1 \leftrightarrow \hat{\mathbb{P}}^2(1) \leftrightarrow \hat{\mathbb{P}}^2(2) \leftrightarrow \dots \leftrightarrow \hat{\mathbb{P}}^2(d) \leftrightarrow \dots \leftrightarrow \hat{\mathbb{P}}^2(1) \leftrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

The automorphism group of  $V$  admits the structure of an amalgamated product analogous to the one of  $\text{Aut}(\mathbb{C}^2)$ , in which the above automorphisms play the role of triangular automorphisms (see [14]). By conjugating by automorphisms of  $V$  which extend to biregular automorphisms of  $\mathbb{P}^1 \times \mathbb{P}^1$ , one obtains the analogues of de Jonquière's automorphisms (see paragraph 2.1). Similarly as in the case of automorphisms of  $\mathbb{C}^2$ , every automorphism  $f$  of  $V$  admits a reduced decomposition into a sequence of automorphisms of de Jonquière's type with the property that every two successive automorphisms appearing in this decomposition have distinct base points on  $\mathbb{P}^1 \times \mathbb{P}^1$ . It follows again that the factorization of  $f$  given by our algorithm is obtained by concatenating the decompositions of its de Jonquière's type factors.

**2.3. Chain inversions.** It is known that if a smooth quasi-projective surface  $V$  admits a smooth compactification by a chain of rational curves with self-intersections  $(-e_1, \dots, -e_k, -1, 0)$ , where  $e_i \geq -2$  for every  $i = 1, \dots, k$ , then it also admits one by a chain of the same length but with

reversed self-intersections  $(-e_k, \dots, -e_1, -1, 0)$  (see *e.g.* [7]). Furthermore two such compactifications are always related by a sequence of elementary transformations with centers on the boundary (see *e.g.* [6] for explicit log-resolutions of these maps). Starting from such chains one can always produce an admissible compactification of  $V$  by first contracting as many successive  $-1$ -curves as possible to smooth points and then contracting the remaining curves with negative self-intersection to a singular point supported on the strict transform of the initial  $0$ -curve, which becomes the boundary.

Here we consider an example which illustrates how these inversions of chains enter the game when one considers a same automorphism of a quasi-projective surface  $V$  as a birational transformation between various admissible compactifications. We let  $V$  be the smooth affine surface in  $\mathbb{C}^4$  defined by the equations

$$\begin{cases} xz = y(y^3 - 1) \\ yu = z(z - 1) \\ xu = (y^3 - 1)(z - 1) \end{cases}$$

One checks that the birational morphism  $\pi_0 : V \rightarrow \mathbb{P}^2$ ,  $(x, y, z, u) \mapsto [x : y : 1]$  lifts to an open immersion of  $V$  into the smooth projective surface  $\bar{V}_0$  obtained from  $\mathbb{P}^2$  with homogeneous coordinates  $[t_0 : t_1 : t_2]$  by first blowing-up four distinct points on the affine line  $L_{0,0} \setminus \{[0 : 1 : 0]\} = \{t_0 = 0\} \setminus \{[0 : 1 : 0]\}$  with exceptional divisors  $D_{0,0}, D_{0,1}, D_{0,2}, D_{0,3}$ , and then blowing-up a point on  $D_{0,0} \setminus L_{0,0}$  with exceptional divisor  $D_{0,4}$ . The boundary  $\bar{V}_0 \setminus V$  (pictured with plain lines on fig. 6) consists of the union of the strict transforms of  $L_{0,0}$ ,  $D_{0,0}$  and the line at infinity  $L_{0,\infty} = \{t_2 = 0\}$  on  $\mathbb{P}^2$ . By contracting the strict transforms of  $L_{0,0}$  and  $D_{0,0}$ , we obtain an admissible compactification  $S_0$  of  $V$ , with a unique singularity of type  $A_{5,2}$  supported on its boundary  $B_{S_0} = L_{0,\infty}$ .

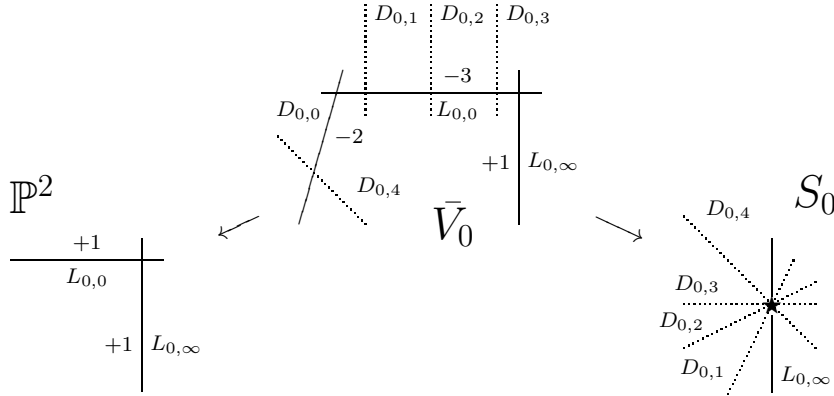


FIGURE 6. Sequence of blow-ups and contractions from  $\mathbb{P}^2$  to  $S_0$ .

A second admissible compactification  $S_2$  of  $V$  can be obtained in a similar way starting from the birational morphism  $\pi_2 : V \rightarrow \mathbb{P}^2$ ,  $(x, y, z, u) \mapsto [u : z : 1]$ . Indeed, one checks that  $\pi_2$  lifts to an open immersion of  $V$  into the smooth projective surface  $\bar{V}_2$  obtained from  $\mathbb{P}^2$  with homogeneous coordinates  $[w_0 : w_1 : w_2]$  by first blowing-up two distinct points on the affine line  $L_{2,0} \setminus \{[0 : 1 : 0]\} = \{w_0 = 0\} \setminus \{[0 : 1 : 0]\}$  with exceptional divisors  $D_{2,0}, D_{2,4}$ , and then

blowing-up three distinct points on  $D_{2,0} \setminus L_{2,0}$  with exceptional divisors  $D_{2,1}$ ,  $D_{2,2}$  and  $D_{2,3}$ . The boundary  $\bar{V}_2 \setminus V$  consists of the union of the strict transforms of  $L_{2,0}$ ,  $D_{2,0}$  and the line at infinity  $L_{2,\infty} = \{w_2 = 0\}$  on  $\mathbb{P}^2$ . By contracting the strict transforms of  $L_{2,0}$  and  $D_{2,0}$ , we obtain an admissible compactification  $S_2$  of  $V$ , with a unique singularity of type  $A_{3,1}$  supported on its boundary  $B_{S_2} = L_{2,\infty}$ .

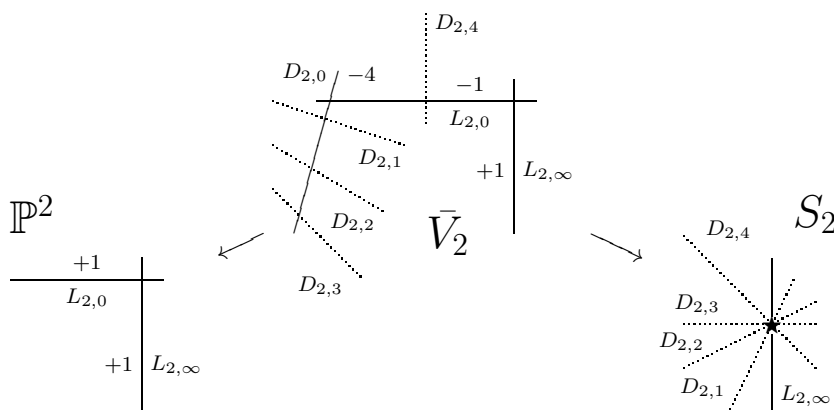


FIGURE 7. Sequence of blow-ups and contractions from  $\mathbb{P}^2$  to  $S_2$ .

The identity morphism  $\text{id} : V \rightarrow V$  induces a birational map  $\sigma : S_0 \dashrightarrow S_2$ . The relations

$$\begin{cases} z = x^{-1}y(y^3 - 1) \\ u = x^{-1}(y^3 - 1)(z - 1) = x^{-2}(y^3 - 1)(y(y^3 - 1) - x) \end{cases}$$

in the function field of  $V$  imply that there exists a commutative diagram

$$\begin{array}{ccc} S_0 & \xrightarrow{\sigma} & S_2 \\ \downarrow & & \downarrow \\ \mathbb{P}^2 & \xrightarrow{g} & \mathbb{P}^2 \end{array}$$

where the vertical arrows denote the natural birational morphisms obtained from the construction of  $S_0$  and  $S_2$  and where  $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$  is the birational map defined by

$$\begin{aligned} g : [t_0 : t_1 : t_2] & \dashrightarrow [w_0 : w_1 : w_2] \\ & = [(t_1^3 - t_2^3)(t_1(t_1^3 - t_2^3) - t_0t_2^3) : t_0t_1t_2^2(t_1^3 - t_2^3) : t_0^2t_2^5] \end{aligned}$$

The point  $p = [1 : 0 : 0] \in L_{0,\infty}$  is a unique base point at infinity of  $g$  and a resolution of  $g$  is obtained by first blowing-up  $p$  with exceptional divisor  $\text{Aux}$ , then blowing-up the point  $\text{Aux} \cap L_{0,\infty}$  with exceptional divisor  $E_1$  and finally blowing-up  $\text{Aux} \cap E_1$  with exceptional divisor  $E_2$ . This resolution lifts to a log-resolution of  $\sigma : S_0 \dashrightarrow S_2$  by performing the same sequence of blow-ups over a nonsingular point of  $B_{S_0} = L_{0,\infty}$  and then taking a minimal resolution of the singularity  $A_{5,2}$  of  $S_0$  by a chain of two rational curves  $C_1, C_2$  (see figure 8).

It follows that the factorization of  $\sigma : S_0 \dashrightarrow S_2$  consists of two links  $S_0 \leftrightarrow S_1 \leftrightarrow S_2$ . Note that the intermediate surface  $S_1$  has two singularities of type  $A_{3,2}$  and  $A_{2,1}$  respectively. By successive



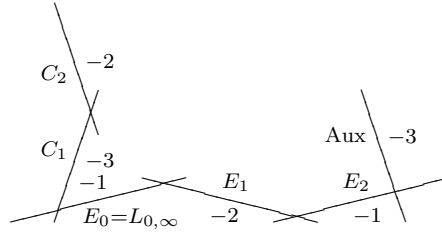


FIGURE 8. Resolution of  $\sigma : S_0 \dashrightarrow S_2$ .

blow-ups, one can obtain from  $S_0$  and  $S_2$  two distinct compactifications of  $V$  by chains of type  $(-2, -4, -1, 0)$  and  $(-4, -2, -1, 0)$  respectively. The birational map  $\sigma : S_0 \dashrightarrow S_2$  corresponds by construction to an inversion of these chains.

Now let  $h : V \xrightarrow{\sim} V$  be the unique automorphism of  $V$  lifting the triangular automorphism  $(u, z) \mapsto (u, z + u^2)$  of  $\mathbb{C}^2$  via the birational morphism  $V \rightarrow \mathbb{C}^2$ ,  $(x, y, z, u) \mapsto (u, z)$ . The birational map  $h_2 = h : S_2 \dashrightarrow S_2$  admits a resolution by four blow-ups with the first one on the singularity. From this, we get a factorization into two links  $S_2 \leftrightarrow S_3 \leftrightarrow S_2$ . One checks that the intermediate surface  $S_3$  is obtained from the weighted projective plane  $\mathbb{P}^2(2)$  by performing a sequence of blow-ups and contractions similar to the one used to construct  $S_2$  from  $\mathbb{P}^2$ , and  $h$  extends to a biregular automorphism of  $S_3$ .

One can also consider  $h$  as a birational transformation  $h_0 = h : S_0 \dashrightarrow S_0$ . Using again the fact that  $h_0$  can be interpreted as a lifting via the natural birational map  $S_0 \dashrightarrow \mathbb{P}^2$  of a suitable birational transformation of  $\mathbb{P}^2$ , one checks that the boundary of a minimal log-resolution of  $h_0$  has the structure pictured on figure 9.

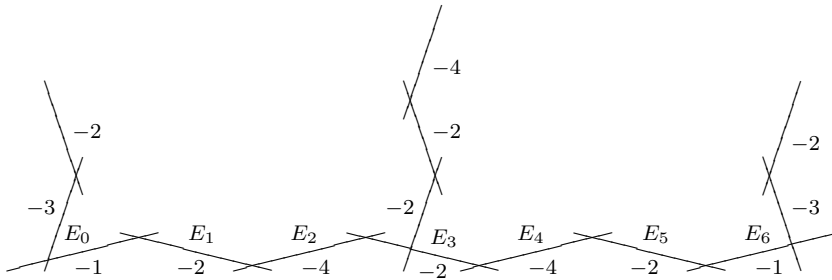


FIGURE 9. Resolution of  $h_0 : S_0 \dashrightarrow S_0$ .

We deduce from this description that the factorization of  $h_0$  consists of six elementary links

$$\underbrace{S_0 \leftrightarrow S_1 \leftrightarrow S_2 \leftrightarrow S_3 \leftrightarrow S_2 \leftrightarrow S_1 \leftrightarrow S_0}_{h_0}$$

$\underbrace{S_0 \leftrightarrow S_1 \leftrightarrow S_2}_{\sigma} \quad \underbrace{S_2 \leftrightarrow S_3 \leftrightarrow S_2}_{h_2} \quad \underbrace{S_2 \leftrightarrow S_1 \leftrightarrow S_0}_{\sigma^{-1}}$

obtained by concatenating the factorizations of  $\sigma : S_0 \dashrightarrow S_2$ ,  $h_2 : S_2 \dashrightarrow S_2$  and  $\sigma^{-1} : S_2 \dashrightarrow S_0$ .

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