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# EXPLICIT PARAMETRIX AND LOCAL LIMIT THEOREMS FOR SOME DEGENERATE DIFFUSION PROCESSES

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*Abstract*

For a class of degenerate diffusion processes of rank 2, i.e. when only Poisson brackets of order one are needed to span the whole space, we obtain a parametrix representation of the density from which we derive some explicit Gaussian controls that characterize the additional singularity induced by the degeneracy.

We then give a local limit theorem with the usual convergence rate for an associated Markov chain approximation. The key point is that the "weak" degeneracy allows to exploit the techniques first introduced in Konakov and Molchanov [KM85] and then developed in [KM00] that rely on Gaussian approximations.

## 1. Introduction.

1.1. *Global overview.* Let us consider in $\mathbb{R}^d,\ d \geq 1$ the Markov diffusion process with generator

$$L = \frac{1}{2} \sum_{i,j \in [\![1,d]\!]^2} a_{ij}(x) \partial^2_{x_i x_j} + \sum_{i \in [\![1,d]\!]} b_i(x) \partial_{x_i}.$$

If the coefficients of $L$ are smooth enough, say $C^1(\mathbb{R}^d)$, bounded, and the diffusion matrix $A(x) = (a_{ij}(x))$ is uniformly elliptic ($\forall \lambda \in \mathbb{R}^d, \langle A\lambda, \lambda \rangle \in [\delta, \delta^{-1}]$ for an appropriate $\delta > 0$) then the associated process $(X_t)_{t \geq 0}$ has a transition density $p(t,x,y)$ which is the fundamental solution of the parabolic

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problem $\partial_t p(.) = L_x p(.),\ p(0,x,y) = \delta_y(x)$. Of course, one also has $\partial_t p(.) = L_y^* p(.),\ p(0,x,y) = \delta_x(y)$.

Moreover, this density satisfies uniformly in $t \in ]0,T]$ the following Gaussian bounds

$$\frac{M^{-1}}{t^{d/2}} \exp\left(-M\frac{|x-y|^2}{t}\right) \leq p(t,x,y) \leq \frac{M}{t^{d/2}} \exp\left(-\frac{|x-y|^2}{Mt}\right),$$

where the constant $M$ depends on $T$, $d$, the ellipticity constant and the norms of the coefficients in $C^1(\mathbb{R}^d)$, see e.g. Aronson [Aro67] or Stroock [Str88].

The above estimations express the following physically obvious fact: if the process starts from $x_0 \in \mathbb{R}^d$, then for small $t > 0$, in the neighborhood of $x_0$ it is "almost Gaussian" with the "frozen" diffusion tensor $A(x_0)$ and the drift $b(x_0)$.

The justification of this fact requires the solution of the perturbative integral equation for $p(\cdot)$ (so-called *Parametrix equation*), where the leading term of the perturbation theory for $p(\cdot)$ is *exactly* the Gaussian kernel $p_0(\cdot)$ corresponding to the "frozen" coefficients at $x_0$. For details concerning *Parametrix equations* we refer the reader to Mc Kean and Singer [MS67], Friedman [Fri64] or [KM85].

If the matrix $A(x)$ degenerates, but the coefficients $a, b$ are still smooth, the diffusion process $(X_t)_{t\geq 0}$ with generator $L$ exists (one can use the Itô calculus for the direct construction of the trajectories), but has generally speaking no density.

Consider now generators of the form $L = \sum_{i=1}^{k} X_i^2 + Y, k < d$, where $(X_i)_{i\in[\![1,k]\!]}, Y$ are first order operators (vector fields) on $\mathbb{R}^d$ (or more generally on smooth manifolds) with $C^\infty$ coefficients. Sufficient conditions for the existence of the density can be formulated in terms of the structure of the Lie algebra of the vector fields on $\mathbb{R}^d$, with usual linear operations and the Poisson bracketing $[.,.]$. Namely, if $\dim(\mathrm{Lie}((X_i)_{i\in[\![1,k]\!]}, Y)) = d$ then the density exists. This result is due to Hörmander [Hör67], see also Norris [Nor86] for a Malliavin calculus based probabilistic proof. Operators having the previous property are said to be hypoelliptic. Also, in [Hör67], Hörmander stressed that the seed of the idea of hypoellipticity goes back to Kolmogorov's note [Kol34].

A. Kolmogorov made the following important observation. Let $d = 2$. For the generator $L = \frac{1}{2}\partial_{xx}^2 + ax\partial_y,\ a \neq 0$, the solution of the associated SDE writes $(X_t, Y_t) = (x_0 + W_t, y_0 + a(x_0 t + \int_0^t W_s ds))$, where $W$ is

a standard one dimensional Brownian motion. Thus $(X_t, Y_t)$ has two dimensional Gaussian distribution with mean $(x_0, y_0 + ax_0t)$ and covariance matrix $C = \begin{pmatrix} t & \frac{at^2}{2} \\ \frac{at^2}{2} & \frac{a^2t^3}{3} \end{pmatrix}$. Note that the transition density for small $t$ has higher singularity than the usual heat kernel. In Hörmander's form $L = \frac{1}{2}X_1^2 + Y$, $X_1 = \partial_x, Y = ax\partial_y$ so that $[X_1, Y] = a\partial_y$ and thus, $X_1, [X_1, Y]$ have together rank 2.

The natural development of the Kolmogorov example consists in taking operators of the form

$$\begin{aligned} L &= \frac{1}{2}\sigma^2(x)\partial_{xx}^2 + b(x)\partial_x + F(x)\partial_y = \frac{1}{2}X_1^2 + Y, \\ X_1 &= \sigma(x)\partial_x,\ Y = (b(x) - \frac{(\sigma\partial_x\sigma)(x)}{2})\partial_x + F(x)\partial_y, \end{aligned}$$

for a uniformly elliptic $\sigma$. One has:

$$[X_1, Y] = \sigma(x)\partial_x(b(x) - (\sigma\partial_x\sigma)(x))\partial_x + \sigma(x)\partial_x F(x)\partial_y.$$

The first term is irrelevant since $\sigma$ is uniformly elliptic. Now, the condition $0 < \delta < \partial_x F(x) \leq \delta^{-1}$ will guarantee the uniform hypoellipticity of $L$ with only the first order brackets.

For a fixed point $x'$ the natural parametrix for $L$ is the operator

$$L_{x'} = \frac{1}{2}\sigma^2(x')\partial_{xx}^2 + b(x')\partial_x + [F(x') + \partial_x F(x')(x - x')]\partial_y.$$

The corresponding transition density $p_{x'}$ has, up to trivial changes, the same nature as in the Kolmogorov example. Anyhow, for the parametrix approach to work, we need to introduce a "compensated" operator $\widetilde{L}_{x,x'} = L_x - F(x')\partial_y$. The term $F(x')\partial_y$ is removed in order to get rid of non-integrable singularities, see Section 2 for details. The analysis of the Volterra type integral equation for the fundamental solution of $\partial_t p = Lp$, based on the identity $p(t, ., *) = p_0(t, ., *) + \int_0^t \int K(s, ., z)p(t - s, z, *)dzds$ for a suitable kernel $K$ is then, up to this compensation more or less standard.

In this paper, we present the corresponding analysis and some associated local limit theorems in the following natural generality.

1.2. *Statement of the problem.* We consider $\mathbb{R}^d \times \mathbb{R}^k$-valued diffusion processes, $k \leq d$, that follow the dynamics

$$\left\{ \begin{array}{l} X_t = x + \displaystyle\int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \\ Y_t = \displaystyle\int_0^t F(X_s)ds, \end{array} \right. \tag{1.1}$$

where $(W_t)_{t\geq 0}$ is a standard $d$-dimensional Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual assumptions. We assume that $b, \sigma, F$ satisfy conditions that guarantee the existence and uniqueness of a strong solution to (1.1).

Concerning the applications, this type of process appears for instance in mathematical finance when dealing with Asian options. In this framework, $X$ represents the dynamics of the underlying asset and $Y$ is involved in the option payoff, see [BPV01] and [Tem01]. Also, if $X$ describes the speed of a particle, the couple, $(X, Y)$ is associated to a speed-position dynamics, see Nelson [Nel67] or Bismut [Bis81].

As mentioned above, equation (1.1) also provides one of the simplest forms of degenerated processes. In a hypoelliptic setting, some authors have studied the behavior of the density, see e.g. Cattiaux [Cat90, Cat91], or Ben Arous and Léandre [BL91] but much remains to be done. In particular none of the above references deals with the simple case of (1.1). The main results are proved under the "strong" Hörmander condition that involves the Poisson brackets of the diffusive part of the process. A characteristic feature of (1.1) is that there is no Brownian term in $Y$. Therefore the strong Hörmander assumption breaks down.

We will work under assumptions that guarantee that Hörmander's theorem is satisfied taking only the first Poisson brackets between the vector fields associated to the drift and the diffusive part in (1.1). Namely, we generalize the sufficient condition $\delta^{-1} \geq \partial_x F(x) \geq \delta > 0$ of the previous paragraph to our current framework. Then, using a parametrix approach, we give an explicit expression of the density. From the parametrix expansion we finally derive some explicit Gaussian bounds that emphasize the additional singularity due to the degeneracy. These bounds are the natural extension to the multidimensional setting of Kolomogorov's example introduced in Section 1.1. In particular the processes $X$ and $Y$ have different characteristic scales.

A natural question then concerns the Markov chain approximation of (1.1). For non degenerated processes this aspect has been widely studied, see e.g. [KM00] for local limit theorems. In [BT96], using Malliavin calculus techniques, Bally and Talay obtain an expansion at order one w.r.t. the time step for the difference of the densities of the diffusion and a perturbed Euler scheme, i.e. the stochastic integrals are approximated by Gaussian variables and an artificial viscosity is added to ensure the discrete scheme has a density. This rate corresponds to the usual "weak error" bound. Since we follow the local limit theorem approach we can handle a wider class of random variables in the approximation but also obtain a rate of order $1/2$ w.r.t the

time step. Similarly to [BT96], we need to introduce an artificial viscosity to ensure the existence of a density for the underlying degenerate Markov chain. We then develop a parametrix approach to express the density of the Markov chain in term of the density of an auxiliary frozen random walk. The random walk is degenerated as well, but we obtain the existence of the density, without any additional perturbation contrarily to the Markov chain, after a sufficiently large number of time steps, see Appendix C for details. Anyhow, this yields to consider two time scales: a "micro" one needed to obtain a density enjoying good properties and a "macro" one, corresponding to the iterations of the "micro" one.

The paper is organized as follows. Our main working assumptions are given in Section 1.3. We fix some notations in Section 1.4. Then, since the form of the Markov chain approximation strongly relies on the proof of our results for the diffusion we choose to divide this paper into two parts. Sections 2 and 3 deal with the results for the diffusion and their proofs. Sections 4 and 5 are dedicated to the Markov chain approximation of (1.1), the associated convergence results and their proofs. The proofs of the most technical parts are postponed to the Appendices.

1.3. *Assumptions.* In the following, for the vector valued function $F = (F_1, ..., F_k)^*$ appearing in (1.1) we denote

$$\overrightarrow{\nabla}_x F(x) \quad = \quad (\nabla_x F_1(x), \nabla_x F_2(x) \cdots \nabla_x F_k(x))^* \in \mathbb{R}^k \otimes \mathbb{R}^d.$$

We also suppose that the coefficients of equation (1.1) satisfy the following assumptions.

**(UE)** $\exists (\lambda_{\min}, \lambda_{\max}) \in (0,\infty)^2,\ \forall z \in \mathbb{R}^d,\ \lambda_{\min}|z|^2 \leq \langle \sigma\sigma^*(x)z, z\rangle \leq \lambda_{\max}|z|^2$.

**(B)** The coefficients $b, \sigma$ in (1.1) are uniformly Lipschitz continuous and bounded.

**(G)** The function $F$ is twice continuously differentiable in $x$ and has bounded derivatives, i.e. $\exists M > 0$, s.t. $\forall x \in \mathbb{R}^d$, $|\overrightarrow{\nabla}_x F(x)| + \sup_{l \in [\![1,k]\!]} |H_{F_l}(x)| \leq M$, where $|.|$ denotes the usual Euclidean norm and $H_{F_l}$ stands for the $\mathbb{R}^d \otimes \mathbb{R}^d$ Hessian matrix of $F_l$.

Also, the Gram matrix $G(x)$

$$G(x) := \begin{pmatrix} \langle \nabla_x F_1(x), \nabla_x F_1(x)\rangle & \cdots & \langle \nabla_x F_1(x), \nabla_x F_k(x)\rangle \\ \vdots & \cdots & \vdots \\ \langle \nabla_x F_k(x), \nabla_x F_1(x)\rangle & \cdots & \langle \nabla_x F_k(x), \nabla_x F_k(x)\rangle \end{pmatrix}$$

is uniformly non degenerated, i.e.

$$\exists (\alpha_{\min}, \alpha_{\max}) \in (0,\infty)^2,\ \forall z \in \mathbb{R}^k,\ \alpha_{\min}|z|^2 \leq \langle G(x)z, z\rangle \leq \alpha_{\max}|z|^2.$$

From now on, unless otherwise indicated we assume **(UE)**, **(B)**, **(G)** are in force.

1.4. *Notations.* Throughout the paper we consider the running diffusion (1.1) up to a fixed final time $T > 0$. We denote by $C$ a generic positive constant that may change from line to line and only depends on $T$, and the parameters appearing in **(UE)**, **(B)**, **(G)**. We reserve the notation $c$ for constants that only depend on parameters from **(UE)**, **(B)**, **(G)**. Other possible dependencies are explicitly indicated.

**2. Explicit parametrix and associated controls for the density of the diffusion.** The assumptions of Section 1.3 guarantee that Hörmander's Theorem, see e.g. Nualart [Nua98], holds true, and therefore that $\forall t > 0,\ (X_t, Y_t)$ has a density w.r.t. the Lebesgue measure. Introduce the vector fields

$$A_0(x) = \begin{pmatrix} b_1(x) \\ \vdots \\ b_d(x) \\ F_1(x) \\ \vdots \\ F_k(x) \end{pmatrix},\ \forall j \in [\![1, d]\!],\ A_j(x) = \begin{pmatrix} \sigma_{1j}(x) \\ \vdots \\ \sigma_{dj}(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.1}$$

We have the following result.

**Proposition 2.1** *For all $x \in \mathbb{R}^d,\ \exists i^*(x) = (i_1^*(x), ..., i_k^*(x)) \in \mathbb{R}^k, 1 \leq i_1^*(x) < i_2^*(x) < ... < i_k^*(x) \leq d$ s.t.*

$$\mathrm{Span}(A_1(x), ..., A_d(x), [A_0(x), A_{i_1^*(x)}(x)], ..., [A_0(x), A_{i_k^*(x)}(x)]) = \mathbb{R}^{d+k},$$

*where $\forall (i,j) \in [\![0, d]\!]^2,\ [A_i, A_j] = A_i \nabla A_j - A_j \nabla A_i$ denotes the Poisson bracket.*

Fix $T > 0$ and $0 \leq s < t \leq T$, $(x, y) \in \mathbb{R}^d \times \mathbb{R}^k$. Since, we now know that $(X_t, Y_t)$ has a transition density, i.e. $\mathbb{P}[X_t \in dx', Y_t \in dy' | X_s = x, Y_s = y] = p(s, t, (x, y), (x', y'))dx'dy'$, our aim is to develop a parametrix for (1.1) to obtain an explicit representation of this density. To this end, as usual with the parametrix techniques we need to introduce a "frozen" diffusion process, $(\widetilde{X}_t, \widetilde{Y}_t)_{t \in [s,T]}$ below. It will be derived from an additional auxiliary process $(X_t, \widehat{Y}_t)_{t \in [s,T]}$ easily related to $(X_t, Y_t)_{t \in [s,T]}$. Namely, for any $s \in [0, T], (x, y) \in \mathbb{R}^d \times \mathbb{R}^k,\ x' \in \mathbb{R}^d$,

$$\begin{cases} dX_t = \sigma(X_t)dW_t + b(X_t)dt,\ X_s = x, \\ d\widehat{Y}_t = [F(X_t) - F(x')]\, dt,\ \widehat{Y_s} = Y_s = y. \end{cases} \tag{2.2}$$

Thus, $\widehat{Y}_t = Y_t - F(x')(t-s), t \in [s,T]$. Clearly, for fixed $(x',y')$ the transition densities $p(s,t,(x,y),(z,v))$ and $\widehat{p}(s,t,(x,y),(z,v))$ of $(X_t,Y_t)$ and $(X_t,\widehat{Y}_t)$ are simply related. Indeed,

$$p(s,t,(x,y),(z,v)) = \widehat{p}(s,t,(x,y),(z,v-F(x')(t-s))).$$

In particular, for $(z,v)=(x',y')$ we obtain

$$p(s,t,(x,y),(x',y')) = \widehat{p}(s,t,(x,y),(x',y'-F(x')(t-s))). \tag{2.3}$$

A first order Taylor approximation in (2.2) then yields the dynamics of the "frozen" compensated process $(\widetilde{X}_t,\widetilde{Y}_t)_{t\in[s,T]}$:

$$\begin{cases} d\widetilde{X}_t = \sigma(x')dW_t + b(x')dt, \widetilde{X}_s = x, \\ d\widetilde{Y}_t^i = \langle \nabla_x F_i(x'), \widetilde{X}_t - x'\rangle dt, \widetilde{Y}_s^i = y^i, \ \forall i \in [\![1,k]\!]. \end{cases} \tag{2.4}$$

Define for all $x\in\mathbb{R}^d$, $a(x) := \sigma\sigma^*(x)$. The processes $(X_t,\widehat{Y}_t)$ and $(\widetilde{X}_t,\widetilde{Y}_t), t\in[s,T]$, have the following generators: $\forall (x,y)\in\mathbb{R}^d\times\mathbb{R}^k$, $\psi\in C^2(\mathbb{R}^d\times\mathbb{R}^k)$,

$$\begin{aligned} \widehat{L}\psi(x,y) &= \Big(\frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\partial^2_{x_ix_j} + \sum_{i=1}^d b_i(x)\partial_{x_i} \\ &\quad + \sum_{i=1}^k [F_i(x)-F_i(x')]\partial_{y_i}\Big)\psi(x,y), \end{aligned} \tag{2.5}$$

$$\begin{aligned} \widetilde{L}\psi(x,y) &= \Big(\frac{1}{2}\sum_{i,j=1}^d a_{ij}(x')\partial^2_{x_ix_j} + \sum_{i=1}^d b_i(x')\partial_{x_i} \\ &\quad + \sum_{i=1}^k \langle \nabla_x F_i(x'), x-x'\rangle \partial_{y_i}\Big)\psi(x,y). \end{aligned} \tag{2.6}$$

From these operators we define for $0\le s<t\le T, ((x,y),(x',y'))\in(\mathbb{R}^d\times\mathbb{R}^k)^2$ the kernel $H$ by

$$H(s,t,(x,y),(x',y')) = (\widehat{L}-\widetilde{L})\widetilde{p}(s,t,(x,y),(x',y')).$$

The next proposition gives the expression of the density $\widehat{p}$ in terms of an infinite sum involving iterated convolutions of the density $\widetilde{p}$ with the kernel $H$. Namely,

**Proposition 2.2 (Parametrix expansion for the compensated process)**

*For all* $0 \le s < t \le T, ((x,y),(x',y')) \in (\mathbb{R}^d \times \mathbb{R}^k)^2$,

$$\widehat{p}(s,t,(x,y),(x',y')) = \sum_{r=0}^{+\infty} \widetilde{p} \otimes H^{(r)}(s,t,(x,y),(x',y')), \tag{2.7}$$

*where* $f \otimes g(s,t,(x,y),(x',y'))$

$$= \int_s^t du \int_{\mathbb{R}^d \times \mathbb{R}^k} f(s,u,(x,y),(z,v)) g(u,t,(z,v),(x',y')) dz dv,$$

$\tilde{p} \otimes H^{(0)} = \tilde{p}$ *and* $H^{(r)} = H \otimes H^{(r-1)}$, $r > 0$ *denotes the* $r$*-fold convolution of the kernel* $H$*.*

The previous Proposition is a direct consequence of the usual parametrix recurrence relations. For the sake of completeness we provide its proof in Section 3, see also [KM00] for details.

Now, since $(\tilde{X}_t, \tilde{Y}_t)_{t \in [s,T]}$ is a Gaussian process, $\widetilde{p}$ and its derivatives are well controlled. The previous expression is the starting point to derive the following

**Theorem 2.1 (Parametrix expansion and associated control)**
*For all* $0 \le s < t \le T, ((x,y),(x',y')) \in (\mathbb{R}^d \times \mathbb{R}^k)^2$, *one has:*

$$\begin{aligned} p(s,t,(x,y),(x',y')) &= \widehat{p}(s,t,(x,y),(x',y' - F(x')(t-s))) \\ &= \sum_{r=0}^{\infty} \widetilde{p} \otimes H^{(r)}(s,t,(x,y),(x',y' - F(x')(t-s))), \end{aligned}$$

*and*

$$\begin{aligned} \exists c, C > 0, \quad & p(s,t,(x,y),(x',y')) \le C(t-s)^{-(d+3k)/2} \\ & \times \exp\left( -c \left[ \tfrac{|x'-x|^2}{t-s} + \tfrac{|y'-y-F(x')(t-s)|^2}{(t-s)^3} \right] \right). \end{aligned} \tag{2.8}$$

## 3. Proof of the main results: diffusion process.

3.1. *Proof of Proposition 2.1.* From (2.1) one has $\forall x \in \mathbb{R}^d$, $\forall j \in [\![1,d]\!]$,

$$[A_0(x), A_j(x)] = \begin{pmatrix} \langle b(x), \nabla_x \sigma_{1j}(x) \rangle - \left\langle \sigma^{(j)}(x), \nabla_x b_1(x) \right\rangle \\ \vdots \\ \langle b(x), \nabla_x \sigma_{dj}(x) \rangle - \left\langle \sigma^{(j)}(x), \nabla_x b_d(x) \right\rangle \\ - \left\langle \nabla_x F_1(x), \sigma^{(j)}(x) \right\rangle \\ \vdots \\ - \left\langle \nabla_x F_k(x), \sigma^{(j)}(x) \right\rangle \end{pmatrix}$$

where $\sigma^{(j)}(x)$ denotes the $j^{\text{th}}$ column of $\sigma(x)$. Thus, according to **(UE)** and the previous expression, to prove the proposition it is sufficient to show that for any $x \in \mathbb{R}^d$ there exists $i^* := i^*(x) \in \mathbb{R}^k,\ 1 \leq i_1^* < i_2^* < ... < i_k^* \leq d$ such that

$$\det \begin{pmatrix} \left\langle \nabla_x F_1(x), \sigma^{(i_1^*)}(x) \right\rangle & \cdots & \left\langle \nabla_x F_1(x), \sigma^{(i_k^*)}(x) \right\rangle \\ \vdots & \dots & \vdots \\ \left\langle \nabla_x F_k(x), \sigma^{(i_1^*)}(x) \right\rangle & \cdots & \left\langle \nabla_x F_k(x), \sigma^{(i_k^*)}(x) \right\rangle \end{pmatrix} \neq 0. \tag{3.1}$$

We prove (3.1) by induction on $k$. For $k = 1$, Assumptions **(UE)** and **(G)** imply that for any $x \in \mathbb{R}^d$ there exists $i^* = i^*(x)$ such that

$$\left\langle \nabla_x F_1(x), \sigma^{(i^*)}(x) \right\rangle \neq 0.$$

Suppose first that for $k = n-1$ and any $x \in \mathbb{R}^d$ (3.1) holds true. Suppose now that for $k = n$ (3.1) does not hold, that is for some $x_0 \in \mathbb{R}^d$

$$\det \begin{pmatrix} \left\langle \nabla_x F_1(x_0), \sigma^{(i_1)}(x_0) \right\rangle & \cdots & \left\langle \nabla_x F_1(x_0), \sigma^{(i_n)}(x_0) \right\rangle \\ \vdots & \dots & \vdots \\ \left\langle \nabla_x F_n(x_0), \sigma^{(i_1)}(x_0) \right\rangle & \cdots & \left\langle \nabla_x F_n(x_0), \sigma^{(i_n)}(x_0) \right\rangle \end{pmatrix} = 0, \tag{3.2}$$

for any $1 \leq i_1 < i_2 < ... < i_n \leq d$ . In particular, we can take $i_1 = i_1^*(x_0), i_2 = i_2^*(x_0), ..., i_{n-1} = i_{n-1}^*(x_0)$ where $i^*(x_0) \in \mathbb{R}^{n-1}$ is the index s.t. (3.1) holds true for $n-1$. Developing the determinant (3.2) in the last column we get

$$\forall i_n \in [\![1, d]\!],\ \left\langle \sigma^{(i_n)}(x_0), \sum_{j=1}^{n} M_j(x_0) \nabla_x F_j(x_0) \right\rangle = 0,$$

where for $j \in [\![1, n]\!]$ the $M_j(x_0)$ are the corresponding minors. The linear independence of the vectors $(\sigma^{(i)}(x_0))_{i \in [\![1,d]\!]}$ implies that $\sum_{j=1}^{n} M_j(x_0) \times \nabla_x F_j(x_0) = 0$. The linear independence of the vectors $(\nabla_x F_j(x_0))_{j \in [\![1,n]\!]}$ then yields $M_j(x_0) = 0,\ \forall j \in [\![1, n]\!]$. In particular $M_n(x_0) =$

$$\det \begin{pmatrix} \left\langle \nabla_x F_1(x_0), \sigma^{(i_1^*)}(x_0) \right\rangle & \cdots & \left\langle \nabla_x F_1(x_0), \sigma^{(i_{n-1}^*)}(x_0) \right\rangle \\ \vdots & \dots & \vdots \\ \left\langle \nabla_x F_{n-1}(x_0), \sigma^{(i_1^*)}(x_0) \right\rangle & \cdots & \left\langle \nabla_x F_{n-1}(x_0), \sigma^{(i_{n-1}^*)}(x_0) \right\rangle \end{pmatrix} = 0$$

which contradicts that (3.1) holds true for $k = n-1$. Thus, (3.1) holds for any $x \in \mathbb{R}^d$. □

3.2. *Proof of Proposition 2.2: parametrix expansion of the compensated process.* From the forward and backward Kolmogorov equations associated to $(\widehat{X},\widehat{Y})$, $(\widetilde{X},\widetilde{Y})$ and denoting by $\widehat{L}^*$ the adjoint of $\widehat{L}$, we have

$$\begin{aligned}
&\widehat{p}(s,t,(x,y),(x',y')) - \widetilde{p}(s,t,(x,y),(x',y')) \\
&= \int_s^t du \frac{\partial}{\partial u} \int_{\mathbb{R}^d\times\mathbb{R}^k} dwdz \widehat{p}(s,u,(x,y),(w,z))\widetilde{p}(u,t,(w,z),(x',y')) \\
&= \int_s^t du \int_{\mathbb{R}^d\times\mathbb{R}^k} dwdz \Big[ \frac{\partial \widehat{p}(s,u,(x,y),(w,z))}{\partial u} \widetilde{p}(u,t,(w,z),(x',y')) \\
&\qquad + \widehat{p}(s,u,(x,y),(w,z)) \times \frac{\partial \widetilde{p}(u,t,(w,z),(x',y'))}{\partial u} \Big] \\
&= \int_s^t du \int_{\mathbb{R}^d\times\mathbb{R}^k} dwdz \Big[ \widehat{L}^* \widehat{p}(s,u,(x,y),(w,z))\widetilde{p}(u,t,(w,z),(x',y')) \\
&\qquad - \widetilde{L}\widetilde{p}(u,t,(w,z),(x',y'))\widehat{p}(s,u,(x,y),(w,z)) \Big] \\
&= \int_s^t du \int_{\mathbb{R}^d\times\mathbb{R}^k} dwdz \widehat{p}(s,u,(x,y),(w,z))(\widehat{L}-\widetilde{L})\widetilde{p}(u,t,(w,z),(x',y')) \\
&= \widehat{p}\otimes H(s,t,(x,y),(x',y')).
\end{aligned}$$

A simple iteration completes the proof. □

3.3. *Proof of Theorem 2.1.* We prove the result for $\widehat{p}$. The statement of the theorem then follows from the explicit shift relation between $\widehat{p}$ and $p$.

The proof is divided into two parts. First an elementary control on the density of $(\widetilde{X},\widetilde{Y})$ is stated in Lemma 3.1. Then, this control is used to control the kernel $H$ and the convolution.

**Step 1: Gaussian control for $(\widetilde{X},\widetilde{Y})$.**

**Lemma 3.1** *There exist constants $c>0, C>0$, s.t. for all multi-index $\alpha,\beta,\gamma$, $|\alpha|\le 3, |\beta|\le 2, |\gamma|\le 1$, $\forall 0\le u<t\le T$, $\forall (w,z),(x',y')\in\mathbb{R}^d\times\mathbb{R}^k$*

$$\begin{aligned}
&|\partial_w^\alpha \partial_z^\beta \partial_{y'}^\gamma \widetilde{p}(u,t,(w,z),(x',y'))| \le C \exp\Big(-\frac{c}{t-u}|x'-w|^2\Big) \\
&\exp\Big(-\frac{c}{(t-u)^3}|y'-z|^2\Big)(t-u)^{-\{(d+3k)/2+|\alpha|/2+3(|\beta|+|\gamma|)/2\}}.
\end{aligned}$$

The proof is postponed to the end of the section.

**Step 2: Control of the kernel.**

To estimate the kernel $H(u,t,(w,z),(x',y'))$ we have to estimate

$$\begin{aligned}&\left(a_{ij}(w)-a_{ij}(x')\right)\partial^2_{w_iw_j}\widetilde{p}(u,t,(w,z),(x',y')),\ (i,j)\in[\![1,d]\!]^2\\&\left(b_i(w)-b_i(x')\right)\partial_{w_i}\widetilde{p}(u,t,(w,z),(x',y')),\ i\in[\![1,d]\!]\end{aligned}$$

and

$$\left[F_i(w)-F_i(x')-\left\langle\nabla_xF_i(x'),w-x'\right\rangle\right]\partial_{z_i}\widetilde{p}(u,t,(w,z),(x',y')),\ i\in[\![1,k]\!].$$

It is easy to get from Lemma 3.1 and **(B)**, i.e. Lipschitz condition for $b(x)$ and $a(x)$, that

$$\begin{aligned}&\left|\left(b_i(w)-b_i(x')\right)\partial_{w_i}\widetilde{p}(u,t,(w,z),(x',y'))\right|\\&\leq\frac{C}{(t-u)^{(d+3k)/2}}\exp\left(-c\left[\frac{|x'-w|^2}{t-u}+\frac{|y'-z|^2}{(t-u)^3}\right]\right),i\in[\![1,d]\!],\\&\left|\left(a_{ij}(w)-a_{ij}(x')\right)\partial^2_{w_iw_j}\widetilde{p}(u,t,(w,z),(x',y'))\right|\\&\leq\frac{C}{(t-u)^{1/2}(t-u)^{(d+3k)/2}}\exp\left(-c\left[\frac{|x'-w|^2}{t-u}+\frac{|y'-z|^2}{(t-u)^3}\right]\right),(i,j)\in[\![1,d]\!]^2,\\&\left|\left[F_i(w)-F_i(x')-\left\langle\nabla_xF_i(x'),w-x'\right\rangle\right]\partial_{z_i}\widetilde{p}(u,t,(w,z),(x',y'))\right|\\&\leq\frac{C}{(t-u)^{1/2}(t-u)^{(d+3k)/2}}\exp\left(-c\left[\frac{|x'-w|^2}{t-u}+\frac{|y'-z|^2}{(t-u)^3}\right]\right),\ i\in[\![1,k]\!].\end{aligned}\tag{3.3}$$

Concerning the convolution w.r.t the second variable below, we note that for $u\in[s,t]$, $\frac{(t-s)^3}{8}<(u-s)^3+(t-u)^3<(t-s)^3$. We finally obtain

$$\begin{aligned}&\left|\widetilde{p}\otimes H(s,t,(x,y),(x',y'))\right|\\&\leq\int_s^t du\int_{\mathbb{R}^d\times\mathbb{R}^k}\widetilde{p}(s,u,(x,y),(w,z))\left|H(u,t,(w,z),(x',y'))\right|dwdz,\\&\leq\int_s^t du\int_{\mathbb{R}^d\times\mathbb{R}^k}\frac{C^2}{(u-s)^{(d+3k)/2}}\exp\left(-c\left[\frac{|w-x|^2}{u-s}+\frac{|z-y|^2}{(u-s)^3}\right]\right)\\&\times\frac{1}{\sqrt{t-u}(t-u)^{(d+3k)/2}}\exp\left(-c\left[\frac{|x'-w|^2}{t-u}+\frac{|y'-z|^2}{(t-u)^3}\right]\right)dwdz\end{aligned}$$

$$\leq C^2\rho B(1,\frac{1}{2})(t-s)^{-(d+3k)/2}\exp\left(-c\left[\frac{|x'-x|^2}{t-s}+\frac{|y'-y|^2}{(t-s)^3}\right]\right),$$

up to a modification of $C$ in the last inequality, where $\rho=(t-s)^{1/2}$ and $B(m,n)=\int_0^1 duu^{m-1}(1-u)^{n-1}$ denotes the $\beta$-function. By induction in $r$,

$$\left|\widetilde{p}\otimes H^{(r)}(s,t,(x,y),(x',y'))\right|\leq C^{r+1}\rho^r B(1,\frac{1}{2})B(\frac{3}{2},\frac{1}{2})\times ...\times B(\frac{r+1}{2},\frac{1}{2})$$

$$\times(t-s)^{-(d+3k)/2}\exp\left(-c\left[\frac{|x'-x|^2}{t-s}+\frac{|y'-y|^2}{(t-s)^3}\right]\right),r\in\mathbb{N}^*. \tag{3.4}$$

This implies that the series representing the density $\widehat{p}(s,t,(x,y),(x',y'))$

$$\widehat{p}(s,t,(x,y),(x',y'))=\sum_{r=0}^{\infty}\widetilde{p}\otimes H^{(r)}(s,t,(x,y),(x',y'))$$

is absolutely convergent and the following estimate holds

$$\begin{aligned}|\widehat{p}(s,t,(x,y),(x',y'))| &\leq C(t-s)^{-(d+3k)/2}\\ &\quad\times\exp\left(-c\left[\frac{|x'-x|^2}{t-s}+\frac{|y'-y|^2}{(t-s)^3}\right]\right).\end{aligned}$$

By the shift relation (2.3) the proof is complete. □

**Proof of Lemma 3.1**. We prove the lemma for $|\alpha|=|\beta|=|\gamma|=0$, i.e. without derivation. The bounds for the derivatives can be deduced in a similar way, recall that $(\widetilde{X},\widetilde{Y})$ is Gaussian, see e.g. Friedman [Fri64]. We get from (2.4) with $x=w$ that for all $s\leq u\leq t\leq T$,

$$\begin{aligned}\widetilde{Y}_t =& w+\int_u^t\overrightarrow{\nabla}_x F(x')(w-x'+b(x')(v-u))dv\\ &+\int_u^t\overrightarrow{\nabla}_x F(x')\sigma(x')(W_v-W_u)dv:=m_{2,u,t}+A_{u,t},\end{aligned}$$

$$m_{2,u,t}=\overrightarrow{\nabla}_x F(x')(w-x')(t-u)+\frac{(\overrightarrow{\nabla}_x Fb)(x')}{2}(t-u)^2. \tag{3.5}$$

For all $(p,q)\in[\![1,d]\!]\times[\![1,k]\!]$ one has

$$Cov(\widetilde{X}_t^p,\widetilde{Y}_t^q)=\mathbb{E}\left[\sum_{l=1}^d\sigma_{pl}(x')(W_t^l-W_u^l)\times\sum_{j=1}^d\mu_{qj}\int_u^t\left(W_\tau^j-W_u^j\right)d\tau\right],$$

where $\mu_i = \mu_i(x') = \sigma^*(x')(\nabla_x F_i(x'))^*, \mu_i = (\mu_i^1, ..., \mu_i^d)^*, |\mu_i| > 0$. Simple calculations imply that

$$\left\langle \mu_i, \int_u^t (W_\tau - W_u) d\tau \right\rangle \backsim \mathcal{N}(0, \frac{(t-u)^3}{3} |\mu_i|^2).$$

Hence, denoting by $\sigma_{(p)}$ the $p-th$ row of the matrix $\sigma(x')$, we have

$$Cov(\widetilde{X}_t^p, \widetilde{Y}_t^q) = \frac{(t-u)^2}{2} \left\langle \mu_q, \sigma_{(p)} \right\rangle.$$

In a similar way, we obtain for all $(j,l) \in [\![1,k]\!]^2$,

$$Cov(\widetilde{Y}_t^j, \widetilde{Y}_t^l) = \frac{(t-u)^3}{3} \langle \mu_j, \mu_l \rangle.$$

Finally we obtain that the covariance matrix $\Sigma_{d+k}$ of the vector $(\widetilde{X}_t, \widetilde{Y}_t)$ is equal to

$$\Sigma_{d+k} \quad = \quad \begin{pmatrix} a(x')(t-u) & \frac{(t-u)^2}{2}\Theta(x') \\ \frac{(t-u)^2}{2}\Theta^*(x') & \frac{(t-u)^3}{3}\mu(x') \end{pmatrix}$$

where $a(x') = \sigma\sigma^*(x')$, $\Theta(x') = \sigma(x')(\mu_1(x') \cdots \mu_k(x')), \forall (i,j) \in [\![1,k]\!]^2$, $(\mu(x'))_{i,j} = \langle \mu_i, \mu_j \rangle(x')$ or equivalently $\mu(x') = (\overrightarrow{\nabla}_x F a \overrightarrow{\nabla}_x F^*)(x')$.

The mean vector of $(\widetilde{X}_t, \widetilde{Y}_t)$ is equal to $(m_{1,u,t}, m_{2,u,t})$, with $m_{1,u,t} = w + b(x')(t-u)$ and $m_{2,u,t}$ as in (3.5). Note that

$$\det \Sigma_{d+k} = \frac{(t-u)^{d+3k}}{4^k} \det \begin{pmatrix} a(x') & \Theta(x') \\ \Theta^*(x') & \frac{4}{3}\mu(x') \end{pmatrix}.$$

Considering the $(d+i)-$th columns, $i \in [\![1,k]\!]$ as the linear combination of the first $d$ columns whose coefficients are components of the vector $\nabla_x F_i(x')$ and the last $(d+i)-$th rows, $i \in [\![1,k]\!]$ as the linear combination of the first $d$ rows with the same coefficients we obtain from the elementary properties of the determinants

$$\det \Sigma_{d+k} = \frac{(t-u)^{d+3k}}{12^k} \det \begin{pmatrix} a(x') & 0 \\ \Theta^*(x') & \mu(x') \end{pmatrix}.$$

Finally, we obtain from **(UE)** and **(G)** that

(3.6)

$$\det \Sigma_{d+k} = \frac{(t-u)^{d+3k}}{12^k} \times \left\{ \prod_{i=1}^d \lambda_i(x') \right\} \times \det(\mu(x')) \geq \frac{(t-u)^{d+3k}}{12^k} \lambda_{\min}^{d+k} \alpha_{\min}^k.$$

To calculate $\exp\left(-\frac{1}{2}\left\langle \Sigma_{d+k}^{-1}Z,Z\right\rangle\right)$, where $Z=(x'-m_{1,u,t},y'-m_{2,u,t})^*\in\mathbb{R}^d\times\mathbb{R}^k$, the main idea is to use a suitable change of variable in order to de-correlate the components associated to $\widetilde{X},\widetilde{Y}$. This also permits us to separate the two different scales for these processes.

Note that $\Sigma_{d+k}=(t-u)T\mathcal{A}(x')T^*$, where

$$T^* = \begin{pmatrix} I_d & \frac{t-u}{2}\overrightarrow{\nabla}_x F(x')^* \\ 0 & \frac{t-u}{2\sqrt{3}}I_k \end{pmatrix}, \mathcal{A}(x') = \begin{pmatrix} a(x') & 0 \\ 0 & \mu(x') \end{pmatrix}$$

Hence,

$$\begin{aligned} \Sigma_{d+k}^{-1} &= \frac{1}{t-u}\left(T^*\right)^{-1}\mathcal{A}^{-1}(x')T^{-1} = \frac{1}{t-u}\begin{pmatrix} I_d & -\sqrt{3}\overrightarrow{\nabla}_x F^*(x') \\ 0 & \frac{2\sqrt{3}}{t-u}I_k \end{pmatrix} \\ &\quad \mathcal{A}^{-1}(x')\begin{pmatrix} I_d & 0 \\ -\sqrt{3}\overrightarrow{\nabla}_x F(x') & \frac{2\sqrt{3}}{t-u}I_k \end{pmatrix}, \\ \mathcal{A}^{-1}(x') &= \begin{pmatrix} a^{-1}(x') & 0 \\ 0 & \mu^{-1}(x') \end{pmatrix}. \end{aligned}$$

Now we have

$$\mathcal{E} := -\left\langle \Sigma_{d+k}^{-1}Z,Z\right\rangle = \frac{1}{(t-u)}\left\langle \mathcal{A}^{-1}(x')(T^{-1}Z),T^{-1}Z\right\rangle.$$

We have

$$\begin{aligned} Z &= (Z_1,Z_2), Z_1 = x'-(w+b(x')(t-u)), \\ Z_2 &= y'-(z+\overrightarrow{\nabla}_x F(x')(w-x')(t-u)+\frac{(\overrightarrow{\nabla}_x Fb)(x')}{2}(t-u)^2). \end{aligned}$$

Thus,

$$T^{-1}Z = \begin{pmatrix} x'-w-b(x')(t-u) \\ \frac{2\sqrt{3}}{t-u}(y'-z+\frac{1}{2}\overrightarrow{\nabla}_x F(x')(x'-w)(t-u)) \end{pmatrix}. \tag{3.7}$$

Exploiting **(UE)** and **(G)**, equation (3.7) then yields

$$\begin{aligned} \mathcal{E} &\leq -\frac{\lambda_{\max}^{-1}}{t-u}|x'-w-b(x')(t-u)|^2 \\ &\quad -\frac{12(\alpha_{\max}\lambda_{\max})^{-1}}{(t-u)^3}|y'-z+\frac{1}{2}\overrightarrow{\nabla}_x F(x')(x'-w)(t-u)|^2. \end{aligned}$$

From **(B)** (boundedness of $b$) and using Young inequalities (i.e. $|ab| \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2},\ \forall \varepsilon > 0,\ (a,b) \in \mathbb{R}^2$), we derive that there exist $c, C > 0$ s.t.

$$\mathcal{E} \leq C - c\left[\frac{|x'-w|^2}{t-u} + \frac{|y'-z|^2}{(t-u)^3}\right]$$

which gives the statement for $|\alpha| = |\beta| = |\gamma| = 0$. □

**4. Markov Chain approximation and associated convergence results.** Because of the degeneracy, one of the main problems in the Markov chain approximation of system (2.2) is to have a density for the discrete models. Following the approach of [KM00], we aim at giving a parametrix expansion of the density of the Markov chain using iterated convolutions of a discrete kernel and the density of a frozen Markov chain. We manage to obtain a density, and the associated required controls for the error analysis, for the natural frozen Markov chain deriving from (2.4) after a sufficient number of time steps, see Proposition C.1. We therefore consider a "macro scale" model corresponding to this number of time steps. For the initial Markov chain at "macro scale", we add an "artificial" noise on the second component to guarantee the existence of the density.

Now, fix $T > 0$, $\widetilde{N} \in \mathbb{N}^*$ and let $\tilde{h} = T/\widetilde{N}$ be the "micro" time discretization step. Let $n \in \mathbb{N}^*$ be large enough so that the natural "frozen" chain associated to (2.4) has a density, see Proposition C.1, and define the "macro" scale time step $h = n\tilde{h}$ and set $N = \widetilde{N}/n \in \mathbb{N}^*$ the total number of "macro" time steps over $[0,T]$.

For all $j \in [\![0, N]\!]$ set $t_j := jh$. For any $(x,y),(x',y') \in \mathbb{R}^d \times \mathbb{R}^k$, $(j,j') \in [\![0,N]\!]^2,\ j < j'$, we define on the time grid $\{t_j, ..., t_{j'}\}$ an $\mathbb{R}^d \times \mathbb{R}^k$ valued Markov chain $(Z^h_{t_i})_{i\in[\![j,j']\!]} = ((X^h_{t_i}, Y^h_{t_i})^*)_{i\in[\![j,j']\!]}$ whose dynamics is given by

$$\begin{aligned} Z^h_{t_j} &= (x,y)^*, \text{ and } \forall i \in [\![j, j'-1]\!], \\ X^h_{t_{i+1}} &= X^h_{t_i} + b(X^h_{t_i})h + \sigma(X^h_{t_i})\sqrt{h}\eta^1_{i+1}, \\ Y^h_{t_{i+1}} &= Y^h_{t_i} + F(X^h_{t_i} + \frac{\gamma_n}{2}b(X^h_{t_i})h + \sigma(X^h_{t_i})\sqrt{h}\eta^2_{i+1})h + h^{3/2+\varepsilon}\eta^3_{i+1}, \end{aligned} \tag{4.1}$$

where $\gamma_n := (1+\frac{1}{n})$ and $\varepsilon > 0$ is an arbitrarily small parameter. The variables $(\vartheta_i)_{i\in(j,j']\!]} := (\eta^1_i, \eta^2_i, \eta^3_i)_{i\in(j,j']\!]}$ are i.i.d. centered $2d+k$-dimensional random variables s.t. for all $i \in (j,j']\!]$, $\eta^3_i$ is independent of $(\eta^1_i, \eta^2_i)$. The density $q_n(\eta_1,\eta_2,\eta_3) = f_n(\eta_1,\eta_2)q(\eta_3)$ of $\vartheta_{j+1}$ satisfies

**(A1)** $\mathbb{E}[\vartheta_{j+1}] = 0$, and $Cov(\vartheta_{j+1}) = \begin{pmatrix} \mathbf{I}_{d\times d} & \frac{1}{2}\gamma_n\mathbf{I}_{d\times d} & \mathbf{0}_{d\times k} \\ \frac{1}{2}\gamma_n\mathbf{I}_{d\times d} & \frac{1}{3}\gamma_n(1+\frac{1}{2n})\mathbf{I}_{d\times d} & \mathbf{0}_{d\times k} \\ \mathbf{0}_{k\times d} & \mathbf{0}_{k\times d} & \mathbf{I}_{k\times k} \end{pmatrix}$.

**(A2)** There exist a positive integer $S'$ and a function $\psi : \mathbb{R}^{2d+k} \to \mathbb{R}$ with $\sup_{u\in\mathbb{R}^{2d+k}} \psi(u) < \infty$ and $\int \|u\|^{2S-6} \psi(u)du < \infty$ for $S = 2(d+k)S' + 4$ such that

$$|D_u^\nu q_n(u)| \leq \psi(u)$$

for all $|\nu| \in [\![0, 4]\!]$.

Also, additionally to **(UE)**, **(G)**, **(A1)** and **(A2)**, we reinforce **(B)** and now assume

**(BS)** The elements of $b(x), \overrightarrow{\nabla}_x F(x), \sigma(x)$ and their first derivatives are continuous and bounded (uniformly in $x$). All these functions are Lipschitz continuous w.r.t. $x$.

**Remark 4.1** *The random variable $\eta^3$ appearing in the $Y^h$ component of equation* (4.1) *is "artificial". Indeed, it is only needed to guarantee the existence of a density for the Markov chain at every time step. Observe that at every time step it yields a negligible contribution in the covariance matrix of $(X^h_{t_i}, Y^h_{t_i})_{i\in(j,j'\rceil}$, see computations below. Similar "artificial viscosity" terms had previously been employed by Bally and Talay [BT96] for degenerated Euler schemes. This is somehow a standard approach in the analysis of discretization schemes for which we do not have easily the existence of the density.*

Now, similarly to the diffusion case we first introduce a compensated Markov chain. For $(x, y) \in \mathbb{R}^d \times \mathbb{R}^k$, $x' \in \mathbb{R}^d$, $(j, j') \in [\![0, N]\!]^2$, $j < j'$ we define $(\widehat{Z}^h_{t_i})_{i\in[\![j,j']\!]} = ((X^h_{t_i}, \widehat{Y}^h_{t_i})^*)_{i\in[\![j,j']\!]}$ by

$$\begin{aligned}
\widehat{Z}^h_{t_j} &= (x, y)^*, \text{ and } \forall i \in [\![j, j' - 1]\!], \\
X^h_{t_{i+1}} &= X^h_{t_i} + b(X^h_{t_i})h + \sigma(X^h_{t_i})\sqrt{h}\widehat{\eta}^1_{i+1}, \\
\widehat{Y}^h_{t_{i+1}} &= \widehat{Y}^h_{t_i} + \left\{ F(X^h_{t_i} + \frac{\gamma_n}{2} b(X^h_{t_i})h + \sigma(X^h_{t_i})\sqrt{h}\widehat{\eta}^2_{i+1}) - F(x') \right\} h \\
&\quad + h^{3/2+\varepsilon}\widehat{\eta}^3_{i+1}, \qquad (4.2)
\end{aligned}$$

where the i.i.d. variables $(\widehat{\vartheta}_i)_{i\in(j,j'-1]} := (\widehat{\eta}^1_i, \widehat{\eta}^2_i, \widehat{\eta}^3_i)_{i\in(j,j'-1]}$ have density $q_n(.)$.

Note that, analogously to the continuous case, the following relation holds in law between the initial Markov chain $(Z^h_{t_i})_{i\in[\![j,j']\!]}$ in (4.1) and the compensated one $(\widehat{Z}^h_{t_i})_{i\in[\![j,j']\!]}$ in (4.2):

$$\forall i \in [\![j, j']\!], \ \widehat{Z}^h_{t_i} \overset{\text{law}}{=} Z^h_{t_i} - \begin{pmatrix} \mathbf{0} \\ (t_i - t_j)F(x') \end{pmatrix}, \qquad (4.3)$$

and therefore, denoting by $p_h(t_j, t_{j'}, (x,y), .)$ (resp. $\widehat{p}_h(t_j, t_{j'}, (x,y), .)$) the density of $Z^h_{t_{j'}}$ (resp. $\widehat{Z}^h_{t_{j'}}$) one has

$$p_h(t_j, t_{j'}, (x,y), (x',y')) = \widehat{p}_h(t_j, t_{j'}, (x,y), (x', y' - \rho^2 F(x'))), \tag{4.4}$$

where $\rho^2 = t_{j'} - t_j$.

We finally need a "frozen" Markov chain, or random walk. For $(x,y) \in \mathbb{R}^d \times \mathbb{R}^k$, $x' \in \mathbb{R}^d$, $(j,j') \in [\![0,N]\!]^2$ we define $\widetilde{Z}^h = (\widetilde{X}^h, \widetilde{Y}^h)$ by

$$\begin{aligned}
\widetilde{Z}^h_{t_j} &= (x,y)^*, \text{ and } \forall i \in [\![j, j'-1]\!], \\
\widetilde{X}^h_{t_{i+1}} &= \widetilde{X}^h_{t_i} + b(x')h + \sigma(x')\sqrt{h}\widetilde{\eta}^1_{i+1}, \\
\widetilde{Y}^h_{t_{i+1}} &= \widetilde{Y}^h_{t_i} + \overrightarrow{\nabla}_x F(x') \left\{ (\widetilde{X}^h_{t_i} - x')h + \frac{\gamma_n}{2} b(x')h^2 + \sigma(x')h^{3/2}\widetilde{\eta}^2_{i+1} \right\} \\
&\quad + h^{3/2+\varepsilon}\widetilde{\eta}^3_{i+1}.
\end{aligned} \tag{4.5}$$

The i.i.d. variables $(\tilde{\eta}^1_i, \tilde{\eta}^2_i, \tilde{\eta}^3_i)_{i \in (j,j']\!]}$ have density $q_n(.)$.

**Remark 4.2** *Note that the models introduced in* (4.2) *and* (4.5) *can seem awkward at first sight. They actually derive from computations that yield the existence of the density for the natural frozen Markov chain associated to* (2.4) *after n "micro" time steps $\tilde{h}$, i.e at the "macro" level with time step $h$. This is developed in Appendix C. The additional perturbation of scale $h^{3/2+\varepsilon}$ in* (4.5) *is needed for the comparison step between the "discrete" generators introduced below.*

From now on, $\widehat{p}_h(t_j, t_{j'}, (x,y), (x',y'))$ and $\widetilde{p}_h(t_j, t_{j'}, (x,y), (x',y'))$ denote the transition densities of the "compensated" Markov chain (4.2) and "frozen" Markov chain (4.5) respectively. Introducing a discrete "analogue" to the generators we derive from the Markov property a relation similar to (2.7) between $\widehat{p}_h$ and $\widetilde{p}_h$.

For a sufficiently smooth function $f$, define $\widehat{L}_h$ and $\widetilde{L}_h$ by

$$\begin{aligned}
&\widehat{L}_h f(t_j, t_{j'}, (x,y), (x',y')) = \\
&h^{-1} \left[ \int \widehat{p}_{h,j}((x,y),(u,v)) f(t_{j+1}, t_{j'}, (u,v), (x',y')) du dv \right. \\
&\qquad \left. - f(t_{j+1}, t_{j'}, (x,y), (x',y')) \right],
\end{aligned}$$

$$\begin{aligned}
&\widetilde{L}_h f(t_j, t_{j'}, (x,y), (x',y')) = \\
&h^{-1} \left[ \int \widetilde{p}^{x'}_{h,j}((x,y),(u,v)) f(t_{j+1}, t_{j'}, (u,v), (x',y')) du dv \right. \\
&\qquad \left. - f(t_{j+1}, t_{j'}, (x,y), (x',y')) \right],
\end{aligned}$$

where $\widehat{p}_{h,j}((x,y),(u,v)) = \widehat{p}_h(t_j, t_{j+1}, (x,y),(u,v))$ and $\widetilde{p}^{x'}_{h,j}((x,y),\cdot)$ denotes the conditional density of $\widetilde{Z}^h_{t_{j+1}}$ given $\widetilde{Z}^h_{t_j} = (x,y)^*$. Note that because of technical reasons, there is a shift in time in the above definitions, i.e. the time is $t_{j+1}$, instead of the "expected" $t_j$, in the right hand side of the previous equations.

A discrete analogue $H_h$ of the kernel $H$ is defined as

$$H_h(t_j, t_{j'}, (x,y),(x',y')) = (\widehat{L}_h - \widetilde{L}_h)\widetilde{p}_h(t_j, t_{j'}, (x,y),(x',y')),\ j < j'.$$

From the previous definition

$$H_h(jh, j'h, (x,y),(x',y')) = h^{-1} \times$$
$$\int \left[\widehat{p}_{h,j}((x,y),(u,v)) - \widetilde{p}^{x'}_{h,j}((x,y),(u,v))\right] \widetilde{p}_h(t_{j+1}, t_{j'}, (u,v),(x',y'))dudv.$$

Analogously to Lemma 3.6 in [KM00] we obtain the following result.

**Proposition 4.1 (Parametrix for Markov chain)** .
*Assume* **(UE)**, **(BS)**, **(G)**, **(A1-2)** *are in force. Then, for* $0 \le t_j < t_{j'} \le T$,

$$\widehat{p}_h(t_j, t_{j'}, (x,y),(x',y')) = \sum_{r=0}^{j'-j} \left(\widetilde{p}_h \otimes_h H_h^{(r)}\right)(t_j, t_{j'}, (x,y),(x',y')), \tag{4.6}$$

*where the discrete time convolution type operator* $\otimes_h$ *is defined by*

$$(g \otimes_h f)(t_j, t_{j'}, (x,y),(x',y'))$$
$$= \sum_{i=j}^{j'-1} h \int g(t_j, t_i, (x,y),(u,v)) f(t_i, t_{j'}, (u,v),(x',y'))dudv,$$

$\widetilde{p}_h \otimes_h H_h^{(0)} = \widetilde{p}_h$ *and* $H_h^{(r)} = H_h \otimes_h H_h^{(r-1)}$ *denotes the* $r$*-fold discrete convolution of the kernel* $H_h$*. W.r.t. to the above definition, we use the convention that* $\widetilde{p}_h \otimes_h H_h^{(r)}(t_j, t_j, x, y) = 0, r \ge 1$.

Now (4.6) and (2.7) have the same form. Comparing these two expressions we obtain the following local limit Theorem.

**Theorem 4.1 (Local limit Theorem for the densities)** .
*Assume* **(UE)**, **(BS)**, **(G)**, **(A1-2)** *hold true. Then,*

$$\sup_{(x,y),(x',y')\in\mathbb{R}^d\times\mathbb{R}^k} \chi_{\sqrt{T}}(x'-x, y'-y-TF(x'))^{-1} \times |(p_h - p)(0,T,(x,y),(x',y'))|$$
$$= O(h^{1/2}),$$

*where $p_h$ denotes the density of the Markov chain* (4.1) *and* $\forall(\rho,u,v)\in\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^k$,

$$\chi_\rho(u,v)=\rho^{-(d+3k)}\chi(u/\rho,v/\rho^3),\ \chi(u,v)=\left(1+(|u|^2+|v|^2)^{S'-1}\right)^{-1}.$$

Note from the above result that the bigger is $S'$, the better is the control on the tails.

**5. Proof of the local limit Theorem for the Markov Chain.** This section is devoted to the proof of Theorem 4.1. Our aim is now to compare (2.7) and (4.6). Thanks to (2.3) and (4.4), it is sufficient to prove the theorem for the compensated diffusion and Markov chain.

**Step 1.** The first step consists in comparing the discrete and continuous frozen densities $\widetilde{p}_h(t_j,t_{j'},(x,y),(x',y'))$ and $\widetilde{p}(t_j,t_{j'},(x,y),(x',y'))$.

**Lemma 5.1** *There exists $C>0$, s.t. for all $(j,j')\in[\![0,N]\!]^2,\ j<j',\rho^2:=t_{j'}-t_j$,*

$$\left|(\widetilde{p}_h-\widetilde{p})(t_j,t_{j'},(x,y),(x',y'))\right|\le Ch^{1/2}\rho^{-1}\zeta_\rho(x'-x,y'-y),\tag{5.1}$$

*where $\zeta_\rho(u,v)=\rho^{-(d+3k)}\zeta(u/\rho,v/\rho^3),\zeta(u,v)=\frac{1}{1+[|u|^2+|v|^2]^{(S-4)/2}}$, $S$ being introduced in* **(A2)**.

**Proof.** Iterating (4.5) from $t_j$ till $t_{j'}$ we get

$$\begin{aligned}
\widetilde{X}^h_{t_{j'}} &= x+b(x')\rho^2+\sigma(x')\rho\{\frac{1}{(j'-j)^{1/2}}\sum_{k=0}^{j'-j-1}\widetilde{\eta}^1_{j+k+1}\}\\
\widetilde{Y}^h_{t_{j'}} &= y+\overrightarrow{\nabla}_x F(x')(x-x')\rho^2+\frac{\rho^4}{2}\overrightarrow{\nabla}_x F(x')b(x')(1+\frac{1}{n(j'-j)})\\
&\quad+\overrightarrow{\nabla}_x F(x')\sigma(x')\rho^3\left\{\frac{1}{(j'-j)^{1/2}}\sum_{k=0}^{j'-j-1}\widetilde{\eta}^2_{j+k+1}\frac{1}{j'-j}\right.\\
&\quad\left.+\frac{1}{(j'-j)^{1/2}}\sum_{k=0}^{j'-j-1}\widetilde{\eta}^1_{j+k+1}(1-\frac{k+1}{j'-j})\right\}+\frac{\rho^3h^\varepsilon}{(j'-j)^{3/2}}\sum_{k=0}^{j'-j-1}\widetilde{\eta}^3_{j+k+1}.
\end{aligned}\tag{5.2}$$

Introduce

$$\begin{aligned}
m_{j,j'} &= \begin{pmatrix} x+b(x')\rho^2\\ y+\overrightarrow{\nabla}_x F(x')(x-x')\rho^2+\frac{\rho^4}{2}\overrightarrow{\nabla}_x F(x')b(x')\gamma_{n,j,j'}\end{pmatrix}\\
&:= \begin{pmatrix} m^1_{j,j'}\\ m^2_{j,j'}\end{pmatrix},\ \gamma_{n,j,j'}:=1+\frac{1}{n(j'-j)},
\end{aligned}$$

and

$$\Theta_{j,j'} := \left( \begin{array}{c} \sigma(x')\{\frac{1}{(j'-j)^{1/2}} \sum_{k=0}^{j'-j-1} \widetilde{\eta}^1_{j+k+1}\} \\ \overrightarrow{\nabla}_x F(x')\sigma(x') \left\{ \frac{1}{(j'-j)^{1/2}} \sum_{k=0}^{j'-j-1} \widetilde{\eta}^2_{j+k+1} \frac{1}{j'-j} \right. \\ \left. + \frac{1}{(j'-j)^{1/2}} \sum_{k=0}^{j'-j-1} \widetilde{\eta}^1_{j+k+1}(1 - \frac{k+1}{j'-j}) \right\} + \frac{h^\varepsilon}{(j'-j)^{3/2}} \sum_{k=0}^{j'-j-1} \widetilde{\eta}^3_{j+k+1} \end{array} \right).$$

The dynamics of (4.5) thus writes

$$\left( \begin{array}{c} \widetilde{X}^h_{t_{j'}} \\ \widetilde{Y}^h_{t_{j'}} \end{array} \right) = m_{j,j'} + \left( \begin{array}{cc} \rho \mathbf{I}_{d\times d} & \mathbf{0}_{d\times k} \\ \mathbf{0}_{k\times d} & \rho^3 \mathbf{I}_{k\times k} \end{array} \right) \Theta_{j,j'}.$$

Note now that

$$V_{j,j'} := Cov(\Theta_{j,j'}) = \tag{5.3}$$
$$\left( \begin{array}{cc} a(x') & \frac{\gamma_{n,j,j'} a(x') \overrightarrow{\nabla}_x F(x')^*}{2} \\ \frac{\gamma_{n,j,j'} \overrightarrow{\nabla}_x F(x') a(x')}{2} & \mu(x')(\frac{1}{3} + \frac{1}{2(j'-j)n}(1 + \frac{1}{3(j'-j)n})) + \frac{h^{2\varepsilon}}{(j'-j)^2} \mathbf{I}_{k\times k} \end{array} \right)$$

where $\mu(x') = \overrightarrow{\nabla}_x F(x') a(x') \overrightarrow{\nabla}_x F(x')^*$. Thus, for $h$ small enough, the covariance matrix $V_{j,j'}$ is uniformly invertible w.r.t. the parameters $n, j, j' \in \mathbb{N}^*$. Indeed,

$$\begin{aligned} V_{j,j'} &= \widetilde{T}_n \left( \begin{array}{cc} a(x') & \mathbf{0}_{d\times k} \\ \mathbf{0}_{k\times d} & \mu(x') + \frac{12 h^{2\varepsilon}}{(j'-j)^2(1+\alpha)^2} \mathbf{I}_{k\times k} \end{array} \right) \widetilde{T}_n^*, \\ \widetilde{T}_n &= \left( \begin{array}{cc} \mathbf{I}_{d\times d} & \mathbf{0}_{d\times k} \\ \frac{\overrightarrow{\nabla}_x F(x') \gamma_{n,j,j'}}{2} & \frac{1+\alpha}{2\sqrt{3}} \mathbf{I}_{k\times k} \end{array} \right), \end{aligned}$$

where $1 + \alpha = \left(1 - \frac{1}{n^2(j'-j)^2}\right)^{1/2}$. Hence, setting

$$\begin{aligned} V_{j,j'}^{-1/2} &= \left( \begin{array}{cc} (\sigma)^{-1}(x') & \mathbf{0}_{d\times k} \\ \mathbf{0}_{k\times d} & (\mu_\varepsilon)^{-1/2}(x') \end{array} \right) \tilde{T}_n^{-1}, \\ \widetilde{T}_n^{-1} &= \left( \begin{array}{cc} \mathbf{I}_{d\times d} & \mathbf{0}_{d\times k} \\ -\frac{\sqrt{3} \overrightarrow{\nabla}_x F(x') \gamma_{n,j,j'}}{(1+\alpha)} & \frac{2\sqrt{3}}{(1+\alpha)} \mathbf{I}_{k\times k} \end{array} \right) \end{aligned}$$

where $\mu_\varepsilon^{1/2}(\mu_\varepsilon^{1/2})^*(x') = \mu(x') + \frac{h^{2\varepsilon}}{(j'-j)^2(1+\alpha)^2}\mathbf{I}_{k\times k}$, and denoting by $g_n$ the density of the normalized sum $V_{j,j'}^{-1/2}\Theta_{j,j'}$ we derive

$$\widetilde{p}_h(t_j,t_{j'},(x,y),(x',y')) = \frac{1}{\rho^{d+3k}\det(V_{j,j'}^{1/2})} g_n\left(V_{j,j'}^{-1/2}\begin{pmatrix}\frac{x'-m^1_{j,j'}}{\rho}\\ \frac{y'-m^2_{j,j'}}{\rho^3}\end{pmatrix}\right).$$

Applying the Edgeworth expansion for $g_n$ (see Lemma 3.8 in [KM00] for the details, the key tool is the normal approximation of Bhattacharya and Rao, Theorem 19.3 in [BR76]) we obtain

$$\left|\widetilde{p}_h(t_j,t_{j'},(x,y),(x',y')) - \frac{1}{\rho^{d+3k}\det(V_{j,j'}^{1/2})} g_G\left(V_{j,j'}^{-1/2}\begin{pmatrix}\frac{x'-m^1_{j,j'}}{\rho}\\ \frac{y'-m^2_{j,j'}}{\rho^3}\end{pmatrix}\right)\right| \leq Ch^{1/2}\rho^{-1}\zeta_\rho(x'-x,y'-y), \tag{5.4}$$

where $g_G$ stands for the standard $d+k$ dimensional Gaussian density. To conclude the proof, recall from the proof of Lemma 3.1 that

$$\widetilde{p}(t_j,t_{j'},(x,y),(x',y')) = \frac{1}{\rho^{d+3k}\det(C_{j,j'}^{1/2})} g_G\left(C_{j,j'}^{-1/2}\begin{pmatrix}\frac{x'-m^1_{C,j,j'}}{\rho}\\ \frac{y'-m^2_{C,j,j'}}{\rho^3}\end{pmatrix}\right) \tag{5.5}$$

where

$$\begin{aligned} m_{C,j,j'} &= \begin{pmatrix} x + b(x')\rho^2 \\ y + \overrightarrow{\nabla}_x F(x')(x-x')\rho^2 + \frac{\rho^4}{2}\overrightarrow{\nabla}_x F(x')b(x')\end{pmatrix} \\ &:= \begin{pmatrix} m^1_{C,j,j'} \\ m^2_{C,j,j'}\end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} C_{j,j}^{-1/2} &= \begin{pmatrix} (\sigma)^{-1}(x') & \mathbf{0}_{d\times k} \\ \mathbf{0}_{k\times d} & (\mu)^{-1/2}(x')\end{pmatrix}\widetilde{T}^{-1}, \\ \widetilde{T}^{-1} &= \begin{pmatrix} \mathbf{I}_{d\times d} & \mathbf{0}_{d\times k} \\ -\sqrt{3}\overrightarrow{\nabla}_x F(x') & 2\sqrt{3}\mathbf{I}_{k\times k}\end{pmatrix}. \end{aligned}$$

The result eventually follows from (5.4), (5.5) and standard computations, involving the explicit expression of $1+\alpha$, $\gamma_{n,j,j'}$ and the mean value theorem. □

**Step 2. Difference of the kernels**

From now on, we use the following notations for multi-indices and powers. For $\nu = (\nu_1, ..., \nu_{d+k}) \in \mathbb{N}^{d+k}$, $(x, y) = (x_1, ..., x_d, y_1, ..., y_k)^*$ set

$$|\nu| = \nu_1 + ... + \nu_{d+k},\ \nu! = \nu_1!...\nu_{d+k}!,$$
$$(x,y)^\nu = x_1^{\nu_1}... x_d^{\nu_d} y_1^{\nu_{d+1}}... y_k^{\nu_{d+k}}, D^\nu = D_{x_1}^{\nu_1}...D_{x_d}^{\nu_d} D_{y_1}^{\nu_{d+1}}...D_{y_k}^{\nu_{d+k}}.$$

We first give some controls for the kernel $H_h(t_j, t_{j'}, (x, y), (x', y'))$. Namely, the following Lemma states that the difference between $H_h$, $(\widetilde{L} - \widehat{L})\widetilde{p}_h$ and an additional remainder term $M_h$ is small, i.e. has the order announced in Theorem 4.1.

**Lemma 5.2 (Control of the discrete kernel)**

$$\left|(H_h - K_h - M_h)(t_j, t_{j'}, (x, y), (x', y'))\right| \leq Ch^{1/2}\rho^{-1}\zeta_\rho(x' - x, y' - y) \tag{5.6}$$

*where $\zeta_\rho$ is as in Lemma 5.1 and for $j < j' - 1$,*

$$K_h(t_j, t_{j'}, (x, y), (x', y')) = (\widehat{L} - \widetilde{L})\widetilde{p}_h(t_j, t_{j'}, (x, y), (x', y')),$$

*i.e. $K_h$ is the difference of the generators associated to the compensated and frozen diffusion processes between $t_j$ and $t_{j'}$,*

$$M_h(t_j, t_{j'}, (x, y), (x', y')) = \sum_{k=1}^{4} M_h^k(t_j, t_{j'}, (x, y), (x', y')), \tag{5.7}$$

*where the $(M_h^k)_{k\in[\![1,4]\!]}$ are defined in the appendix.*

*For $j = j' - 1$ we set $K_h(t_j, t_{j+1}, (x, y), (x', y')) = 0$,*

$$M_h(t_j, t_{j+1}, (x, y), (x', y')) = H_h(t_j, t_{j+1}, (x, y), (x', y')).$$

The proof is postponed to the appendix. From this proof one also derives that the terms appearing in Lemma 5.2 are controlled with the following:

**Lemma 5.3** *There exists a constant $C$ s.t. for all $j < j'$, for all $(x, y)$ and $(x', y')$ in $\mathbb{R}^d \times \mathbb{R}^k$*

$$\begin{aligned}(|K_h| \quad + \quad & |M_h| + \sum_{i=1}^{4} |M_h^i| + |H_h|)(t_j, t_{j'}, (x, y), (x', y') \\ & \leq C(\rho^{-1}\mathbb{I}_{t_{j'} > t_j + h} + \rho^{-(1+2\varepsilon)}\mathbb{I}_{t_{j'} = t_j + h})\zeta_\rho\left(x' - x, y' - y\right),\end{aligned}$$

*with $\zeta_\rho$ as in Lemma 5.1. Here again $\rho = \sqrt{t_{j'} - t_j}$.*

The key fact is that the previous bound provides an integrable singularity in $\rho$.

**Step 3. Comparison of the parametrix expansions for the compensated diffusion and Markov chain.** We first state an auxiliary result concerning the behavior of the iterated discrete kernel applied to the density of the frozen Markov chain.

**Lemma 5.4** *There exists a constant $C$ (that does not depend on $(x,y)$ and $(x',y')$) such that, for all $j<j'$, $r\in[\![0,j'-j]\!]$,*

$$\left|\left(\widetilde{p}_h\otimes_h H_h^{(r)}\right)(t_j,t_{j'},(x,y),(x',y'))\right|\leq\frac{C^{r+1}\ \rho^{r(1-2\varepsilon)}}{\Gamma\left(1+\frac{r}{2}-r\varepsilon\right)}\chi_\rho\left(x'-x,y'-y\right)$$

*for $0\leq j<j'\leq N$, where $Nh=T$, $\chi_\rho$ and $S'$ are as in Theorem 4.1.*

To prove the lemma it is sufficient to repeat the proof of Lemma 3.11 in [KM00] with obvious modifications concerning the additional arguments $y'-y$ and taking into account the control of Lemma 5.3 for $H_h$ that yields a different statement compared to the quoted Lemma.

**Lemma 5.5** *For $0\leq j<j'\leq N$ the following formula holds:*

$$\widehat{p}_h(t_j,t_{j'},(x,y),(x',y'))=\sum_{r=0}^{j'-j}\left(\widetilde{p}\otimes_h\left(M_h+K_h\right)^{(r)}\right)(t_j,t_{j'},(x,y),(x',y'))+R,$$

*where $|R|\leq Ch^{1/2}\rho^{-1}\chi_\rho(x'-x,y'-y)$ for some constant $C$. The function $\chi_\rho$ is as in Theorem 4.1.*

The proof follows from Lemmas 5.1 and 5.3 and is analogous to the proof of Lemma 3.13. in [KM00]. □

Let us now compare the parametrix expansions of the compensated diffusion and Markov chain for $t_j=0,t_{j'}=T$. From Proposition 2.2, (3.4) and Stirling's asymptotic formula for the $\Gamma$ function we have

$$\widehat{p}(0,T,(x,y),(x',y'))=\sum_{r=0}^{N}\left(\widetilde{p}\otimes H^{(r)}\right)(0,T,(x,y),(x',y'))+R_1, \tag{5.8}$$

where $|R_1|\leq Ch^{1/2}\Lambda_{\sqrt{T}}(x'-x,y'-y)$, $\forall(u,v)\in\mathbb{R}^d\times\mathbb{R}^k$, $\Lambda_{\sqrt{T}}(u,v)=T^{-(d+3k)/2}\exp\left(-C[\left|\frac{u}{T^{1/2}}\right|^2+\left|\frac{v}{T^{3/2}}\right|^2]\right)$, and by Lemma 5.5

$$\widehat{p}_h(0,T,(x,y),(x',y'))=\sum_{r=0}^{N}\left(\widetilde{p}\otimes_h\left(M_h+K_h\right)^{(r)}\right)(0,T,(x,y),(x',y'))+R_2 \tag{5.9}$$

where

$$|R_2| \le Ch^{1/2}T^{-1/2}\chi_{\sqrt{T}}(x'-x, y'-y).$$

Because of (5.8) and (5.9), to prove the theorem it remains to show that

$$|\Delta_N| := \left| \left( \sum_{r=0}^{N} \left( \widetilde{p} \otimes H^{(r)} \right) - \sum_{r=0}^{N} \left( \widetilde{p} \otimes_h (M_h + K_h)^{(r)} \right) \right) (0, T, (x, y), (x', y')) \right|$$

$$\le Ch^{1/2}\chi_{\sqrt{T}}(x'-x, y'-y). \tag{5.10}$$

Note that $|\Delta_N| \le S_1 + S_2 + S_3$, where

$$S_1 = \left| \left( \sum_{r=0}^{N} \left( \widetilde{p} \otimes H^{(r)} \right) - \sum_{r=0}^{N} \left( \widetilde{p} \otimes_h H^{(r)} \right) \right) (0, T, (x, y), (x', y')) \right|,$$

$$S_2 = \left| \left( \sum_{r=0}^{N} \left( \widetilde{p} \otimes_h H^{(r)} \right) - \sum_{r=0}^{N} \left( \widetilde{p} \otimes_h (M_h + H)^{(r)} \right) \right) (0, T, (x, y), (x', y')) \right|,$$

$$S_3 = \left| \left( \sum_{r=0}^{N} (\widetilde{p} \otimes_h (M_h + H)^{(r)} - \sum_{r=0}^{N} \left( \widetilde{p} \otimes_h (M_h + K_h)^{(r)} \right) \right) (0, T, (x, y), (x', y')) \right|.$$

We shall show

$$S_i \le Ch^{1/2}\chi_{\sqrt{T}}(x'-x, y'-y), \ i = 1, 2, 3. \tag{5.11}$$

This is done in Appendix B.

**Conclusion.** So far, we have considered the case when only Poisson brackets of order one were involved to ensure the hypoellipticity. In our model, this implied that the frozen diffusion process was Gaussian, and so was the related limit Theorem. The bound in Theorem 2.1 remains homogeneous to a Gaussian probability density. The existence of an accurate similar lower bound is still an open question. Indeed, the lower bound holds in small time and a global bound can be obtained using convolutions and convexity inequalities, but in that case the constants degenerate. Also, when brackets of higher order are needed to have hypoellipticity, i.e. when formally Wiener chaos of order strictly greater than 1 appear in the frozen process, the upper bound of the density in terms of another probability density as well as the associated limit theorem are still to be investigated. This will concern further research.

## APPENDIX A: PROOF OF LEMMAS 5.2 AND 5.3

*Proof of Lemma 5.2.*
For $j = j' - 1$ we have $\rho = \sqrt{(j'-j)h} = \sqrt{h}$. By definition of $H_h$

$$H_h(t_j, t_{j+1}, (x,y), (x',y')) = \rho^{-2}\left[\widehat{p}_{h,j}((x,y),(x',y')) - \widetilde{p}^{x'}_{h,j}((x,y),(x',y'))\right].$$

Thus, from **(A2)**, **(BS)**, (B.12), (B.13) and standard computations

$$\begin{aligned} |H_h(t_j, t_{j+1}, (x,y), (x',y'))| &= |M_h(t_j, t_{j+1}, (x,y), (x',y'))| \\ &\leq C\rho^{-(1+2\varepsilon)}\zeta_\rho(x-x', y-y'). \end{aligned} \tag{A.1}$$

For $j < j' - 1$, we proceed like in the proof of Lemma 3.9 in [KM00]. We get that

$$H_h(t_j, t_{j'}, (x,y), (x',y')) = (\widehat{H}_h - \widetilde{H}_h)(t_j, t_{j'}, (x,y), (x',y'))$$

where

$$\begin{aligned} \widehat{H}_h(t_j, t_{j'}, (x,y), (x',y')) = h^{-1}\int f_n(\theta_1,\theta_2)\, q(\theta_3) \times \\ \left[\lambda(x + \widehat{\gamma}^1(\theta_1), y + \widehat{\gamma}^2(\theta_2,\theta_3)) - \lambda(x,y)\right] d\theta_1 d\theta_2 d\theta_3, \end{aligned} \tag{A.2}$$

$$\begin{aligned} \widetilde{H}_h(t_j, t_{j'}, (x,y), (x',y')) = h^{-1}\int f_n(\theta_1,\theta_2) \times \\ \left[\lambda(x + \widetilde{\gamma}^1(\theta_1), y + \widetilde{\gamma}^2(\theta_2,\theta_3)) - \lambda(x,y)\right] d\theta_1 d\theta_2, \end{aligned} \tag{A.3}$$

with $\lambda(u,v) = \widetilde{p}_h(t_{j+1}, t_{j'}, (u,v), (x',y'))$,

$$\begin{aligned} \widehat{\gamma}^1(\theta_1) &= hb(x) + \sqrt{h}\sigma(x)\theta_1, \\ \widehat{\gamma}^2(\theta_2,\theta_3) &= \left\{F(x + \frac{b(x)\gamma_n h}{2} + \sqrt{h}\sigma(x)\theta_2) - F(x')\right\} h + h^{3/2+\varepsilon}\theta_3, \end{aligned}$$

and

$$\begin{aligned} \widetilde{\gamma}^1(\theta_1) &= hb(x') + \sqrt{h}\sigma(x')\theta_1, \\ \widetilde{\gamma}^2(\theta_2,\theta_3) &= \overrightarrow{\nabla}_x F(x')(x-x')h + \frac{\overrightarrow{\nabla}_x F(x')b(x')\gamma_n h^2}{2} \\ &\quad + h^{3/2}\overrightarrow{\nabla}_x F(x')\sigma(x')\theta_2 + h^{3/2+\varepsilon}\theta_3. \end{aligned}$$

Using a Taylor expansion at order three for $\lambda$ in (A.2) and (A.3) we obtain

$$
\begin{aligned}
H_h(t_j,t_{j'},(x,y),(x',y')) &= \Big\{\frac{1}{2}\sum_{l,m=1}^{d}(a_{lm}(x)-a_{lm}(x'))\partial_{x_lx_m}\lambda(x,y)\\
&+\sum_{l=1}^{d}(b_l(x)-b_l(x'))\partial_{x_l}\lambda(x,y)+\sum_{l=1}^{k}\Big(F_l(x)-F_l(x')-\langle\nabla_xF_l(x'),x-x'\rangle\\
&+(\nabla_xF_l(x)b(x)-\nabla_xF(x')b(x')))\frac{\gamma_nh}{2}\\
&+\int_0^1 d\alpha\langle H_{F_l}(x+\alpha(\frac{h\gamma_nb(x)}{2}+\sqrt{h}\sigma(x)\theta_2))\frac{h\gamma_nb(x)}{2}+\sqrt{h}\sigma(x)\theta_2,\\
&\frac{h\gamma_nb(x)}{2}+\sqrt{h}\sigma(x)\theta_2\rangle\Big)\partial_{y_l}\lambda(x,y)\Big\}\\
&+\Big\{\frac{h}{2}\left(\langle H_x\lambda(x,y)b(x),b(x)\rangle-\langle H_x\lambda(x,y)b(x'),b(x')\rangle\right)\\
&+\frac{h^2}{2}\Big(\mathrm{tr}(H_y\lambda(x,y)\overrightarrow{\nabla}_xF(x')(a(x)-a(x'))\overrightarrow{\nabla}_xF(x')^*)\Big)\\
&+\frac{h^2}{2}\gamma_n\langle H_y\lambda(x,y)\overrightarrow{\nabla}_xF(x')(x-x'),\overrightarrow{\nabla}_xF(x')(b(x)-b(x'))\rangle\\
&+\frac{\gamma_n^2h^3}{8}\Big(\langle H_y\lambda(x,y)\overrightarrow{\nabla}_xF(x')b(x),\overrightarrow{\nabla}_xF(x')b(x)\rangle-\\
&\langle H_y\lambda(x,y)\overrightarrow{\nabla}_xF(x')b(x'),\overrightarrow{\nabla}_xF(x')b(x')\rangle\Big)\Big\}+\\
&(M_h^1+R_h^1)(t_j,t_{j'},(x,y),(x',y'))+\{h^{-1}\int d\theta_1d\theta_2d\theta_3f_n(\theta_1,\theta_2)q(\theta_3)\\
&\Big(\langle H_{y,x}\lambda(x,y)\widehat{\gamma}^1(\theta_1),\widehat{\gamma}^2(\theta_2,\theta_3)\rangle-\langle H_{y,x}\lambda(x,y)\widetilde{\gamma}^1(\theta_1),\widetilde{\gamma}^2(\theta_2,\theta_3)\rangle\Big)\}
\end{aligned}
$$

$$
\begin{aligned}
&+3h^{-1}\sum_{|\nu|=3}\int d\theta_1d\theta_2d\theta_3\int_0^1 d\delta(1-\delta)^2f_n(\theta_1,\theta_2)q(\theta_3)\frac{(\widehat{\gamma}^1(\theta_1),\widehat{\gamma}^2(\theta_2,\theta_3))^\nu}{\nu!}\\
&\qquad D^\nu\lambda(x+\delta\widehat{\gamma}^1(\theta_1),y+\delta\widehat{\gamma}^2(\theta_2,\theta_3))\\
&-3h^{-1}\sum_{|\nu|=3}\int d\theta_1d\theta_2d\theta_3\int_0^1 d\delta(1-\delta)^2f_n(\theta_1,\theta_2)q(\theta_3)\frac{(\widetilde{\gamma}^1(\theta_1),\widetilde{\gamma}^2(\theta_2,\theta_3))^\nu}{\nu!}\\
&\qquad\times D^\nu\lambda(x+\delta\widetilde{\gamma}^1(\theta_1),y+\delta\widetilde{\gamma}^2(\theta_2,\theta_3))\\
&:= I+II+(M_h^1+R_h^1)(t_j,t_{j'},(x,y),(x',y'))+III+IV-V,
\end{aligned}
\tag{A.4}
$$

where we denote $H_x\lambda(x,y)$ (resp. $H_y\lambda(x,y)$, $H_{y,x}\lambda(x,y)$) the $\mathbb{R}^d\otimes\mathbb{R}^d$ (resp. $\mathbb{R}^k\otimes\mathbb{R}^k$, $\mathbb{R}^k\otimes\mathbb{R}^d$) matrix $(\partial_{x_i,x_j}\lambda(x,y))_{(i,j)\in[\![1,d]\!]^2}$ (resp. $(\partial_{y_i,y_j}\lambda(x,y))_{(i,j)\in[\![1,k]\!]^2}$,

$(\partial_{y_i,x_j}\lambda(x,y))_{(i,j)\in[\![1,k]\!]\times[\![1,d]\!]^2})$. Also, setting $\widehat{m}_2(x,x') = x - x' + \frac{b(x)\gamma_n h}{2}$, $\forall l \in [\![1,k]\!]$, $R_F^{2,l} = \int d\theta_1 d\theta_2 f_n(\theta_1,\theta_2) \int_0^1 d\alpha(1-\alpha)\langle H_{F_l}(x' + \alpha(\widehat{m}_2(x,x') + \sqrt{h}\sigma(x)\theta_2))\left(\widehat{m}_2(x,x') + \sqrt{h}\sigma(x)\theta_2\right), \left(\widehat{m}_2(x,x') + \sqrt{h}\sigma(x)\theta_2\right)\rangle$ one has:

$$
\begin{aligned}
R_h^1(t_j,t_{j'},(x,y),(x',y')) &= \frac{1}{2}h\langle H_y\lambda(x,y)R_F^2,R_F^2\rangle,\\
M_h^1(t_j,t_{j'},(x,y),(x',y')) &= h\langle H_y\lambda(x,y)[\overrightarrow{\nabla}_x F(x')(\widehat{m}_2(x,x')\\
&\quad +\sigma(x)\sqrt{h}\theta_2], R_F^2\rangle. \qquad \text{(A.5)}
\end{aligned}
$$

Note that $M_1^h$ and $R_1^h$ correspond to a remainder associated to the second order term in the Taylor development of $F$ around $x'$ that is used when considering the second order derivatives in $y$ for the kernel $\widehat{H}_h$.

In the sequel, a useful result is the following. There exists $C > 0$ s.t. for multi-indices $\alpha, \beta$, $|\alpha| \le 3, |\beta| \le 3$,

$$
|\partial_x^\alpha \partial_y^\beta \lambda(x,y)| \le C\rho^{-(|\alpha|+3|\beta|)}\zeta_\rho\left(x'-x, y'-y\right). \qquad \text{(A.6)}
$$

This assertion can be proved similarly to Lemma 3.7 in [KM00].

From (A.6) and (A.5) we directly derive

$$
\begin{aligned}
|R_h^1(t_j,t_{j'},(x,y),(x',y'))| &\le C\sqrt{h}\rho^{-1}\zeta_\rho(x-x',y-y'),\\
|M_h^1(t_j,t_{j'},(x,y),(x',y'))| &\le C\rho^{-1}\zeta_\rho(x-x',y-y'). \qquad \text{(A.7)}
\end{aligned}
$$

Note now that

$$
\begin{aligned}
I &= (\widehat{L}-\widetilde{L})\widetilde{p}_h(t_j,t_{j'},(x,y),(x',y')) + \left\{\frac{h\gamma_n}{2}\sum_{l=1}^k(\nabla F_l(x)b(x)\right.\\
&\left.-\nabla F_l(x')b(x'))\partial_{y_l}\lambda(x,y)\right\}\\
&+\left\{(\widehat{L}-\widetilde{L})(\lambda(x,y) - \widetilde{p}_h(t_j,t_{j'},(x,y),(x',y'))) + \right.\\
&\sum_{l=1}^k\int_0^1 d\alpha\langle H_{F_l}(x+\alpha(\frac{h\gamma_n b(x)}{2}+\sqrt{h}\sigma(x)\theta_2))\left(\frac{h\gamma_n b(x)}{2}+\sqrt{h}\sigma(x)\theta_2\right),\\
&\left.\frac{h\gamma_n b(x)}{2}+\sqrt{h}\sigma(x)\theta_2\rangle\right)\partial_{y_l}\lambda(x,y)\Big\}\\
&:= (K_h + R_h^2 + M_h^2)(t_j,t_{j'},(x,y),(x',y')).
\end{aligned}
$$

From the above equation and (A.6) we get

$$\begin{aligned} |R_h^2(t_j,t_{j'},(x,y),(x',y'))| &\leq C\sqrt{h}\rho^{-1}\zeta_\rho(x-x',y-y'),\\ |M_h^2(t_j,t_{j'},(x,y),(x',y'))| &\leq C\rho^{-1}\zeta_\rho(x-x',y-y'). \end{aligned} \tag{A.8}$$

Using similarly (A.6) and tedious but elementary calculations, one can split in $II, III$ the terms that give the expected order, i.e. bounded by $C\sqrt{h}\rho^{-1}\zeta_\rho(x-x',y-y')$ and denoted below by $R_h^3(t_j,t_{j'},(x,y),(x',y'))$, and those that give an integrable singularity in time, i.e. bounded by $C\rho^{-1}\zeta_\rho(x-x',y-y')$ and denoted below by $M_h^3(t_j,t_{j'},(x,y),(x',y'))$.

It remains to estimate $IV-V$ in (A.4). To this end write,

$$\begin{aligned} IV-V = 3h^{-1}\sum_{|\nu|=3}\frac{1}{\nu!}\int d\theta_1 d\theta_2 d\theta_3 \int_0^1 d\delta(1-\delta)^2 f_n(\theta_1,\theta_2)q(\theta_3)\Big\{ \\ ((\widetilde{\gamma}^1(\theta_1),\widetilde{\gamma}^2(\theta_2,\theta_3))^\nu - (\widehat{\gamma}^1(\theta_1),\widehat{\gamma}^2(\theta_2,\theta_3))^\nu)D^\nu\lambda(x+\delta\widetilde{\gamma}^1(\theta_1),y+\delta\widetilde{\gamma}^2(\theta_2,\theta_3)) \\ +(\widehat{\gamma}^1(\theta_1),\widehat{\gamma}^2(\theta_2,\theta_3))^\nu\sum_{|\mu|=1}\int_0^1 d\alpha D^{\nu,\mu}\lambda(x+\delta\widehat{\gamma}^1(\theta_1)+\alpha\delta(\widetilde{\gamma}^1-\widehat{\gamma}^1)(\theta_1), \\ y+\delta\widehat{\gamma}^2(\theta_2,\theta_3)+\alpha\delta(\widetilde{\gamma}^2(\theta_2,\theta_3)-\widehat{\gamma}^2(\theta_2,\theta_3))) \\ \Big(\delta(\widetilde{\gamma}^1-\widehat{\gamma}^1)(\theta_1),\delta(\widetilde{\gamma}^2(\theta_2,\theta_3)-\widehat{\gamma}^2(\theta_2,\theta_3))\Big)^\mu\Big\} := M_h^4(t_j,t_{j'},(x,y),(x',y')). \end{aligned}$$

Computations involving (A.6) yield

$$|M_h^4(t_j,t_{j'},(x,y),(x',y'))| \leq C\rho^{-1}\zeta_\rho(x'-x,y'-y). \tag{A.9}$$

We refer to the proof of (3.80) p. 584 in [KM00] and Appendix B.2 for additional details. This completes the proof. □

The proof of Lemma 5.3 then follows from the previous proof, (A.6), (A.7), (A.8), (A.9) and (A.4) for $j'>j+1$ and (A.1) for $j'=j+1$.

## APPENDIX B: CONTROL OF THE $(S_I)_{I\in[\![1,3]\!]}$

**B.1. Control of $S_1$.** Set

$$\widehat{p}_d(0,T,(x,y),(x',y')) = \sum_{r=0}^{\infty}\tilde{p}\otimes_h H^{(r)}(0,T,(x,y),(x',y')).$$

From Proposition 2.2 one has

$$\begin{aligned} (\widehat{p}-\widehat{p}_d)(0,T,(x,y),(x',y')) &= (\widehat{p}\otimes H-\widehat{p}\otimes_h H)(0,T,(x,y),(x',y')) \\ &\quad +(\widehat{p}-\widehat{p}_d)\otimes_h H(0,T,(x,y),(x',y')). \end{aligned}$$

Iterating the previous identity we get

$$(\widehat{p}-\widehat{p}_d)(0,T,(x,y),(x',y')) = (\widehat{p}\otimes H - \widehat{p}\otimes_h H)\otimes_h \varphi(0,T,(x,y),(x',y')), \tag{B.10}$$

where $\forall j \in [\![0,N-1]\!],\ \forall (u,v)\in\mathbb{R}^d\times\mathbb{R}^k$,

$$\varphi(t_j,T,(u,v),(x',y')) = \sum_{r=0}^{\infty} H_h^{(r)}(t_j,T,(u,v),(x',y')).$$

Let us first give a bound for $P_j(u,v) := (\widehat{p}\otimes H - \widehat{p}\otimes_h H)(0,t_j,(x,y),(u,v)),\ j\in[\![0,N]\!],\ (u,v)\in\mathbb{R}^d\times\mathbb{R}^k$. First, from the previous definitions of the continuous and discrete convolution operators, $P_0(u,v)=0$, in the sense of generalized functions. For $j\geq 1$ write

$$\begin{aligned} P_j(u,v) &= \sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}} dt \int_{\mathbb{R}^d\times\mathbb{R}^k} dwdz\lambda_{(u,v)}(t,(w,z)) - \lambda_{(u,v)}(t_i,(w,z)),\\ \lambda_{(u,v)}(t,(w,z)) &:= \widehat{p}(0,t,(x,y),(w,z))H(t,t_j,(w,z),(u,v)). \end{aligned}$$

A first order Taylor expansion and Fubini's theorem give

$$\begin{aligned} P_j(u,v) &= \sum_{i=1}^{j-1}\int_{t_i}^{t_{i+1}} dt(t-t_i)\int_0^1 d\delta Q_i^{\delta}(u,v,s) + T_j^0,\\ Q_i^{\delta}(u,v,s) &:= \int_{\mathbb{R}^d\times\mathbb{R}^k} dwdz\partial_s\lambda_{(u,v)}(s,(w,z))_{s=t_i+\delta(t-t_i)},\ i\in[\![1,j-1]\!].\\ T_j^0 &:= \int_0^h dt\int_{\mathbb{R}^d\times\mathbb{R}^k} dwdz\widehat{p}(0,t,(x,y),(w,z))\\ &\quad\times(H(t,t_j,(w,z),(u,v)) - H(0,t_j,(x,y),(u,v))). \end{aligned} \tag{B.11}$$

From Lemma 3.1, Theorem 2.1 and standard computations for Gaussian convolutions we obtain

$$T_j^0 \leq C\sqrt{h}t_j^{-(d+3k)/2}\exp(-c\left[\frac{|u-x|^2}{t_j}+\frac{|v-y|^2}{t_j^3}\right]).$$

Now, Kolmogorov's equations derived from the definitions (2.5) and (2.6) yield

$$\begin{aligned} Q_i^{\delta}(u,v,s) = \int_{\mathbb{R}^d\times\mathbb{R}^k} dwdz\widehat{p}(0,s,(x,y),(w,z))(\widehat{L}^2 - 2\widehat{L}\widetilde{L} + \widetilde{L}^2)\\ \tilde{p}(s,t_j,(w,z),(u,v)). \end{aligned}$$

From Lemma 3.1, the most singular term in the above equation is the one involving two derivatives w.r.t $z$. Namely, we have to control

$$Q_i^{\delta,S}(u,v,s) := \sum_{l,m=1}^{k} \int_{\mathbb{R}^d\times\mathbb{R}^k} dwdz\widehat{p}(0,s,(x,y),(w,z))\Big( [F_l(w) \\ -F_l(u)]\,[F_m(w)-F_m(u)-2\langle\nabla_u F_m(u), w-u\rangle] \\ +\langle\nabla_u F_l(u), w-u\rangle \times \langle\nabla_u F_m(u), w-u\rangle\Big)\partial_{z_l z_m}\tilde{p}(s,t_j,(w,z),(u,v)).$$

As for $T_j^0$, Lemma 3.1, Theorem 2.1, a second order Taylor expansion for $F_l$,$F_m$ and standard computations for Gaussian convolutions yield

$$\begin{aligned} |Q_i^{\delta,S}(u,v,s)| &\leq C(t_j-s)^{-3/2}t_j^{-\frac{d+3k}{2}} \times \exp\left(-c\left\{\frac{|x-u|^2}{t_j}+\frac{|y-v|^2}{t_j^3}\right\}\right) \\ &\leq C(t_j-s)^{-3/2}\Lambda_{\sqrt{t_j}}(x-u,y-v). \end{aligned}$$

The same bound holds for $Q_i^{\delta}(u,v,s)$, up to a multiplicative finite constant. Plug now the above control in (B.11), we get

$$\begin{aligned} P_j(u,v) &\leq C\Lambda_{\sqrt{t_j}}(x-u,y-v)(h^{1/2}+h^2\sum_{i=1}^{j-2}t_i^{-3/2}) \\ &\leq Ch^{1/2}\Lambda_{\sqrt{t_j}}(x-u,y-v). \end{aligned}$$

Hence, from (B.10) and a suitable version of (3.4) for the discrete convolution operator we derive

$$|(\widehat{p}-\widehat{p}_d)(0,T,(x,y),(x',y'))| \leq Ch^{1/2}\Lambda_{\sqrt{T}}(x'-x,y'-y).$$

The bound for $S_1$ can be derived using once again (3.4) for both the continuous and discrete convolution operators and the asymptotics of the Gamma function.

**B.2. Control of $S_2$.** For $r=1$ we have to control

$$\widetilde{p}\otimes_h M_h(0,T,(x,y),(x',y')) = \\ \sum_{i=1}^{4} h\sum_{j=0}^{N-1}\int dudv\widetilde{p}(0,t_j,(x,y),(u,v))M_h^i(t_j,T,(u,v),(x',y')) \\ := h\sum_{i=1}^{4}\sum_{j=0}^{N-2} I_{i,j} + hI_{N-1}.$$

The term $hI_{N-1}$ needs to be handled by a different technique than the other ones. Write

$$hI_{N-1} = \int dudv\widetilde{p}(0, T-h, (x,y), (u,v)) \left(\widetilde{p}_h - \widehat{p}_h\right)(T-h, T, (u,v), (x',y')).$$

From the definitions of the models (4.2) and (4.5) for all $(u,v) \in \mathbb{R}^d \times \mathbb{R}^k$:

$$\begin{aligned}\widehat{p}_h(T-h, T, (u,v), (x',y')) &= h^{-(d+k(3+2\varepsilon))/2}\det(a(u))^{-1/2} \\ &\quad \times \int d\theta_2 f_n\left(\frac{\sigma^{-1}(u)(x'-u-b(u)h)}{h^{1/2}}, \theta_2\right) \\ \times q\left(\frac{y'-v-(F(u+\frac{\gamma_n b(u)h}{2}+\sigma(u)\sqrt{h}\theta_2)-F(x'))h}{h^{3/2+\varepsilon}}\right), \end{aligned} \tag{B.12}$$

$$\begin{aligned}\widetilde{p}_h(T-h, T, (u,v), (x',y')) &= h^{-(d+k(3+2\varepsilon))/2}\det(a(x'))^{-1/2} \\ &\quad \times \int d\theta_2 f_n\left(\frac{\sigma^{-1}(x')(x'-u-b(x')h)}{h^{1/2}}, \theta_2\right) \\ \times q\left(\frac{y'-v-(\overrightarrow{\nabla}_x F(x')\{u-x'+\frac{\gamma_n b(x')h}{2}+\sigma(x')\sqrt{h}\theta_2\})h}{h^{3/2+\varepsilon}}\right). \end{aligned} \tag{B.13}$$

Setting

$$\begin{aligned} -u' &:= \frac{\sigma^{-1}(x')\left[x'-u-b(x')h\right]}{h^{1/2}}, \\ -v' &:= \frac{(y'-v-(\overrightarrow{\nabla}_x F(x')\{u-x'+\frac{\gamma_n b(x')h}{2}+\sigma(x')\sqrt{h}\theta_2\})h)}{h^{3/2+\varepsilon}} \\ g(x',u') &:= x'+\sigma(x')u'h^{1/2}-b(x')h, \\ \overrightarrow{\nabla}_x F(x',u',\theta_2) &:= \overrightarrow{\nabla}_x F(x')\left\{\sigma(x')h^{3/2}(u'+\theta_2)-\frac{b(x')h^2}{2}\left(1-\frac{1}{n}\right)\right\}, \end{aligned}$$

one gets

$$\begin{aligned} hI_{N-1} = \int du'dv'd\theta_2\widetilde{p}(0, T-h, (x,y), g(x',u'), y'+h^{3/2+\varepsilon}v' - \overrightarrow{\nabla}_x F(x',u',\theta_2)) \\ \times f_n(-u',\theta_2)q(-v') - \frac{\det(a(x'))^{1/2}}{\det(a(g(x',u')))^{1/2}} \times \\ \widetilde{p}(0, T-h, (x,y), g(x',u'), y'+h^{3/2+\varepsilon}v' - (\overrightarrow{\nabla}_x F(x',u',\theta_2)+R_1)) \\ \times f_n(-u'+R_2,\theta_2)q(-v'), \end{aligned}$$

where using Young's inequality $|R_1| \leq C([|u'|^2 + |\theta_2|^2]h^2 + h^3), |R_2| \leq C(|u'|^2 h^{1/2} + h^{3/2})$. Note also from **(UE)** and **(G)** that $|1 - \frac{\det(a(x'))^{1/2}}{\det(a(g(x',u')))^{1/2}}| \leq Ch^{1/2}(|u'| + h^{1/2})$. Standard computations, the above controls, Lemma 3.1 and **(A2)** yield

$$|hI_{N-1}| \leq Ch^{1/2} \int du'dv'd\theta_2 \exp\left(-c\left\{\frac{|x - g(x',u')|^2}{T-h}\right\}\right)$$
$$\exp\left(-c\left\{\frac{|y' + h^{3/2+\varepsilon}v' - (\overrightarrow{\nabla}_x F(x',u',\theta_2) + \lambda R_1)|^2}{(T-h)^3}\right\}\right)(1 + |u'|^2 + |\theta_2|^2)$$
$$\times \psi(-u', \theta_2, -v'),$$
$$\leq Ch^{1/2} \int du'dv'd\theta_2 \left(1 + \left\{\frac{|x - g(x',u')|^2}{T-h} + \right.\right.$$
$$\left.\left.\frac{|y' - y + h^{3/2+\varepsilon}v' - (\overrightarrow{\nabla}_x F(x',u',\theta_2) + \lambda R_1)|^2}{(T-h)^3}\right\}^{Z/2}\right)^{-1} (1 + |u'|^2 + |\theta_2|^2)$$
$$\times \psi(-u', \theta_2, -v')$$

where $\lambda := \lambda(u', v', \theta_2) \in [0,1]$, $Z \in \mathbb{N}^*$. For $V \in \mathbb{R}^{d+k}$, $|V| \leq \eta$, apply now the inequality $\frac{1}{1+|U+V|^Z} \leq \frac{\max(2^Z,(2\eta)^Z+1)}{1+|U|^Z}$, with $U = \left(\frac{x-x'}{(T-h)^{1/2}}, \frac{y'-y}{(T-h)^{3/2}}\right)$,

$$V = \left(\frac{\sigma(x')h^{1/2}u' - b(x')h}{(T-h)^{1/2}}, \frac{v'h^{3/2+\varepsilon} - (\overrightarrow{\nabla}_x F(x',u',\theta_2) + \lambda R_1)}{(T-h)^{3/2}}\right),$$

and $\eta = Ch^{1/2}(1 + |u'|^2 + |\theta_2|^2 + |v'|^2 h^{1+\varepsilon})$, for $C$ large enough, one gets

$$\begin{aligned} |hI_{N-1}| &\leq \frac{Ch^{1/2}}{1+|U|^Z} \int du'dv'd\theta_2 (1 + |(u', v', \theta_2)|^{2Z+2}) \psi(-u', \theta_2, -v') \\ &\leq Ch^{1/2} \zeta_{\sqrt{T-h}}(x' - x, y' - y) \leq Ch^{1/2} \zeta_{\sqrt{T}}(x' - x, y' - y), \end{aligned}$$

taking $Z = S - 4$ for the last inequality. Hence the exponent in **(A2)**.

Also, from the definitions of the $(M_h^i)_{i \in [\![1,4]\!]}$ in the previous section and using freely its notations, we derive for all $j \in [\![0, N-2]\!]$:

$$\begin{aligned} |M_h^1(t_j, T, (u,v), (x',y'))| &\leq h^2 (T - t_j)^{-5/2} \zeta_\rho(x' - u, y' - v), \\ |M_h^2(t_j, T, (u,v), (x',y'))| &\leq h(T - t_j)^{-3/2} \zeta_\rho(x' - u, y' - v), \end{aligned}$$

from which one gets $h \sum_{i=1}^{2} \sum_{j=0}^{N-2} |I_{i,j}| \leq Ch^{1/2} \zeta_{\sqrt{T}}(x' - x, y' - y)$. The terms in $M_h^3$ coming from $II$ in (A.4) can be handled as $(M_h^i)_{i \in [\![1,2]\!]}$. For those coming

from $III$, i.e. crossed derivatives, the contribution associated to $j = 0$ is easily analyzed and for $j > 1$ an integration by part w.r.t. $u$ leads to the same control. The trickiest term to analyze is $M_h^4$. Exploiting thoroughly (A.6) and Lemma 3.1, the proof is similar to the one in [KM00], see p.578 control of (3.45), that relies on suitable integration by parts. We omit the details here. Actually, for $r \geq 1$ it can be shown by induction that

$$\left|\left(\widetilde{p} \otimes_h H^{(r)} - \widetilde{p} \otimes_h (H+M)^{(r)}\right)(0,T,(x,y),(x',y'))\right| \leq \frac{h^{1/2} C^{r+1}}{\Gamma([r+2]/2)} \chi_{\sqrt{T}}(x'-x, y'-y),$$

which gives the control.

**B.3. Control of $S_3$.** One can show that Lemma 5.1 is still valid for the derivatives of the frozen densities. Using this result and Lemma 5.3, the proof is then similar to the one of [KM00].

## APPENDIX C: EXISTENCE OF THE DENSITY FOR THE AGGREGATED FROZEN PROCESS

Let $h_0 > 0$ be a given fixed time step. For $i \in \mathbb{N}$ set $t_i := ih_0$. We consider the frozen model defined by $\widetilde{X}_0^{h_0} = x, \widetilde{Y}_0^{h_0} = y$ and for all $i \in \mathbb{N}$,

$$\begin{aligned}
\widetilde{X}_{t_{i+1}}^{h_0} &= \widetilde{X}_{t_i}^{h_0} + b(x')h_0 + \sigma(x')\sqrt{h_0}\widetilde{\xi}_{i+1}, \\
\widetilde{Y}_{t_{i+1}}^{h_0} &= \widetilde{Y}_{t_i}^{h_0} + \overrightarrow{\nabla}_x F(x')(\widetilde{X}_{t_{i+1}}^{h_0} - x')h_0 \\
&= \widetilde{Y}_{t_i}^{h_0} + h_0 \overrightarrow{\nabla}_x F(x')(\widetilde{X}_{t_i}^{h_0} - x') + h_0^2 \overrightarrow{\nabla}_x F(x') b(x') \\
&\quad + h_0^{3/2} \overrightarrow{\nabla}_x F(x')\sigma(x')\widetilde{\xi}_{i+1}, \qquad \text{(C.14)}
\end{aligned}$$

where $(\widetilde{\xi}_i)_{i\in\mathbb{N}^*}$ are i.i.d. with smooth densities, satisfying the growth and smoothness properties introduced in **(A2)**. The aim of this section is to show that for $i$ large enough $(\widetilde{X}_{t_i}^{h_0}, \widetilde{Y}_{t_i}^{h_0})$ admits a density. We refer the reader to the work of Yurinski [Yur72] or Molchanov and Varchenko [MV77] for related topics.

Conditionally to $\begin{pmatrix} \widetilde{X}_{t_i}^{h_0} = x^* \\ \widetilde{Y}_{t_i}^{h_0} = y^* \end{pmatrix}$ and iterating the frozen model we get

$$\begin{aligned}
\widetilde{X}_{t_{i+n}}^{h_0} &= x^* + (nh_0)b(x') + \sigma(x')\sqrt{nh_0}\widetilde{\xi}_{i,n}^{(1)}, \\
\widetilde{Y}_{t_{i+n}}^{h_0} &= y^* + (nh_0)\overrightarrow{\nabla}_x F(x')(x^* - x') + \frac{\gamma_n}{2}(nh_0)^2 \overrightarrow{\nabla}_x F(x') b(x') \\
&\quad + (nh_0)^{3/2} \overrightarrow{\nabla}_x F(x')\sigma(x')\widetilde{\xi}_{i,n}^{(2)}, \qquad \text{(C.15)}
\end{aligned}$$

where we recall $\gamma_n = (1 + \frac{1}{n})$ and

$$\begin{aligned}\widetilde{\xi}^{(1)}_{i,n} &= \frac{1}{\sqrt{n}}\left(\widetilde{\xi}_{i+1} + \widetilde{\xi}_{i+2} + ... + \widetilde{\xi}_{i+n}\right),\\ \widetilde{\xi}^{(2)}_{i,n} &= \frac{1}{\sqrt{n}}\left(\widetilde{\xi}_{i+1} + (1-\frac{1}{n})\widetilde{\xi}_{i+2} + ... + (1-\frac{n-1}{n})\widetilde{\xi}_{i+n}\right).\end{aligned}$$

We have

$$\begin{aligned}Var(\widetilde{\xi}^{(2)}_{i,n}) = \frac{(1-\frac{n-1}{n})^2 + ... + 1^2}{n} &= \frac{2n^2+3n+1}{6n^2} = \frac{1}{3}\gamma_n(1+\frac{1}{2n}),\\ Cov(\widetilde{\xi}^{(1)}_{i,n}, \widetilde{\xi}^{(2)}_{i,n}) = \frac{(1-\frac{n-1}{n}) + ... + 1}{n} &= \frac{n+1}{2n} = \frac{\gamma_n}{2}.\end{aligned}$$

Hence, the covariance matrix of the $2d$ dimensional vector $\left(\widetilde{\xi}^{(1)}_{i,n}, \widetilde{\xi}^{(2)}_{i,n}\right)^*$ is non-degenerate for $n \geq 2$.

Estimating the characteristic function $\varphi_n(\tau_1, \tau_2)$ of the vector $\left(\widetilde{\xi}^{(1)}_{i,n}, \widetilde{\xi}^{(2)}_{i,n}\right)^* \in \mathbb{R}^{2d}$ we derive the following

**Proposition C.1** *Let $\phi(\tau) := \mathbb{E}\left[\exp\left(i\langle\widetilde{\xi}_1, \tau\rangle\right)\right]$, $\tau \in \mathbb{R}^d$ denote the characteristic function of the $(\widetilde{\xi}_i)_{i\in\mathbb{N}^*}$. If for all multi index $\beta$, $|\beta| = 2S - 6 + (2d+1)$, $|D^\beta\phi(\tau)| \leq C(1+|\tau|^{4+2d+1})^{-1}$, then for $n$ large enough and for all multi index $\alpha$, $|\alpha| \leq 4$, one has*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |(\tau_1,\tau_2)|^{|\alpha|} |D^{2S-6+(2d+1)}\varphi_n(\tau_1,\tau_2)| d\tau_1 d\tau_2 < \infty.$$

*In particular, by Fourier inversion the density*

$$\text{(C.16)}\quad f_n(\theta_1,\theta_2) = \frac{1}{(2\pi)^{2d}}\int \exp(-i\langle(\theta_1,\theta_2)^*, (\tau_1,\tau_2)^*\rangle)\varphi_n(\tau_1,\tau_2)d\tau_1 d\tau_2$$

*exists and there exists $C$ s.t. for all multi index $\nu$, $|\nu| \leq 4$,*

$$|D^\nu f_n(\theta_1,\theta_2)| \leq \frac{C}{1+|(\theta_1,\theta_2)|^{2S-6+2d+1}} := \psi_n(\theta_1,\theta_2).$$

*Proof.* Write

$$\varphi_n(\tau_1,\tau_2) = \mathbb{E}\left[\exp\left\{i\left\langle\tau_1, \widetilde{\xi}^{(1)}_{i,n}\right\rangle + i\left\langle\tau_2, \widetilde{\xi}^{(2)}_{i,n}\right\rangle\right\}\right] = \prod_{j=0}^{n-1}\phi\left(\frac{\tau_1 + (1-\frac{j}{n})\tau_2}{\sqrt{n}}\right). \quad \text{(C.17)}$$

We partition the space $\mathbb{R}^{2d}$ into the following disjoint sets

$$\begin{aligned} A_0 &:= \left\{(\tau_1,\tau_2)\in\mathbb{R}^{2d} : |\tau_1| \geq (1-\frac{1}{n})\,|\tau_2|\right\},\\ A_i &:= \left\{(\tau_1,\tau_2)\in\mathbb{R}^{2d} : (1-\frac{i+1}{n})\,|\tau_2| \leq |\tau_1| < (1-\frac{i}{n})\,|\tau_2|\right\}, i\in[\![1,n-2]\!],\\ A_{n-1} &:= \left\{(\tau_1,\tau_2)\in\mathbb{R}^{2d} : |\tau_1| < \frac{1}{n}\,|\tau_2|\right\}. \end{aligned}$$

If $(\tau_1,\tau_2)\in A_0$ then for $i\in[\![2,n-2]\!]$

$$\begin{aligned} \left|\frac{\tau_1+(1-\frac{i}{n})\tau_2}{\sqrt{n}}\right| &\geq \frac{1}{\sqrt{n}}\left(|\tau_1|-(1-\frac{i}{n})\,|\tau_2|\right)\\ &\geq \frac{1}{\sqrt{n}}\left((1-\frac{1}{n})\,|\tau_2|-(1-\frac{i}{n})\,|\tau_2|\right)=\frac{i-1}{n\sqrt{n}}\,|\tau_2| \end{aligned}$$

and similarly $\left|\frac{\tau_1+(1-\frac{i}{n})\tau_2}{\sqrt{n}}\right| \geq \frac{i-1}{n\sqrt{n}}\,|\tau_1|$. Hence,

$$\left|\frac{\tau_1+(1-\frac{i}{n})\tau_2}{\sqrt{n}}\right|^{2d+1} \geq \frac{(i-1)^{2d+1}}{2n^{3d+3/2}}|(\tau_1,\tau_2)|^{2d+1}. \tag{C.18}$$

If $(\tau_1,\tau_2)\in A_{i^*}$ for some $i^*$, $i^*\in[\![1,n-2]\!]$ and $l\in[\![2,n-1-i^*]\!]$ then elementary computations yield similarly

$$\left|\frac{\tau_1+(1-\frac{i^*+l}{n})\tau_2}{\sqrt{n}}\right|^{2d+1} \geq \frac{(l-1)^{2d+1}}{2n^{3d+3/2}}\,|(\tau_1,\tau_2)|^{2d+1}, \tag{C.19}$$

and for $l\in[\![1,i^*-1]\!]$

$$\left|\frac{\tau_1+(1-\frac{i^*-l}{n})\tau_2}{\sqrt{n}}\right|^{2d+1} \geq \frac{l^{2d+1}}{2n^{3d+3/2}}\,|(\tau_1,\tau_2)|^{2d+1}. \tag{C.20}$$

If $(\tau_1,\tau_2)\in A_{n-1}$ then for $i\in[\![1,n-1]\!]$

$$\left|\frac{\tau_1+(1-\frac{i}{n})\tau_2}{\sqrt{n}}\right|^{2d+1} \geq \frac{1}{2n^{d+1/2}}\left(1-\frac{i+1}{n}\right)^{2d+1}|(\tau_1,\tau_2)|^{2d+1}. \tag{C.21}$$

Use now the growth assumption on $\phi$ and the inequality $1+\sum_{j=1}^N p_j \leq$

$\prod_{j=1}^{N}(1+p_j)$ where $p_j \geq 0$, to derive from (C.17)

$$|\varphi_n(\tau_1,\tau_2)| = \left|\prod_{j=0}^{n-1}\phi\left(\frac{\tau_1+(1-\frac{j}{n})\tau_2}{\sqrt{n}}\right)\right| \leq \frac{C^n}{\prod_{j=0}^{n-1}\left(1+\left|\frac{\tau_1+(1-\frac{j}{n})\tau_2}{\sqrt{n}}\right|^{2d+1}\right)}$$

$$\leq \frac{C^n}{1+\sum_{j=0}^{n-1}\left|\frac{\tau_1+(1-\frac{j}{n})\tau_2}{\sqrt{n}}\right|^{2d+1}}.$$

Now equations (C.18), (C.19), (C.20), (C.21) yield that there exists $n$ large enough s.t.

$$|\varphi_n(\tau_1,\tau_2)| \leq \frac{C(n)}{1+|(\tau_1,\tau_2)|^{2d+1}},$$

where $C(n) \underset{n}{\to} +\infty$. Anyhow, for such a fixed $n$, one has $\varphi_n \in L^1(\mathbb{R}^{2d})$ which implies the existence of the density $f_n$ of the vectors $\left(\widetilde{\xi}_{i,n}^{(1)}, \widetilde{\xi}_{i,n}^{(2)}\right)^* \in \mathbb{R}^{2d}$. The properties concerning the growth and derivatives of $f_n$ are derived from (C.16) and the growth and smoothness properties of $\phi$. □

Hence we can set $(\eta_i^1, \eta_i^2) := (\widetilde{\xi}_{i,n}^{(1)}, \widetilde{\xi}_{i,n}^{(2)})$ where $(\widetilde{\xi}_{i,n}^{(1)}, \widetilde{\xi}_{i,n}^{(2)})$ are as in the above proposition. Introducing a "macro" scale time step $h = nh_0$, the discrete model (4.5) corresponds to the "aggregated" dynamics of (C.15) up to the additional "artificial viscosity term" needed for the comparison, see Appendix A. Now, from (4.1), $(\eta_i^3)_{i\in(j,j']\!]}$ are "artificial" additional i.i.d. variables. We can thus arbitrarily choose their density $q$. s.t. $\exists C,\ \forall \nu,\ |\nu| \leq 4,\ |D^\nu q(\theta_3)| \leq C(1+|\theta_3|^{k+1})^{-1} := \psi_q(\theta_3)$. Set for all $(\theta_1,\theta_2,\theta_3) \in \mathbb{R}^{2d+k},\ \psi(\theta_1,\theta_2,\theta_3) := \psi_n(\theta_1,\theta_2)\psi_q(\theta_3)$. With the notations of Section 4 one derives that $q_n(\theta_1,\theta_2,\theta_3) = f_n(\theta_1,\theta_2)q(\theta_3)$ satisfies **(A2)** with the above $\psi$.

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