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On the Kertész line: Some rigorous bounds*†

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Abstract

We study the Kertész line of the q–state Potts model at (inverse) temperature β, in presence of an external magnetic field h. This line separates two regions of the phase diagram according to the existence or not of an infinite cluster in the Fortuin-Kasteleyn representation of the model. It is known that the Kertész line $h_K(\beta)$ coincides with the line of first order phase transition for small fields when q is large enough. Here we prove that the first order phase transition implies a *jump* in the density of the infinite cluster, hence the Kertész line remains below the line of first order phase transition. We also analyze the region of large fields and prove, using techniques of stochastic comparisons, that $h_K(\beta)$ equals $\log(q-1)-\log(\beta-\beta_p)$ to the leading order, as β goes to $\beta_p = -\log(1-p_c)$ where p_c is the threshold for bond percolation.

One important feature of the Fortuin–Kasteleyn representation of Ising and Potts models [1] (the random cluster model), is that the geometrical transition, i.e. the apparition of an infinite cluster, corresponds precisely to the phase transition leading to a spontaneous magnetization in the absence of an external field [2]. In [3], Kertész pointed out that this property is lost in the Ising model when an external field h is introduced: while thermodynamic quantities are analytic for any $h > 0$, a geometric transition appears in the corresponding random cluster model and there is a whole percolation transition line extending from the Curie point ($h = 0$) to infinite fields. As Kertész explained, the analyticity of thermodynamic quantities and the existence of the percolation transition are not contradictory because the free energy remains analytic.

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The Kertész line can be considered as well in the Potts model. There, for large q, the first order transition extends to small, positive fields h and it is an important issue to understand whether or not the Kertész line coincides with the line of phase transition. Such a property was established in [4] for small h (and q large enough) and hence extends the relevance of the random cluster representation for the analysis of the phase transition in the corresponding region. Here we address some of the remaining issues: we prove the existence of the line, show that the first order phase transition results in a discontinuity of the percolation density, and provide bounds on the Kertész line that are particularly precise in the region of large fields.

In the Potts model, the spin variables σ_i associated with lattice sites take values in the discrete set $\{1, \ldots, q\}$. Considering a spin configuration in a finite box $\Lambda \subset \mathbb{Z}^d$ ($d \geqslant 2$), the Potts model at inverse temperature β, subject to an external ordering field h, is defined by the Gibbs measure

$$\mu_\Lambda^{\mathrm{Potts}}(\boldsymbol{\sigma}) = \frac{1}{Z_\Lambda^{\mathrm{Potts}}} \prod_{\langle i,j \rangle} e^{\beta(\delta_{\sigma_i,\sigma_j}-1)} \prod_i e^{h\delta_{\sigma_i,1}} \tag{1}$$

Here the first product is over nearest neighbor pairs of Λ, the second runs over sites of Λ, $Z_\Lambda^{\mathrm{Potts}}$ denotes the partition function (normalizing factor) and δ is the Kronecker symbol.

To study the behavior of clusters, in the sense of FK clusters, we turn to the corresponding Edwards–Sokal formulation [5], given by the joint measure

$$\mu_\Lambda^{\mathrm{ES}}(\boldsymbol{\sigma}, \boldsymbol{\eta}) = \frac{1}{Z_\Lambda^{\mathrm{ES}}} \prod_{\langle i,j \rangle} \left(e^{-\beta}\delta_{\eta_{ij},0} + (1-e^{-\beta})\delta_{\eta_{ij},1}\delta_{\sigma_i,\sigma_j} \right) \prod_i e^{h\delta_{\sigma_i,1}}. \tag{2}$$

This model can be thought as follows. Given a spin configuration, between two neighboring sites with $\sigma_i = \sigma_j$, one put an edge ($\eta_{ij} = 1$) with probability $1 - e^{-\beta}$ and no edge w.p. $e^{-\beta}$; for $\sigma_i \neq \sigma_j$, no edge (or bond) is present. When the field is infinite all spins take the value one and we are left with the classical bond percolation problem. At finite fields, the spins are not uniformly equal to one yet we will see that percolation in the edge variable $\boldsymbol{\eta}$ still occurs at some finite temperature.

We call $\mu_{\Lambda,\boldsymbol{f}}^{\mathrm{RC}}$ the marginal law of $\boldsymbol{\eta}$ under $\mu_\Lambda^{\mathrm{ES}}$. This measure can be considered as well for non-integer $q \geqslant 1$ (see (12)). Our first result concerns the existence of the Kertész line.

Theorem 1 *Let $\beta, h \geqslant 0$ and $q \geqslant 1$.*

i. The infinite volume limit

$$\mu_{\boldsymbol{f}}^{\mathrm{RC}} = \lim_{\Lambda \nearrow \mathbb{Z}^d} \mu_{\Lambda,\boldsymbol{f}}^{\mathrm{RC}} \tag{3}$$

exists.

ii. The probability

$$\theta = \mu_{\boldsymbol{f}}^{\mathrm{RC}}\,(\textit{the origin belongs to an infinite cluster of } \boldsymbol{\eta}) \tag{4}$$

increases with β and h, and decreases with q.

iii. Hence the Kertész line

$$h_K(\beta) = \inf\{h \geqslant 0 : \theta > 0\} \tag{5}$$

exists, and $h_K(\beta)$ decreases with β.

Note that $h_K(\beta) = 0$ if $\beta \geqslant \beta_c$, where β_c is the critical inverse temperature for the phase transition with no field, while $h_K(\beta) = +\infty$ if $\beta \leqslant \beta_p$, where

$$\beta_p = -\log(1 - p_c) \tag{6}$$

is the critical inverse temperature for percolation at infinite fields and p_c the threshold for bond percolation on $\mathbb{Z}^d$.

Then we examine the consequences of the first order transition on the density θ of the infinite cluster:

Theorem 2 *A discontinuity in the parameter β in the mean energy*

$$e_{\boldsymbol{f}} = \frac{1}{1 - e^{-\beta}} \mu_{\boldsymbol{f}}^{\mathrm{RC}}\,(\eta_{ij}) \tag{7}$$

where i, j are neighboring sites, or in the magnetization, implies a discontinuity in the density θ of percolation.

This means that θ has a *jump* on the line of first order phase transition. Consequently, the Kertész line cannot be found above the line of first order phase transition. It is known [4] that both lines coincide at small fields when q is large, hence the question remains whether they coincide up to the other extremity of the line of first order phase transition. In the corresponding mean field analysis [6] we proved the existence of a cusp as soon as $q > 2$, that is, whenever appears a line of first order phase transition. However, in the two dimensional Potts model no bifurcation was noted numerically [4].

We conclude our exposition of the results with upper and lower bounds on the Kertész line, which are particularly efficient when β is taken slightly above $\beta_p = -\log(1 - p_c)$, corresponding to the regime of large fields. The idea that led to the next theorem is that the model can be understood as independent bond percolation over a *random media* : the spins not equal to 1 are considered as *defects*, which become rare when $h \to +\infty$. Our proofs are reminiscent of [7] in which similar methods were employed to provide necessary and sufficient conditions for the phase transition in the dilute Ising model, see also [8] for beautiful results on mixed percolation.

Theorem 3 *For any $d \geqslant 2$, $q > 1$ and $\beta > \beta_p$, one has*

$$h_K(\beta) \leqslant -\log \frac{\sqrt{\frac{e^{\beta}-1}{e^{\beta_p}-1}} - 1}{q-1} \tag{8}$$

$$= -\log(\beta - \beta_p) + \log(2p_c(q-1)) + O_{\beta \to \beta_p^+}(\beta - \beta_p) \tag{9}$$

while

$$h_K(\beta) \geqslant -\log \frac{e^{-\beta_p} - \mathrm{e}^{-\beta}}{p_c(q-1)} - 2\beta d \tag{10}$$

$$= -\log(\beta - \beta_p) + \log(p_c(q-1)) - (2d-1)\beta_p + O_{\beta \to \beta_p^+}(\beta - \beta_p) \tag{11}$$

Thus, to the leading order, $h_K(\beta) \simeq -\log(\beta - \beta_p) + \log(q-1)$ when $\beta \to \beta_p^+$. The upper and lower asymptotes differ from the constant $\log(2) - (2d-1)\log(1-p_c)$ that does not depend on q.

These upper and lower bounds are presented in Fig. 1 together with the numerical results of [4].

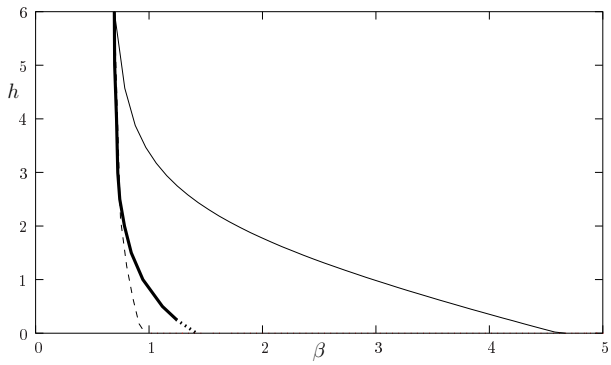

Figure 1: A comparison between upper and lower bounds with the numerical results of [4] for $d = 2$ and $q = 10$.

To summarize, we have shown that for the lattice Potts model subject to an external field, the Kertész line is well defined. We have presented upper and lower bounds on this line. These bounds are very precise at high fields and complement the previous study [4] in which a precise approximation at low field was given. In addition, we have shown that a jump of the mean energy or of the magnetization implies a jump in the percolation density of the clusters associated to the corresponding FK representation of the model. This last result does not exclude the presence of an intermediate regime of the field where when decreasing the temperature, one first encounters the percolation transition and then for a lower temperature the percolation density would exhibits a jump.

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A Appendix

A.1 A random cluster representation

For any $\beta, h \geqslant 0$ and $q \geqslant 1$ we define a random cluster model that takes into account the external field. Edge configurations $\boldsymbol{\eta}$ have $\eta_{ij} \in \{0,1\}$ for all $< i,j >$ nearest neighbor pairs in the domain. For Λ a finite subset of $\mathbb{Z}^d$ and $\boldsymbol{\pi}$ a boundary condition on Λ, that is an edge configuration on $\mathbb{Z}^d$ which restriction to Λ has no open edge, we consider

$$\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{\pi}}(\boldsymbol{\eta}) = \frac{1}{Z^{\mathrm{RC}}_{\Lambda,\boldsymbol{\pi}}} \prod_{\langle i,j\rangle} \left(e^{-\beta}\delta_{\eta_{ij},0} + (1-e^{-\beta})\delta_{\eta_{ij},1}\right) \prod_{C\in\mathcal{C}^{\boldsymbol{\pi}}_{\Lambda}(\boldsymbol{\eta})} w(S(C)) \tag{12}$$

where

$$w(S) = 1 + (q-1)e^{-hS}.$$

The first product runs over $\langle i,j\rangle$ nearest neighbor pairs in Λ. The second one is over all connected components (clusters) $C \in \mathcal{C}^{\boldsymbol{\pi}}_{\Lambda}(\boldsymbol{\eta})$, where $\mathcal{C}^{\boldsymbol{\pi}}_{\Lambda}(\boldsymbol{\eta})$ is the set of clusters of $\mathbb{Z}^d$ under the wiring $\boldsymbol{\pi} \vee \boldsymbol{\eta}$ (the edge configuration defined by $(\pi \vee \eta)_{ij} = \max(\pi_{ij}, \eta_{ij})$), that own some site of Λ. For any such cluster, $S(C)$ stands for its number of sites.

The variable $\boldsymbol{\eta}$ under the joint measure $\mu^{\mathrm{ES}}_{\Lambda}$ defined at (2) follows the law $\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{f}}$, where $\boldsymbol{f}$ stands for the *free boundary condition*, that is the edge configuration with no edge open. Conditionally on $\boldsymbol{\eta}$, for integer $q \in \{2,3,\ldots\}$ the distribution of $\boldsymbol{\sigma}$ under the joint measure $\mu^{\mathrm{ES}}_{\Lambda}$ is as follows: the spin is constant on each cluster of $\boldsymbol{\eta}$, and a cluster with S sites obtains the color 1 with probability $e^{hS}/(e^{hS}+q-1)$, any other of the $q-1$ colors with probability $1/(e^{hS}+q-1)$, independently of other clusters. This conditional distribution accounts for the definition (18) of the magnetization.

Let us now compare our representation with that of [4]. There, as an alternative to the ghost spin scheme, a colored version of the Edwards–Sokal representation was introduced. This representation was defined by

$$\begin{aligned}\mu^{\mathrm{CES}}_{\Lambda}(\boldsymbol{\sigma},\boldsymbol{n}) \quad = \quad & \frac{1}{Z^{\mathrm{CES}}_{\Lambda}} \prod_{\langle i,j\rangle} [e^{-\beta}\delta_{n_{ij},0} + (1-e^{-\beta})\delta_{n_{ij},1}\chi(\sigma_i=\sigma_j=1) \\ & +(1-e^{-\beta})\delta_{\eta_{ij},2}\chi(\sigma_i=\sigma_j\neq 1)] \prod_i e^{h\delta_{\sigma_i,1}}\end{aligned} \tag{13}$$

with the edges variables n_{ij} taking values in the set $\{0,1,2\}$. Let us consider, for a while, thermodynamics limits (the existence of thermodynamics limits will be proven at the next section). We want to emphasize that the question of percolation for $\boldsymbol{\eta}$ under $\mu^{\mathrm{RC}}_{\boldsymbol{f}}$ and for the color 1 in $\boldsymbol{n}$ under μ^{CES} are *equivalent*. Indeed, any infinite cluster for $\boldsymbol{\eta}$ under μ^{ES} will be given the color $\sigma = 1$ with probability one as soon as $h > 0$ (resp. w. p. $1/q$ if $h = 0$). Hence, relabelling $\boldsymbol{\eta}$ into $\boldsymbol{n}$ according to the spin of clusters we obtain in fact an infinite cluster for the color 1 in $\boldsymbol{n}$, w. p. 1 (w. p. $1/q$ if $h = 0$) and this shows that the

probability of percolation from the origin θ under $\mu^{\mathrm{RC}}_{\boldsymbol{f}}$ and θ_1 for the color 1 in $\boldsymbol{n}$ under μ^{CES} satisfy

$$\theta_1 = \begin{cases} \theta/q & \text{if } h = 0 \\ \theta & \text{if } h > 0. \end{cases}$$

A.2 Conditional probabilities and infinite volume limit

Here we give the proof of Theorem 1. Like the usual random cluster representation, the measures $\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{\pi}}$ satisfy the DLR equations, which means that, given any $\Lambda' \subset \Lambda$, the restriction of $\boldsymbol{\eta}$ to Λ' under the measure $\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{\pi}}$ conditioned on $\boldsymbol{\eta} = \boldsymbol{\omega}$ outside of Λ' has law $\mu^{\mathrm{RC}}_{\Lambda',\boldsymbol{\omega}\vee\boldsymbol{\pi}}$. Consequently, the measures $\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{\pi}}$ are characterized by the law of $\boldsymbol{\eta}$ on a single edge ij given the boundary condition $\boldsymbol{\pi}$:

$$\mu^{\mathrm{RC}}_{\{ij\},\boldsymbol{\pi}}(\boldsymbol{\eta}_{ij} = 1) = p^{\boldsymbol{\pi}}_{ij} \tag{14}$$

where $p^{\boldsymbol{\pi}}_{ij} = p \stackrel{\mathrm{def.}}{=} 1 - e^{-\beta}$ if $\boldsymbol{\pi}$ connects i and j, otherwise

$$p^{\boldsymbol{\pi}}_{ij} = \frac{p}{p + (1-p)\frac{w(S^{\boldsymbol{\pi}}_i)w(S^{\boldsymbol{\pi}}_j)}{w(S^{\boldsymbol{\pi}}_i + S^{\boldsymbol{\pi}}_j)}} \tag{15}$$

where $S^{\boldsymbol{\pi}}_i$ (resp. $S^{\boldsymbol{\pi}}_j$) is the number of sites of the cluster containing i (resp. j) under the connections $\boldsymbol{\pi}$.

It is easily verified that $p^{\boldsymbol{\pi}}_{ij}$ is an increasing function of β, h and $\boldsymbol{\pi}$, decreasing with $q \geqslant 1$. Thanks to the DLR equations, the hypothesis of Holley's Lemma (see for instance Theorem 4.8 in [9] or Theorems 2.1 and 2.6 in [10]) are verified and this implies that $\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{\pi}}$ stochastically increases with β, h and $\boldsymbol{\pi}$, and stochastically decreases with q. Using again the DLR equations, we see that the measure $\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{f}}$ stochastically increases as $\Lambda \nearrow \mathbb{Z}^d$, proving the existence of the weak limit $\mu^{\mathrm{RC}}_{\boldsymbol{f}}$ at (3). Point *ii* of the theorem follows from the variations of $\mu^{\mathrm{RC}}_{\boldsymbol{f}}$ with β, h and q which are the same than those of $\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{\pi}}$ while point *iii* is an immediate consequence of *ii*.

A.3 First order transitions

Theorem 2 is essentially a consequence of the uniqueness of infinite volume measures under the condition that the infinite cluster has the same density under both infinite volume limits for free and wired boundary conditions (Theorem 4 below). We adapt here the classical argument at $h = 0$ to our setting $h \geqslant 0$.

By the stochastic comparison argument, one can consider as well the infinite volume limit $\mu^{\mathrm{RC}}_{\boldsymbol{w}}$ of $\mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{\pi}}$ under the wired boundary condition $\boldsymbol{w}$, that has all edges open. As in [11, 12, 13] it happens that:

Lemma 1 *Given $h \geqslant 0, q \geqslant 1$, the set of β at which $\mu^{\mathrm{RC}}_{\boldsymbol{f}} \neq \mu^{\mathrm{RC}}_{\boldsymbol{w}}$ is at most countable.*

Proof Let

$$y_\Lambda^{\boldsymbol{\pi}} = \frac{1}{|\Lambda|} \log \left[\prod_{\langle i,j \rangle} \left(e^\beta - 1\right)^{\eta_{ij}} \prod_{C \in \mathcal{C}_\Lambda^{\boldsymbol{\pi}}(\boldsymbol{\eta})} w(S(C)) \right]. \tag{16}$$

When $\boldsymbol{\pi} = \boldsymbol{f}$, the quantity $y_\Lambda^{\boldsymbol{\pi}}$ is sub-additive – when one cluster of size S is cut into two clusters of size S_1, S_2 with $S_1 + S_2 = S$, then $w(S) \leqslant w(S_1)w(S_2)$. Hence $y_\Lambda^{\boldsymbol{f}}$ converges to some $y(\beta, h)$ as $\Lambda \to \mathbb{Z}^d$. The influence of the boundary condition $\boldsymbol{\pi}$ on $y_\Lambda^{\boldsymbol{\pi}}$ is of order $|\partial\Lambda|/|\Lambda|$: for any configuration $\boldsymbol{\eta}$ the product $\prod_{C \in \mathcal{C}_\Lambda^{\pi}(\boldsymbol{\eta})} w(S(C))$ decreases with $\boldsymbol{\pi}$ and conversely,

$$\prod_{C \in \mathcal{C}_\Lambda^{\boldsymbol{f}}(\boldsymbol{\eta})} w(S(C)) \leqslant (w(1))^{|\partial\Lambda|} \prod_{C \in \mathcal{C}_\Lambda^{\boldsymbol{w}}(\boldsymbol{\eta})} w(S(C)) \tag{17}$$

because $\mathcal{C}_\Lambda^{\boldsymbol{f}}(\boldsymbol{\eta})$ contains at most $|\partial\Lambda|$ clusters not present in $\mathcal{C}_\Lambda^{\boldsymbol{w}}(\boldsymbol{\eta})$, each of them having size $S \geqslant 1$. Hence for any sequence $\boldsymbol{\pi}_\Lambda$, any sequence of cubes $\Lambda \to \mathbb{Z}^d$, we have $y_\Lambda^{\boldsymbol{\pi}} \to y$. Now we show that $y_\Lambda^{\boldsymbol{\pi}}$ is a convex function of $\lambda = \log(e^\beta - 1)$. Indeed, the derivative

$$\frac{\partial y_\Lambda^{\boldsymbol{\pi}}}{\partial \lambda} = \mu_{\Lambda,\boldsymbol{\pi}}^{\mathrm{RC}} \left(\frac{1}{|\Lambda|} \sum_{<i,j>} \eta_{ij} \right)$$

is an increasing function of β, hence of λ, and the convexity holds for both $y_\Lambda^{\boldsymbol{\pi}}$ and its limit y. Therefore y is derivable at all $\beta \notin \mathcal{D}_h$ where $\mathcal{D}_h$ (that depends on h) is finite or countable. When this occurs, by convexity of the $y_\Lambda^{\boldsymbol{\pi}}$ we have

$$\lim_\Lambda \frac{\partial y_\Lambda^{\boldsymbol{f}}}{\partial \lambda} = \frac{\partial y_\Lambda}{\partial \lambda} = \lim_\Lambda \frac{\partial y_\Lambda^{\boldsymbol{w}}}{\partial \lambda}$$

which implies that the probability of opening a given edge is the same under both free and wired boundary conditions: $\mu_{\boldsymbol{f}}^{\mathrm{RC}}(\eta_{ij}) = \mu_{\boldsymbol{w}}^{\mathrm{RC}}(\eta_{ij})$. Because of the stochastic domination $\mu_{\boldsymbol{f}}^{\mathrm{RC}} \underset{\text{stoch}}{\leqslant} \mu_{\boldsymbol{w}}^{\mathrm{RC}}$, the conclusion $\mu_{\boldsymbol{f}}^{\mathrm{RC}} = \mu_{\boldsymbol{w}}^{\mathrm{RC}}$ follows. □

On the other hand we introduce the magnetization

$$m_{\boldsymbol{w}} = \mu_{\boldsymbol{w}}^{\mathrm{RC}} \left(\frac{1}{1 + (q-1)e^{-hS_i^{\boldsymbol{\eta}}}} \right) - \frac{1}{q} \tag{18}$$

under the infinite volume measure $\mu_{\boldsymbol{w}}^{\mathrm{RC}}$ with wired boundary condition, for any $h > 0$, where $S_i^{\boldsymbol{\eta}}$ is the number of sites of the cluster of $\boldsymbol{\eta}$ that contains i. We let $m_{\boldsymbol{f}}$ the same quantity under $\mu_{\boldsymbol{f}}^{\mathrm{RC}}$. We consider also $e_{\boldsymbol{f}}$, the mean energy as in (7) and $\theta_{\boldsymbol{f}} = \theta$ (see (4)) the density of the percolating cluster under the measure $\mu_{\boldsymbol{f}}^{\mathrm{RC}}$, and call $e_{\boldsymbol{w}}$ and $\theta_{\boldsymbol{w}}$ the corresponding quantities under $\mu_{\boldsymbol{w}}^{\mathrm{RC}}$. We can write $e_{\boldsymbol{f}}$ and $m_{\boldsymbol{f}}$ as increasing limits and $e_{\boldsymbol{w}}, m_{\boldsymbol{w}}$ and $\theta_{\boldsymbol{w}}$ as decreasing limits of continuous, increasing functions of β. For instance,

$$e_{\boldsymbol{w}} = \frac{1}{p} \lim_{\Lambda \nearrow \mathbb{Z}^d} \mu_{\Lambda,\boldsymbol{w}}^{\mathrm{RC}}(\eta_{ij})$$

and

$$\theta_{\boldsymbol{w}} = \lim_{\Delta \nearrow \mathbb{Z}^d} \lim_{\Lambda \nearrow \mathbb{Z}^d} \mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{w}}(0 \overset{\boldsymbol{\eta}}{\leftrightarrow} \partial\Delta)$$

are decreasing limits while the functions $\beta \mapsto \mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{w}}(\eta_{ij})$ and $\beta \mapsto \mu^{\mathrm{RC}}_{\Lambda,\boldsymbol{w}}(0 \overset{\boldsymbol{\eta}}{\leftrightarrow} \partial\Delta)$ are continuous, increasing. Hence:

Lemma 2 *For any $h \geqslant 0$ and $q \geqslant 1$, $e_{\boldsymbol{f}}$ and $m_{\boldsymbol{f}}$ are left-continuous functions of β, while $e_{\boldsymbol{w}}, m_{\boldsymbol{w}}$ and $\theta_{\boldsymbol{w}}$ are right-continuous.*

As a consequence of Lemma 1 the equalities $e_{\boldsymbol{f}} = e_{\boldsymbol{w}}$, $m_{\boldsymbol{f}} = m_{\boldsymbol{w}}$ and $\theta_{\boldsymbol{f}} = \theta_{\boldsymbol{w}}$ hold true for all but countably many β. In view of Lemma 2, the energy (resp. the magnetization) is continuous at some β if and only if it has the same value under both $\mu^{\mathrm{RC}}_{\boldsymbol{f}}$ and $\mu^{\mathrm{RC}}_{\boldsymbol{w}}$. Hence, at a point of discontinuity it is the case that $\mu^{\mathrm{RC}}_{\boldsymbol{f}} \neq \mu^{\mathrm{RC}}_{\boldsymbol{w}}$. But at such points we cannot have $\theta_{\boldsymbol{f}} = \theta_{\boldsymbol{w}}$ in view of Theorem 4 below and Theorem 2 follows.

Theorem 4 *The equality $\theta_{\boldsymbol{f}} = \theta_{\boldsymbol{w}}$ implies the uniqueness of random cluster measures – in other words, $\mu^{\mathrm{RC}}_{\boldsymbol{f}} = \mu^{\mathrm{RC}}_{\boldsymbol{w}}$ when $\theta_{\boldsymbol{f}} = \theta_{\boldsymbol{w}}$.*

Theorem 4 was proven in [12] in the case of $h = 0$ (Theorem 5.2 in [12] ; see also Theorem 5.16 in [10] for the complete construction). The proof given in [10] applies verbatim in the present setting.

The reader might be interested as well in a simpler proof of the fact that $\theta_{\boldsymbol{w}} = 0$ implies the uniqueness of random cluster measures (Theorem A.2 in [14]) which shows as well that the Kertész line remains below the line of discontinuous phase transition.

A.4 An upper bound on the Kertész line

Our upper bound is based directly on the conditional probabilities (14) and (15). Since

$$\inf_{\boldsymbol{\pi}} \mu^{\mathrm{RC}}_{\{ij\},\boldsymbol{\pi}}(\boldsymbol{\eta}_{ij} = 1) = \mu^{\mathrm{RC}}_{\{ij\},\boldsymbol{f}}(\boldsymbol{\eta}_{ij} = 1) = \tilde{p} \overset{\mathrm{def.}}{=} \frac{p}{p + (1-p)w(1)^2/w(2)},$$

the measure $\mu^{\mathrm{RC}}_{\boldsymbol{f}}$ stochastically dominates independent bond percolation of parameter $\tilde{p}$, and $\tilde{p} > p_c$ ensures that percolation occurs, i.e. that $\theta > 0$. We recall the notation $\beta_p = -\ln(1-p_c)$, which yields

$$\begin{aligned} \theta > 0 \quad &\Leftarrow \quad \tilde{p} > p_c \\ &\Leftrightarrow \quad \frac{\left(1 + (q-1)e^{-h}\right)^2}{1 + (q-1)e^{-2h}} < \frac{e^{\beta} - 1}{e^{\beta_p} - 1} \\ &\Leftarrow \quad \left(1 + (q-1)e^{-h}\right)^2 < \frac{e^{\beta} - 1}{e^{\beta_p} - 1} \end{aligned} \tag{19}$$

and the upper bound (8) follows.

A.5 A lower bound on the Kertész line

The former method yields here the only information that $h_K(\beta) = +\infty$ for all $\beta \leqslant \beta_p$. Hence we consider another point of view : we use a joint measure analogous to $\mu_\Lambda^{\mathrm{ES}}$ (2) and compare the spins which are not of color 1 to *random defects*, which have a vanishing density in the limit $h \to \infty$.

As we aim at a lower bound that holds for non-integer $q \geqslant 1$, we consider a modified (monochrome) version of $\mu_\Lambda^{\mathrm{ES}}$ that gives only two colors to spin configurations $\boldsymbol{s}$. The color 1 plays effectively the role of a color in the Potts model, and undergoes the external field. The color 0 condensates all $q-1$ other colors (in the case of integer q). Let

$$\mu_\Lambda^M(\boldsymbol{s}, \boldsymbol{\eta}) = \frac{1}{Z_\Lambda^M} \omega(\boldsymbol{s}, \boldsymbol{\eta}) \tag{20}$$

where

$$\omega(\boldsymbol{s}, \boldsymbol{\eta}) = \prod_{\langle i,j \rangle} \left(e^{-\beta} \delta_{\eta_{ij},0} + (1 - e^{-\beta}) \delta_{\eta_{ij},1} \delta_{s_i,s_j} \right) \times \prod_i e^{h \delta_{s_i,1}} \times (q-1)^{N_0(\boldsymbol{s},\boldsymbol{\eta})}, \tag{21}$$

and $N_0(\boldsymbol{s}, \boldsymbol{\eta})$ is the number of clusters of $\boldsymbol{\eta}$ that have spin $s = 0$. The marginal law of $\boldsymbol{\eta}$ equals $\mu_{\Lambda,\boldsymbol{f}}^{\mathrm{RC}}$, while the conditional law of $\boldsymbol{\eta}$ knowing $\boldsymbol{s}$ is the following: $\boldsymbol{\eta}$ has all edges closed between regions of $\boldsymbol{s}$ of different colors, while its restriction to the regions with $s = 1$ follows a bond percolation process of parameter $p = 1 - e^{-\beta}$, and its restriction to the regions with $s = 0$ follows the usual random cluster measure of parameters $p = 1 - e^{-\beta}$ and $q' = q - 1$ with free boundary conditions.

We compare now the structure of spins of color 0 to independent site percolation of low density. Let $\bar{\boldsymbol{s}}$ a spin configuration with $\bar{s}_i = 1$, and call $\tilde{\boldsymbol{s}}$ the modified configuration with $\tilde{\boldsymbol{s}}_i = 0$. For any $\boldsymbol{\eta}$ such that $\eta_{ij} = 0$ for all j adjacent to i, one has

$$\omega(\tilde{\boldsymbol{s}}, \boldsymbol{\eta}) = \omega(\bar{\boldsymbol{s}}, \boldsymbol{\eta})(q-1)e^{-h}. \tag{22}$$

Therefore,

$$\begin{aligned} \frac{\mu_\Lambda^M(\tilde{\boldsymbol{s}})}{\mu_\Lambda^M(\bar{\boldsymbol{s}})} &\geqslant (q-1)e^{-h} \frac{\sum_{\boldsymbol{\eta}:\eta_{ij}=0,\forall j \sim i} \omega(\bar{\boldsymbol{s}}, \boldsymbol{\eta})}{\sum_{\boldsymbol{\eta}} \omega(\bar{\boldsymbol{s}}, \boldsymbol{\eta})} \\ &= (q-1)e^{-h} \times \mu_\Lambda^M \left(\eta_{ij} = 0, \text{ for all } j \text{ adjacent to } i | \boldsymbol{s} = \bar{\boldsymbol{s}} \right). \end{aligned} \tag{23}$$

But the latter probability is at least $e^{-2\beta d}$ and (23) implies that

$$\sup_{\bar{\boldsymbol{s}}} \mu \left(s_i = 1 | s_j = \bar{s}_j, \forall i \neq j \right) \leqslant \rho \stackrel{\mathrm{def.}}{=} \frac{1}{1 + (q-1)e^{-2\beta d} e^{-h}}. \tag{24}$$

Hence we have a lower bound on the density of defects : the process of good sites ($s_i = 1$) is stochastically dominated by site percolation of parameter ρ, and percolation cannot occur (i.e. $\theta = 0$) if the mixed percolation process [8] of

site density ρ and edge density $p = 1 - e^{-\beta}$ does not percolate, that is, if there is no infinite cluster after the removal of closed sites and closed bonds.

The order in which sites and bonds are close does not modify the properties of the mixed percolation process. Here we shall consider that the edge percolation at density p is done first, giving the diluted graph G made of the open edges and their vertices, and that the site percolation of parameter ρ is realized afterwards. It has been known for a long time that bond percolation of parameter ρ on G is more likely to succeed than site percolation (see [15, 16] for inductive proofs and [17], proof of Lemma 5 for a *dynamical coupling*). But the process of bond percolation with intensity ρ on the diluted graph G boils down to the classical bond percolation on $\mathbb{Z}^d$ with parameter $p \times \rho$ and we have shown that

$$\begin{aligned} \theta = 0 \quad &\Leftarrow \quad p \times \rho < p_c \\ &\Leftrightarrow \quad e^{-\beta_p} - e^{-\beta} < p_c(q-1)e^{-2\beta d}e^{-h} \end{aligned} \tag{25}$$

which leads to the lower bound (10).

References

[1] C. M. Fortuin and P. W. Kasteleyn. On the random cluster model I: Introduction and relation to other models. *Physica*, **57**:536, 1972.

[2] A. Coniglio and W. Klein. Clusters and Ising critical droplets: a renormalisation group approach. *J. Phys. A: Math. Gen.*, **13**:2775, 1980.

[3] J. Kertész. Exitence of weak singularities when going around the liquid–gas critical point. *Physica A*, **161**:58, 1989.

[4] Ph. Blanchard, D. Gandolfo, L. Laanait, J. Ruiz, and H. Satz. On the Kertész line: Thermodynamic versus geometric criticality. *arXiv 0711.1148, to appear in J. Phys. A: Math. Gen.*, 2008.

[5] R.G. Edwards and A.D. Sokal. Generalization of the Fortuin–Kasteleyn–Swendsen–Yang representation and Monte Carlo algorithm. *Phys. Rev. D*, **38**:2009–2012, 1988.

[6] Ph. Blanchard, D. Gandolfo, J. Ruiz, and M. Wouts. Thermodynamic versus topological phase transitions: Cusp in the Kertész line. 2008.

[7] M. Aizenman, J. T. Chayes, L. Chayes, and C. M. Newman. The phase boundary in dilute and random Ising and Potts ferromagnets. *J. Phys. A*, 20(5):L313–L318, 1987.

[8] L. Chayes and R. H. Schonmann. Mixed percolation as a bridge between site and bond percolation. *Ann. Appl. Probab.*, 10(4):1182–1196, 2000.

[9] H.-O. Georgii, O. Häggström, and C. Maes. The random geometry of equilibrium phases. In *Phase transitions and critical phenomena*, volume 18, pages 1–142. Academic Press, San Diego, CA, 2001.

[10] G. R. Grimmett. *The random-cluster model*, volume 333 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 2006.

[11] J. L. Lebowitz. Coexistence of phases in Ising ferromagnets. *J. Stat. Phys.*, 16(6):463–476, 1977.

[12] G. R. Grimmett. The stochastic random-cluster process and the uniqueness of random-cluster measures. *Ann. Probab.*, 23(4):1461–1510, 1995.

[13] M. Wouts. A coarse graining for the Fortuin-Kasteleyn measure in random media. *Stochastic Process. Appl.*, 2007.

[14] M. Aizenman, J. T. Chayes, L. Chayes, and C. M. Newman. Discontinuity of the magnetization in one-dimensional $1/|x-y|^2$ Ising and Potts models. *J. Stat. Phys.*, 50(1-2):1–40, 1988.

[15] J. M. Hammersley. Comparison of atom and bond percolation processes. *J. Math. Phys.*, 2(5):728–733, 1961.

[16] J. G. Oxley and D. J. A. Welsh. On some percolation results of J. M. Hammersley. *J. Appl. Probab.*, 16(3):526–540, 1979.

[17] G. R. Grimmett and A. M. Stacey. Critical probabilities for site and bond percolation models. *Ann. Probab.*, 26(4):1788–1812, 1998.