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# A KINETIC APPROACH TO THE STUDY OF OPINION FORMATION

LAURENT BOUDIN AND FRANCESCO SALVARANI

ABSTRACT. In this work, we use the methods of nonequilibrium statistical mechanics in order to derive an equation which models some mechanisms of opinion formation. After proving the main mathematical properties of the model, we provide some numerical results.

## 1. INTRODUCTION

The methods of nonequilibrium statistical mechanics, classically used in the framework of the kinetic theory of gases, can also be fruitfully applied to all the contexts that study the collective behaviour of a large enough number of individuals, where none of which has a preponderant role with respect to the others. For example, kinetic-type equations have been introduced in order to describe a simple market economy with a constant growth mechanism [11, 5, 4], showing the formation of steady states with (overpopulated) Pareto tails.

Such an approach seems also to be fitted to sociophysics or physics of politics. For instance, we can refer to [14] (see also [1]), which studies the opinion dynamics in a closed community. The modeling of opinion dynamics has already been the subject of numerous works, mainly for its application in politics, to predict the behaviour of voters during an election process or the public opinion tendencies [9, 6, 7, 8].

Nevertheless, this approach is not the only possible one: a wide literature is based, for example, on Ising models [13, 12]. The main difference between such a description and kinetic-based models lies in the possibility, for the latter, to also treat opinions with different degrees of strength.

In the present work, based on a kinetic description, we propose and study a model for the process of opinion formation. The dynamics of the model is given by the competition of two opposite effects. The first one is the exchanges of ideas between individuals, treated as binary collisions, which implies a modification of the agents opinions. The second one is the process of self-thinking and is modelled by a diffusion term.

The sociological dynamics taken into account are the same as in [14], but they are here treated in a different way. The modifications of the agent opinion due to the self-thinking are modelled as a weighted linear diffusion which vanishes on the boundary of the opinion space, whereas the binary interactions between individuals are defined as an exchange of opinions with

respect to their average value before the collision. The opinion of individuals is represented by a one-dimensional real variable between  $(-1)$  and  $(+1)$ . The choice of the closed interval  $[-1, 1]$  instead of  $\mathbb{R}$  means that extreme opinions can actually be reached, and not only asymptotically.

This paper is organized as follows. We first describe our model, including the processes of binary opinion exchanges and self-thinking. Then we prove an existence result for the considered problem. Finally, we present some numerical tests.

## 2. THE MODEL

We aim to describe the time evolution of the opinion set of an isolated population about a particular statement. The process of opinion formation is partially governed by the interactions between the individuals of the population, who exchange their viewpoints and who are then influenced by the others.

We shall only consider opinions as regards binary questions. Even if there are many situations where the opinion cannot be easily expressed in terms of “yes” or “no”, many important situations (e.g. a referendum) require that, at the end of a process of opinion formation, the individuals should accept or reject, without reserve, a very precise statement.

In the following,  $\Omega$  denotes the open interval  $(-1, 1)$ . We label with  $x = -1$  and  $x = 1$  the two extreme answers to the statement, i.e. “yes” or “no” without reserve, and describe the opinion by the continuous variable  $x \in \bar{\Omega}$ . Note that any intermediate value between the two extremes (zero excluded) means that the corresponding individual partially agrees with the opinion labelled with the same sign, with a degree of conviction which is proportional to  $|x|$ . If  $x = 0$ , the corresponding individual has no preference with respect to the question.

The unknown of our model is the density (or distribution function)  $f = f(t, x)$ , defined on  $\mathbb{R}_+ \times \bar{\Omega}$ , whose time evolution is described, as shown later, by an integro-differential equation. The precise meaning of  $f$  is the following. Once individuated the population to study, if the opinions are defined on a sub-domain  $D \subseteq \bar{\Omega}$ , the integral

$$\int_D f(t, x) dx$$

represents the number of individuals with opinion included in  $D$  at time  $t \geq 0$ . Note that, in order to give a sense to the previous considerations, it would be adequate if  $f$  satisfies  $f(t, \cdot) \in L^1(\Omega)$  for all  $t \in \mathbb{R}_+$ .

As sketched in the introduction, we shall take into account only two processes of opinion formation. The first one is given by the binary interaction between individuals, which exchange their points of view and adjust their opinions on the ground of each other’s belief, whereas the second behaviour we take into account is the self-thinking process.

**2.1. Exchange of opinions inside the population.** We model this process by borrowing the collisional mechanism of a typical interaction in the kinetic theory of gases: whereas in rarefied gas dynamics the particles exchange momentum and energy in such a way that the principles of classical mechanics are satisfied, here the “collision” between individuals allows the exchange of opinions.

Let  $x, x_* \in \bar{\Omega}$  the opinions of two individuals before an interaction. We suppose that the opinions after the interaction change according to the following rule. Once found the average opinion before the interaction, each of the two individuals will approach it in a way which guarantees that stronger opinions are less attracted towards the average than weaker opinions through the following formula:

$$(1) \quad \begin{cases} x' = \frac{x + x_*}{2} + \eta(x) \frac{x - x_*}{2}, \\ x'_* = \frac{x + x_*}{2} + \eta(x_*) \frac{x_* - x}{2}. \end{cases}$$

The function  $\eta : \bar{\Omega} \rightarrow \mathbb{R}$ , which we henceforth name the *attraction coefficient*, is a smooth function which describes the degree of attraction of the average opinion with respect to the starting opinion of the agent. In order to make the model unaffected by the change of label of the two extreme opinions, we shall always suppose that  $\eta$  is an even function. In the sequel, we need some more assumptions on the attraction coefficient  $\eta$ .

**Definition 2.1.** *Let  $\eta : \bar{\Omega} \rightarrow \mathbb{R}$  be an even function of class  $C^1(\bar{\Omega})$ . The attraction coefficient is admissible if  $0 \leq \eta < 1$ ,  $\eta'(x) > 0$  for all  $x > 0$ , and if the Jacobian of the collision mechanism (1)*

$$\begin{aligned} J(x, x_*) &= \frac{1}{2}[\eta(x) + \eta(x_*)] + \frac{1}{4}[\eta'(x) - \eta'(x_*)](x - x_*) \\ &\quad + \frac{1}{4}[\eta'(x)\eta(x_*) - \eta(x)\eta'(x_*)](x - x_*) - \frac{1}{4}\eta'(x)\eta'(x_*)(x - x_*)^2 \end{aligned}$$

*is uniformly lower bounded by a nonnegative constant, i.e. there exists  $\xi > 0$  such that  $J(x, x_*) \geq \xi$ , for any  $x, x_* \in \bar{\Omega}$ .*

The first property avoids that the interaction destroys the bounds of the interval  $\Omega$ . The second one translates the assumption that stronger opinions are less attracted towards the average than weaker ones. The third one ensures that the inverse of the collision rule (1) is well defined.

Note that the collision mechanism (1) is different from the mechanism proposed by Toscani in [14], since here the post-collisional opinions are defined from the mean value of the two pre-collisional opinions. Moreover, it differs from the standard collisional rules of dissipative kinetic models, which are usually linear.

**Remark 2.2.** *By using the properties listed in Definition 2.1, it is not difficult to also prove that, for any  $x, x_* \in \bar{\Omega}$ ,*

$$x' - x'_* = \frac{1}{2}(\eta(x) + \eta(x_*))(x - x_*),$$

and, since  $0 \leq \eta < 1$ ,

$$|x' - x'_*| < |x - x_*|.$$

It is then clear that the lateral bounds are not violated, i.e.

$$\max\{|x'|, |x'_*|\} < \max\{|x|, |x_*|\}.$$

We note that the set of admissible attraction coefficients is not empty. A possible choice of  $\eta$  is the following:

$$(2) \quad \eta(x) = \lambda(1 + x^2), \quad 0 < \lambda < 1/2.$$

It is easy to see, when  $\eta$  has the form (2), that the properties listed in Definition 2.1 are satisfied. The first and the second requirements in Definition 2.1 are obviously fulfilled. Easy but tedious computations finally show that the Jacobian  $J$  of the collision rule (1), with  $\eta$  defined by (2), is given by

$$\begin{aligned} J(x, x_*) &= \frac{\lambda}{2}(2 + x^2 + x_*^2) + \frac{\lambda}{2}(x - x_*)^2 \\ &\quad + \frac{\lambda^2}{2}[x(1 + x_*^2) - (1 + x^2)x_*](x - x_*) - \lambda^2 x x_* (x - x_*)^2, \end{aligned}$$

and satisfies

$$J \geq \lambda.$$

**Remark 2.3.** *The transformation of type (1) can be locally dissipative. As a matter of fact, if  $\eta$  is, for example, defined by (2), with  $\lambda \in (0, (\sqrt{2}-1)/2]$ , we have*

$$\begin{aligned} x'^2 + x_*'^2 &= x^2 + x_*^2 \\ &\quad + \frac{(x - x_*)^2}{4} [\lambda^2(1 + x^2)^2 + \lambda^2(1 + x_*^2)^2 + 2\lambda(x + x_*)^2 - 2] \\ &\leq x^2 + x_*^2 + \frac{1}{2}(4\lambda^2 + 4\lambda - 1)(x - x_*)^2 \\ &\leq x^2 + x_*^2. \end{aligned}$$

Once defined the collision rule (1), the interaction between individuals and the corresponding exchange of opinions is described by a collisional integral of Boltzmann type.

The collisional integral, which will be henceforth denoted as  $Q$ , has the classical structure of the dissipative Boltzmann kernels. It can be viewed as composed of two parts: a *gain term*  $Q^+$ , which quantifies the exchanges of opinion between individuals which give, after the interaction with another individual, the opinion  $x$ , and a *loss term*  $Q^-$ , which quantifies the exchanges of opinion where an individual with pre-collisional opinion  $x$  experiences an interaction with another member of the population.

It is apparent that, in general, the existence of a pre-collisional pair which restitutes the post-collisional pair  $(x, x_*)$  through a collision of type (1) is not guaranteed, unless we suppose that the collisional rule is a diffeomorphism of  $\bar{\Omega}^2$  onto itself. Unfortunately, the collisional mechanism (1) does not verify this property. For instance, there is no  $(x, x_*) \in \bar{\Omega}^2$  which gives, after collision, the couple of extreme opinions  $(-1, 1)$ .

In order to overcome this difficulty, the natural framework for such a collision rule is given by the weak form. Two choices are possible. We may either build a model in a weak form with respect to  $x$  only, or to work in a weak setting with respect to the whole set of independent variables. We choose the first option, which seems to be the correct framework for such kind of models.

Let  $\phi = \phi(x)$  be a suitably regular test function. We define the weak form of the collision kernel as

$$(3) \quad \langle Q(f, f), \phi \rangle = \beta \iint_{\Omega^2} f(t, x) f(t, x_*) [\phi(x') - \phi(x)] dx_* dx.$$

Note that the particular form of the collision rule (1) only enters through the test function  $\phi(x')$ . The cross section  $\beta > 0$  is a parameter which governs the probability that an exchange of opinions can occur. In our model, we suppose that  $\beta$  is purely constant. This is the simplest possible assumption, which means that the probability of interaction of two individuals does not depend on their respective opinions. Of course, other choices, based on sociological considerations, are possible.

It is also clear that the operator  $Q$  only acts on the variable of opinion, and not on time.

The explicit form of the change of variables (1) allows also to give the following alternative formulations of the collision kernel:

$$\begin{aligned} \langle Q(f, f), \phi \rangle &= \beta \iint_{\Omega^2} f(t, x) f(t, x_*) [\phi(x'_*) - \phi(x_*)] dx_* dx \\ &= \frac{\beta}{2} \iint_{\Omega^2} f(t, x) f(t, x_*) [\phi(x') + \phi(x'_*) - \phi(x) - \phi(x_*)] dx_* dx. \end{aligned}$$

**Remark 2.4.** *At least formally, we have  $\langle Q(f, f), 1 \rangle = 0$ .*

The form of the collisional integral given by (3) is nevertheless not completely satisfactory for the gain term because of the intricate dependence of the argument of the test function with respect to the variables  $x, x_*$ . We therefore consider the weak form of the gain term

$$(4) \quad \langle Q^+(f, f), \phi \rangle = \beta \iint_{\Omega^2} f(t, x) f(t, x_*) \phi(x') dx_* dx.$$

Let us denote

$$D_\eta = \left\{ (x, x') \in \mathbb{R} \times \bar{\Omega} \mid \frac{x' - 1}{2} + \eta(x') \frac{x' + 1}{2} \leq x \leq \frac{x' + 1}{2} + \eta(x') \frac{x' - 1}{2} \right\}$$

and

$$K_\eta(x, x') = \frac{2\beta}{1 - \eta(x')} \chi_{D_\eta}(x, x'), \quad \forall x, x' \in \bar{\Omega},$$

where  $\chi_{D_\eta}$  is the characteristic function of the set  $D_\eta$ . Since  $\eta$  is an admissible attraction coefficient, it is clear that  $D_\eta \subseteq \bar{\Omega}^2$ .

We then perform the change of variable  $x_* \mapsto x'$  in (4), for a fixed  $x$ . It is easy to see that

$$dx_* = \frac{2}{1 - \eta(x)} dx' \quad \text{and} \quad x_* = \frac{2x' - x - \eta(x)x}{1 - \eta(x)}.$$

Then, after permuting  $x$  and  $x'$ , we obtain the following weak form of the gain term:

$$\langle Q^+(f, f), \phi \rangle = \iint_{\Omega^2} K_\eta(x, x') f\left(t, \frac{2x - x' - \eta(x')x'}{1 - \eta(x')}\right) f(t, x') \phi(x) dx' dx.$$

A new weak form of the collision operator immediately comes:

$$(5) \quad \langle Q(f, f), \phi \rangle = \langle Q^+(f, f), \phi \rangle - \beta \iint_{\Omega^2} f(t, x) f(t, x') \phi(x) dx' dx,$$

which will be henceforth our definition of the collisional kernel for our model.

It is then straightforward to prove the following lemma.

**Lemma 2.5.** *Let  $f(t, \cdot) \in L^1(\Omega)$ . Then both  $Q^+(f, f)(t, \cdot)$  and  $Q(f, f)(t, \cdot)$  are of class  $L^1(\Omega)$ , and we have, for a.e.  $t$ ,*

$$(6) \quad \|Q^+(f, f)(t, \cdot)\|_{L^1(\Omega)} \leq \frac{2\beta}{1 - \max \eta} \|f(t, \cdot)\|_{L^1(\Omega)}^2,$$

$$(7) \quad \|Q(f, f)(t, \cdot)\|_{L^1(\Omega)} \leq \left( \frac{2}{1 - \max \eta} + 1 \right) \beta \|f(t, \cdot)\|_{L^1(\Omega)}^2.$$

**2.2. Self-thinking.** As explained in Section 1, the self-thinking phenomenon is described, in our model, by a diffusive term obeying to a non-uniform Fourier law, with Fourier term  $\alpha = \alpha(x)$ . This means that we introduce a term which has the structure  $(\alpha f_x)_x$ , which will compete with the kinetic term describing the interactions between the agents.

We suppose that the non homogeneous Fourier coefficient  $\alpha$  satisfies the assumptions listed in the following definition.

**Definition 2.6.** *Let  $\alpha : \bar{\Omega} \rightarrow \mathbb{R}$  be a nonnegative function of class  $C^1(\bar{\Omega})$ . We say that  $\alpha$  is admissible if  $\alpha(x) = \alpha(-x)$  for all  $x \in \bar{\Omega}$  and  $\alpha(-1) = \alpha(1) = 0$ .*

The symmetry property allows to treat in an equivalent way positive and negative opinions. The second one forces the diffusive process to respect the bounds of the opinion space  $\bar{\Omega}$ .

**Remark 2.7.** Note that  $(\alpha f_x)_x$  has a meaning in the distributional sense if  $f(t, \cdot) \in L^1(\Omega)$ . Indeed, since  $\alpha \in W^{1,\infty}(\Omega)$  (in fact  $C^1(\bar{\Omega})$ ), we can write, in the distributional sense:

$$(\alpha f_x)_x = (\alpha f)_{xx} - (f \alpha_x)_x.$$

A possible choice of  $\alpha$  is, for example,  $\alpha(x) = \kappa(1 - x^2)$ , with  $\kappa > 0$ . This profile translates the idea that individuals with a stronger opinion are more stable in their convictions. As in the case of the attraction coefficient, other choices, based on psychological studies, are obviously possible.

**2.3. Combining the two phenomena.** We are then able to write down the whole model, by considering both the interaction effect and the self-thinking. The evolution law of the unknown  $f = f(t, x)$  results in a partial integro-differential equation of second order with respect to the opinion variable:

$$(8) \quad \int_{\Omega} f_t(t, x) \varphi(x) dx = \int_{\Omega} [\alpha(x) \varphi_x(x)]_x f(t, x) dx + \langle Q(f, f), \varphi \rangle$$

posed in  $(t, x) \in [0, T] \times \Omega$ ,  $T > 0$ , for all  $\varphi \in C^2(\bar{\Omega})$ , with initial condition

$$(9) \quad f(0, x) = f^{\text{in}}(x) \text{ for all } x \in \bar{\Omega}.$$

### 3. MAIN MATHEMATICAL PROPERTIES

This section is devoted to state and study some mathematical properties of Equations (8)–(9). We first obtain some a priori estimates and then deduce a theorem which proves the existence of weak solutions to (8)–(9).

Our model guarantees the conservation of the total number of individuals of the population. By borrowing the kinetic theory language, the following result is also named the conservation of the total mass.

**Proposition 3.1.** *Let  $f = f(t, x)$  be a nonnegative weak solution of (8)–(9), with a nonnegative initial condition  $f^{\text{in}} \in L^1(\Omega)$ . Then we have*

$$\|f(t, \cdot)\|_{L^1(\Omega)} = \|f^{\text{in}}\|_{L^1(\Omega)} \quad \text{for a.e. } t \geq 0.$$

*Proof.* We simply consider Equation (8) with test function  $\varphi \equiv 1$ ; the thesis immediately follows.  $\square$

Mass conservation is not realistic if we consider long-time forecasts. Indeed, in such situations, we should also consider processes of birth and death, which would lead to oscillations in the total number of individuals.

But usually, as in the case of elections or referendums, the interest of such models is to deduce short-term forecast by using, as an initial datum, the result of some opinion poll. We also have to point out that the poll is in fact the comparison at a given time  $t$  between the two integrals

$$\int_{\Omega_-} f(t, x) dx \quad \text{and} \quad \int_{\Omega_+} f(t, x) dx,$$



where  $\Omega_- = (-1, 0)$  and  $\Omega_+ = (0, 1)$ . That is why we are also interested in computing both integrals in the numerical experiments described in Section 4.

Moreover, since  $|x| \leq 1$ , from the conservation of mass, we immediately deduce that all the moments of  $f$  are bounded.

**Corollary 3.2.** *Let  $f = f(t, x)$  be a nonnegative weak solution of problem (8)–(9), with nonnegative initial condition  $f^{\text{in}} \in L^1(\Omega)$ . Then*

$$\int_{\Omega} x^n f(t, x) dx \leq \|f^{\text{in}}\|_{L^1(\Omega)}, \quad \text{a.e. } t \geq 0.$$

In order to prove the existence of weak solutions of our model, we need the following result.

**Proposition 3.3.** *Consider the initial-boundary problem for the unknown  $v = v(t, x)$ ,  $x \in \Omega$  and  $t \in [0, T]$ ,*

$$(10) \quad v_t - [\alpha(x)v_x]_x + \mu v = g, \quad \mu \geq 0,$$

*with initial condition*

$$(11) \quad v(0, \cdot) = v^{\text{in}}$$

*and boundary conditions*

$$(12) \quad \lim_{x \rightarrow \pm 1} \alpha(x)v_x(t, x) = 0 \quad \text{a.e. } t,$$

*where  $v^{\text{in}} \in L^1(\Omega)$ ,  $g \in C([0, T]; L^1(\Omega))$  are nonnegative functions. Then (10)–(12) admits a unique solution  $v \in C^0([0, T]; L^1(\Omega))$ , and  $v$  is nonnegative.*

*Proof.* The result, when  $g \equiv 0$ , has been proved in [3]. Proposition 3.3 easily follows, using the Duhamel principle.  $\square$

As pointed out in [3], the singular Neumann boundary conditions (12) are the only ones which allow a unified treatment of the initial-boundary value problem for the evolution equation (10) independently on the degeneracy of  $\alpha$ . In particular, they are automatically satisfied if  $\alpha$  is highly degenerate and reduce to the classical Wentzel conditions of boundary regularity of the solutions for low order zeros.

Thanks to the previous result, we are able to prove the following existence theorem.

**Theorem 3.4.** *Let  $f^{\text{in}}$  a nonnegative function of class  $L^1(\Omega)$ . Then there exists a nonnegative weak solution  $f \in L^\infty(0, T; L^1(\Omega))$  of problem (8)–(9), that is of*

$$\int_{\Omega} f_t(t, x)\varphi(x) dx = \int_{\Omega} [\alpha(x)\varphi_x(x)]_x f(t, x) dx + \langle Q(f, f), \varphi \rangle,$$

*for all  $\varphi \in C^2(\bar{\Omega})$ , where the equation takes sense in  $\mathcal{D}'(-T, T)$  with initial condition  $f(0, x) = f^{\text{in}}(x)$  for all  $x \in \Omega$ .*

*Proof.* Let us set

$$\rho = \int_{\Omega} f^{\text{in}}(x_*) dx_*,$$

and consider the sequence  $(f^n)_{n \in \mathbb{N}}$  defined by finite induction as  $f^0 \equiv 0$  and as solutions of

$$(13) \quad \int_{\Omega} f_t^{n+1} \varphi dx - \int_{\Omega} (\alpha \varphi_x)_x f^{n+1} dx + \beta \rho \int_{\Omega} f^{n+1} \varphi dx = \langle Q^+(f^n, f^n), \varphi \rangle,$$

for all  $\varphi \in C^2(\bar{\Omega})$ , together with the initial conditions

$$f^n(0, \cdot) = f^{\text{in}}, \quad n \geq 1,$$

and the boundary conditions

$$\lim_{x \rightarrow \pm 1} \alpha(x) f_x^n(t, x) = 0 \quad \text{a.e. } t, \quad n \geq 1.$$

Thanks to Lemma 2.5, we can apply Proposition 3.3 and deduce, by induction, that  $f^n$  exists, belongs to  $C^0([0, T]; L^1(\Omega))$  and is nonnegative.

If we pick  $\varphi \equiv 1$  in (13), it comes

$$\frac{d}{dt} \int_{\Omega} f^{n+1} dx + \beta \rho \int_{\Omega} f^{n+1} dx = \beta \left( \int_{\Omega} f^n dx \right)^2.$$

Therefore, by finite induction, we immediately get

$$\int_{\Omega} f^n dx \leq \rho, \quad n \geq 1.$$

Moreover,  $(f^n)$  is a non-decreasing sequence. Indeed, for  $n \geq 1$ , if we subtract Equation (13) for  $f^n$  from Equation (13) for  $f^{n+1}$ , the quantity  $(f^{n+1} - f^n)$  satisfies the following equation, which is of type (10):

$$\begin{aligned} & \int_{\Omega} (f^{n+1} - f^n)_t \varphi dx - \int_{\Omega} (\alpha \varphi_x)_x (f^{n+1} - f^n) dx \\ &= \langle Q^+(f^n, f^n), \varphi \rangle - \langle Q^+(f^{n-1}, f^{n-1}), \varphi \rangle - \beta \rho \int_{\Omega} (f^{n+1} - f^n) \varphi dx, \end{aligned}$$

with initial datum equal to 0 and singular Neumann boundary conditions (12). We know that  $f^0 \equiv 0$ , and that  $f^1 \geq 0$  by Proposition 3.3. If we suppose, by induction, that  $f^n \geq f^{n-1}$ , then the second member of the previous equation is nonnegative, and hence Proposition 3.3 implies that  $(f^{n+1} - f^n) \geq 0$ , i.e.  $f^{n+1} \geq f^n$ .

Therefore, by monotone convergence, there exists  $f \in L^\infty(0, T; L^1(\Omega))$  such that  $(f^n)$  converges to  $f$  almost everywhere and in  $L^\infty(0, T; L^1(\Omega))$ .

Equality (9) is clear. Let us prove now that  $f$  solves (8) in  $\mathcal{D}'_t$ . First of all, for all  $\varphi \in C^2(\bar{\Omega})$  and  $\psi \in \mathcal{D}(-T, T)$ , the quantities

$$\int_0^T \int_{\Omega} f^{n+1}(t, x) \varphi(x) \psi(t) dx dt + \int_{\Omega} f^{n+1}(0, x) \varphi(x) \psi(0) dx$$

and

$$\int_0^T \int_{\Omega} [\alpha(x) \varphi_x(x)]_x f^{n+1}(t, x) \psi(t) dx dt$$

are clearly convergent when  $n$  goes to  $+\infty$ , respectively towards

$$\int_0^T \int_{\Omega} f(t, x) \varphi(x) \psi(t) dx dt + \int_{\Omega} f^{\text{in}}(x) \varphi(x) \psi(0) dx$$

and

$$\int_0^T \int_{\Omega} [\alpha(x) \varphi_x(x)]_x f(t, x) \psi(t) dx dt$$

for all  $\varphi \in C^2(\bar{\Omega})$  and  $\psi \in \mathcal{D}(-T, T)$ .

For the collision term, the loss part easily is treated because of mass conservation given by Lemma 3.1

$$\beta \rho \int_0^T \int_{\Omega} f^{n+1}(x) \varphi(x) \psi(t) dx dt \rightarrow \beta \int_0^T \iint_{\Omega^2} f(x_*) f(x) \varphi(x) \psi(t) dx dx_* dt$$

when  $n$  goes to  $+\infty$ , for all  $\varphi \in C^2(\bar{\Omega})$  and  $\psi \in \mathcal{D}(-T, T)$ .

Finally, we are concerned with the nonlinear limit, when  $n$  goes to  $+\infty$ , of

$$\beta \int_0^T \iint_{\Omega^2} f^n(x) f^n(x_*) \varphi(x') \psi(t) dx dx_* dt,$$

with  $\varphi \in C^2(\Omega)$  and  $\psi \in \mathcal{D}(-T, T)$ . For the sake of simplicity, we assume that  $\|\varphi\|_{L^\infty(\Omega)} = 1$ . We successively have

$$\begin{aligned} & \iint_{\Omega^2} |f^n(x) f^n(x_*) - f(x) f(x_*)| |\varphi(x')| dx dx_* \\ & \leq \iint_{\Omega^2} |f^n(x) - f(x)| f^n(x_*) dx dx_* + \iint_{\Omega^2} |f^n(x_*) - f(x_*)| f(x) dx dx_* \\ & \leq 2\rho \|f^n(t, \cdot) - f(t, \cdot)\|_{L^1(\Omega)} \end{aligned}$$

which obviously goes to 0 when  $n$  goes to  $+\infty$ .

By collecting the previous results, we hence deduce that  $f$  satisfies (8). The proof is therefore complete.  $\square$

**Remark 3.5.** *The previous theorem gives an existence proof of the problem (8)–(9), without uniqueness, and does not provide either a characterization of the solutions in term of boundary data, even if they have been built as limit of solutions of problem (10)–(12), which satisfy boundary conditions of type (12). This happens because we have limited ourselves to build a theory in  $L^1(\Omega)$ , and therefore the meaning of the trace of the solution on the space boundary is not guaranteed.*

*Of course, if one supposes that  $f(t, \cdot)$  belongs to the class  $C^2(\bar{\Omega})$ , we would automatically have that  $\lim \alpha(x) f_x^n(t, x) = 0$  when  $x \rightarrow \pm 1$ . In the next section, we shall treat the solution as a smooth function of class  $C^1([0, T]; C^2(\bar{\Omega}))$ , and therefore the selected numerical solution satisfies in a natural way the singular Neumann conditions (12).*

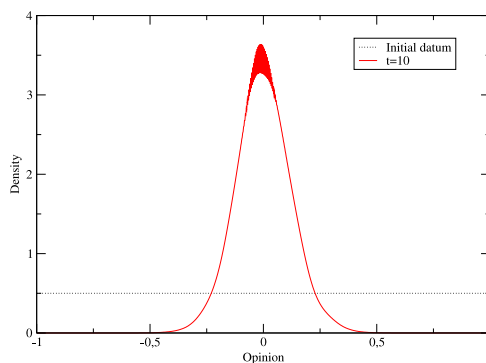


FIGURE 1. Constant initial datum

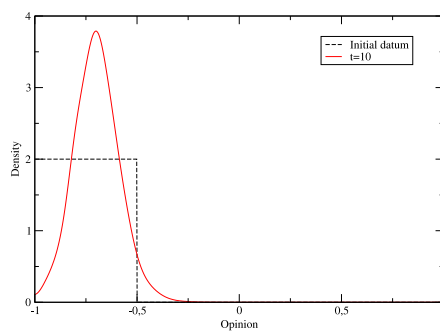


FIGURE 2. Monomodal initial datum

#### 4. NUMERICAL TESTS

In this section, we present some numerical results. The computations were performed using the Inria software `Scilab` [10]. We consider a regular subdivision  $(x_1, \dots, x_N)$  of  $\Omega$ , with  $N \geq 2$ . The function  $f$  is computed at the center  $x_{i+1/2}$  of each interval  $[x_i, x_{i+1}]$ ,  $1 \leq i \leq N - 1$ .

The scheme itself conserves the total opinion, i.e.  $\|f(t)\|_{L_x^1}$ , and is time-splitted into a diffusion part and a collision one. The diffusion part is here a standard explicit scheme and the collision one uses a slightly modified Bird method [2]. In particular, an opinion which is numerically going out from  $[-1, 1]$  is instantaneously relocated in  $x_1$  or  $x_N$  regarding its location with respect to  $\pm 1$ . Hence the quantity  $\|f(t)\|_{L_x^1}$  is numerically conserved at each time step, during the whole computation. Moreover, since the diffusive scheme is explicit, the diffusion time step is very small with respect to the

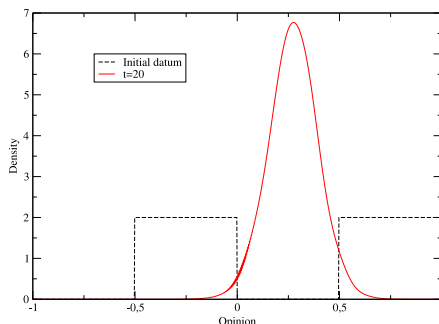


FIGURE 3. Bimodal initial datum

collision time step, so that we need several diffusion steps for one collision step.

For the four following computations, we set  $N = 1001$ , the diffusion coefficient  $\alpha(x) = .1(1 - x^2)^{1/3}$  and the collision frequency  $\beta = 50$ , so that neither the diffusion nor the collisions are preponderant.

As already explained in Section 3, we are also interested in the computations of the integrals of  $f$  over  $(-1, 0)$  and  $(0, 1)$ . They are here respectively denoted  $I_-(t)$  and  $I_+(t)$ .

**4.1. Constant initial datum.** We first choose a constant initial datum equal to  $1/2$ . Fast enough, here at time  $t = 10$ ,  $f$  is very different from the constant profile (see Figure 1). Moreover, it appears in this particular computation, that  $I_-(t) = 0.507$  and  $I_+(t) = 0.493$ . In terms of poll, the uniform opinion is here a little bit modified, with 50.7% more favorable for opinion  $-1$  and 49.3% nearer opinion  $1$ .

We can also note the profile of  $f$  is almost centered, but not completely. That also explains why there is that difference between  $I_-(t)$  and  $I_+(t)$ .

**4.2. Monomodal profile.** Here we choose a monomodal initial datum. More precisely, from  $-1$  to  $-1/2$ ,  $f^{\text{in}}$  equals  $2$ , and anywhere else equals  $0$ . The solution  $f$  is fastly regularized (see Figure 2). We also note that the support of  $f$  is larger than the one of its initial condition. Hence, at time  $10$ , the average opinion still favours opinion  $1$  with  $100\%$  of the individuals.

**4.3. Uncentered bimodal profile.** Here we deal with an uncentered bimodal initial datum. More precisely, from  $-1/2$  to  $0$ , and from  $1/2$  to  $1$ ,  $f^{\text{in}}$  is set to  $2$ , and elsewhere equals  $0$ . Once again, the solution is fastly regularized (see Figure 3). The support of  $f$  is then widely modified, since it is a segment almost centered on the middle of the two components of the support of  $f^{\text{in}}$ . The result of the poll is very different from the initial datum. Only  $1\%$  of the individuals can be considered as favorable to opinion  $-1$ ,

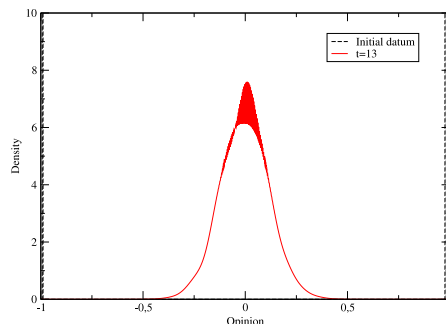


FIGURE 4. Dirac masses initial datum

and the huge majority favours opinion 1, whereas the average opinions were initially well-balanced.

**4.4. Dirac masses.** We finally propose two Dirac masses in  $-0.99$  and  $0.98$  as the initial datum for  $f$ . The support of the solution significantly changes. Figure 4 shows the situation of  $f$  at time  $t = 13$ . Note that obviously the initial datum and the current solution do not share the same scale of density. Indeed one can only see a small part of the initial datum in the figure. Before this time, the diffusion seems to be the leading process. Some perturbations due to the collisions then appear. Further computations show in fact that the collisions begin to disturb the behaviour of  $f$  after time  $t = 13$ , as shown in Figure 4 on the upper part of the graph.

The opinions are initially equally distributed. At time  $t = 13$ , they are not anymore, since  $I_-(t) = 50.2\%$  and  $I_+(t) = 49.8\%$ .

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