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# A NEW APPROACH OF POINT ESTIMATION AND ITS APPLICATION TO TRUNCATED DATA SITUATIONS

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## Abstract

We propose a new approach for estimating the parameters of a probability distribution. It consists on combining two new methods of estimation. The first is based on the definition of a new distance measuring the difference between variations of two distributions on a finite number of points from their support and on using this measure for estimation purposes by the method of minimum distance. For the second method, given an empirical discrete distribution, we build up an auxiliary discrete theoretical distribution having the same support of the first and depending on the same parameters of the parent distribution of the data from which the empirical distribution emanated. We estimate then the parameters from the empirical distribution by the usual statistical methods. In practice, we propose to compute the two estimations and we choose the second, of known theoretical properties, and let the first being as a control of the effectiveness of the obtained estimation, so we have also a criterion on the quality of the information contained in the observations. We apply the approach to truncated data situations to give the flavour on the effectiveness of the approach. We give also some interesting perspectives of the approach including model selection from truncated data, estimation of the initial trial value in the celebrate *EM* algorithm in the case of truncation and merged normal populations, a test of goodness of fit based on the new distance, quality of estimations and data, optimum number of classes in grouping data from continuous distributions.

Key words and phrases: EM algorithm, Minimum distance, Model selection from truncated data, Point estimation, Truncated data.

## 1 Introduction

Point estimation is the most popular forms of statistical inference (see Lehmann and Casella (1998)). We introduce in this paper a new statistical point estimation approach which found be useful in special practical situations such as truncated data. The data are said to be truncated when measuring devices fail to report observations below and

above certain readings. For example, truncated data frequently arise in the statistical analysis of astronomical observations ( see Efron and Petrosian (1999)) and in medical data (see Klein and Zhang (1996)), and if the truncation is ignored this can cause considerable bias in the estimation. There exist in the literature many approaches of estimation from "incomplete data" such as maximum likelihood based approach of the EM algorithm (Hartley (1958), Dempster et al (1977)), or nonparametric methods such as Kaplan-Meier (Kaplan and Meier (1958)) or Lynden-Bell estimators (Lynden-Bell (1971)). The purpose of the present paper is to investigate another approach which consists on combining two new methods of estimation.

In the first method, we remark that in estimation problems we deal in general with three functions: a theoretical probability law  $f(\cdot, \theta)$  of a random variable  $X$ , depending on a parameter  $\theta$  (real or vector valued), an empirical distribution  $\hat{f}$  constructed from a sample of observations drawn from the random variable  $X$ , and an estimation  $\tilde{f}$  (from an estimation  $\tilde{\theta}$  of  $\theta$ ) obtained through the empirical law  $\hat{f}$ . The empirical distribution  $\hat{f}$  is considered as a representative distribution of  $f$ , but in practice it is reduced to only few of its characteristics such as the mean and variance. The variational aspect of  $\hat{f}$  is often neglected while its importance. We can easily find, for instance, two distributions having the same support, mean and variance while their variations differ significantly, or conversely having the same variations but their supports and characteristic parameters are different. But two probability distributions with same support and same variations in each subset of the support are necessarily the same. We introduce then a new distance which measures the difference between variations of two distributions on a finite number of points and to use it for estimation purposes by the method of minimum distance. Since the new measure is not equivalent to classical ones it will give new insights that could not be investigated by classical distances.

In the second method, we remark that the empirical distribution arising from a sample of observations can be viewed in fact as a conditional distribution as it is built from the knowledge of the data. It will be then an estimation of the theoretical conditional distribution with respect to the observations before being an estimation for the parent distribution. This theoretical conditional distribution is represented by the auxiliary distribution introduced in this paper. To determine this distribution in discrete case, we have simply to take the conditional distribution with respect to the observed values and we proceed analogously for the continuous case. It should be noted that in discrete case it is known as the truncated distribution which is the conditional distribution given a truncation (see for example Shaw (1988)) but it is presented here in a general framework. We have to deal with two discrete probability distributions having the same finite support, a theoretical distribution and its empirical representation with respect to the observations. The parameters of the former are those of the parent distribution and the aim is to estimate them from the first instead of the parent one as commonly used. We use classical tools such as the method of moments or maximum likelihood principle. The setting that seems to us most suitable for illustrating our approach is the one of truncated data. In usual practical problems, truncation can be on left or right or in either situations, and the "cut off" can be deterministic or random. In our approach, the truncation may be on any part of the range of the distribution so that the setting is more general. Also, classical approaches for truncated data are in general custom-made

depending on specific problems and distributions, or subjective based methods. Instead, our approach is quite general and might be used in any situation where the underlying complete data come from a known family of distributions. We confine ourselves as a first presentation to deterministic truncations.

In the subsequent section, we propose a variational distance between probability distributions. In Section 3, we define a truncation of data and associated empirical and theoretical distributions and we use two different methods for estimation from truncation, a first method using minimum of the new distance introduced in this paper and a second method based on traditional tools of estimations such as the method of maximum likelihood. In Section 4, we present the new approach and we illustrate the procedure by three examples: a binomial probability law, a normal distribution and a Gamma density function. We present also a basic feature of the new approach which prove the accuracy of the method and some illustrative examples. In Section 5, we give some elements of comparison with the classical approach of estimation. In Section 6, we list some perspectives of the new approach: model selection from truncated data using the new distance, estimation of the first trial value in the celebrate *EM* algorithm for incomplete data in the case of truncation and merged normal distributions, a goodness of fit test based on the new distance, decision making about the quality of estimations and data, optimum number of classes in grouping data from continuous distributions. Finally, concluding remarks are made some pointing to other possible extensions and applications.

## 2 A New Distance Between Probability Distributions

As is usual, given a sample of  $n$  independent and identically distributed observations,  $(x_1, \dots, x_n)$ , drawn from an unknown discrete random variable  $X$  falling in a discrete family of probability laws  $\mathcal{P} = \{f(\cdot, \theta), \theta \in \mathbb{R}^r\}$  depending on a parameter  $\theta$  (real or vector valued), i.e.,  $f(x, \theta) = \mathbb{P}(X = x)$ , one can summarize the sample into  $k$  couples  $(y_1, \hat{f}_1), \dots, (y_k, \hat{f}_k)$ ,  $k \leq n$ , where the  $y_i$  are the different values taken by the sample and  $\hat{f}$  is the empirical law  $\hat{f}_j = n_j/n$ , where  $n_j$  represents the absolute frequency of the value  $y_j$ ,  $j = 1, \dots, k$ .

Usually, it is hoped that  $\hat{f}_j \approx f(y_j, \theta)$ , in a certain probabilistic sense. But if the empirical distribution arises from truncated data, we do not hope in general having  $\hat{f}(x) \approx f(x, \theta)$ , for the values  $x$  in the support of  $\hat{f}$ , since the complete sample size  $n$  is usually not reported. However, we expect reasonably to have approximately

$$\frac{\hat{f}(x)}{\hat{f}(y)} \approx \frac{f(x, \theta)}{f(y, \theta)}, \quad (1)$$

for any points in its support, only if the sample has serious irregularities.

Introduce the following *distance of proportional variations* between  $f(\cdot, \theta)$  and  $\hat{f}$

$$d_v(\hat{f}, f(\cdot, \theta)) = \sum_{i,j \in \{1, \dots, k\}} \left| \frac{\hat{f}_i}{\hat{f}_j} - \frac{f(y_i, \theta)}{f(y_j, \theta)} \right|. \quad (2)$$

It turns out that this new distance, as we will show, measures the variations between probability distributions.

In continuous case also, any sample  $x_1, \dots, x_n$  is summarized into  $k$  couples  $(y_1, \widehat{f}_1), \dots, (y_k, \widehat{f}_k)$ ,  $k \leq n$ . This can be done uniquely, by grouping for example the sample in classes where the  $y_i$  are the mid-classes (or class means) and  $\widehat{f}_i = \widehat{f}(y_i)$  where  $\widehat{f}$  is a kernel density estimator. The proportional variational distance  $d_v$  in this case, between the density  $f(x, \theta)$  of  $X$  and its empirical law  $\widehat{f}$ , is thus defined as (2). One of its main powerful feature is that when using traditional distances we have to use the sample size  $n$  through the expression of  $\widehat{f}_i = n_i/(nh_n)$ , where  $h_n$  is the size of class intervals ; but sometimes, as for truncated data situations where measuring devices fail to report even the number of sample points in certain ranges, then the real size  $n$  is not known, but a truncated sample size  $n_t$  is instead used. Using the ratios  $\widehat{f}_i/\widehat{f}_j$  will clear up the effect of the truncated sample size which can lead to considerable bias in the estimation.

Note that  $d_v$  possesses the properties of symmetry and triangle inequality. But in the identity property  $d_v(f, g)(x, y) = 0 \iff f \equiv g$ , the equality between  $f$  and  $g$  must be understood in the sense that  $f$  and  $g$  have the same variations on the points  $x$  and  $y$ . It should be stressed that this new measure is not equivalent to classical ones and should then give new insights and information about other characteristics and features of probability distributions.

From now on  $f$  shall represent a theoretical probability law in both discrete or continuous cases and  $\widehat{f}$  shall represent the corresponding empirical law in both cases. Denote by  $\Omega = \{x \in \mathbb{R}, f(x, \theta) > 0\}$  the set of *atoms* of  $f$  or *support*. Let  $\mathcal{F}$  be the  $\sigma$ -algebra generated by sets  $A = B \cap \omega$  where the  $\omega$  are the Borel sets of  $\mathbb{R}$  and  $B \subset \Omega$ . For all  $A \in \mathcal{F}$ , we have  $\mathbb{P}(A) = \int_A f(x, \theta) \mu(dx)$ , where  $\mu$  is the Lebesgue measure on  $\mathbb{R}$ . In discrete case, we have  $\mathbb{P}(A) = \sum_{x \in A} f(x, \theta)$ .

For all  $i \geq 1$ , we set  $\Omega_i = \Omega$ ,  $\mathcal{F}_i = \mathcal{F}$  and  $\mathbb{P}_i = \mathbb{P}$ . Let  $\Omega^n = \Omega_1 \times \dots \times \Omega_n$ ,  $\mathcal{F}^{(n)} = \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$  and  $\mathbb{P}^{(n)} = \mathbb{P}_1 \otimes \dots \otimes \mathbb{P}_n$ . The probability space  $(\Omega^n, \mathcal{F}^{(n)}, \mathbb{P}^{(n)})$  represents the space of samples of size  $n$  from the random variable  $X$ . We omit the subscript  $n$  in  $(\Omega^n, \mathcal{F}^{(n)}, \mathbb{P}^{(n)})$  for notational convenience and shall denote the sample space as  $(\Omega, \mathcal{F}, \mathbb{P})$ .

Let be given a classical distance  $d$  between two functions  $f$  and  $g$  which associates for points  $x$  and  $y$  from the intersection of their domain of definitions, the quantity  $d(f, g)(x, y) = |f(x) - g(x)| + |f(y) - g(y)|$ .

**Proposition 1** *We have the following properties for the distance  $d_v$  :*

1.  $d(f, g)(x, y) = 0 \implies d_v(f, g)(x, y) = 0$ , the converse is not always true.
2. Let  $\widehat{f}$  be a kernel density estimation. Then  $\lim_{n \rightarrow \infty} d_v(\widehat{f}, f) = 0$  in probability.
3. Let  $f$  and  $g$  be two functions defined on  $\mathbb{R}$  and  $E \subset \mathbb{R}$  satisfying:

$$\forall (x, y) \in E \times E, d_v(f, g)(x, y) = 0.$$

If

$$\int_{\mathbb{R}} f d\mu = \int_{\mathbb{R}} g d\mu = 1,$$

where  $\mu$  is the Lebesgue measure on  $\mathbb{R}$ , then

$$\mu(\overline{E}) = 0 \implies f = g \quad \mu - \text{almost surely on } \mathbb{R}.$$

**Proof. 1.** Follows directly from the definitions of  $d$  and  $d_v$ .

**2.** Follows from the fact  $\lim_{n \rightarrow \infty} d(\hat{f}, f) = 0$  in probability (see Parzen (1962)), then  $\lim_{n \rightarrow \infty} d_v(\hat{f}, f) = 0$  in the same probabilistic notion of convergence.

**3.** Fix  $y_0 \in E$ , we have  $f(x)/f(y_0) = g(x)/g(y_0)$  for all  $x \in E$ . This implies that

$$\int_E f(x)dx = 1 \iff \int_E f(y_0) \frac{g(x)}{g(y_0)} dx = \frac{f(y_0)}{g(y_0)} \int_E g(x)dx = 1.$$

We deduce that  $f(y_0) = g(y_0)$ , and the result follows. ■

### 3 Truncated Data

The truncated data specification, or generally *incomplete data*, implies the existence of two sample spaces  $\mathcal{X}_o$  and  $\mathcal{X}_t$ , such that the complete sample space is given by  $\Omega = \mathcal{X}_o \cup \mathcal{X}_t$ . The observed data  $\mathbf{x}_o = (x_1, \dots, x_{n_t})$ , where  $n_t$  is the truncated sample size, are a realization from  $\mathcal{X}_o$  and the unobserved data  $\mathbf{z} = (x_1^*, \dots, x_{n-n_t}^*)$ , where  $n$  is the complete unknown sample size, are from  $\mathcal{X}_t$ . The complete data  $\mathbf{x} = \mathbf{x}_o \cup \mathbf{z}$  is known only through the observed data  $\mathbf{x}_o$  (see Dempster, Laird and Rubin (1977) for further explanations about incomplete data specification).

Consider a sample of observations  $x_1, \dots, x_n$  drawn from a theoretical probability law  $f(\cdot, \theta)$ , depending on a parameter  $\theta \in \mathbb{R}^r$ . As usual, the data are summarized, in discrete or continuous cases (as shown in Section 2), into  $k$  couples  $(y_1, \hat{f}_1), \dots, (y_k, \hat{f}_k)$ ,  $k \leq n$ , and let  $\Delta = \{u_1, \dots, u_m\}$  a part from the set  $\{y_1, \dots, y_k\}$ ,  $m \leq k$ , which we will call *truncation*. The observed data is summarized by a truncation  $\Delta_o = \{u_1, \dots, u_m\}$  and an empirical estimation  $\hat{f}_o$  and assume that the unobserved data is also summarized by a set  $\Delta_t = \{u_1^*, \dots, u_p^*\}$  and  $\hat{f}_t$ .

The structure of the new distance  $d_v$  allows the following decomposition property:

$$d_v(\hat{f}, f(\cdot, \theta)) = d_v(\hat{f}_o, f(\cdot, \theta)) + d_v(\hat{f}_t, f(\cdot, \theta)) + \sum_{\substack{u_i \in \Delta_o \\ u_j^* \in \Delta_t}} \left| \frac{\hat{f}_o(u_i)}{\hat{f}_t(u_j^*)} - \frac{f(u_i, \theta)}{f(u_j^*, \theta)} \right| + \sum_{\substack{u_i \in \Delta_o \\ u_j^* \in \Delta_t}} \left| \frac{\hat{f}_t(u_j^*)}{\hat{f}_o(u_i)} - \frac{f(u_j^*, \theta)}{f(u_i, \theta)} \right|. \quad (3)$$

The following proposition is typical for the new distance and is useful for using the minimum of distance  $d_v$ .

**Proposition 2** *Let be given a truncated data  $\Delta_o$  with corresponding empirical estimation  $\hat{f}_o$ . Then  $\lim_{n_t \rightarrow \infty} d_v(\hat{f}_o, f) = 0$  in probability.*

**Proof.** We have from Proposition 1 that  $\lim_{n \rightarrow \infty} d_v(\hat{f}, f) = 0$  in probability. Then, from the decomposition property (3) we obtain  $\lim_{n \rightarrow \infty} d_v(\hat{f}_o, f) = \lim_{n_t \rightarrow \infty} d_v(\hat{f}_o, f) = 0$  in probability. ■

### 3.1 An Auxiliary Distribution

Define the empirical distribution  $\tilde{f}$  corresponding to a given truncation  $\Delta$  by:

$$\tilde{f}(x) = \begin{cases} \tilde{f}_i & \text{if } x = u_i, \quad i = 1, \dots, m, \\ 0 & \text{otherwise,} \end{cases}$$

where the  $\tilde{f}_i$  satisfy the following set of proportional allocation equations  $\tilde{f}_i/\tilde{f}_j = \hat{f}_i/\hat{f}_j$ , for  $i, j = 1, \dots, m$  and  $\tilde{f}_1 + \dots + \tilde{f}_m = 1$ .

Define the following auxiliary distribution from  $f(\cdot, \theta)$ , which is akin to the proportional allocation procedure for missing values (see Hartley (1958)).

$$h(x, \theta) = \begin{cases} \frac{f(x, \theta)}{f(u_1, \theta) + f(u_2, \theta) + \dots + f(u_m, \theta)} & \text{if } x = u_i, \quad i = 1, \dots, m, \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

**Remark 3** *If the truncation is random, that is, there exists a random variable  $T$  such that we observe, for example, the random variable  $X$  only if  $X > T$  or  $X < T$ , then the probability law used in (4) is replaced by the conditional law of  $X$  with respect to  $\{X > T\}$  or  $\{X < T\}$  respectively.*

The auxiliary distribution  $h$  was found be useful for estimation problems in truncated data. Indeed, it is well known in classical estimation from truncated data (see Hartley (1958)) that missing values could be recovered by "proportional allocation" procedures, then the auxiliary distribution  $h$ , which is already based on proportional allocation, will be an intuitive and natural tool for estimation purposes from truncated data. The function  $h$  is a theoretical probability distribution depending on the same parameters of those of  $f$ . It has also the same *support* as that of  $f$ .

**Definition 4** *We call  $\tilde{f}$  and  $h(\cdot, \theta)$  the empirical and theoretical distributions of a given truncation  $\Delta = \{u_1, \dots, u_m\}$  from a sample of observations  $(x_1, \dots, x_n)$ .*

## 4 The Approach of Estimation

We will use mainly two methods of estimation. The first method is a minimum distance estimation using the metric  $d_v$  between the empirical and theoretical distributions  $\tilde{f}$  and  $f(\cdot, \theta)$ . The second is similar to traditional ones such as the method of substitution or maximum likelihood principle, by considering  $\tilde{f}$  as an empirical estimation of  $h(\cdot, \theta)$ . The first is based on variational difference between distributions and the second in the sense of an euclidean difference and hence they treat different aspects of the sample of observations. If for a given data they give different estimations, we cannot suspect the approaches but we can say that the data do not restore in a coherent way all aspects of the probability distribution from which it emanated. If on the other hand they give significantly the same estimations we can assert that the estimation is credible since through different aspects it has given the same distribution. That is the distribution which fits the best the empirical distribution. Practically, we propose to calculate the estimations by the two methods and take the second one since based on maximum

likelihood principle of good known theoretical properties. We use then the first as a tool of decision on whether the estimation is credible or not. The estimation will then be considered as credible in cases where the two methods give approximately the same estimation.

In order to test the performance of the proposed approach, we use synthetic data sets which were generated by simulation from three examples of probability law: binomial law, normal density and a Gamma distribution. The examples were selected from various simulation studies from different family of probability distributions and the two methods have shown their effectiveness and never deviate significantly from the true parameter. The reason for using synthetic data sets is that the true parameters for the synthetic datasets are known and the accuracy of results obtained by using the two new methods can be compared.

#### 4.1 Examples

**Binomial distribution.** We generated a synthetic data set of size 500 from a binomial law  $\mathcal{B}(n, p)$  with  $n = 10$  and  $p = 0.3$ . The data are summarized in the following table.

Table 1.

$y_i$	0	1	2	3	4	5	6	7
$n_i$	15	71	108	134	97	47	23	5

Our aim is to estimate the parameter  $p$ , with the knowledge of  $n = 10$ , from different truncation of data.

For illustrating the two methods, consider the truncation  $\Delta = \{2, 3, 4, 5\}$  with truncated sample size  $n_t = 386$ . We have then a truncation proportion of  $Q = (n - n_t)/n = 22,8 \%$  in data. For the first method, we have to search the value of the parameter  $p$  which minimizes the distance  $d_v$ , that is:

$$\begin{aligned} \min_p d_v(\hat{f}, f) = \min_p & \left[ \left| \frac{C_{10}^2 p^2 (1-p)^8}{C_{10}^3 p^3 (1-p)^7} - \frac{108}{134} \right| + \left| \frac{C_{10}^3 p^3 (1-p)^7}{C_{10}^2 p^2 (1-p)^8} - \frac{134}{108} \right| + \right. \\ & \left| \frac{C_{10}^2 p^2 (1-p)^8}{C_{10}^4 p^4 (1-p)^6} - \frac{108}{97} \right| + \left| \frac{C_{10}^4 p^4 (1-p)^6}{C_{10}^2 p^2 (1-p)^8} - \frac{97}{108} \right| + \left| \frac{C_{10}^2 p^2 (1-p)^8}{C_{10}^5 p^5 (1-p)^5} - \frac{108}{47} \right| + \\ & \left| \frac{C_{10}^5 p^5 (1-p)^5}{C_{10}^2 p^2 (1-p)^8} - \frac{47}{108} \right| + \left| \frac{C_{10}^3 p^3 (1-p)^7}{C_{10}^4 p^4 (1-p)^6} - \frac{134}{97} \right| + \left| \frac{C_{10}^4 p^4 (1-p)^6}{C_{10}^3 p^3 (1-p)^7} - \frac{97}{134} \right| + \\ & \left| \frac{C_{10}^3 p^3 (1-p)^7}{C_{10}^5 p^5 (1-p)^5} - \frac{134}{47} \right| + \left| \frac{C_{10}^5 p^5 (1-p)^5}{C_{10}^3 p^3 (1-p)^7} - \frac{47}{134} \right| + \left| \frac{C_{10}^4 p^4 (1-p)^6}{C_{10}^5 p^5 (1-p)^5} - \frac{97}{47} \right| + \\ & \left. \left| \frac{C_{10}^5 p^5 (1-p)^5}{C_{10}^4 p^4 (1-p)^6} - \frac{47}{97} \right| \right]. \end{aligned}$$

Using computer algebra package, we obtain the result  $\tilde{p}_1 = 0.2991$ .

For the second method, the empirical distribution  $\tilde{f}$  given the truncation  $\Delta = \{2, 3, 4, 5\}$  is given by  $\tilde{f}(2) = 108/386$ ,  $\tilde{f}(3) = 134/386$ ,  $\tilde{f}(4) = 97/386$ ,  $\tilde{f}(5) = 47/386$  and  $\tilde{f}(x) = 0$  if  $x \notin \Delta$ .



Let  $f(x, p) = C_n^x p^x (1-p)^{n-x}$  the binomial probability law. The auxiliary distribution  $h(\cdot, p)$  is given by:

$$h(x, p) = \begin{cases} \frac{f(x, p)}{f(2, p) + f(3, p) + f(4, p) + f(5, p)} & \text{if } x = u_i, \quad u_i \in \{2, 3, 4, 5\} \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

By the method of substitution, the estimation of  $p$  is obtained by solving the equation:

$$\sum_{u_i \in \{2, 3, 4, 5\}} u_i \times h(u_i, p) = \sum_{u_i \in \{2, 3, 4, 5\}} u_i \times \tilde{f}(u_i) \quad (6)$$

that is

$$\frac{2 \times C_{10}^2 p^2 (1-p)^8 + 3 \times C_{10}^3 p^3 (1-p)^7 + 4 \times C_{10}^4 p^4 (1-p)^6 + 5 \times C_{10}^5 p^5 (1-p)^5}{C_{10}^2 p^2 (1-p)^8 + C_{10}^3 p^3 (1-p)^7 + C_{10}^4 p^4 (1-p)^6 + C_{10}^5 p^5 (1-p)^5} = \frac{1241}{386}. \quad (7)$$

Using a computer algebra package we obtain the result  $\tilde{p}_2 = 0.3001$ .

In the following table we present the estimations  $\tilde{p}_1$  from the first method using minimum distance approach using the distance  $d_v$ , and  $\tilde{p}_2$  from the auxiliary distribution, of the parameter  $p$ , for known  $n$ , according to the truncation  $\Delta = \{u_1, \dots, u_m\}$  considered.

Table 2. The estimations  $\tilde{p}_1$  and  $\tilde{p}_2$  by the new approach of the parameter  $p$  of the binomial probability law  $\mathcal{B}(n, p)$  with  $p = 0.3$  and known  $n = 10$ .

$n^\circ$	$\Delta$	Truncated sample size $n_t$	Proportion of truncation $Q$ (%)	$\tilde{p}_1$	$\tilde{p}_2$
1	{0, 1, 2, 3, 4, 5, 6, 7}	500	0	0.3053	0.2978
2	{0, 1, 2, 3, 4, 5}	472	5.6	0.2952	0.2926
3	{1, 2, 3, 4, 5}	457	8.6	0.2877	0.2922
4	{0, 1, 2, 3, 4}	425	15	0.2952	0.2928
5	{1, 2, 3, 4}	410	18	0.2868	0.2922
6	{0, 2, 3, 4, 5}	401	19.8	0.2952	0.2983
7	{2, 3, 4, 5}	386	22.8	0.2991	0.3001
8	{0, 1, 3, 4, 5}	364	27.2	0.2952	0.2892
9	{0, 2, 3, 4}	354	29.2	0.2952	0.3009
10	{1, 3, 4, 5}	349	30.2	0.2870	0.2872
11	{2, 3, 4}	339	32.2	0.3049	0.3052
12	{0, 3, 4, 5}	293	41.4	0.2952	0.2932
13	{2, 4, 5, 6, 7}	280	44	0.3077	0.3071
14	{0, 1, 2, 5, 6, 7}	269	46.2	0.2981	0.2985
15	{0, 1, 4, 5, 6, 7}	258	48.4	0.3013	0.2945
16	{0, 4, 5, 6, 7}	187	62.6	0.3071	0.3018
17	{0, 5, 6, 7}	90	82	0.3014	0.3011
18	{0, 5}	62	87.6	0.2937	0.2937

As previously said, the two estimations by the new approach,  $\tilde{p}_1$  and  $\tilde{p}_2$ , are accurate in all cases and close to each other. Furthermore, the truncation proportion has no effect on the quality of estimations. The two estimations are also not sensitive to small cell probabilities as for truncations including the value  $y_8 = 7$ . It should be noted that the classical estimation by maximum likelihood without truncation is  $\hat{p} = 0.297$ , and considering our approach we obtained the estimations  $\tilde{p}_1 = 0.3053$  for the first method and  $\tilde{p}_2 = 0.2978$  for the second.

**Normal distribution.** Consider a sample of size 400 drawn from a normal population with mean  $m = 0$  and standard deviation  $\sigma = 1$ . Consider the data falling in 11 fixed class intervals as shown in the following table, with mid-classes  $u_i$  and absolute frequencies  $n_i$

Table 3.

$u_i$	-2.5808	-2.057	-1.5331	-1.0092	-0.4853	0.0386	0.5625	1.0863
$n_i$	5	8	23	48	71	89	72	43
	1.6102	2.1341	2.658					
	25	10	6					

In the following table we show the estimations of  $m$  according to the truncation  $\Delta$  considered, where  $\tilde{m}_1$  represents the estimation by the minimum of the variational distance  $d_v$ , and  $\tilde{m}_2$  that of the second method.

Table 4. The estimations  $\tilde{m}_1$  and  $\tilde{m}_2$  by the new approach of the parameter  $m$  of the standard normal distribution with known standard deviation  $\sigma = 1$ .

$n^\circ$	$\Delta$	$n_t$	$Q$ (%)	$\tilde{m}_1$	$\tilde{m}_2$
1	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}\}$	400	0	0.0651	0.0521
2	$\{u_3, u_4, u_5, u_6, u_7, u_8, u_9\}$	371	7.25	0.0592	0.0325
3	$\{u_4, u_5, u_6, u_7, u_8, u_9\}$	348	13	0.0515	0.0216
4	$\{u_5, u_6, u_7, u_8, u_9\}$	300	25	0.0520	0.0119
5	$\{u_5, u_6, u_7\}$	232	42	0.052	0.0518
6	$\{u_3, u_4, u_5, u_6\}$	231	42.25	0.1137	0.1092
7	$\{u_6, u_7, u_8, u_9\}$	229	42.75	0.0165	-0.0402
8	$\{u_6, u_7, u_8\}$	204	49	-0.1319	-0.1284
9	$\{u_3, u_5, u_7\}$	166	58.5	0.0593	0.0574
10	$\{u_6, u_7\}$	161	59.75	-0.1040	-0.1039
11	$\{u_2, u_3, u_4, u_5\}$	150	62.5	0.1179	0.0845
12	$\{u_3, u_4, u_5\}$	142	64.5	0.0665	0.0529
13	$\{u_7, u_8, u_9\}$	140	65	0.0767	0.0323
14	$\{u_4, u_5\}$	119	70.25	$6 \times 10^{-6}$	0.0003
15	$\{u_2, u_3, u_4\}$	79	80.25	0.1769	0.1631
16	$\{u_8, u_9, u_{10}\}$	78	80.5	0.2181	0.2436

For the normal distribution the two estimations  $\tilde{m}_1$  and  $\tilde{m}_2$  by the new approach are somewhat close to each other for all the truncations considered, they are also close to the true value  $m = 0$ . When we do not consider truncation in data the empirical mean is given by  $\bar{x} = 0.0517$ , and the two new estimations are  $\tilde{m}_1 = 0.0651$  and  $\tilde{m}_2 = 0.0521$ . Here also, the truncation proportion has no effect on the quality of estimations.

### 4.1.1 Gamma probability density

Consider a sample of size 800 drawn from a Gamma distribution  $G(a, b)$  with density given by

$$f(x | a, b) = \frac{1}{b^a \Gamma(a)} x^{a-1} \exp\left(-\frac{x}{b}\right), \quad x \geq 0, \quad (8)$$

and parameters  $a = 7$  and  $b = 3$ . Consider the data falling in 16 fixed class intervals as shown in the following table, with mid-classes  $u_i$  and absolute frequencies  $n_i$  :

Table 5.

$u_i$	5.89	8.72	11.56	14.39	17.23	20.06	22.89	25.73	28.56	31.39
$n_i$	11	40	60	108	118	104	100	74	63	53
	34.23	37.06	39.89	42.73	45.56	48.39				
	27	21	11	5	3	2				

In the following table we show the estimations  $\tilde{b}_1$  from the minimum of distance  $d_v$  and  $\tilde{b}_2$  by the second method for the parameter  $b$ , with known  $a = 10$ , according to the truncation  $\Delta$  considered.

Table 6. The estimations  $\tilde{b}_1$  and  $\tilde{b}_2$  by the new approach of the parameter  $b$  of the Gamma probability distribution  $G(a, b)$  with  $b = 3$  and known  $a = 7$ .

$n^\circ$	$\Delta$	$n_t$	$Q$ (%)	$\tilde{b}_1$	$\tilde{b}_2$
1	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}\}$	800	0	3.0183	3.0541
2	$\{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}$	787	1.625	2.9802	3.0652
3	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}\}$	779	2.625	3.0121	3.0676
4	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$	731	8.625	2.8948	3.0586
5	$\{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$	720	10	3.0628	3.0751
6	$\{u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$	680	15	3.1574	3.1192
7	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9\}$	678	15.25	2.8636	3.0016
8	$\{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9\}$	667	16.625	2.9778	3.0181
9	$\{u_3, u_4, u_5, u_6, u_7, u_8, u_9\}$	627	21.625	3.0860	3.0615
10	$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$	615	23.125	2.8587	2.9595
11	$\{u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$	604	24.5	2.9082	2.9773
12	$\{u_4, u_5, u_6, u_7, u_8, u_9\}$	567	29.125	3.0462	3.0156
13	$\{u_2, u_3, u_4, u_5, u_6, u_7\}$	530	33.75	2.9082	2.9776
14	$\{u_2, u_3, u_4, u_5, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}\}$	443	44.625	3.0175	3.0795
15	$\{u_1, u_2, u_3, u_4, u_5, u_6\}$	441	44.875	2.7753	2.89414
16	$\{u_1, u_2, u_3, u_4, u_8, u_9, u_{10}, u_{11}, u_{15}\}$	439	45.125	2.9691	3.0480
17	$\{u_1, u_2, u_3, u_4, u_5, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}\}$	406	50.75	3.0183	3.0312
18	$\{u_1, u_2, u_3, u_4, u_5\}$	337	57.875	2.7882	2.9306
19	$\{u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}\}$	256	67.625	2.9896	3.2119
20	$\{u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}\}$	122	84.75	2.8944	2.8219

The estimations from the two methods are also accurate in this case of gamma distribution for the parameter  $b$ . Here also the truncation proportion does not affect the

quality of estimations. When we consider the complete data, the classical estimation is  $\tilde{b} = 3.04$  and the two new estimations are  $\tilde{b}_1 = 3.0183$  and  $\tilde{b}_2 = 3.0541$ .

As it was noticed in the examples above, the two methods lead to approximately the same estimation results. Nevertheless, if the two estimations are significantly different, it seems related to the quality of the selected data. An important feature of this new approach is that of the uninfluence of the truncation proportion on the quality of estimations. The following section will give further insights of the new approach.

## 4.2 A Basic Feature of the New Approach

The preceding results have shown the effectiveness of the new approach and worked well in simulation experiments. Furthermore, the proposition below will give an insight of a major feature of the new approach by considering the one parameter exponential family. We will prove that for all truncation considered formed by more than two points, from a sample of observations; if the ratios of the relative frequencies of the  $u_i$  are equal to the theoretical ones, then we may obtain the true value of the parameter. We may conjecture that when considering an arbitrary law of probability depending on  $r$  parameters, such that we have a truncation composed by  $r + 1$  points having exact empirical ratios of the relative frequencies then we obtain the true values of the  $r$  parameters.

**Proposition 5** Consider a probability distribution  $f$  from the one-parameter exponential family with density

$$f(x, \theta) = K(x) \times \exp[\theta T(x) - A(\theta)], \quad (9)$$

where  $\theta \in \mathbb{R}$  is the parameter,  $T$  a statistic,  $K(x)$  a function of  $x$  and  $A$  is a function of the parameter  $\theta$ . Assume that we wish to estimate the parameter  $\theta$ . If we consider a truncation having two points  $x$  and  $y$  with empirical frequencies  $f_1$  and  $f_2$  satisfying  $f_1/f_2 = f(x, \theta)/f(y, \theta)$ , then, using the approach considered here, we obtain the true value of  $\theta$ .

**Proof.** 1. If we consider the minimum of distance  $d_v$  the result is immediate.  
2. Consider now the second method to estimate  $m$ . Consider two values  $x$  and  $y$  from the exponential family with density given by (9), with  $\tilde{\theta}$  being the estimation by the new approach, and assume that their empirical frequencies  $f_1$  and  $f_2$  are such that

$$\frac{f_1}{f_2} = \frac{f(x, \tilde{\theta})}{f(y, \tilde{\theta})}.$$

We obtain

$$\bar{u} = x f_1 + y f_2 = \frac{x K(x) \exp(\tilde{\theta} T(x)) + y K(y) \exp(\tilde{\theta} T(y))}{K(x) \exp(\tilde{\theta} T(x)) + K(y) \exp(\tilde{\theta} T(y))}.$$

Then, we solve on  $\theta$  the following equation:

$$\left( x - \frac{x K(x) \exp(\tilde{\theta} T(x)) + y K(y) \exp(\tilde{\theta} T(y))}{K(x) \exp(\tilde{\theta} T(x)) + K(y) \exp(\tilde{\theta} T(y))} \right) K(x) \exp(\theta T(x))$$

$$+ \left( y - \frac{xK(x) \exp(\tilde{\theta}T(x)) + yK(y) \exp(\tilde{\theta}T(y))}{K(x) \exp(\tilde{\theta}T(x)) + K(y) \exp(\tilde{\theta}T(y))} \right) K(y) \exp(\theta T(y)) = 0,$$

after straightforward algebra we obtain

$$(x - y) \exp(\tilde{\theta}T(y) + \theta T(x)) + (y - x) \exp(\tilde{\theta}T(x) + \theta T(y)) = 0,$$

yielding the true value  $\tilde{\theta} = \theta$ . The proof is complete. ■

**Remark 6** Note that the frequencies  $f_1$  and  $f_2$  need not be exact, that is  $f_1$  may be different from  $f(x, \theta)$  and also  $f_2$ , but we require only that their ratio is equal to the theoretical one  $f(x, \theta)/f(y, \theta)$ .

### Examples

**Binomial distribution.** Consider again the binomial distribution  $\mathcal{B}(n, p)$  with  $n = 10$  and  $p = 0.3$  and assume  $n$  is known and we wish to estimate  $p$ . Assume we have the following truncation with only two points  $\Delta = \{0, 1\}$ . The exact ratio of their probability distribution is given by  $f(0, p)/f(1, p) = 7/30$ , which is a rational value that will simplify the example. Choose the absolute frequencies of the two values considered as being  $n_1 = 7$  and  $n_2 = 30$  for the values  $u_1 = 0$  and  $u_2 = 1$  respectively, in order for having  $f_1/f_2 = f(x, p)/f(y, p) = 7/30$ . Using the first approach, that of the minimum of distance  $d_v$ , we have to solve

$$\min_p d_v(\hat{f}, f) = \min_p \left[ \left| \frac{C_{10}^0(1-p)^{10}}{C_{10}^1 p(1-p)^9} - \frac{7}{30} \right| + \left| \frac{C_{10}^1 p(1-p)^9}{C_{10}^0(1-p)^{10}} - \frac{30}{7} \right| \right],$$

and we get the true value  $\tilde{p}_1 = 0.3$ .

Using the second method we have to solve the following equation on  $p$

$$\frac{0 \times C_{10}^0(1-p)^{10} + 1 \times C_{10}^1 p(1-p)^9}{C_{10}^0(1-p)^{10} + C_{10}^1 p(1-p)^9} = \frac{30}{37},$$

and we obtain also the exact result  $\tilde{p}_2 = 0.3$ .

**Gamma distribution.** Consider the Gamma probability distribution  $G(a, b)$  with  $a = 10$  and  $b = 5$ . Assume that  $a$  is known and we wish to estimate  $b$ . Consider the truncation  $\Delta = \{u_3, u_8\}$  with  $u_3 = 30.13$  and  $u_8 = 60.02$ . We have the following value of the ratio  $f(u_3, b)/f(u_8, b) \approx 0.7993$  (the result is an approximate result since for probability density functions it is difficult to get an exact rational value but we will show that the estimations are very close to the true value). Consider the absolute frequencies  $n_3 = 79.93$  (or 80) and  $n_8 = 100$  for the values  $u_3 = 30.13$  and  $u_8 = 60.02$  respectively. We have then  $n_3/n_8 \approx f(u_3, b)/f(u_8, b)$ . Using the minimum of distance  $d_v$ , we have to solve

$$\begin{aligned} \min_b d_v(\hat{f}, f) = \min_b & \left[ |(79.93/100) - ((30.13/60.02)^9 \times \exp(-(1/b) \times (30.13 - 60.02)))| \right. \\ & \left. + |(100/79.93) - ((60.02/30.13)^9 \times \exp(-(1/b) \times (60.02 - 30.13)))| \right], \end{aligned}$$

and we get the result  $\tilde{b}_1 \approx 5$ .

From the second method, we compute  $\bar{u} = 46.7438$  and solve on  $b$  the following equation

$$(30.13 - 46.7438) \times 30.13^9 \times \exp(-30.13/b) \\ + (60.02 - 46.7438) \times 60.02^9 \times \exp(-60.02/b) = 0.$$

The result is  $\tilde{b}_2 \approx 5$ .

## 5 Elements of Comparison with the Classical Approach

Our aim here is not to give a detailed comparison study which needs to be investigated thoroughly, but only some elements of appreciation. A major feature which characterizes this new approach from the others is that when we have exact ratios of frequencies we obtain the true parameter and when their difference from the theoretical ratios decrease the quality of estimation increase even if we are using only a part from the sample of observations. This is not the case for classical approaches. In classical approaches, quality considerations are only viewed through mean properties of estimators or their asymptotic behaviour. By combining the two proposed methods we have in fact a point criterion. Another characteristics is that the new approach works even the sample may be selected randomly or not and the proportion of truncation has any effect on the quality of estimations. The first method uses a well known method of minimum distance but with a new one which has an important advantage of being symmetric, the property of which many traditional distances do not have. However, the estimations are obtained in this case implicitly so it is difficult to find explicit expressions and study their properties to compare them with classical ones. Furthermore, the new distance may open new perspectives in handling small sample cases. Using the new distance we hope having fast convergent estimators since we expect that the influence of the errors in the frequencies will be slight in the new approach as we are using ratios of frequencies. But it still an open problem which needs a rigorous proof. Consider now the second method of the new approach. We use classical procedures of estimation such as the maximum likelihood principle using the auxiliary distribution. We may obtain the estimators and study their properties as commonly used and then preserves the advantages of classical methods. In classical approach, given a sample, the estimation of certain parameters such as the mean and variance do not change according to the family of parent distributions. The latter information is not used and this disadvantages the approach. However, in the new approach the estimations of the mean and variance change according to the distribution from which the data emanated.

The following two examples show the effectiveness of using the auxiliary distribution.

**Example.** Consider the following frequency table:

Table 7.

$x_i$	2	3	Total
$n_i$	$n_1$	$n_2$	$n$
$\hat{f}(x_i) = f_i$	$f_1 = (n_1/n)$	$f_2 = (n_2/n)$	1

Any sample of observations that satisfies the preceding frequency table may belong from one of the following distributions:

$$g_1(x) = \begin{cases} \frac{x}{6} & \text{if } x \in \{1, 2, 3\}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{or} \quad g_2(x) = \begin{cases} \frac{x-1}{6} & \text{if } x \in \{2, 3, 4\}, \\ 0 & \text{otherwise.} \end{cases}$$

The decision for determining which of the two distributions is more appropriate for table 7, depends intuitively on the values  $n_1$  and  $n_2$  (or  $f_1$  and  $f_2$ ). However, if we use the classical maximum likelihood, we obtain that the samples of observations were generated from distribution  $h_1$  whatever the values of  $n_1$  and  $n_2$ , that is:

$$\left(\frac{1}{6}\right)^{n_1} \times \left(\frac{2}{6}\right)^{n_2} < \left(\frac{2}{6}\right)^{n_1} \times \left(\frac{3}{6}\right)^{n_2}.$$

We will show by using the new approach that the decision is more relevant. Determine first the auxiliary distributions,  $h_1$  and  $h_2$ , based on the truncation  $\Delta = \{2, 3\}$ , for  $g_1$  and  $g_2$  respectively. We obtain

$$h_1(x) = \begin{cases} 2/5 & \text{if } x = 2, \\ 3/5 & \text{if } x = 3, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad h_2(x) = \begin{cases} 1/3 & \text{if } x = 2, \\ 2/3 & \text{if } x = 3, \\ 0 & \text{otherwise.} \end{cases}$$

By using the maximum likelihood for  $h_1$  and  $h_2$ , we have to decide according to the quantities  $(2/5)^{n_1} \times (3/5)^{n_2}$  and  $(1/3)^{n_1} \times (2/3)^{n_2}$ . Solving the following inequality

$$\left(\frac{2}{5}\right)^{n_1} \times \left(\frac{3}{5}\right)^{n_2} \leq \left(\frac{1}{3}\right)^{n_1} \times \left(\frac{2}{3}\right)^{n_2},$$

which is equivalent to  $(6/5)^\alpha (9/10)^{1-\alpha} \leq 1$ , where  $\alpha = n_1/n_2$ , we obtain  $0 < \alpha \leq -\log(9/10)/\log(4/3) = x_0 \approx 0.36624$ . If  $0 < \alpha < x_0$ , the data were generated from  $g_2$  and if  $x_0 < \alpha < 1$ , the data were generated from  $g_1$ . We cannot make any decision about the case  $\alpha = x_0$ .

**Example.** Consider a binomial distribution with parameters  $n = 4$  and  $p$  is unknown, from which we consider some samples of observations of size 15 given in table 8 by their absolute frequencies and chosen in order for having  $\bar{x} = 8/15$ .

Table 8.

samples	Values					$\hat{p}$	$\tilde{p}$
	0	1	2	3	4		
1	7	8	0	0	0	0.1333	0.2222
2	9	5	0	1	0	0.1333	0.1838
3	9	4	2	0	0	0.1333	0.1394
4	10	3	1	1	0	0.1333	0.1336
5	10	4	0	0	1	0.1333	0.2157
6	12	0	2	0	1	0.1333	0.1957
7	13	0	0	0	2	0.1333	0.3851

It is clear that the information given by the samples are not the same, nevertheless the classical estimation method gives us the same estimation  $\hat{p} = 8/(15 \times 4) \approx 0.1333$ . If we use the second method of the new approach, we have to solve the following equation for each sample:

$$0 \times h(0, p) + 1 \times h(1, p) + 2 \times h(2, p) + 3 \times h(3, p) + 4 \times h(4, p) = \bar{x},$$

where  $h(x, p)$  is the corresponding auxiliary distribution. The estimations given by the new method differ from sample to another as shown in the latest column of table 8, which is natural since each sample provides a different information about the parent distribution. We can also use the minimum of distance  $d_v$  and we get also the same conclusion.

## 6 Perspectives for the New Approach

### 6.1 Model Selection From Truncated Data

The fact that the distance  $d_v$  is a metric allow to propose various applications of this new measure. We can use it for model selection amongst different probability families. We choose two or more possible candidate parametric families of distributions, and for each alternative family, estimate the parameters to select a specific candidate. Determine the distance between the specific candidate and the empirical distribution using the new metric  $d_v$ . Finally, select the family which yields the minimum distance. In view of the new approach this can also be done in case of truncated data as opposed to classical approaches (see for example Cox (1961, 1962), Taylor and Jakeman (1985)) for model selection which can be used, from the best of our knowledge, only for complete data.

To investigate this perspective thoroughly, samples of various sizes from known distributions should be simulated, and the method for model selection applied, we can score the selection as correct or not after repeating the process a large number of times, the probability of correct selection could be estimated according to a given sample size.

We can also use the new distance in cases where classical goodness of fit tests cannot reject two candidate families. We can choose the one which yields the minimum of distance  $d_v$ .

In the following examples, we shall select, in the first, between a Normal and a Student distribution from a left truncated data. In the second example, we select between a Weibull, a Gamma and a Log-normal distributions from right truncated data.

**Selection from a normal and a Student distributions.** Consider a data set of size 300 simulated from a standard normal distribution and truncated on the left by taking only observations below the level  $-0.5$ . The resulting truncated data has a truncated sample size  $n_t = 204$  and is summarized in 8 classes in table

Table 9. Truncated data set from  $\mathcal{N}(0, 1)$ .

$u_i$	-0.318	0.032	0.383	0.733	1.083	1.434	1.784	2.134
$n_i$	38	46	42	30	24	14	6	4



Using the new distance  $d_v$  we compute the distance between the empirical distribution  $\widehat{f}$  of the data in table 9 and the standard normal density  $f_N$  and also with the student distribution with 5 degree of freedom  $f_{t_5}$ . We also drawn a sample of size 300 from a Student distribution with 5 degree of freedom and truncated it on left by taking only observations greater than  $-0.5$ . The truncated sample of size  $n_t = 200$  is summarized in the following table.

Table 10. Truncated data set from  $t_5$ .

$u_i$	-0.215	0.342	0.898	1.454	2.01	2.567	3.123	3.679
$n_i$	54	73	39	18	10	3	1	2

The results are given in the following table

Table 11

Truncated Data	$d_v(\widehat{f}, f)$	
	$f \equiv \mathcal{N}(0, 1)$	$f \equiv t_5$
$\mathcal{N}(0, 1)$	<b>18, 41</b>	34, 38
$t_5$	2847, 2	<b>215, 02</b>

Table 11 shows that in each cases the distance has selected the correct distribution even with a truncated data set.

**Selection between Weibull, Gamma and log-normal distributions.** We simulated three samples of sizes 400 from the Weibull  $\mathcal{W}(1.2, 1.5)$ , the Gamma  $G(2, 0.5)$  and the log-normal  $LN(0, 0.5)$ . The data were truncated on right by considering only observations above the cut-off 1.25. Each truncated sample was summarized into classes. The following table 12 summarizes the truncated dataset from  $\mathcal{W}(1.2, 1.5)$ .

Table 12. Truncated data from  $\mathcal{W}(1.2, 1.5)$ .

$u_i$	0.067	0.192	0.316	0.44	0.564	0.689	0.813	0.937	1.061	1.185
$n_i$	20	36	37	43	45	38	35	27	26	19

The truncated data of the  $G(2, 0.5)$  is given in table 13.

Table 13. Truncated data from  $G(2, 0.5)$ .

$u_i$	0.087	0.209	0.332	0.454	0.576	0.698	0.82	0.942	1.065	1.187
$n_i$	14	33	31	37	37	28	37	27	16	23

For the  $LN(0, 0.5)$  we obtain the following table for the truncated dataset.

Table 14. Truncated data from  $LN(0, 0.5)$ .

$u_i$	0.342	0.437	0.533	0.628	0.723	0.819	0.914	1.01	1.105	1.201
$n_i$	10	12	34	34	38	27	40	37	23	24

The following table 15 shows that the distance  $d_v$  has selected the correct distribution in each combination between the three distributions.

Table 15

Truncated Data	$d_v(\hat{f}, f)$		
	$f \equiv \mathcal{W}(1.2, 1.5)$	$f \equiv G(2, 0.5)$	$f \equiv LN(0, 0.5)$
$\mathcal{W}(1.2, 1.5)$	<b>8.96</b>	23.13	355.84
$G(2, 0.5)$	26.42	<b>20.21</b>	89079
$LN(0, 0.5)$	67.07	59.9	<b>26.96</b>

We can also find, before selecting between distribution, the best fit from the family of gamma distributions  $G(a, b)$  of the truncated data (table 12 for example) from a given probability density say  $\mathcal{W}(1.2, 1.5)$ . We have then to solve an optimization problem of finding the minimum of a function of two variables,  $\min_{a,b} d_v(\hat{f}, f)$  where  $\hat{f}$  is the empirical distribution from table 12 and  $f \equiv G(a, b)$ , using well known methods such as Lavenberg-Marquardt using a computer algebra package.

## 6.2 Estimation of the initial trial value in *EM* Algorithm

The initial starting value is of great importance in convergence behaviour of algorithms such as *EM* Algorithm. Usually, as for the latter, the initial trial value is guessed. Surprisingly, we will show that our procedure gives an estimation of the starting value instead of having to guess. The approach will be illustrated by the following classical example which was the basis of the *EM* algorithm.

**Example of Hartley (1958) revisited.** Hartley (1958) used an algorithmic procedure to estimate the parameter of a Poisson distribution from data on the pollution of a sort of seeds by the presence of noxious weed seeds quoted from Snedecor (1956) and truncated them by missing the frequencies of the values 0 and 1 as shown in the following table 16 (Table 1 in Hartley (1958))

Table 16.

Values	missing		0 1						
	observed		2	3	4	5	6	7	9
frequencies $n_i$			26	16	18	9	3	5	1

Hartley (1958) has guessed the frequencies of the missing values 0 and 1 by taking  $n_0 = 4$  and  $n_1 = 14$ , and after 4 steps of his algorithmic procedure, which has been the basis of the well known EM algorithm for incomplete data (Dempster, Laird and Rubin (1977)), has reached the estimation  $\hat{\lambda} = 3.026$  (see table 1 p.177 Hartley (1958)). Using the second method, we get the estimation  $\tilde{\lambda}_2 = 3.1149$ . And by proportional allocation procedure we can see that the frequencies we get are  $n_0 = 4.29$  and  $n_1 = 13.38$  which are close to the guessed values. Using the distance  $d_v$  we obtain the estimation  $\tilde{\lambda}_1 = 3.8447$ , and by removing the last value which has a small frequency  $n_7 = 1$ , we obtain a better result  $\tilde{\lambda}_1 = 3.4441$ , which are also appreciable as starting values since in practice the true parameter is unknown.

**Initial trial value for mixture Normal Populations.** We shall present an application of the previous method used for truncated data in the situation where we have

a mixture population of two normal distributions. In classical methods, we use the merged distribution  $f = \alpha f_1 + (1 - \alpha) f_2$  and we estimate the parameters  $\alpha$ ,  $m_1$  and  $m_2$  using for example the EM algorithm which is based on maximizing the complete likelihood of the merged distribution by an algorithmic procedure from a guessed initial trial value. However, the problem of occurrence of several local maxima is well-known for the setting of EM algorithm. Also, Seidel, Mosler and Alker (2000) pointed out that the likelihood-ratio test in mixture models depends on the choice of the initial trial value for the EM algorithm. If the initial trial value is close to the true value it is clear that the algorithm will converge in few steps to the true local maximum. We will show that using the new approach we get an accurate estimated initial trial value.

Assume we have a merged sample from two samples of observations of sizes  $n_1$  and  $n_2$  from two normal distributions  $f_1 = N(m_1, \sigma_1)$  and  $f_2 = N(m_2, \sigma_2)$ , with  $m_1 \neq m_2$ . By assuming that  $\sigma_1$  and  $\sigma_2$  are known, our aim is to estimate the means  $m_1$  and  $m_2$ , and also the merging proportion  $\alpha$  of each population.

We will use a method based on truncations. The main idea being to split the range of the merged sample into three suitably chosen parts. A central part where the observations are highly merged, a left and right truncated parts where the observations become mainly from one of the distributions considered. If for example  $m_1 < m_2$ , then to estimate  $m_1$  we have to use the chosen right truncated part (left truncation  $\Delta$ ).

The procedure is summarized as follows:

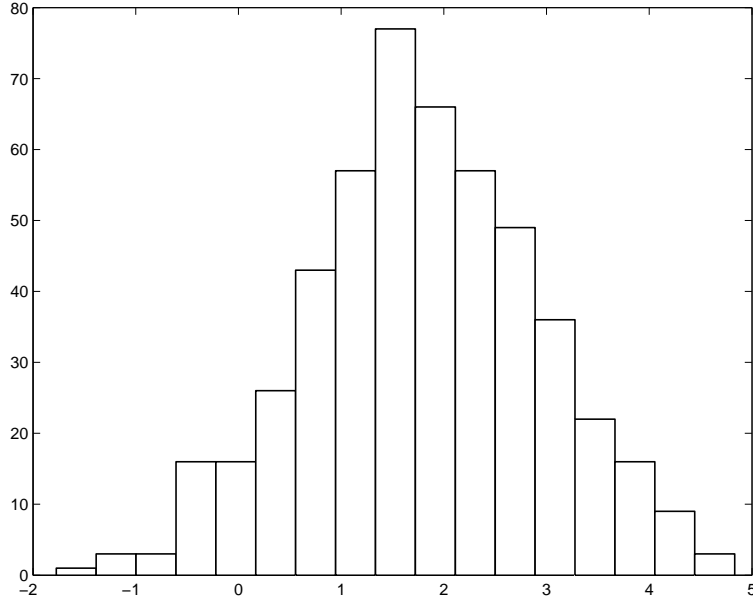
1. We compute the sample mean  $m_g$  of the merged observations.
2. For determining the location of the two means  $m_1$  and  $m_2$ , we compute the empirical standard deviation  $S_l$  of the observations less than  $m_g$ , and  $S_r$  for those that are greater. Assume that  $S_l < S_r$ , in this case if  $\sigma_1 < \sigma_2$  then we deduce that  $m_1$  is situated on the left of  $m_g$ . Otherwise, it will be assumed to be on its right. We follow the same idea for the case  $S_l > S_r$ . If  $\sigma_1 = \sigma_2$  we pass directly to the third step.

3. Assume that  $m_1$  is on the left. It is well known that for a normal distribution  $N(m, \sigma)$  we have  $P(|m - \sigma, m + \sigma|) \simeq 0.68$ . We hope that on the left of  $\sup_l = m_g - \sigma_2$  the number of observations generated from  $N(m_2, \sigma_2)$  is negligible, and on the right of  $\min_r = m_g + \sigma_1$  the number of observations generated from  $N(m_1, \sigma_1)$  is also negligible. Hence, to estimate  $m_1$ , we consider only the part of observations situated on the left of  $m_g - \sigma_2$ , and to estimate  $m_2$  we consider the part situated on the right of  $m_g + \sigma_1$ .

The following example will provide some feel for the accuracy of the procedure.

**Example.** We consider the case where  $\sigma_1 = \sigma_2$ . consider two samples of observations generated from  $N(m_1, \sigma_1)$  and  $N(m_2, \sigma_2)$ , where  $m_1 = 1.3$  and  $m_2 = 2.4$ , with known  $\sigma_1 = \sigma_2 = 1$  and sizes  $n_1 = 300$  and  $n_2 = 200$ . We combine them to obtain a merged sample of size  $n = 500$ . We have chosen the distributions in such a way that the histogram (Fig.1) of the merged sample does not show directly the existence of a mixture of two distributions. When the histogram of the merged population is bimodal the situation is more easier, since when taking a suitably left (or right) part we get more accurate estimation from the situation that this part will have a negligible number of observations from the second distribution.

Fig 1. Merged histogram of two normal distributions  $N(1.3, 1)$  and  $N(2.4, 1)$ .



It should be stressed that the histogram is one modal and does not show at first glance any mixture situation. Following the steps of the procedure we begin by calculating the mean of the resulting merged sample and we obtain  $m_g = 1.8046$ . Since the standard deviations are assumed to be equal then we compute directly  $\text{sup}_l = m_g - \sigma_2 = 0.8046$ . By grouping the observations on the left of  $\text{sup}_l$  (which constitute the chosen right truncated part) in 7 classes we obtain the following table:

Table 17.

$u_i$	-1.5589	-1.1294	-0.6998	-0.2703	-0.1593	0.5888
$n_i$	1	3	6	17	24	41

Using the distance  $d_v$  we obtain for all the truncation  $\tilde{m}_1^{(dv)} = 1.244$  and by deleting  $u_1$  we get the value  $\tilde{m}_1^{(dv)} = 1.2516$ .

The sample mean of the observations on the left of  $\text{sup}_l$  is given by  $\bar{u}_l = 0.1483$ . Using the second method we have to solve on  $m$  the following formula

$$\frac{u_1 \times \exp\left[\frac{-(u_1-m)^2}{2\sigma^2}\right] + u_2 \times \exp\left[\frac{-(u_2-m)^2}{2\sigma^2}\right] + \dots + u_k \times \exp\left[\frac{-(u_k-m)^2}{2\sigma^2}\right]}{\exp\left[\frac{-(u_1-m)^2}{2\sigma^2}\right] + \exp\left[\frac{-(u_2-m)^2}{2\sigma^2}\right] + \dots + \exp\left[\frac{-(u_k-m)^2}{2\sigma^2}\right]} = \bar{u}_l. \quad (10)$$

we obtain the estimation  $\tilde{m}_1 = 1.2646$ . By deleting the first value  $u_1$  which has a weak frequency  $n_1 = 1$ , that is using the truncation  $\Delta = \{u_2, u_3, u_4, u_5, u_6\}$ , (we compute again  $\bar{u}_l = 0.1734$ ) we obtain a better estimation  $\tilde{m}_1 = 1.3011$ , which is very close to the true value  $m_1 = 1.3$ .

To estimate  $m_2$ , we consider the part situated on the right of  $\text{min}_r = m_g + \sigma_1 = 2.8046$ . Grouping the observations on the right of  $\text{inf}_d$  (which constitute the chosen right part) in 7 classes we obtain the following table:

Table 18.

$u_i$	2.9791	3.3159	3.6527	3.9896	4.3264	4.6632
$n_i$	38	25	15	9	7	3

Using the distance  $d_v$  for all the truncation we get  $\tilde{m}_2^{(dv)} = 2.3971$ . The sample mean of the observations on the right part is given by  $\bar{u}_d = 3.5227$ . Using formula (10) with  $\bar{u}_d$ , we obtain the result  $\tilde{m}_2 = 2.245$ . Deleting the extreme values  $u_1$  and  $u_6$  we obtain  $\tilde{m}_2 = 2.4124$ .

The mixture proportion  $\alpha$  can easily be estimated using the formula  $\alpha \times \tilde{m}_1 + (1 - \alpha) \times \tilde{m}_2 = m_g$ .

Considering the estimations obtained, which are close to the true values of  $m_1$  and  $m_2$ , it is clear that the EM algorithm will converge fastly to the unique solutions.

### 6.3 Test of Goodness of Fit Based on the New Distance

We can obtain empirical quantile estimations of  $d_v$  using Montecarlo or Bootstrapping technics, and use them in a test of goodness of fit for a specified probability distribution. We simulate  $N$  samples of the same size from the specified probability distribution and calculate the distances  $d_v^{(1)}, \dots, d_v^{(N)}$ . We can then estimate the asymptotic distribution of  $d_v$  by

$$F_{d_v}(d) = \frac{\#d_v^{(i)} < d}{N}. \quad (11)$$

Consequently, for a sample of the same size we compute  $d_v^{(obs)}$  and we reject the hypothesis that it belongs from the specified distribution if  $F_{d_v}(d_v^{(obs)}) > (1 - \alpha)$  for a given level of significance  $\alpha$ .

The values  $d_v^{(1)}, \dots, d_v^{(N)}$  may be obtained from the empirical distribution function  $F_n$  of the sample.

### 6.4 Quality of Data

The fact that the new measure  $d_v$  is not equivalent to classical ones means that it treats other aspects not investigated by the latter. This may open new perspectives such as making decision about the accuracy of an estimation in cases where the classical and new estimations are close to each others. In cases where the classical estimation and the new one using  $d_v$  are significantly different then we can say that the sample of observations considered does not restore coherently all necessary information about the parent distribution from which it emanated.

### 6.5 Optimum number of classes

We define the optimum number of classes for grouping data in continuous setting as the one that express the best the variations of the parent distribution. A criterion can be developed using the minimum of the distance  $d_v$ . A computer program may easily be implemented in statistical packages for determining the best number of classes in continuous case using the new distance.

## 7 Concluding Remarks

In the foregoing study, we have presented a new statistical point estimation method which found be useful in truncated data situations. A new distance between probability distributions was introduced. It measures the difference between the variations of two given probability distributions. We introduced an auxiliary distribution based on a truncation, from a chosen family of probability distributions. This new distribution will have the same parameters to estimate as the parent one. We use then statistical methods to estimate the parameters of the random variable under study using the empirical and new auxiliary distribution in the region that captures the data, from which we determine the corresponding parent distribution. The later is the estimation by the new method. Using the new distance introduced we also estimate by the minimum distance approach and use the resulting estimation as a control on the accuracy of estimation obtained by the former method. We have obtained a result which states that if we have to estimate the parameter of a probability distribution from the one parameter exponential family, then it suffices to have two points with exact ratio of frequencies, that is equal to the theoretical one expressed by the ratio of the value of the probability distribution on these two points, to obtain the true value of the parameter. We have conjectured that if we have in general  $r$  parameters, then it suffices to have  $r + 1$  points with exact ratios of their frequencies to obtain the  $r$  true parameters exactly. The later result need to be proved rigorously in a general setting for other distributions than the class considered. A large comparative study between the classical and new methods should also be investigated. We presented some perspectives of the new approach such as model selection from truncated data using the new distance, estimation of the first trial value in the celebrate *EM* algorithm in the case of truncation and for mixture of two normal populations, a test of goodness of fit based on the new distance, decision making about the quality of estimations and data, optimum number of classes in grouping data from continuous distributions. A computer program should be implemented in statistical packages for determining the best number of classes in continuous case.

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