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To cite this version:

Guy Barles, Sheetal Dharmatti, Mythily Ramaswamy. Unbounded Viscosity Solutions of Hybrid Control Systems. ESAIM: Control, Optimisation and Calculus of Variations, 2010, 16 (01), pp.176–193. 10.1051/cocv:2008076 . hal-00255763

HAL Id: hal-00255763

https://hal.science/hal-00255763v1

Submitted on 14 Feb 2008

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hal-00255763, version 1 - 14 Feb 2008

UNBOUNDED VISCOSITY SOLUTIONS OF HYBRID CONTROL SYSTEMS

GUY BARLES*, SHEETAL DHARMATTI†, AND MYTHILY RAMASWAMY‡

Abstract. We study a hybrid control system in which both discrete and continuous controls are involved. The discrete controls act on the system at a given set interface. The state of the system is changed discontinuously when the trajectory hits predefined sets, namely, an autonomous jump set A or a controlled jump set C where controller can choose to jump or not. At each jump, trajectory can move to a different Euclidean space. We allow the cost functionals to be unbounded with certain growth and hence the corresponding value function can be unbounded. We characterize the value function as the unique viscosity solution of the associated quasivariational inequality in a suitable function class. We also consider the evolutionary, finite horizon hybrid control problem with similar model and prove that the value function is the unique viscosity solution in the continuous function class while allowing cost functionals as well as the dynamics to be unbounded.

Key words. Dynamic programming Principle, viscosity solution, quasivariational inequality, hybrid control

AMS subject classifications. 34H05, 34K35, 49L20, 49L25

1. Introduction. The hybrid control systems are those involving continuous and discrete dynamics and continuous and discrete controls. Typical examples of such systems are constrained robotic systems [1] and automated highway systems [12]. For some more examples of such systems, see [6],[7] and references therein.

In [6], Branicky, Borkar and Mitter have presented a model for a general hybrid control system in which continuous controls are present and in addition discrete controls act at a given set interface, which corresponds to the logical decision making process as in the above examples. The state of the system is changed discontinuously when the trajectory hits these predefined sets, namely, an autonomous jump set A or a controlled jump set C where controller can choose to jump or not. They prove right continuity of the value function corresponding to this hybrid control problem. Using dynamic programming principle they arrive at the partial differential equation satisfied by the value function which turns out to be quasivariational inequality, referred hereafter as QVI.

In an earlier work [9], the authors have studied this problem and proved the local Hölder continuity of the value function and have characterized it as the unique viscosity solution of the QVI. There the cost functionals were assumed to be bounded and hence the value function was also bounded. This was essential in the uniqueness proof, as the auxiliary function had to be modified a finite number of times to get the comparison result. In this paper we allow the cost functions to be unbounded. We use a different method to compare unbounded value functions, namely a suitable change of variable to reduce the unbounded function to a bounded one (See [3], for example). We also tackle the uniqueness for the time dependent problem, using a test-function similar to the one introduced in [4].

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The paper is organized as follows: In section 2, we introduce the notations, assumptions and quasivariational inequality (QVI) satisfied by the value function. In section 3, we show that the value function is continuous. The next section deals with uniqueness of the solution of QVI. We give a comparison principle proof characterizing the value function as unique viscosity solution of QVI , for the stationary case in section 4 and then for the time dependent case in section 5.

2. Preliminaries . In a hybrid control system, as in [6], the state vector during continuous evolution, is given by the solution of the following problem

$$\dot{X}(t) = f_i(X(t), u(t)) \tag{2.1}$$

$$X(0) = x \tag{2.2}$$

where $x \in \Omega_i$, a connected subset of $I\!R^{d_i}$, $d_i \in Z_+$ for $i \in Z_1 \subset Z_+$. Here $f_i : \Omega_i \times \mathcal{U} \to \Omega_i$ and the continuous control set is :

$$\mathcal{U} = \{u : [\ 0, \infty) \to U \mid u \ \text{ measurable, } U \text{ compact metric space. } \}$$

The trajectory undergoes discrete jump when it hits predefined sets A_i the autonomous jump set and C_i, the controlled jump set, both subsets of $I\!R^{d_i}$. The trajectory starting from $x \in \Omega_i$, on hitting A_i, can jump to a predefined destination set D_j in another Euclidean space Ω_j and continue the evolution there. This jump is given by prescribed transition map $g : A_i \times \mathcal{V} \to \bigcup_i D_i$, where $\mathcal{V}$ is the discrete control set. On hitting C_i the controller can choose either to jump or not to jump. If the controller chooses to jump, then the trajectory is moved to a new point in D_k, possibly in another space Ω_k. This gives rise to a sequence of hitting times of $\bigcup_j A_j$, which we denote by σ_i and sequence of hitting times of $\bigcup_j C_j$, where the controller chooses to jump which is denoted by ξ_i.

We introduce the state space

$$\Omega := \bigcup_i \Omega_i \times \{i\}, \quad i \in Z_1 \subset Z_+,$$

and the dynamics $f : \Omega \times \mathcal{U} \to \Omega$. Actually, $f = f_i$ with the understanding, $\dot{X}(t) = f_i(X(t), u(t))$ whenever $x \in \Omega_i$. The predefined sets are

$$\begin{aligned} A &= \bigcup_i A_i \times \{i\} \quad A_i \subseteq \overline{\Omega_i} \subseteq I\!R^{d_i} \ ; \\ C &= \bigcup_i C_i \times \{i\} \quad C_i \subseteq \Omega_i \subseteq I\!R^{d_i} \ ; \\ D &= \bigcup_i D_i \times \{i\} \quad D_i \subseteq \Omega_i \subseteq I\!R^{d_i} \ . \end{aligned}$$

The trajectory of this problem, is composed of continuous evolution given by (2.1) between two hitting times and discrete jumps at the hitting times. We denote $X(\sigma_i^-, u(\cdot))$ by x_i, the point before an autonomous jump and $g(X(\sigma_i^-), v)$ by x_i', after the jump. The controlled jump destination of $X(\xi_i^-, u(\cdot))$ is $X(\xi_i^-)'$, or simply, $X(\xi_i)'$. For example, the dynamics for $\sigma_i < \xi_k < \sigma_{i+1}$, is given by

$$\begin{aligned} \dot{X}(t) &= f(X(t), u(t)) \quad \sigma_i < t < \xi_k, \\ X(\sigma_i) &= X(\sigma_i^+) = g(X(\sigma_i^-), v) = x_i', \end{aligned}$$

and then

$$\dot{X}(t) = f(X(t), u(t)) \quad \xi_k < t < \sigma_{i+1},$$
$$X(\xi_k) = X(\xi_k^+) = X(\xi_k^-)',$$

We give the inductive limit topology on Ω namely,

$$(x_n, i_n) \in \Omega \text{ converges to } (x, i) \in \Omega \quad \text{if for some } N \text{ large and } \forall\, n \geq N$$

$$i_n = i \;\; x, x_n \in \Omega_i, \quad \Omega_i \subseteq I\!R^{d_i}, \text{for some i, and } \|x_n - x\|_{I\!R^{d_i}} < \varepsilon.$$

With the understanding of above topology we suppress the second variable i from Ω. We follow the same for A, C and D. We also denote by $|x|$ the quantity which is equal to $\|x\|_{I\!R^{d_i}}$ if $x \in \overline{\Omega_i}$.

We make the following basic assumptions on the sets A, C, D and on functions f and g:
(A1): Each $\overline{\Omega_i}$ is closure of a connected, open subset of $I\!R^{d_i}$.
(A2): A_i, C_i, D_i are closed, and for all i and for all $x \in D_i$, $|x| < R$. ∂A_i,∂C_i are C^2 and $\partial A_i \supseteq \partial \Omega_i \quad \forall\, i$.
(A3): $g : A \times \mathcal{V} \to D$ is bounded, uniformly Lipschitz continuous map, with Lipschitz constant G with the understanding that $g = \{g_i\}$ and $g_i : A_i \times \mathcal{V} \to \cup_j D_j$.
(A4): Vector field f is Lipschitz continuous with Lipschitz constant L in the state variable x and uniformly continuous in control variable u. Also,

$$|f(x, u)| \leq F \quad \forall\, x \in \Omega \text{ and } \forall\, u \in U. \tag{2.3}$$

(A5): Each ∂A_i is compact for all i, and for some $\xi_0 > 0$, following transversality condition holds:

$$f(x_0, u) \cdot \zeta(x_0) \;\leq -2\xi_0 \;\; \forall \;\; x_0 \;\in\; \partial A_i \;\; \forall\, u \in U \tag{2.4}$$

where $\zeta(x_0)$ is the unit outward normal to ∂A_i at x_0. We assume similar transversality condition on ∂C_i.
(A6): We assume that

$$\inf_i d(A_i, C_i) \geq \beta \quad \text{and} \quad \inf_i \; d(A_i, D_i) \geq \beta > 0 \tag{2.5}$$

where d is the appropriate Euclidean distance. Note that, above rules out infinitely many jumps in finite time.

(A7): The control sets U and $\mathcal{V}$ are assumed to be compact metric spaces.

Now $\theta := (u(\cdot), v, \xi_i, X(\xi_i)')$ is the control and total discounted cost is given by

$$J(x, u(\cdot), v, \xi_i, X(\xi_i)') = \int_0^\infty K(X(t), u(t)) e^{-\lambda t} dt + \sum_{i=0}^\infty C_a(X(\sigma_i^-), v) e^{-\lambda \sigma_i}$$
$$+ \sum_{i=0}^\infty C_c(X(\xi_i^-), X(\xi_i)') e^{-\lambda \xi_i} \tag{2.6}$$

where λ is the discount factor, $K : \Omega \times \mathcal{U} \to I\!R_+$ is the running cost, $C_a : A \times \mathcal{V} \to I\!R_+$ is the autonomous jump cost and $C_c : C \times D \to I\!R_+$ is the controlled jump cost. The value function V is then defined as:

$$V(x) = \inf_{\theta \in (\mathcal{U} \times \mathcal{V} \times [0,\infty) \times D)} J(x, u(\cdot), v, \xi_i, X(\xi_i)') \tag{2.7}$$

We assume the following conditions on the cost functionals:

(C1): K is nonnegative, continuous in the x variable with at most polynomial growth of degree k , with k and the growth being independent of i. K is uniformly continuous in u variable.

(C2): $C_a(x, v)$ and $C_c(x, x')$ are continuous in both variables, uniformly continuous in x, uniformly with respect to v and x' respectively, and bounded below by $C' > 0$. Moreover C_a and C_c have at most polynomial growth of degree k in the first variable, with k and the growth being independent of i.

Note that under (C1) and (C2), value function is always non-negative and hence bounded below by 0. Using dynamic programming principle, one can show that the value function satisfies the following QVI in viscosity sense:

THEOREM 2.1. *[*Quasivariational Inequality*] Under the assumptions* $(A1 - A7)$, $(C1), (C2)$ *and if* $\lambda > kL$, *the value function* V *described in (2.7) is continuous and has at most a polynomial growth of degree* k. *Moreover it satisfies the following quasivariational inequality in the viscosity sense:*

$$V(x) = \begin{cases} MV(x) & \forall\, x \in A \\ \min\{NV(x), -H(x, DV(x))\} & \forall\, x \in C \\ -H(x, DV(x)) & \forall\, x \in \Omega \setminus A \cup C \end{cases} \tag{QVI}$$

For ϕ, *a function defined on* Ω *and bounded below,* M, N *and* H *are given by*

$$\begin{aligned} M\phi(x) &= \inf_{v \in \mathcal{V}} \{\phi(g(x, v)) + C_a(x, v)\}, \\ N\phi(x) &= \inf_{x' \in D} \{\phi(x') + C_c(x, x')\}, \\ H(x, p) &= \sup_{u \in U} \left\{ \frac{-K(x, u) - f(x, u) \cdot p}{\lambda} \right\}. \end{aligned}$$

The condition $\lambda > kL$ in Theorem 2.1 is used (and needed in the general case) to ensure that the value function is locally bounded and has a polynomial growth. For the uniqueness result for the quasivariational inequality (cf. Section 4), such condition is not playing any role.

Now we consider the hybrid control problem, with autonomous and controlled jumps as before, but in finite horizon, namely for $t \in [0, T]$. In this case the dynamics depends on the time variable t and we also allow cost functionals to depend on t. We relax the global Lipschitzness assumption on f and assume only local Lipschitzness. Further relaxing boundedness of f, we allow it to grow linearly.

In order to simplify we assume that the cost C_a, C_c are as above independent of time. It is not difficult to check that, besides additional technical details, the case when C_a, C_c depend on t can be treated in a similar way, with suitable adaptations.

The total cost in the time dependent case is given by,

$$J(s,x,u(\cdot),v,\xi_i,X(\xi_i)') = \int_s^T K(t,X(t),u(t))dt + \sum_{s\le\sigma_i<T} C_a(X(\sigma_i^-),v)$$
$$+ \sum_{s\le\xi_i<T} C_c(X(\xi_i^-),X(\xi_i)') + h(X_x(T)) \tag{2.8}$$

where $K : [0,T]\times\Omega\times\mathcal{U} \to I\!R_+$ is the running cost, $C_a : A\times\mathcal{V} \to I\!R_+$ is the autonomous jump cost, $C_c : C\times D \to I\!R_+$ is the controlled jump cost and h is the terminal cost. The value function V is then defined as:

$$V(s,x) = \inf_{\theta\in(\mathcal{U}\times\mathcal{V}\times[s,T)\times D)} J(s,x,u(\cdot),v,\xi_i,X(\xi_i)') \tag{2.9}$$

The precise assumptions are as follows: as before we assume (A1)-(A3), (A6) and (A7). Because f and K are time dependent we further assume the following:
(A4)': Vector field $f(t,x,u)$ is locally Lipschitz continuous in the state variable x and uniformly continuous in the rest of the variables. Its linear growth is given by

$$|f(t,x,u)| \le F(1+|x|) \quad \forall\, x\in\Omega \text{ and } \forall\,(t,u)\in[0,T]\times U.$$

(A5)': Each ∂A_i is compact for all i, and for some $\xi_0 > 0$, following transversality condition holds:

$$f(t,x_0,u)\cdot\zeta(x_0) \le -2\xi_0 \;\; \forall \;\; x_0 \in \partial A_i \;\; \forall\,(t,u)\in[0,T]\times U \tag{2.10}$$

where $\zeta(x_0)$ is the unit outward normal to ∂A_i at x_0. We assume similar transversality condition on ∂C_i. The cost functionals are assumed to satisfy
(C1)': K is nonnegative and continuous in all the three variables; moreover K is uniformly bounded for bounded $|x|$.

One can derive the DPP for time dependent problem which is given as follows:
For s' such that $s \le s' < T$,

$$V(s,x) = \inf_{(u,\xi_1)} \Bigg[1_{s'<(\sigma_1\wedge\xi_1)} \left\{ \int_s^{s'} K(t,X(t),u(t))dt + V(s',X(s')) \right\}$$
$$+1_{\sigma_1<(s'\wedge\xi_1)} \left\{ \int_s^{\sigma_1} K(t,X(t),u(t))dt + V(\sigma_1,X(\sigma_1{}^-)) \right\}$$
$$+1_{\xi_1<(s'\wedge\sigma_1)} \left\{ \int_s^{\xi_1} K(t,X(t),u(t))dt + V(\xi_1,X(\xi_1{}^-)) \right\} \Bigg]$$

It can be verified that the quasivariational inequality (QVI-T) satisfied by $V(s,x)$, is given by

$$\left.\begin{array}{rcll} V - MV & = & 0 & \text{in } (0,T)\times A \\ \max\{V-NV, -V_s + H(s,x,DV)\} & = & 0 & \text{in } (0,T)\times C \\ -V_s + H(s,x,DV) & = & 0 & \text{in } (0,T)\times\Omega\setminus A\cup C \end{array}\right\} \quad (QVI-T)$$

where for ϕ, a function defined on $([0,T]\times\Omega)$ and bounded below, M, N and H are given by

$$\begin{aligned} M\phi(t,x) &= \inf_{v\in V}\{\phi(t,g(x,v)) + C_a(x,v)\} \quad \forall\, x \ \in A \\ N\phi(t,x) &= \inf_{x'\in D}\{\phi(t,x') + C_c(x,x')\} \quad \forall\, x \ \in C \\ H(t,x,p) &= \sup_{u\in U}\{-K(t,x,u) - f(t,x,u)\cdot p\} \end{aligned}$$

For the terminal condition, it turns out to be more complicated than it is usually the case because of the possible (or imposed) jumps. In fact, the terminal data is obtained by solving the stationary problem

$$\left.\begin{array}{rcl} V(T,x) - MV(T,x) &=& 0 \quad \forall\, x\in A \\ \max\{V(T,x) - NV(T,x), V(T,x) - h(x)\} &=& 0 \quad \forall\, x\in C \\ V(T,x) - h(x) &=& 0 \quad \forall\, x\in \Omega\setminus A\cup C. \end{array}\right\} \tag{2.11}$$

In order to have a continuous solution V (and therefore a continuous terminal $V(T,x)$), we have to assume

(D1): Equation (2.11) has a (unique) bounded from below solution $\tilde{h}$ which is uniformly continuous for bounded $|x|$.

We point out that the key property in this assumption is the existence of such uniformly continuous solution (uniformity means really uniformity in i) while uniqueness is rather easy to obtain. This is really an assumption on h. To go further in this direction, we also remark that if C_c satisfies

$$C_c(x,y) \le C_c(x,z) + C_c(z,y) \quad \text{for any } x\in C, y\in D \text{ and } z\in C\cap D\ ,$$

which means that to jump once is always better that to jump twice, then $\tilde{h}$ can be built in the following way

(i) $\tilde{h}$ is given except on A and C.

(ii) For points in C, set $\tilde{h} = min(h, Nh)$.

The above condition implies that, actually, $Nh \le N(Nh)$ on $C\cap D$ and therefore the right inequality holds in (2.11)

(iii) For points in A, set $\tilde{h} = M\tilde{h}$.

This is well-defined since $\tilde{h}$ is known on D.

Assumption (D1) consists in assuming that this function $\tilde{h}$ is uniformly continuous for bounded $|x|$, which is both a continuity assumption on h and a compatibility condition on its value near ∂A and ∂C. If for example, $C\cap D$ is empty, then assuming $h = Mh$ on ∂A and $h \le Nh$ on ∂C in addition to the uniform continuity of h on bounded subsets of $\overline{\Omega}$, will ensure that of $\tilde{h}$, since $\tilde{h} = h$. One can construct easy examples in one dimension to show that if $C\cap D$ is nonempty, the situation gets complicated and the conditions on h are not that transparent.

Once $\tilde{h}$ is known, we are left with a more classical terminal condition

$$V(T,x) - \tilde{h}(x) = 0 \quad \forall\, x\in\Omega.$$

3. Continuity of the value function. Let the trajectory given by solution of (2.1) and starting from the point x be denoted by $X_x(t,u(\cdot))$. Since $x\in\Omega$, in particular it belongs to some Ω_i. Then we recall from theory of ordinary differential equations:

$$|X_x(t,u(\cdot)) - X_z(t,u(\cdot))| \le e^{Lt}|x-z|, \tag{3.1}$$

$$|X_x(t,u(\cdot)) - X_x(\bar{t},u(\cdot)) \le F|t-\bar{t}|, |X_x(t,u(\cdot))| \le |x|e^{Lt} + \frac{C}{L}(e^{Lt}-1) \tag{3.2}$$

For the derivation of these, see for example, ([5] , Appendix, Chapter 5). Here F and L are as in (A4). Now we can proceed as in [9] using the transversality condition (A5) and conclude the continuity of the value function. Moreover, as running cost K, and discrete jump costs C_a and C_c have polynomial growth of order k, one can see that total cost J also has at most the same growth because of the assumption $kL < \lambda$. Then it follows that the value function V also has a polynomial growth of order k, uniformly in all Ω_i.

For $\eta > 0$, we introduce the following spaces of functions : we denote by $E_\eta(\Omega)$ the space of semicontinuous functions u defined on Ω and bounded below and such that, in each Ω_i

$$\lim_{|x|\to\infty} u(x)e^{-\eta|x|} = 0,$$

the limit being uniform w.r.t. i. Then $E(\Omega)$ is defined by

$$E(\Omega) = \cup_{\eta<\lambda/F} E_\eta \ .$$

In section 4, we prove the uniqueness of viscosity solution of QVI in this function class E to which the value function belongs.

In the case of finite horizon problems, the continuity of the value function is a more delicate problem : indeed, in the infinite horizon case, the fact that time is unbounded allows to use the transversality condition (A5) and to compensate the jumps, perhaps with a delay, and a priori this is not possible anymore in the finite horizon case. This shows that the problem is even more important when t is close to the terminal time T and this explains the above discussion about (D1) and the terminal condition $\tilde{h}$.

To prove directly (and without to many technicalities) that the value function is continuous requires (D1) but also the uniform continuity on bounded subsets of Ω of the functions K, C_a, C_c, which are rather strong assumptions. An alternative proof (but essentially with the same assumptions) is to show that the upper and lower semi-continuous envelopes of V are respectively viscosity sub and supersolutions of (QVI-T) and to use the comparizon result of Section 5; this provides a slightly simpler proof.

4. Uniqueness - Infinite Horizon Problem. In this section, we state and prove a comparison between sub and supersolutions of QVI, which belong to $E(\Omega)$.

THEOREM 4.1. *Assume* $(A1) - (A7)$ *and* $(C1) - (C2)$. *Let* $u_1, u_2 \in E(\Omega)$, *be respectively upper semicontinuous subsolution and lower semicontinuous supersolution of the quasivariational inequality given by (QVI) in the viscosity sense. Then* $u_1 \leq u_2$ *in* Ω.

Proof. We complete the proof in 4 steps. In step 1, we convert the unbounded value functions u_1, u_2 into bounded functions w_1, w_2 by a suitable change of variable. We derive the modified quasivariational inequality satisfied by w_1 and w_2. In the following 2 steps, for fixed $\mu \in (0,1)$, close to 1, we argue on μw_1 and w_2 : in Step 2, we examine the possibility of having approximate suprema at points of $A \cup C$ and the actual consequences of such facts, while, in Step 3, we use the previous results to reach a contradiction. In Step 4, we conclude the comparison of u_1 and u_2 using steps 2 and 3.

Step 1: Let u_1, u_2 be respectively upper and lower semicontinuous sub and supersolutions of the QVI in $E(\Omega)$. Fix $0 < \eta < \frac{\lambda}{F}$ such that both u_1, u_2 lie in E_η, where,

as above, λ is the discount factor and F is the bound on dynamics f. Define

$$w_1(x) = u_1(x)e^{-\eta\xi(x)}, \quad w_2(x) = u_2(x)e^{-\eta\xi(x)},$$

where $\xi : \Omega \to R^+$ is a smooth function such that, $|D\xi| \leq 1$ for all $x \in \Omega$, $\xi(x) = 0$ if $|x| \leq R$ and $\xi(x) = (1 + |x|^2)^{1/2}$ if $|x| > 2R$ where R is given by (A2). Since ξ behaves like $|x|$ at infinity and $u_1, u_2 \in E_\eta$, $w_1(x), w_2(x)$ are bounded and converge to 0 when x tends to infinity in each Ω_i, uniformly w.r.t. i.

The functions w_1, w_2 are respectively upper and lower semicontinuous sub and supersolution of a QVI which can be first written as

$$w(x)e^{\eta\xi(x)} = \begin{cases} M(w(x)e^{\eta\xi(x)}) & \forall\, x \in A \\ \min\left\{N(w(x)e^{\eta\xi(x)}), -H(x, D(w(x)e^{\eta\xi(x)}))\right\} & \forall\, x \in C \\ -H(x, D(w(x)e^{\eta\xi(x)})) & \forall\, x \in \Omega \setminus A \cup C \end{cases}$$

But, recalling the facts that $|x| < R$ if $x \in D$ and that g maps $A \times \mathcal{V}$ into D, we can simplify this QVI in the following way

$$w(x) = \begin{cases} \tilde{M}w(x) & \forall\, x \in A \\ \min\left\{\tilde{N}w(x), -\tilde{H}(x, w(x), Dw(x))\right\} & \forall\, x \in C \\ -\tilde{H}(x, w(x), Dw(x)) & \forall\, x \in \Omega \setminus A \cup C \end{cases}$$

where

$$\begin{aligned} \tilde{M}w(x) &= e^{-\eta\xi(x)} \inf_{v \in \mathcal{V}} \{w(g(x, v)) + C_a(x, v)\} \\ \tilde{N}w(x) &= e^{-\eta\xi(x)} \inf_{x' \in D} \{w(x') + C_c(x, x')\} \\ \tilde{H}(x, w, Dw(x)) &= \frac{1}{\lambda}\left\{\sup_{u \in U}\{-f(x, u) \cdot (Dw(x) + \eta w(x) D\xi(x))\} - e^{-\eta\xi(x)}K(x, u)\}\right\} \end{aligned}$$

Indeed, by definition of ξ and for $i = 1, 2$, $w_i(g(x, v)) = u_i(g(x, v))$ for any $x \in A$ and $v \in \mathcal{V}$, and, in the same way, $w_i(x') = u_i(x')$ for any $x' \in D$.

Using the definition of $\tilde{H}$ and the assumptions on the cost functionals and dynamics we can show that

$$\begin{aligned} |\tilde{H}(y, w_2(y), p_2) - \tilde{H}(x, w_1(x), p_1)| &\leq (L/\lambda)|x - y||p_1| + (F/\lambda)|p_1(x) - p_2(y)| \\ &\quad + (\eta F/\lambda)|w_1(x) - w_2(y)| + (1/\lambda)\omega_{\tilde{K}}(|x - y|) \\ &\quad + (\eta/\lambda)|w_2| \sup_{u \in U}\{|D\xi(x) \cdot f(x, u) - D\xi(y) \cdot f(y, u)|\} \end{aligned} \tag{4.1}$$

where $\tilde{K}(x, u) = e^{-\eta\xi(x)}K(x, u)$ and $\omega_{\tilde{K}}$ is the modulus of continuity of $\tilde{K}$.

Now we fix $\mu \in (0, 1)$, close to 1, and prove the comparison result for μw_1 and w_2. To do so, we assume, by contradiction, that $\sup_{\bar{\Omega}}(\mu w_1(x) - w_2(x)) > 0$ or, in other words

$$\sup_j \sup_{\Omega_j}(\mu w_1(x) - w_2(x)) = m > 0.$$

We first recall that $w_1(x), w_2(x)$ converge to 0 when x tends to infinity in all Ω_j, uniformly w.r.t. j; therefore each $\sup_{\Omega_j}$ is indeed achieved at some point. But since

the set of indices j is infinite, we have anyway to argue with approximate supremums (of course, if this supremum is attained at some finite point then the following proof gets simplified). We consider $\kappa > 0$ small enough (a more precise estimate on its size will be given later on) and let i and x_κ be such that

$$\sup_{\Omega_i}(\mu w_1(x) - w_2(x)) = \mu w_1(x_\kappa) - w_2(x_\kappa) \geq m - \kappa > \frac{m}{2} > 0. \tag{4.2}$$

Note that x_κ remains bounded : indeed we can find R_m such that

$$|\mu w_1(x)| < m/10 \ \text{ and } \ |w_2(x)| < m/10 \ \text{ for } |x| > R_m \text{ for all } x \in \Omega_j \text{ and for all } j.$$

Then, x_κ being the maximum of $\mu w_1 - w_2$ in Ω_i and $m/2 < m - \kappa < (\mu w_1 - w_2)(x_\kappa)$ will imply that $|x_k| < R_m$.

Next we are going to consider three different cases, namely,

1. $x_\kappa \in A_i$,
2. $x_\kappa \in C_i$ and $w_2(x_\kappa) \geq \tilde{N} w_2(x_\kappa)$
3. Either $x_\kappa \in \Omega_i \setminus (A_i \cup C_i)$ or $x_\kappa \in C_i$ and $w_2(x_\kappa) < \tilde{N} w_2(x_\kappa)$.

We consider cases 1 and 2 in the next step and case 3 in Step 3.

Step 2: We first consider the case 1 namely $x_\kappa \in A_i$. For w_1, w_2 , viscosity sub and supersolutions, the conditions on ∂A are satisfied only in the viscosity sense: For the subsolution

$$\min\{w_1 + \tilde{H}(x, Dw_1, w_1), \quad w_1 - \tilde{M} w_1\} \leq 0 \quad \text{ on } \partial A \tag{4.3}$$

For the supersolution,

$$\max\{w_2 + \tilde{H}(x, Dw_2, w_2), \quad w_2 - \tilde{M} w_2\} \geq 0 \quad \text{ on } \partial A. \tag{4.4}$$

Hence we need to rule out the bad inequalities for w_1, w_2, namely

$$w_1 > \tilde{M} w_1 \ \text{ or } \ w_2 < \tilde{M} w_2$$

on ∂A. Our assumption $(A5)$ helps us to avoid the above situation and to conclude that the right inequalities for w_1, w_2 hold on ∂A. To do so, we need the following

Lemma 4.2. *If w_1 and w_2 are viscosity sub and supersolutions of QVI and w_1, w_2 are u.s.c and l.s.c on $\overline{\Omega}$ respectively, then the functions $\tilde{M} w_1, \tilde{N} w_1$ and $\tilde{M} w_2, \tilde{N} w_2$ are respectively upper and lower semi-continuous. Moreover, for any $x \in \partial A$, we have*

$$w_1(x) \leq \tilde{M} w_1(x), \quad w_2(x) \geq \tilde{M} w_2(x)$$

and for any $x \in \partial C$, we have $w_1(x) \leq \tilde{N} w_1(x)$.

We postpone the proof of this lemma to the end of the section.

We now conclude the proof by using Lemma 4.2. If $x_\kappa \in A$ then, using Lemma 4.2 for boundary points, we have

$$w_1(x_\kappa) \leq \tilde{M} w_1(x_\kappa) \quad \text{and} \quad w_2(x_\kappa) \geq \tilde{M} w_2(x_\kappa)$$

By definition of $\tilde{M}$, and compactness of control set $\mathcal{V}$, we can find v_0 such that,

$$\tilde{M} w_2(x_\kappa) = e^{-\eta \xi(x_\kappa)} \{w_2(g(x_\kappa, v_0)) + C_a(x_\kappa, v_0)\}$$

For all controls $v \in \mathcal{V}$ and hence in particular for v_0 we have,

$$\tilde{M}w_1(x_\kappa) \leq e^{-\eta\xi(x_\kappa)}\{w_1(g(x_\kappa, v_0)) + C_a(x_\kappa, v_0)\}$$

Hence,

$$\begin{aligned}\mu w_1(x_\kappa) - w_2(x_\kappa) &\leq e^{-\eta\xi(x_\kappa)}\{\mu w_1(g(x_\kappa, v_0)) - w_2(g(x_\kappa, v_0)) - (1-\mu)C_a(x_\kappa, v_0)\}\\ &\leq e^{-\eta\xi(x_\kappa)}\{\mu w_1(g(x_\kappa, v_0)) - w_2(g(x_\kappa, v_0)) - (1-\mu)C'\}\\ &\leq e^{-\eta\xi(x_\kappa)}\{m - (1-\mu)C'\}\end{aligned}$$

This inequality and (4.2) imply

$$m - \kappa \leq \mu w_1(x_\kappa) - w_2(x_\kappa) \leq e^{-\eta\xi(x_\kappa)}\{m - (1-\mu)C'\} \leq m - (1-\mu)C'$$

This is a contradiction if we choose κ such that

$$(1-\mu)C' > \kappa. \tag{4.5}$$

Thus, for such a choice of κ, x_κ cannot be on A_i.

Next we consider the case when $x_\kappa \in C_i$. If $w_2(x_\kappa) \geq \tilde{N}w_2(x_\kappa)$, we use Lemma 4.2 to deduce $w_1(x_\kappa) \leq \tilde{N}w_1(x_\kappa)$ and a similar argument as above leads to a contradiction under the same condition on κ. Therefore we may assume without loss of generality that $w_2(x_\kappa) < \tilde{N}w_2(x_\kappa)$ which means that we are in the third case.

Step 3: We examine here the third (and last possible) case. To do so, we introduce the function $\Phi : \Omega_i \times \Omega_i \to I\!R$ by

$$\Phi(x, y) = \mu w_1(x) - w_2(y) - \frac{1}{\epsilon}|x-y|^2 - |x - x_\kappa|^2$$

where ϵ is a small positive parameter, devoted to tend to 0. Since Φ goes to $-\infty$ as $|x|$ goes to ∞, there exists a maximum point (x_0, y_0) of Φ. Note that,

$$\frac{m}{2} < \mu w_1(x_\kappa) - w_2(x_\kappa) = \Phi(x_\kappa, x_\kappa) \leq \sup \Phi(x, y) = \Phi(x_0, y_0)\ . \tag{4.6}$$

The following estimates, listed in a lemma, are standard in the theory of viscosity solutions and can be derived from the definition of Φ and (x_0, y_0). For example see [7], Chapter 3, Theorem 2.12. Only the last property stated in the lemma, is non-standard.

Lemma 4.3.

(i) $|x_0 - y_0| \leq \sqrt{C\epsilon}$ *for some constant* C *depending only on the* L^∞ *bounds on* w_1 *and* w_2.

(ii) $\dfrac{|x_0 - y_0|^2}{\epsilon} \to 0$ *as* $\epsilon \to 0$

(iii) $x_0, y_0 \to x_\kappa$, $w_1(x_0) \to w_1(x_\kappa)$, $w_2(y_0) \to w_2(x_\kappa)$ *when* $\epsilon \to 0$,

(iv) If $x_\kappa \in C_i$ *and* $w_2(x_\kappa) < \tilde{N}w_2(x_\kappa)$, $w_2(y_0) < \tilde{N}w_2(y_0)$ *if* ϵ *is small enough.*

We leave the proof of the point (iv) of this lemma to the reader: it just uses the fact that $w_2(y_0) \to w_2(x_\kappa)$ when $\epsilon \to 0$ together with the lower semi-continuity of Nw_2 coming from Lemma 4.2.

Now define the test functions ϕ_1 and ϕ_2 by

$$\phi_1(x) = w_2(y_0) + \frac{1}{\epsilon}|x-y_0|^2 + |x-x_\kappa|^2 \quad \text{and} \quad \phi_2(y) = \mu w_1(x_0) - \frac{1}{\epsilon}|x_0-y|^2 - |x_0-x_\kappa|^2$$

The function $\mu w_1 - \phi_1$ attains its maximum at x_0 and, for ϵ small enough, $x_0 \in \Omega_i \setminus A_i$ because $x_0 \to x_\kappa \in \Omega_i \setminus A_i$; therefore QVI implies

$$\mu w_1(x_0) + \sup_{u\in U} \frac{1}{\lambda}\{-f(x_0,u)\cdot(\mu D\phi_1(x_0) + \eta\mu w_1(x_0) D\xi(x_0)) - \mu e^{-\eta\xi(x_0)} K(x_0,u)\} \leq 0 \ .$$

On the other hand, $w_2 - \phi_2$ attains its minimum at $y_0 \in \Omega_i \setminus A_i$ and $w_2(y_0) < \tilde{N} w_2(y_0)$ if $y_0 \in C_i$; therefore, by using definition of viscosity supersolution, we have

$$w_2(y_0) + \frac{1}{\lambda} \sup_{u\in U}\{-f(y_0,u)\cdot(D\phi_2(y_0) + \eta w_2(y_0) D\xi(y_0)) - e^{-\eta\xi(y_0)} K(y_0,u)\} \geq 0 \ .$$

Denoting by $p_\epsilon = \dfrac{2(x_0 - y_0)}{\epsilon}$ and substituting $D\phi_1(x_0) = p_\epsilon + 2(x_0 - x_\kappa)$, $D\phi_2(y_0) = p_\epsilon$ in the above and estimating as in (4.1), we get,

$$\begin{aligned}(\lambda - \eta F)(\mu w_1(x_0) - w_2(y_0)) \leq L|x_0 - y_0|\{|p_\epsilon| + \eta|w_2|(F+L)|x_0 - y_0| \\ +(1-\mu)\omega_{\tilde{K}}(|x_0 - y_0|) + 2F|x_0 - x_\kappa| \ .\end{aligned}$$

Now remembering $\lambda - \eta F > 0$,

$$\begin{aligned}(\lambda - \eta F)\Phi(x_0,y_0) &= (\lambda - \eta F)\{\mu w_1(x_0) - w_2(y_0) - \frac{1}{\epsilon}|x_0 - y_0|^2\} \\ &\leq (\lambda - \eta F)(\mu w_1(x_0) - w_2(y_0)) \\ &\leq L|x_0 - y_0|\{|\frac{2(x_0 - y_0)}{\epsilon}\} + \eta(F+L)|w_2||x_0 - y_0| + \\ &\quad (1-\mu)\omega_{\tilde{K}}(|x_0 - y_0|) + 2F|x_0 - x_\kappa|\end{aligned}$$

Using the lemma, we can choose ϵ such that RHS of the above inequality can be made arbitrarily small, which will contradict (4.6) namely,

$$\Phi(x_0,y_0) \geq \frac{m}{2} > 0.$$

Thus, in each of the three steps, we arrive at a contradiction to (4.2) and we conclude that,

$$\sup_i \sup_{\Omega_i}(\mu w_1(x) - w_2(x)) \leq 0.$$

Now sending μ to 1 we get the required comparison between w_1 and w_2, namely, $w_1(x) \leq w_2(x)$ for all x in Ω.

Step 4: From Step 3, we have concluded that

$$w_1(x) - w_2(x) \leq 0 \quad \forall\ x \in \Omega$$

Now, by definition of w_1, w_2,

$$w_1(x) = u_1(x)e^{-\eta\xi(x)}, \quad w_2(x) = u_2(x)e^{-\eta\xi(x)},$$

hence,

$$u_1(x) - u_2(x) = e^{\eta\xi(x)}\{w_1(x) - w_2(x)\} \leq 0 \quad \forall\ x \in \Omega$$

Thus, we have the comparison between u_1 and u_2, which are solutions of the original quasivariational inequality.

In order to complete the proof, we have to provide the

Proof of Lemma 4.2. This uses the idea in lemma 5.2 (page 113) of [3]. Suppose that for some $x \in \partial A_i$

$$w_1(x) > \tilde{M}w_1(x).$$

Let the distance function from A_i be denoted by $d(\cdot)$. Consider the test function,

$$\phi(y) = w_1(y) - \frac{|x-y|^2}{\varepsilon} - Cd(y)$$

for positive parameters ε and C, to be fixed suitably later on. Let y_ε be the max of ϕ in $\overline{\Omega_i}$. Then in particular,

$$w_1(y_\varepsilon) \geq w_1(x)$$

for each ε and $y_\varepsilon \to x$. Hence

$$\liminf w_1(y_\varepsilon) \geq w_1(x).$$

Using the uppersemicontinuous property of w_1,

$$\limsup w_1(y_\varepsilon) \leq w_1(x).$$

Thus

$$\lim w_1(y_\varepsilon) = w_1(x)$$

By our assumption $w_1(x) > \tilde{M}w_1(x)$. Since $\tilde{M}w$ is u.s.c, we have

$$w_1(y_\varepsilon) > \tilde{M}w_1(y_\varepsilon)$$

for ε sufficiently small. If $y_\varepsilon \in \partial A_i$, then we have by (4.3),

$$w_1(y_\varepsilon)+\sup_u \frac{1}{\lambda}\{-\langle f(y_\varepsilon,u), \frac{2(x-y_\varepsilon)}{\varepsilon} - Cn(y_\varepsilon)+\eta w_1(y_\varepsilon)D\xi(y_\varepsilon)\rangle - e^{-\eta\xi(y_\varepsilon)}K(y_\varepsilon,u)\} \leq 0$$

where for the distance function $d(\cdot)$ from A, $Dd(x) = n(x)$. Now, we divide by C and let $\varepsilon \to 0$ and $C \to \infty$, in such a manner that $C\sqrt{\varepsilon} \to \infty$. Recalling that $\frac{|x-y_\varepsilon|}{\sqrt{\varepsilon}} \to 0$, we get

$$\sup_u\{\langle f(x,u), n(x)\rangle\} \leq 0, \quad \Rightarrow \quad \langle f(x,u), n(x)\rangle \leq 0 \quad \forall \quad u \in \mathcal{U}.$$

As $n(x) = -\zeta(x)$, appearing in $(A5)$ this conclusion contradicts $(A5)$. Hence our assumption cannot hold.

In a similar manner arguing with the test function.

$$\phi(y) = w_2(y) + \frac{|x-y|^2}{\varepsilon} + cd(y)$$

we get the required inequality of w_2. Thus the proof is complete. □

This theorem characterizes the value function of hybrid control problem as the unique viscosity solution of the QVI in the function class E defined earlier. We would like to remark the following.

Remark 4.4. *We can extend the uniqueness result to the hybrid game theory problem with unbounded value functions. The case of bounded value functions was treated in [10].*

5. Uniqueness - Finite Horizon Problem. The comparison proof for finite horizon problem is proved by building test-functions, which go to infinity at the boundary of the domain of comparison and tend to zero in a specific subset of the interior. These are similar to the ones in Ley [11] and Barles, Biton and Ley[4] where they are introduced as "friendly giants", either for proving "finite speed of propagation type results" for first-order equations or for proving that the comparison holds for second-order equations. Here it is extended to quasi-variational inequalities involving HJB equations.

First we announce the uniqueness theorem.

THEOREM 5.1. - *Assume* $(A1-A3), (A4)', (A5)', (A6), (A7)$ *and* $(C1)', (C2)$. *Let* u_1, u_2 *be respectively a u.s.c. subsolution bounded from below and a l.s.c supersolution bounded from below of the quasivariational inequality (QVI-T) in* $(0,T)\times\Omega$. *If* u_1, u_2 *satisfy: for any* $r>0$, u_1, u_2 *are bounded on* $[0,T]\times\{x\in\Omega\,;\,|x|\le r\}$ *and*

$$\limsup_{t\to T}\ \sup_{|x|\le r}\ [u_1(t,x)-u_2(t,x)]\le 0\ , \tag{5.1}$$

then $u_1\le u_2$ *in* $(0,T)\times\Omega$.

We point out that (5.1) is a non trivial assumption since we work in $\Omega=\bigcup_i \Omega_i\times\{i\}$ because of the dependence in i; roughly speaking, it means that the terminal data is assumed with a certain uniformity w.r.t i. This assumption is satisfied by the value function of the control problem under mild additional assumptions on the data as we show it at the end of the section.

Proof. Since u_1, u_2 are both bounded from below, we can assume as well that they are positive since we can add the same positive constant to both of them and the resulting functions still solve (QVI-T) in $(0,T)\times\Omega$ (of course, (5.1) also holds.)

We want to prove that for fixed μ close to 1 in $(0,1)$,

$$w=\mu u_1-u_2\le 0\ \text{ in }\ [0,T]\times\Omega\ .$$

For that, we divide the interval $[0,T]$ into finite number of sub-intervals of length less than a certain constant and prove the comparison on this subinterval first and then extend the result to $[0,T]$ by repeating the argument on each of the subintervals.

Let us set $\bar{K}:=F(1+R+\beta)$, where F is as in $(A4)'$, R is the bound for D, introduced in $(A2)$ and β is as in $(A6)$. We choose T_0 such that

$$T-T_0\ <\ \min\{\frac{1}{2F},\frac{\beta}{2\bar{K}}\} \tag{5.2}$$

The first step consists in proving that $w\le 0$ on $[T_0,T]\times D$. To do so, we introduce a neighbourhood of D, namely

$$\tilde{D}:=\{x\in\Omega\ :\ d(x,D)<\beta\ \}$$

for $\beta>0$ as before and we set

$$\tilde{m}(T_0):=\sup_{[T_0,T]\times D} w(t,x).$$

Because of assumption (5.1) and since

$$w(t,x)=(\mu-1)u_1(t,x)+(u_1-u_2)(t,x)\le(u_1-u_2)(t,x),$$

there exists $\tau,\ T_0 \le \tau \le T$ such that

$$\tilde{m}(\tau)\ \le (1-\mu)\frac{C'}{2} \tag{5.3}$$

where C' is the lower bound for the cost functionals introduced in (**C2**). We will show that w is a subsolution of the variational inequality

$$\min\{-w_t(t,x) - F(1+|x|)|Dw|\ ,\ w(t,x)\} = 0 \tag{5.4}$$

in $\tilde{D}_\tau := (\tau,T)\times\tilde{D}$ and from that we will conclude $w \le 0$ on $[\tau,T]\times D$ by using "finite speed of propagation type properties" (cf. for example Ley [11]). Once this is true for every such τ, it will follow that $w \le 0$ in $[T_0,T]\times D$.

From this first result, we will deduce, in a second step, that $w \le 0$ on $[T_0,T]\times A$. Finally the third step will be devoted to proving first that w is a subsolution of (5.4) in $[T_0,T]\times\Omega\setminus A$ and then $w \le 0$ in $[T_0,T]\times\Omega$.

Proof of Step 1 : Let $\Phi \in C^2((0,\infty)\times\Omega_\kappa)$ and let $(\bar{t},\bar{x}) \in \tilde{D}_\tau$ be a strict local maximum point of $w-\Phi = \mu u_1 - u_2 - \Phi$ in $\tilde{D}_\tau$. We want to prove that

$$\min\{-\Phi_t(\bar{t},\bar{x}) - F(1+|\bar{x}|)|D\Phi(\bar{t},\bar{x})|\ ,\ w(\bar{t},\bar{x})\} \le 0\ .$$

If $w(\bar{t},\bar{x}) \le 0$, there is nothing to prove.

If $w(\bar{t},\bar{x}) > 0$, since $C\cap D$ need not be empty, we first remark that

$$u_2(\bar{t},\bar{x}) < Nu_2(\bar{t},\bar{x})\ \text{ if }\ \bar{x}\in C_\kappa \tag{5.5}$$

Indeed, otherwise, since $u_1(\bar{t},\bar{x}) \le Nu_1(\bar{t},\bar{x})$, we would have, for some $x' \in D$

$$(\mu u_1 - u_2)(\bar{t},\bar{x}) \le (\mu u_1 - u_2)(\bar{t},x') + (\mu-1)C_c(\bar{x},x') \le \tilde{m}(\tau) + (\mu-1)C',$$

which contradicts the fact that $w(\bar{t},\bar{x}) > 0$, because of our choice of τ, satisfying (5.3).

For $\varepsilon > 0$, we consider for some ρ such that $(\bar{t}-\rho,\bar{t}+\rho)\times B(\bar{x},\rho) \subset \tilde{D}_\tau$,

$$\max_{(\bar{t}-\rho,\bar{t}+\rho)\times B(\bar{x},\rho)^2}\{\mu u_1(t,x) - u_2(t,y) - \Phi(t,x) - \frac{|x-y|^2}{\varepsilon^2}\}\ .$$

It can be shown that the above maximum is achieved at points $(x_\varepsilon,y_\varepsilon,t_\varepsilon)$ such that

$$(t_\varepsilon,x_\varepsilon,y_\varepsilon) \to (\bar{t},\bar{x},\bar{x}) \quad \text{and} \quad \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0 \text{ as } \varepsilon \to 0,$$

$$u_2(t_\varepsilon,y_\varepsilon) \to u_2(\bar{t},\bar{x}).$$

Then, thanks to (5.5) and the lower semicontinuity of Nu_2 and the above convergence, we have, for ε small enough,

$$u_2(t_\varepsilon,y_\varepsilon) < Nu_2(t_\varepsilon,y_\varepsilon).$$

Therefore, at these points $(t_\varepsilon,y_\varepsilon)$, u_2 is a viscosity supersolution of the HJB equation given by

$$-v_t + \sup_{u\in U}\{-K(t,x,u) - f(t,x,u)\cdot Dv\} = 0\ ,$$

while $v = \mu u_1$ is a viscosity subsolution of the (slightly different) HJB equation

$$-v_t + \sup_{u\in U}\{-\mu K(t,x,u) - f(t,x,u)\cdot Dv\} = 0\ .$$

By the results of the User's guide [8], there exists $a \in I\!R$ such that

$$\begin{aligned}(a, p_\varepsilon) &\in D^- u_2(t_\varepsilon, y_\varepsilon)\\ (a + \Phi_t(t_\varepsilon, x_\varepsilon), p_\varepsilon + D\Phi(t_\varepsilon, x_\varepsilon)) &\in D^+ u_1(t_\varepsilon, x_\varepsilon)\end{aligned}$$

where $p_\varepsilon := \dfrac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}$ and therefore

$$\begin{aligned}-a - \Phi_t(t_\varepsilon, x_\varepsilon) + \sup_{u\in U}\{-\mu K(t_\varepsilon, x_\varepsilon, u) - f(t_\varepsilon, x_\varepsilon, u)\cdot(D\Phi(t_\varepsilon, x_\varepsilon) + p_\varepsilon)\} &\le 0\ ,\\ -a + \sup_{u\in U}\{-K(t_\varepsilon, y_\varepsilon, u) - f(t_\varepsilon, y_\varepsilon, u)\cdot p_\varepsilon\} &\ge 0\ .\end{aligned}$$

Subtracting these two viscosity inequalities, we get

$$\begin{aligned}-\Phi_t(t_\varepsilon, x_\varepsilon) &- \sup_{u\in U}\{-K(t_\varepsilon, y_\varepsilon, u) - f(t_\varepsilon, y_\varepsilon, u)\cdot p_\varepsilon\}\\ &+ \sup_{u\in U}\{-\mu K(t_\varepsilon, x_\varepsilon, u) - f(t_\varepsilon, x_\varepsilon, u)\cdot(D\Phi(t_\varepsilon, x_\varepsilon) + p_\varepsilon)\} \le 0\end{aligned}$$

Using that $\sup(\cdots) - \sup(\cdots) \ge \inf(\cdots - \cdots)$ and the continuity properties of K and f, we are lead to

$$-\Phi_t(t_\varepsilon, x_\varepsilon) - F(|1 + |x_\varepsilon|)|D\phi(t_\varepsilon, x_\varepsilon)| - \mu\omega_K(|x_\varepsilon - y_\varepsilon|) - \omega_f(|x_\varepsilon - y_\varepsilon||p_\varepsilon|) \le 0$$

Thus, as $\varepsilon \to 0$, noticing that $|x_\varepsilon - y_\varepsilon||p_\varepsilon| = 2\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0$, we obtain

$$-\Phi_t(\bar t, \bar x) - F(1 + |\bar x|)|D\phi(\bar t, \bar x)| \le 0$$

proving the claim that $w = \mu u_1 - u_2$ is a viscosity subsolution of (5.4) in $\tilde D_\tau$.

Next we deduce that $w \le 0$ on $[\tau, T] \times D$. We first remark that, by **(A2)**, if $x \in \tilde D$, $|x| \le R + \beta$. Recall that $\bar K := F(1 + R + \beta)$. Hence w is as well a subsolution of

$$\min\{-w_t(t,x) - \bar K|Dw|, w(t,x)\} = 0 \quad \text{in } \tilde D_\tau\ . \tag{5.6}$$

We now construct a strict supersolution for this equation using "friendly giant" functions. For that, we consider an increasing C^∞ function $\psi : (-\infty, \beta) \to I\!R$ such that

$$\psi(s) \equiv 0 \ \text{ if } s \le \beta/2 \quad \text{ and } \quad \psi(s) \to +\infty \text{ as } s \to \beta.$$

We claim that, for any κ, $x_0 \in D_\kappa$, $\eta > 0$ and T_0 satisfying (5.2), the function

$$\chi_{x_0}(t,x) := \psi(\bar K(T - t) + |x - x_0|) + \eta(T - t)$$

is a strict (smooth) supersolution of (5.6) in the domain

$$\mathcal{C}_{x_0} := \{(t,x) \in \tilde D_\tau \ :\ \bar K(T - t) + |x - x_0| < \beta\,;\ \bar K(T - t) \le \beta/2\}.$$

In fact this property can be checked by an immediate computation noticing that, for $x = x_0$, $\bar K(T - t) + |x - x_0| = \bar K(T - t) \le \beta/2$ and therefore all the derivatives of ψ are equal to 0 at that point. Finally, for any κ, set

$$\chi(t,x) = \inf_{x_0} \chi_{x_0}(t,x) = \psi(\bar K(T - t) + d(x, D)) + \eta(T - t).$$

Then χ is well-defined and is still a strict supersolution of (5.6) in the domain

$$\mathcal{C} := \{(t,x) \in \tilde{D}_\tau \ : \ \bar{K}(T-t) + d(x,D) < \beta\,;\ \bar{K}(T-t) \le \beta/2\}.$$

Now we want to compare w and χ in $\mathcal{C}$. From the definition of χ, it is clear that it tends to infinity on the boundary of the domain $\mathcal{C}$. Using this remark together with standard comparison arguments yield that $w \le \chi$ in $\mathcal{C}$. For any $x \in D$,

$$w(t,x) \le \chi(t,x) = \psi(\bar{K}(T-t)) + \eta(T-t) = \eta(T-t)$$

since $\bar{K}(T-t) \le \beta/2$. Letting η tend to 0, we deduce that for $x \in D$ and $\tau \le t \le T$

$$w(x,t) \le \ 0.$$

Hence $\tilde{m}(\tau) \le 0$ for all $T_0 \le \tau < T$. Thus $\tilde{m}(T_0) \le 0$ and the proof of the first step is complete.

Proof of Step 2 : By the definition of M and N, and in particular the fact that g takes values in D, we deduce that, if $x \in A$, then for any v_0

$$\mu M u_1(t,x) \le \mu u_1(t, g(x,v_0)) + \mu C_a(x_\kappa, v_0) \le u_2(t, g(x,v_0)) + C_a(x_\kappa, v_0) + (\mu - 1)C' \ ,$$

and therefore, taking the infimum on v_0, $\mu M u_1(t,x) \le M u_2(t,x) + (\mu-1)C'$. A similar property holds for Nu_1 and Nu_2. For the boundary points of A and C, we have to argue on the same lines as in Lemma 4.2, to show that the correct inequalities hold.

We immediately deduce that $w(t,x) \le 0$ for $T_0 \le t \le T$ and $x \in A$ because $\mu u_1(t,x) - u_2(t,x) \le \mu M u_1(t,x) - M u_2(t,x) \le 0$.

Proof of Step 3 : It remains to show that $w \le 0$ in $[T_0, T] \times (\Omega \setminus A)$. To do so, we first prove that w is a subsolution of (5.4) in the set $\{w > 0\}$. We only sketch this proof since it follows arguments we already used above.

The only difficulty comes from the points (t,x) in C since we may have the inequality $u_2(x,t) \ge Nu_2(x,t)$ (and not the inequality associated to the HJB equation). But, if this is the case, then, by Step 2,

$$\mu u_1(t,x) - u_2(t,x) \le \mu N u_1(t,x) - N u_2(t,x) \le (\mu - 1)C' < 0 \ .$$

Therefore this case cannot happen on the set $\{w > 0\}$ and the same arguments as in Step 1 above show that w is a subsolution of (5.4) in the set $\{w > 0\}$.

To conclude, we have to modify the comparison argument by building "friendly giant functions" as the function χ above but in a slightly different way in order to take into account the $|x|$-dependence in a suitable way. This also allows us to use a "local" comparison argument.

For any interior point $x_0 \in \Omega_\kappa \setminus A_\kappa$ and for $r, c > 0$, we introduce the set

$$\mathcal{O}(x_0, r) = \{(t,x) \in [T_0, T] \times \Omega \ : \ |x - x_0| < r\sqrt{c(t - T_0)}\} \ .$$

Next we define $\chi : \mathcal{O}(x_0, r) \to I\!R$ by

$$\chi(t,x) = \frac{1}{r^2 c(t - T_0) - |x - x_0|^2} \ .$$

We need to show that χ is a strict supersolution of (5.4) in $\mathcal{O}(x_0, r)$. By simple calculations, we get

$$\chi_t = \frac{-cr^2}{(r^2c(t-T_0) - |x-x_0|^2)^2}$$
$$D\chi = \frac{2(x-x_0)}{(r^2c(t-T_0) - |x-x_0|^2)^2}$$

Since $|x| \leq |x-x_0| + |x_0|$ and $|x-x_0| \leq r\sqrt{c(T-T_0)}$, we have

$$2F(1+|x|)|x-x_0| \leq 2F(1+r\sqrt{c(T-T_0)} + |x_0|)r\sqrt{c(T-T_0)}\ .$$

Finally if we choose $r \geq r(x_0) = (1+|x_0|)$ and c such that $\sqrt{c} > \frac{2F\sqrt{(T-T_0)}}{1-2F(T-T_0)}$, we have

$$2F(1+|x|)|x-x_0| < cr^2.$$

Note that such a choice of c is possible because of (5.2). Substituting and using above inequality we can conclude that χ is a strict supersolution of (5.4) in $\mathcal{O}(x_0, r)$.

By definition, χ goes to ∞ on the lateral boundary of $\mathcal{O}(x_0, r)$ and hence, on that part of the lateral boundary contained inside $[T_0, T] \times (\Omega_\kappa \setminus A_\kappa)$, we have no problem to compare w and χ. Of course, $w \leq \chi$ on $[T_0, T] \times A_\kappa$ since $\chi \geq 0$ and $w \leq 0$ there by step 2. Therefore we can conclude that $w \leq \chi$ in $\mathcal{O}(x_0, r)$.

Now we let r go to infinity to obtain

$$w \leq 0 \quad \text{in } [T_0, T] \times \Omega_\kappa \setminus A_\kappa\ ,$$

for every κ. The proof is now complete since we can iterate the argument in time according to the fact that $T - T_0$ depends only on β and $\bar{K}$. □

Finally we show that the value function naturally satisfies the condition (5.1) under some reasonable additional assumptions. For that it is enough to show that

$$|V(t,x) - \tilde{h}(x)| \to 0$$

as t approaches T, uniformly for bounded $x \in \Omega$.

If t is close to T and $|x| \leq r$, it is easy to see that the trajectory X starting from x at time t, remains in a bounded subset of Ω on the time interval $[t, T]$. This is true even after the jumps, since the jump destination set D is bounded. Because of (C2), the number of jumps is finite and estimated uniformly in x by a fixed number depending on r. Therefore, the integral of K gives a $O(T-t)$ whatever the trajectory is.

Next we estimate $V(t,x) - \tilde{h}(x)$ from below by considering a ε-optimal control $(u(\cdot), v, \xi_i, X(\xi_i)')$ (we drop the dependence with respect to ε for the sake of simplicity of notations); we have

$$V(t,x) + \varepsilon \geq \int_t^T K(t, X(t), u(t))dt + \sum_{s \leq \sigma_i < T} C_a(X(\sigma_i^-), v)$$
$$+ \sum_{s \leq \xi_i < T} C_c(X(\xi_i^-), X(\xi_i)') + h(X_x(T))$$

But, using that

$$\tilde{h}(X(s)) = M\tilde{h}(X(s)) \quad \text{if } X(s) \in A$$

or

$$\tilde{h}(X(s)) = \min(h(X(s), Nh(X(s))) \quad \text{if } X(s) \in C,$$

we deduce

$$C_a(X(\sigma_i^-), v) \geq \tilde{h}(X(\sigma_i^-)) - \tilde{h}(X(\sigma_i^+)) \ ,$$

$$C_c(X(\xi_i^-), X(\xi_i)') \geq \tilde{h}(X(\xi_i^-)) - \tilde{h}(X(\xi_i)') \ .$$

Therefore

$$\begin{aligned} V(t,x) + \varepsilon \geq O(T-t) &+ \sum_{s \leq \sigma_i < T} \Big(\tilde{h}(X(\sigma_i^-)) - \tilde{h}(X(\sigma_i^+)) \Big) \\ &+ \sum_{s \leq \xi_i < T} \Big(\tilde{h}(X(\xi_i^-)) - \tilde{h}(X(\xi_i)') \Big) + \tilde{h}(X_x(T)) \end{aligned}$$

In order to conclude, it is enough to rearrange the terms of the right-hand side and to make it appear as a sum of differences of values $\tilde{h}$ between two jumps (after one jump and before the next one). The uniform continuity of $\tilde{h}$ and the estimates on the f-ode (2.1), lead to the desired property, since ε is arbitrary.

Conversely, for any $u(\cdot)$, we write

$$\begin{aligned} V(t,x) \leq \int_t^T K(t, X(t), u(t))dt &+ \sum_{s \leq \sigma_i < T} C_a(X(\sigma_i^-), v) \\ &+ \sum_{s \leq \xi_i < T} C_c(X(\xi_i^-), X(\xi_i)') + h(X_x(T)) \end{aligned}$$

but this time, when we touch either A or C, we choose the jumps in order to have

$$C_a(X(\sigma_i^-), v) \leq \tilde{h}(X(\sigma_i^-)) - \tilde{h}(X(\sigma_i^+)) + \varepsilon \ ,$$

$$C_c(X(\xi_i^-), X(\xi_i)') \leq \tilde{h}(X(\xi_i^-)) - \tilde{h}(X(\xi_i)') + \varepsilon \ .$$

By the same arguments as above, we again obtain

$$|V(t,x) - \tilde{h}(x)| \leq \hat{m}(T-t) \quad \text{for any } |x| \leq r$$

where $\hat{m}(\tau) \to 0$ as $\tau \to 0$ depend only on the L^∞ norm of K and on the modulus of continuity of $\tilde{h}$ on the bounded subset where the trajectory X lives.

The above inequalities show that V assumes the terminal data $\tilde{h}$ uniformly w.r.t i, which is exactly (5.1).

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