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# On Jacobi Sums in $\mathbb{Q}\left(\zeta_{p}\right)$ 

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#### Abstract

We study the $p$-adic behavior of Jacobi sums for $\mathbb{Q}\left(\zeta_{p}\right)$ and link this study to the $p$-Sylow subgroup of the class group of $\mathbb{Q}\left(\zeta_{p}\right)^{+}$and to some properties of the jacobian of the Fermat curve $X^{p}+Y^{p}=1$ over $\mathbb{F}_{\ell}$ where $\ell$ is a prime number distinct from $p$.


Let $p$ be a prime number, $p \geq 5$. Iwasawa has shown that the $p$-adic properties of Jacobi sums for $\mathbb{Q}\left(\zeta_{p}\right)$ are linked to Vandiver's Conjecture (see [5]). In this paper, we follow Iwasawa's ideas and study the $p$-adic properties of the subgroup $J$ of $\mathbb{Q}\left(\zeta_{p}\right)^{*}$ generated by Jacobi sums.

Let $A$ be the $p$-Sylow subgroup of the class group of $\mathbb{Q}\left(\zeta_{p}\right)$. If $E$ denotes the group of units of $\mathbb{Q}\left(\zeta_{p}\right)$, then if Vandiver's Conjecture is true for $p$, by Kummer Theory, we must have $\frac{A^{-}}{p A^{-}} \hookrightarrow \operatorname{Gal}\left(\mathbb{Q}\left(\zeta_{p}\right)\left({ }^{p} \sqrt{E}\right) / \mathbb{Q}\left(\zeta_{p}\right)\right)$. Note that $J$ is analoguous for the odd part to the group of cyclotomic units for the even part. We introduce a submodule $W$ of $\mathbb{Q}\left(\zeta_{p}\right)^{*}$ which was already considered
by Iwasawa ([6]). This module can be thought as the analogue for the odd part of the group of units for the even part. We observe that $J \subset W$ and if the Iwasawa-Leopoldt Conjecture is true for $p$ then $W\left(Q\left(\zeta_{p}\right)^{*}\right)^{p}=$ $J\left(Q\left(\zeta_{p}\right)^{*}\right)^{p}$. We prove that if $p A^{-}=\{0\}$ then (Corollary 4.8):

$$
\frac{A^{+}}{p A^{+}} \hookrightarrow \operatorname{Gal}\left(\mathbb{Q}\left(\zeta_{p}\right)(\sqrt[p]{W}) / \mathbb{Q}\left(\zeta_{p}\right)\right)
$$

The last part of our paper is devoted to the study of the jacobian of the Fermat curve $X^{p}+Y^{p}=1$ over $\mathbb{F}_{\ell}$ where $\ell$ is a prime number, $\ell \neq p$. It is well-known that Jacobi sums play an important role in the study of that jacobian. Following ideas developped by Greenberg (困), we prove that Vandiver's Conjecture is equivalent to some properties of that jacobian (for the precise statement see Corollary (5.3).

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## 1 Notations

Let $p$ be a prime number, $p \geq 5$. Let $\zeta_{p} \in \mu_{p} \backslash\{1\}$, and let $L=\mathbb{Q}\left(\zeta_{p}\right)$. Set $\mathcal{O}=\mathbb{Z}\left[\zeta_{p}\right]$ and $E=\mathcal{O}^{*}$. Let $\Delta=\operatorname{Gal}(L / \mathbb{Q})$ and let $\widehat{\Delta}=\operatorname{Hom}\left(\Delta, \mathbb{Z}_{p}^{*}\right)$. Let $\mathcal{I}$ be the group of fractional ideals of $L$ which are prime to $p$, and let $\mathcal{P}$ be the group of principal ideals in $\mathcal{I}$. Let $A$ be the $p$-Sylow subgroup of the ideal class group of $L$.

Set $\pi=\zeta_{p}-1, K=\mathbb{Q}_{p}\left(\zeta_{p}\right), U=1+\pi^{2} \mathbb{Z}_{p}\left[\zeta_{p}\right]$. Observe that if $\mathcal{A} \in \mathcal{P}$, then there exists $\alpha \in L^{*} \cap U$ such that $\mathcal{A}=\alpha \mathcal{O}$. If $H$ is a subgroup of $U$, we will denote the closure of $H$ in $U$ by $\bar{H}$. Let $\omega \in \widehat{\Delta}$ be the Teichmüller character, i.e. :

$$
\forall \sigma \in \Delta, \sigma\left(\zeta_{p}\right)=\zeta_{p}^{\omega(\sigma)}
$$

For $\rho \in \widehat{\Delta}$, we set:

$$
e_{\rho}=\frac{1}{p-1} \sum_{\delta \in \Delta} \rho^{-1}(\delta) \delta \in \mathbb{Z}_{p}[\Delta] .
$$

If $M$ is a $\mathbb{Z}_{p}[\Delta]$-module, for $\rho \in \widehat{\Delta}$, we set:

$$
M(\rho)=e_{\rho} M
$$

For $\psi \in \widehat{\Delta}, \psi$ odd, recall that:

$$
B_{1, \psi}=\frac{1}{p} \sum_{a=1}^{p-1} a \psi(a) .
$$

Set:

$$
\theta=\frac{1}{p} \sum_{a=1}^{p-1} a \sigma_{a}^{-1} \in \mathbb{Q}[\Delta]
$$

where $\sigma_{a} \in \Delta$ is such that $\sigma_{a}\left(\zeta_{p}\right)=\zeta_{p}^{a}$. Observe that we have the following equality in $\mathbb{C}[\Delta]$ :

$$
\theta=\frac{N}{2}+\sum_{\psi \in \widehat{\Delta}, \psi \text { odd }} B_{1, \psi^{-1}} e_{\psi},
$$

where $N=\sum_{\delta \in \Delta} \delta$.
Let $M$ be a $\mathbb{Z}[\Delta]$-module, we set:

$$
\begin{gathered}
M^{-}=\left\{m \in M, \sigma_{-1}(m)=-m\right\}, \\
M^{+}=\left\{m \in M, \sigma_{-1}(m)=m\right\} .
\end{gathered}
$$

If $M$ is an abelian group of finite type, we set:

$$
\begin{gathered}
M[p]=\{m \in M, p m=0\}, \\
d_{p} M=\operatorname{dim}_{\mathbb{F}_{p}} \frac{M}{p M}
\end{gathered}
$$

## 2 Background on Jacobi Sums

Let $C l(L)$ be the ideal class group of $L$, then:

$$
C l(L) \simeq \frac{\mathcal{I}}{\mathcal{P}}
$$

Note that we have a natural $\mathbb{Z}[\Delta]$-morphism (see [6], pages 102-103):

$$
\phi:\left(\operatorname{Ann}_{\mathbb{Z}[\Delta]} C l(L)\right)^{-} \rightarrow \operatorname{Hom}_{\mathbb{Z}[\Delta]}\left(C l(L), \frac{E^{+}}{\left(E^{+}\right)^{2}}\right) .
$$

For the convenience of the reader, we recall the construction of $\phi$. Let $x \in$ $\left(\mathrm{Ann}_{\mathbb{Z}[\Delta]} C l(L)\right)^{-}$. Let $\mathcal{A} \in \mathcal{I}$, we have:

$$
\mathcal{A}^{x}=\gamma_{a} \mathcal{O},
$$

where $\gamma_{a} \in L^{*} \cap U$. Now:

$$
\overline{\gamma_{a}}=\varepsilon_{a} \gamma_{a}^{-1}
$$

for some $\varepsilon_{a} \in E^{+} \cap U$. One can prove that we obtain a well-defined morphism of $\mathbb{Z}[\Delta]$-modules: $\phi(x): C l(L) \rightarrow \frac{E^{+}}{\left(E^{+}\right)^{2}}$, class of $\mathcal{A} \mapsto$ class of $\varepsilon_{a}$. In this paragraph, we will study the kernel of the morphism $\phi$.

Let $\mathcal{W}$ be the set of elements $f \in \operatorname{Hom}_{\mathbb{Z}[\Delta]}\left(\mathcal{I}, L^{*}\right)$ such that: - $f(\mathcal{I}) \subset U$,

- there exists $\beta(f) \in \mathbb{Z}[\Delta]$ such that for all $\alpha \in L^{*} \cap U, f(\alpha \mathcal{O})=\alpha^{\beta(f)}$.

One can prove that if $f \in \mathcal{W}$ then $\beta(f)$ is unique, the map $\beta: \mathcal{W} \rightarrow \mathbb{Z}[\Delta]$ is an injective $\mathbb{Z}[\Delta]$-morphism and $\beta(\mathcal{W}) \subset \operatorname{Ann}_{\mathbb{Z}[\Delta]}(C l(L))$ (see [2]). If $\mathcal{B}$ denotes the group of Hecke characters of type $\left(A_{0}\right)$ that have values in $\mathbb{Q}\left(\zeta_{p}\right)$ (see [G]), then one can prove that $\mathcal{B}$ is isomorphic to $\mathcal{W}$.

Lemma 2.1 $\operatorname{Ker} \phi=\beta\left(\mathcal{W}^{-}\right)$.
Proof We just prove the inclusion $\operatorname{Ker} \phi \subset \beta\left(\mathcal{W}^{-}\right)$. Let $x \in \operatorname{Ker} \phi$. Let $\mathcal{A} \in \mathcal{I}$, then there exists an unique $\gamma_{a} \in L^{*} \cap U$ such that $\overline{\gamma_{a}} \gamma_{a}=1$ and:

$$
\mathcal{A}^{x}=\gamma_{a} \mathcal{O}
$$

Let $f: \mathcal{I} \rightarrow L^{*}, \mathcal{A} \mapsto \gamma_{a}$. It is not difficult to see that $f \in \operatorname{Hom}_{\mathbb{Z}[\Delta]}\left(\mathcal{I}, L^{*}\right)$ and $f(\mathcal{I}) \subset U$. Now, let $\alpha \in L^{*} \cap U$, we have:

$$
f(\alpha \mathcal{O})=\alpha^{x} u
$$

for some $u \in E$. Since $x \in \mathbb{Z}[\Delta]^{-}$and $\alpha, f(\alpha \mathcal{O}) \in U$, we must have $u=1$. Threfore $f \in \mathcal{W}^{-}$and $x=\beta(f) . \diamond$

Now, we recall some basic properties of Gauss and Jacobi sums (we refer the reader to [11], paragraph 6.1).

Let $P$ be a prime ideal in $\mathcal{I}$ and let $\ell$ be the prime number such that $\ell \in P$. We fix $\zeta_{\ell} \in \mu_{\ell} \backslash\{1\}$. Set $\mathbb{F}_{P}=\frac{\mathcal{O}}{P}$. Let $\chi_{P}: \mathbb{F}_{P}^{*} \rightarrow \mu_{p}$, such that:

$$
\forall \alpha \in \mathbb{F}_{P}^{*}, \chi_{P}(\alpha) \equiv \alpha^{\frac{1-N P}{p}} \quad(\bmod P)
$$

where $N P=\left|\frac{\mathcal{O}}{P}\right|$. For $a \in \frac{\mathbb{Z}}{p \mathbb{Z}}$, we set:

$$
\tau_{a}(P)=-\sum_{\alpha \in \mathbb{F}_{P}} \chi_{P}^{a}(\alpha) \zeta_{\ell}^{\operatorname{Tr}_{\mathbb{F}_{P} / \mathbb{F}_{\ell}}(\alpha)}
$$

We also set $\tau(P)=\tau_{1}(P)$. For $a, b \in \frac{\mathbb{Z}}{p \mathbb{Z}}$, we set:

$$
j_{a, b}(P)=-\sum_{\alpha \in \mathbb{F}_{P}} \chi_{P}^{a}(\alpha) \chi_{P}^{b}(1-\alpha) .
$$

Then:

- if $a+b \equiv 0 \quad(\bmod p)$,
i) if $a \not \equiv 0(\bmod p), j_{a, b}(P)=1$,
ii) if $a \equiv 0(\bmod p), j_{a, b}(P)=2-N P$,
- if $a+b \not \equiv 0 \quad(\bmod p)$, we have:

$$
j_{a, b}(P)=\frac{\tau_{a}(P) \tau_{b}(P)}{\tau_{a+b}(P)}
$$

Observe that $\tau(P) \equiv 1 \quad(\bmod \pi)$, and therefore (see [5], Theorem 1$)$ :

$$
\forall a, b \in \frac{\mathbb{Z}}{p \mathbb{Z}}, j_{a, b}(P) \in U
$$

Let $\Omega$ be the compositum of the fields $\mathbb{Q}\left(\zeta_{\ell}\right)$ where $\ell$ runs through the prime numbers distinct from $p$. The map $P \mapsto \tau(P)$ induces by linearity a $\mathbb{Z}[\Delta]$ morphism:

$$
\tau: \mathcal{I} \rightarrow \Omega\left(\zeta_{p}\right)^{*}
$$

Let $\mathcal{G}$ be the sub- $\mathbb{Z}[\Delta]$-module of $\operatorname{Hom}_{\mathbb{Z}[\Delta]}\left(\mathcal{I}, \Omega\left(\zeta_{p}\right)^{*}\right)$ generated by $\tau$. We set:

$$
\mathcal{J}=\mathcal{G} \cap \operatorname{Hom}_{\mathbb{Z}[\Delta]}\left(\mathcal{I}, L^{*}\right)
$$

Let $\mathcal{S}$ be the Stickelberger ideal of $L$, i.e. : $\mathcal{S}=\mathbb{Z}[\Delta] \theta \cap \mathbb{Z}[\Delta]$. Then one can prove the following facts (see [2]):
$-\mathcal{J} \subset \mathcal{W}$,

- the map $\beta: \mathcal{W} \rightarrow \mathbb{Z}[\Delta]$ induces an isomorphism of $\mathbb{Z}[\Delta]$-modules :

$$
\mathcal{J} \simeq \mathcal{S}
$$

Lemma 2.2 Let $\mathcal{N} \in \operatorname{Hom}_{\mathbb{Z}[\Delta]}\left(I_{L}, L^{*}\right)$ be the ideal norm map. Then, as a $\mathbb{Z}$-module:

$$
\mathcal{J}=\mathcal{N} \mathbb{Z} \oplus \bigoplus_{n=1}^{(p-1) / 2} j_{1, n} \mathbb{Z}
$$

Proof Recall that, for $1 \leq n \leq p-2$, for a prime $P$ in $\mathcal{I}$, we have:

$$
j_{1, n}(P)=-\sum_{\alpha \in \mathbb{F}_{P}} \chi_{P}(\alpha) \chi_{P}^{n}(1-\alpha)=\frac{\tau(P) \tau_{n}(P)}{\tau_{n+1}(P)}
$$

Thus, for $1 \leq n \leq p-2$, we have:

$$
j_{1, n}=\tau^{1+\sigma_{n}-\sigma_{1+n}}=\frac{\tau \tau_{n}}{\tau_{n+1}}
$$

where for $a \in \mathbb{F}_{p}^{*}, \tau^{\sigma_{a}}=\tau_{a}$. Observe that:

$$
\forall a \in \mathbb{F}_{p}^{*}, \tau_{a} \tau_{-a}=N
$$

Thus $\mathcal{N} \in \mathcal{J}$. Since $\mathcal{J} \simeq \mathcal{S}, \mathcal{J}$ is a $\mathbb{Z}$-module of $\operatorname{rank}(p+1) / 2$. It is not difficult to show that (see [5], Lemma 2):

$$
\mathcal{J}=\tau^{p} \mathbb{Z} \bigoplus_{a=1}^{(p-1) / 2} \tau_{-a} \tau^{a} \mathbb{Z}
$$

Observe also that, for $2 \leq n \leq p-2$, we have:

$$
j_{1, p-n}=j_{1, n-1} .
$$

Let $V$ be the sub-Z-module of $\mathcal{J}$ generated by $\mathcal{N}$ and the $j_{1, n}, 1 \leq n \leq$ $(p-1) / 2$. Then for $1 \leq n \leq p-2, j_{1, n} \in V$. Furthermore:

$$
\prod_{n=1}^{p-2} j_{1, n}=\frac{\tau^{p}}{\mathcal{N}}
$$

Therefore $\tau^{p} \in V$. Since $\tau_{-1} \tau^{1}=\mathcal{N}, \tau_{-1} \tau^{1} \in V$. Now, let $2 \leq r \leq(p-1) / 2$ and assume that we have proved that $\tau_{-(r-1)} \tau^{r-1} \in V$. We have:

$$
j_{1, r-1}=\frac{\tau \tau_{r-1}}{\tau_{r}}=\frac{\mathcal{N} \tau \tau_{1-r}^{-1}}{\mathcal{N} \tau_{-r}^{-1}}
$$

Thus:

$$
\tau_{-r}=j_{1, r-1}^{-1} \tau_{1-r} \tau^{-1}
$$

and

$$
\tau_{-r} \tau^{r}=j_{1, r-1}^{-1} \tau_{-(r-1)} \tau^{r-1}
$$

Thus $\tau_{-r} \tau^{r} \in V$ and the Lemma follows. $\diamond$
Lemma 2.3 Let $\ell$ be a prime number, $\ell \neq p$. Let $P$ be a prime ideal of $\mathcal{O}$ above $\ell$ and let $a \in\{1, \cdots, p-2\}$. Then $\mathbb{Q}\left(j_{1, a}(P)\right)=L$ if and only if $\ell \equiv 1$ $(\bmod p)$ and $a^{2}+a+1 \not \equiv 0(\bmod p)$ if $p \equiv 1(\bmod 3)$.

Proof Since $j_{1, a}(P) \equiv 1 \quad\left(\bmod \pi^{2}\right)$ and $j_{1, a}(P) j_{1, a}(P)^{\sigma_{-1}}=\ell^{f}$ where $f$ is the order of $\ell$ in $(\mathbb{Z} / p \mathbb{Z})^{*}$, we have:

$$
\forall \sigma \in \Delta, j_{1, a}(P)^{\sigma}=j_{1, a}(P) \Leftrightarrow j_{1, a}(P)^{\sigma} \mathcal{O}=j_{1, a}(P) \mathcal{O}
$$

Recall that:

$$
\forall \sigma \in \Delta, j_{1, a}(P)^{\sigma} \mathcal{O}=j_{1, a}(P) \mathcal{O} \Leftrightarrow P^{(\sigma-1)\left(1+\sigma_{a}-\sigma_{1+a}\right) \theta}=\mathcal{O}
$$

Since $j_{1, a}(P)^{\sigma_{\ell}}=j_{1, a}(P)$, we can assume $\ell \equiv 1 \quad(\bmod p)$. Let $\sigma \in \Delta$, we have to consider the following equation in $\mathbb{C}[\Delta]$ :

$$
(\sigma-1)\left(1+\sigma_{a}-\sigma_{1+a}\right) \theta=0 .
$$

This is equivalent to:

$$
\forall \psi \in \widehat{\Delta}, \psi \text { odd, }(\psi(\sigma)-1)(1+\psi(a)-\psi(1+a))=0
$$

Assume that $\omega^{3}(\sigma) \neq 1$. Then:

$$
1+\omega^{3}(a)-\omega^{3}(1+a)=0
$$

This implies:

$$
a^{2}+a \equiv 0 \quad(\bmod p),
$$

which is a contradiction. Thus $\omega^{3}(\sigma)=1$. Let's suppose that $\sigma \neq 1$. We get:

$$
1+\omega(a)=\omega(1+a),
$$

which is equivalent to:

$$
a^{2}+a+1 \equiv 0 \quad(\bmod p) .
$$

Conversely, one can see that if $p \equiv 1(\bmod 3), a^{2}+a+1 \equiv 0(\bmod p)$, $\omega^{3}(\sigma)=1$, then:

$$
\forall \psi \in \widehat{\Delta}, \psi \operatorname{odd},(\psi(\sigma)-1)(1+\psi(a)-\psi(1+a))=0
$$

The Lemma follows. $\diamond$
For $x \in \mathbb{Z}_{p}$, let $[x] \in\{0, \cdots, p-1\}$ such that $x \equiv[x] \quad(\bmod p)$. We set:

$$
\eta=\left(\prod_{n=1}^{p-2} j_{1, n}^{\left[n^{-1}\right]}\right)^{1-\sigma_{-1}} \in \mathcal{J}^{-} .
$$

## Lemma 2.4

a) Let $\psi \in \widehat{\Delta}, \psi \neq \omega, \psi$ odd. Then:

$$
e_{\psi}\left(\sum_{n=1}^{p-2}\left(1+\sigma_{n}-\sigma_{1+n}\right)\left[n^{-1}\right]\right) \in \mathbb{Z}_{p}^{*} e_{\psi}
$$

b) We have:

$$
\frac{1}{p} e_{\omega}\left(\sum_{n=1}^{p-2}\left(1+\sigma_{n}-\sigma_{1+n}\right)\left[n^{-1}\right]\right) \in \mathbb{Z}_{p}^{*} e_{\omega} .
$$

Proof
a) Write $\psi=\omega^{k}, k$ odd, $k \in\{3, \cdots, p-2\}$. We have:

$$
\sum_{n=2}^{p-2}(1+\psi(n)-\psi(1+n))\left[n^{-1}\right] \equiv \sum_{n=1}^{p-1} \frac{1+n^{k}-(1+n)^{k}}{n} \equiv k \quad(\bmod p)
$$

This implies a).
b) We have:

$$
\forall a \in \mathbb{F}_{p}^{*}, \omega(a) \equiv a^{p} \quad\left(\bmod p^{2}\right)
$$

Thus:

$$
\frac{1}{p} \sum_{n=1}^{p-2}(1+\omega(n)-\omega(1+n))\left[n^{-1}\right] \equiv-\sum_{n=1}^{p-1} \sum_{k=1}^{p-1} \frac{p!}{(p-k)!k!p} n^{k-1} \quad(\bmod p)
$$

And we get:

$$
\frac{1}{p} \sum_{n=1}^{p-2}(1+\omega(n)-\omega(1+n))\left[n^{-1}\right] \equiv-1 \quad(\bmod p)
$$

This implies b). $\diamond$
Lemma 2.5 Let $\ell$ be a prime number, $\ell \neq p$. Let $V_{\ell}$ be the sub- $\mathbb{Z}[\Delta]$-module of $L^{*} /\left(L^{*}\right)^{p}$ generated by $\{f(P), f \in \mathcal{J}\}$ where $P$ is some prime of $\mathcal{I}$ above $\ell$. Let $\psi \in \widehat{\Delta}, \psi$ odd and $\psi \neq \omega$. Then:

$$
V_{\ell}(\psi)=\mathbb{F}_{p} e_{\psi} \eta(P) .
$$

Proof Let $E=L\left(\zeta_{\ell}\right)$. Then:

$$
\frac{L^{*}}{\left(L^{*}\right)^{p}}(\psi) \hookrightarrow \frac{E^{*}}{\left(E^{*}\right)^{p}}(\psi) .
$$

Now, in $\frac{E^{*}}{\left(E^{*}\right)^{p}}(\psi)$, we have:

$$
V_{\ell}(\psi)=\mathbb{F}_{p} e_{\psi} \tau(P) .
$$

It remains to apply Lemma 2.4. $\diamond$
Finally, we mention the following Lemma:
Lemma 2.6 We have:

$$
\left(\mathcal{J}^{-}: \mathbb{Z}[\Delta] \eta\right)=2^{\frac{p-3}{2}} \frac{1}{p} \prod_{\psi \in \widehat{\Delta}, \psi \text { odd }}\left(\sum_{n=1}^{p-2}(1+\psi(n)-\psi(1+n))\left[n^{-1}\right]\right)
$$

Furthermore:

$$
\left(\mathcal{J}^{-}: \mathbb{Z}[\Delta] \eta\right) \not \equiv 0 \quad(\bmod p) .
$$

Proof Set $\widetilde{\mathcal{J}}^{-}=\left(1-\sigma_{-1}\right) \mathcal{J} \subset \mathcal{J}^{-}$. Ten (see 11], paragraph 6.4):

$$
\left(\mathcal{J}^{-}: \widetilde{\mathcal{J}}^{-}\right)=2^{\frac{p-3}{2}}
$$

Now, by the same kind of arguments as in [11], paragraph 6.4, we get:

$$
\left(\widetilde{\mathcal{J}}^{-}: \mathbb{Z}[\Delta] \eta\right)=\frac{1}{p} \prod_{\psi \in \widehat{\Delta}, \psi \text { odd }}\left(\sum_{n=1}^{p-2}(1+\psi(n)-\psi(1+n))\left[n^{-1}\right]\right) .
$$

It remains to apply Lemma 2.4. to conclude the proof of this Lemma.

## 3 Jacobi Sums and the Ideal Class Group of $\mathbb{Q}\left(\zeta_{p}\right)$

Recall that the Iwasawa-Leopoldt Conjecture asserts that $A$ is a cyclic $\mathbb{Z}_{p}[\Delta]$-module. This Conjecture is equivalent to:

$$
\forall \psi \in \widehat{\Delta}, \psi \text { odd }, \psi \neq \omega, A(\psi) \simeq \frac{\mathbb{Z}_{p}}{B_{1, \psi^{-1}} \mathbb{Z}_{p}}
$$

It is well-known (see 11, Theorem 10.9) that:

$$
\forall \psi \in \widehat{\Delta}, \psi \text { odd }, \psi \neq \omega, A\left(\omega \psi^{-1}\right)=\{0\} \Rightarrow A(\psi) \simeq \frac{\mathbb{Z}_{p}}{B_{1, \psi^{-1}} \mathbb{Z}_{p}}
$$

In this paragraph, we will study the links between Jacobi sums and the structure of $A^{-}$.

We fix $\psi \in \widehat{\Delta}, \psi$ odd and $\psi \neq \omega$. We set:

$$
m(\psi)=v_{p}\left(B_{1, \psi^{-1}}\right)
$$

Recall that, by [11], paragraph 13.6, we have:

$$
|A(\psi)|=p^{m(\psi)}
$$

Let $p^{k(\psi)}$ be the exponent of the group $A(\psi)$. Then:

$$
B_{1, \psi^{-1}} \equiv 0 \quad\left(\bmod p^{k(\psi)}\right)
$$

Lemma 3.1 Let $P$ be a prime ideal in $\mathcal{I}$ above a prime number $\ell$. Then:

$$
e_{\psi} \eta(P) \mathcal{O}=0 \text { in } \frac{\mathcal{I}}{\mathcal{I}^{p}} \Leftrightarrow \psi(\ell) \neq 1 \text { or } B_{1, \psi^{-1}} \equiv 0 \quad(\bmod p) .
$$

Proof First note that, if $\rho \in \widehat{\Delta}$, then $e_{\rho} P=0$ in $\frac{\mathcal{I}}{\mathcal{I}^{p}}$ if and only if $\rho(\ell) \neq 1$. By the Stickelberger Theorem, we have:

$$
\eta(P) \mathcal{O}=\left(\sum_{n=1}^{p-2}\left(1+\sigma_{n}-\sigma_{1+n}\right)\left[n^{-1}\right]\right)\left(1-\sigma_{-1}\right) \theta P .
$$

Recall that:

$$
e_{\psi} \theta=B_{1, \psi^{-1}} e_{\psi}
$$

The Lemma follows. $\diamond$

Lemma 3.2 Let $f \in \mathcal{W}^{-}$. Then $f$ lies in $\mathcal{W}^{p}$ if and only if for all prime ideal $P \in \mathcal{I}, f(P) \in\left(L^{*}\right)^{p}$.

Proof Let $f \in \mathcal{W}^{-}$such that for all prime ideal $P \in \mathcal{I}, f(P) \in\left(L^{*}\right)^{p}$. Let $\mathcal{A} \in \mathcal{I}$. Then there exists $\gamma_{a} \in L^{*} \cap U$ such that $\gamma_{a} \bar{\gamma}_{a}=1$ and:

$$
f(\mathcal{A})=\gamma_{a}^{p}
$$

Observe that $\beta(f) \in p(\mathbb{Z}[\Delta])^{-}$. Let $g: \mathcal{I} \rightarrow L^{*}, \mathcal{A} \mapsto \gamma_{a}$. Then one can verify that $f=g^{p}$ and $g \in \mathcal{W}^{-} . \diamond$

Let $m \geq 1$ such that $p^{m}>|A|$. Set $n=|C l(L)| /|A|$. Let $e_{m}(\psi) \in \mathbb{Z}[\bar{\Delta}]^{-}$such that:

$$
e_{m}(\psi) \equiv e_{\psi} \quad\left(\bmod p^{m}\right)
$$

Set:

$$
\beta_{\psi}=2 n p^{k(\psi)} e_{m}(\psi) \in \mathbb{Z}[\Delta]^{-} .
$$

Since $n p^{k(\psi)} e_{m}(\psi) \in\left(\operatorname{Ann}_{\mathbb{Z}[\Delta} C l(L)\right)^{-}$, by Lemma 2.1, there exists a unique element $f_{\psi} \in \mathcal{W}^{-}$such that $\beta\left(f_{\psi}\right)=\beta_{\psi}$. Recall that:

$$
\left(\operatorname{Ann}_{\mathbb{Z}_{p}[\Delta]} A\right)(\psi)=p^{k(\psi)} \mathbb{Z}_{p} e_{\psi}
$$

Therefore, for $0 \leq k \leq m, \frac{\mathcal{W}^{-}}{\left(\mathcal{W}^{-}\right)^{p^{k}}}(\psi)$ is cyclic of order $p^{k}$ generated by the image of $f_{\psi}$. We set:

$$
W=\{f(\mathcal{A}), \mathcal{A} \in \mathcal{I}, f \in \mathcal{W}\}
$$

and:

$$
J=\{f(\mathcal{A}), \mathcal{A} \in \mathcal{I}, f \in \mathcal{J}\}
$$

Observe that $J$ is a sub- $\mathbb{Z}[\Delta]$-module of $W$, and it is called the module of Jacobi sums of $\mathbb{Q}\left(\zeta_{p}\right)$. Note that, by Lemma 3.2, we have:

$$
\frac{W\left(L^{*}\right)^{p}}{\left(L^{*}\right)^{p}}(\psi) \neq\{0\}
$$

Theorem 3.3 The map $f_{\psi}$ induces an isomorphism of groups:

$$
A(\psi) \simeq \frac{W\left(L^{*}\right)^{p^{k}(\psi)}}{\left(L^{*}\right)^{p^{k(\psi)}}}(\psi)
$$

Proof First observe that $m \geq k(\psi)+1$. Let $P$ be a prime in $\mathcal{I}$. Then:

$$
f_{\psi}(P) \mathcal{O}=P^{\beta_{\psi}} .
$$

Let $\rho \in \widehat{\Delta}, \rho \neq \psi$. Then:

$$
e_{m}(\rho) e_{m}(\psi) \equiv 0 \quad\left(\bmod p^{m}\right)
$$

Therefore, there exists $\gamma \in L^{*} \cap U$ such that:

$$
P^{\left(1-\sigma_{-1}\right) n e_{m}(\rho) e_{m}(\psi)}=\left(\frac{\gamma}{\sigma_{-1}(\gamma)}\right)^{p} \mathcal{O} .
$$

But $\left(1-\sigma_{-1}\right) e_{m}(\psi)=2 e_{m}(\psi)$. Thus, there exists $\alpha \in L^{*} \cap U, \alpha \sigma_{-1}(\alpha)=1$, and:

$$
f_{\psi}(P)^{e_{m}(\rho)}=\alpha^{p^{k(\psi)+1}}
$$

Therefore, $e_{\rho} f_{\psi}(\mathcal{I})=0$ in $L^{*} /\left(L^{*}\right)^{p^{k(\psi)+1}}$. It is clear that $f_{\psi}$ induces a morphism:

$$
\frac{\mathcal{I}}{(\mathcal{I})^{p^{m}} \mathcal{P}}(\psi) \rightarrow \frac{L^{*}}{\left(L^{*}\right)^{p^{k(\psi)}}}(\psi)
$$

Now, let $P$ be a prime in $\mathcal{I}$ such that $e_{\psi} f_{\psi}(P)=0$ in $\frac{L^{*}}{\left(L^{*}\right) p^{k(\psi)}}(\psi)$. Then, by the above remark, we get:

$$
f_{\psi}(P)=0 \text { in } L^{*} /\left(L^{*}\right)^{p^{k(\psi)}} .
$$

Thus, there exists $\gamma \in L^{*} \cap U$ such that:

$$
P^{\beta_{\psi}}=\gamma^{p^{k(\psi)}} \mathcal{O} .
$$

Thus:

$$
P^{2 n e_{m}(\psi)}=\gamma \mathcal{O} .
$$

This implies:

$$
e_{\psi} P=0 \text { in } \frac{\mathcal{I}}{(\mathcal{I})^{p^{m}} \mathcal{P}}(\psi)
$$

Thus our map is injective. Now, observe that the image of the map induced by $f_{\psi}$ is $\frac{W\left(L^{*}\right)^{k(\psi)}}{\left(L^{*}\right)^{k(\psi)}}(\psi)$ and that:

$$
A(\psi) \simeq \frac{\mathcal{I}}{(\mathcal{I})^{p^{m}} \mathcal{P}}(\psi)
$$

The Theorem follows.
Recall that:

$$
\eta=\left(\prod_{n=1}^{p-2} j_{1, n}^{\left[n^{-1]}\right.}\right)^{1-\sigma_{-1}} \in \mathcal{J}^{-}
$$

Set:

$$
z=\left(1-\sigma_{-1}\right) \sum_{n=1}^{p-2}\left(1+\sigma_{n}-\sigma_{1+n}\right)\left[n^{-1}\right] \in \mathbb{Z}[\Delta]^{-} .
$$

We have:

$$
\beta(\eta)=z \theta
$$

## Corollary 3.4

1) The map $\eta$ induces an isomorphism of groups:

$$
A(\psi) \simeq \frac{J\left(L^{*}\right)^{p^{m(\psi)}}}{\left(L^{*}\right)^{p^{m(\psi)}}}(\psi)
$$

2) $\frac{J\left(L^{*}\right)^{p}}{\left(L^{*}\right)^{p}}(\psi) \neq\{0\}$ if and only if $A(\psi)$ is $\mathbb{Z}_{p}$-cyclic.

Proof

1) Let $P$ be a prime in $\mathcal{I}$. Then one can show that:

$$
f_{\psi}(P)^{z \theta}=\eta(P)^{2 n p^{k(\psi)} e_{m}(\psi)} .
$$

The first assertion follows from Theorem 3.3.
2) Note that:

$$
A(\psi) \text { is } \mathbb{Z}_{p}-\text { cyclic } \Leftrightarrow m(\psi)=k(\psi) .
$$

Thus, if $A(\psi)$ is $\mathbb{Z}_{p}$-cyclic, then:

$$
\frac{J\left(L^{*}\right)^{p}}{\left(L^{*}\right)^{p}}(\psi)=\frac{W\left(L^{*}\right)^{p}}{\left(L^{*}\right)^{p}}(\psi) \neq\{0\}
$$

By the proof of assertion 1), if $k(\psi)<m(\psi)$ and if $P$ is a prime in $\mathcal{I}$, then:

$$
\eta(P)^{e_{m}(\psi)} \in\left(L^{*}\right)^{p} .
$$

Therefore, we get assertion 2). $\diamond$

## 4 The $p$-adic behavior of Jacobi Sums

Let $M$ be a subgroup of $L^{*} /\left(L^{*}\right)^{p}$, we say that $M$ is unramified if $L(\sqrt{M}) / L$ is an unramified extension. Note that Kummer's Lemma asserts that (11], Theorem 5.36):

$$
\forall \rho \in \widehat{\Delta}, \rho \text { even, } \rho \neq 1, \frac{E}{E^{p}}(\rho) \text { is unramified } \Rightarrow B_{1, \rho \omega^{-1}} \equiv 0 \quad(\bmod p)
$$

It is natural to ask if this implication is in fact an equivalence (see [1], [3]). We will say that the converse of Kummer's Lemma is true for the character $\rho$ if we have:

$$
\frac{E}{E^{p}}(\rho) \text { is unramified } \Leftrightarrow B_{1, \rho \omega^{-1}} \equiv 0 \quad(\bmod p)
$$

In this paragraph, we will study this question with the help of Jacobi sums.
Let $F / L$ be the maximal abelian $p$-extension of $L$ which is unramified outside $p$. Set $\mathcal{X}=\operatorname{Gal}(F / L)$. We have an exact sequence of $\mathbb{Z}_{p}[\Delta]$-modules ([11], Corollary 13.6):

$$
0 \rightarrow \frac{U}{\bar{E}} \rightarrow \mathcal{X} \rightarrow A \rightarrow 0
$$

Let $\rho \in \widehat{\Delta}$, observe that:

- if $\rho=1, \omega$ then $\mathcal{X}(\rho) \simeq \mathbb{Z}_{p}$,
- if $\rho$ is even, $\rho \neq 1, \mathcal{X}(\rho) \simeq \operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\rho)$,
- if $\rho$ is odd, $\rho \neq \omega, \mathcal{X}(\rho) \simeq \mathbb{Z}_{p} \oplus \operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\rho)$.

Lemma 4.1 Let $\psi \in \widehat{\Delta}, \psi$ odd, $\psi \neq \omega$. Then:

$$
d_{p} \operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\psi)=d_{p} A\left(\omega \psi^{-1}\right) .
$$

Proof This is a consequence of the proof of Leopoldt's reflection Theorem (【1] , Theorem 10.9). For the convenience of the reader, we give the proof of the above Lemma.

Let $H$ be the Galois group of the maximal abelian extension of $L$ which is unramified outside $p$ and of exponent $p$. Then $H$ is a $\mathbb{Z}_{p}[\Delta]$-module and we have:

- $H(1) \simeq \mathbb{F}_{p}$ and corresponds to $L\left(\zeta_{p^{2}}\right) / L$,
- $H(\omega) \simeq \mathbb{F}_{p}$ and corresponds to $L\left({ }^{p} \sqrt{p}\right) / L$,
- if $\rho$ is even, $\rho \neq 1, d_{p} H(\rho)=d_{p} \operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\rho)$,
- if $\rho$ is odd, $\rho \neq \omega$, then $d_{p} H(\rho)=1+d_{p} \operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\rho)$.

Let $V$ be the sub- $\mathbb{Z}[\Delta]$-module of $L^{*} /\left(L^{*}\right)^{p}$ which corresponds to $H$, i.e. $H=\operatorname{Gal}\left(L\left({ }^{p} \sqrt{V}\right) / L\right)$. Let $M$ be the sub- $\mathbb{Z}[\Delta]$-module of $L^{*} /\left(L^{*}\right)^{p}$ generated by $E$ and $1-\zeta_{p}$. We have an exact sequence:

$$
0 \rightarrow M \rightarrow V \rightarrow A[p] \rightarrow 0
$$

Soit $\psi \in \widehat{\Delta}, \psi$ odd, $\psi \neq \omega$. We have by Kummer theory :

$$
1+d_{p} \operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\psi)=d_{p} V\left(\omega \psi^{-1}\right)
$$

And, by the above exact sequence, we have:

$$
d_{p} V\left(\omega \psi^{-1}\right)=1+d_{p} A\left(\omega \psi^{-1}\right)
$$

The Lemma follows. $\diamond$
Lemma 4.2 Let $\rho \in \widehat{\Delta}$, $\rho$ even and $\rho \neq 1$. If $\frac{E}{E^{p}}(\rho)$ is ramified then $d_{p} A(\rho)=d_{p} A\left(\omega \rho^{-1}\right)$.

Proof We keep the notations of the proof of Lemma 4.1. Let $V^{n r} \subset V$ which corresponds via Kummmer theory to $A / p A$. Then:

$$
V^{n r}(\rho) \simeq \frac{A}{p A}\left(\omega \rho^{-1}\right)
$$

But we have:
$\frac{E}{E^{p}}(\rho)$ is ramified if and only if $V^{n r}(\rho) \hookrightarrow A[p](\rho)$.
Now recall that:

$$
d_{p} A(\rho) \leq d_{p} A\left(\omega \rho^{-1}\right)
$$

The Lemma follows. $\diamond$
Lemma 4.3 There exists an unique $\mathbb{Z}[\Delta]$-morphism $\varphi: K^{*} \rightarrow \mathbb{Z}_{p}[\Delta]$ such that:

$$
\forall x \in K^{*}, \varphi(x) \zeta_{p}=\log _{p}(x)
$$

Furthermore, we have:

$$
\operatorname{Im} \varphi=\bigoplus_{\rho=1, \omega} p \mathbb{Z}_{p} e_{\rho} \bigoplus_{\rho \neq 1, \omega} \mathbb{Z}_{p} e_{\rho}
$$

Proof Let $\lambda \in K^{*}$ such that $\lambda^{p-1}=-p$. Then:

$$
K^{*}=\lambda^{\mathbb{Z}} \times \mu_{p-1} \times \mu_{p} \times U .
$$

Recall that:

- the kernel of $\log _{p}$ on $K^{*}$ is equal to $\lambda^{\mathbb{Z}} \times \mu_{p-1} \times \mu_{p}$,
- $\log _{p} U=\pi^{2} \mathbb{Z}_{p}\left[\zeta_{p}\right]$.

For $\rho \in \widehat{\Delta}$, set:

$$
\tau(\rho)=\sum_{a=1}^{p-1} \rho(a) \zeta_{p} \in \mathbb{Z}_{p}\left[\zeta_{p}\right] .
$$

Then:

$$
e_{\rho} \zeta_{p}=\tau\left(\rho^{-1}\right) .
$$

But recall that that $\mathbb{Z}_{p}\left[\zeta_{p}\right]=\mathbb{Z}_{p}[\Delta] \zeta_{p}$. Thus:

$$
e_{\rho} \mathbb{Z}_{p}\left[\zeta_{p}\right]=\mathbb{Z}_{p} \tau\left(\rho^{-1}\right)
$$

If $\rho=\omega^{k}, k \in\{0 \cdots, p-2\}$, we have:

$$
v_{p}\left(\tau\left(\rho^{-1}\right)\right)=\frac{k}{p-1} .
$$

Therefore:

$$
\pi^{2} \mathbb{Z}_{p}\left[\zeta_{p}\right]=\bigoplus_{\rho=1, \omega} p \mathbb{Z}_{p} \tau\left(\rho^{-1}\right) \bigoplus_{\rho \neq 1, \omega} \mathbb{Z}_{p} \tau\left(\rho^{-1}\right) .
$$

The Lemma follows. $\diamond$
Let $P$ be a prime in $\mathcal{I}$. We fix a generator $r_{P} \in \mathbb{F}_{P}^{*}$ such that:

$$
\chi_{P}\left(r_{P}\right)=\zeta_{p} .
$$

For $x \in \mathbb{F}_{P}^{*}$, let $\operatorname{Ind}(P, x) \in\{0, \cdots, N P-2\}$ such that:

$$
x=r_{p}^{\operatorname{Ind}(P, x)} .
$$

We recall the following Theorem (see also [9] for a statement similar but weaker than part 2) of the following Theorem):

## Theorem 4.4

1) $\varphi\left(1-\zeta_{p}\right)=\sum_{\rho \in \widehat{\Delta}, \rho \neq 1, \rho \text { even }}-(p-1)^{-1} L_{p}(1, \rho) e_{\rho}$.
2) Let $\psi \in \widehat{\Delta}, \psi$ odd, $\psi \neq \omega$. Write $\psi=\omega^{k}, k \in\{2, \cdots, p-2\}$. We have:

$$
e_{\psi} \varphi(\eta(P)) \equiv 2 k \operatorname{Ind}\left(P, \prod_{a=1}^{p-1}\left(\frac{1-\zeta_{p}^{-a}}{1-\zeta_{p}}\right)^{a^{k-1}}\right) e_{\psi} \quad(\bmod p) .
$$

Proof

1) Let $\rho \in \widehat{\Delta}, \rho$ even, $\rho \neq 1$. By [11, Theorem 5.18, we have:

$$
L_{p}(1, \rho) \tau\left(\rho^{-1}\right)=-(p-1) e_{\rho} \log _{p}\left(1-\zeta_{p}\right) .
$$

Thus the first assertion follows.
2) Let $\psi \in \widehat{\Delta}, \psi$ odd, $\psi \neq \omega$. By a beautiful result of Uehara (10], Theorem1), we have:

$$
e_{\psi} \log _{p}(\eta(P)) \equiv 2 k \operatorname{Ind}\left(P, \prod_{a=1}^{p-1}\left(\frac{1-\zeta_{p}^{-a}}{1-\zeta_{p}}\right)^{a^{k-1}}\right) \tau\left(\psi^{-1}\right) \quad(\bmod p) .
$$

This implies the second assertion. $\diamond$
Theorem 4.5 Let $\psi \in \widehat{\Delta}, \psi \neq \omega, \psi$ odd. We have the following exact sequences:

$$
\begin{aligned}
& 0 \rightarrow \operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\psi) \rightarrow A(\psi) \rightarrow \frac{\bar{W}(\psi)}{U^{p^{k(\psi)}}(\psi)} \rightarrow 0, \\
& 0 \rightarrow \operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\psi) \rightarrow A(\psi) \rightarrow \frac{\bar{J}(\psi)}{U^{p^{m(\psi)}}(\psi)} \rightarrow 0 .
\end{aligned}
$$

Proof This Theorem is a consequence of the method developped by Iwasawa in (5). Let's recall briefly this method.

Let $f \in \mathcal{W}$. Set, for $n \geq 2, \mathcal{P}_{n}=\left\{\alpha \mathcal{O}, \alpha \equiv 1\left(\bmod \pi^{n}\right)\right\}$. Observe that:

$$
f\left(\mathcal{P}_{n}\right) \subset 1+\pi^{n} \mathbb{Z}_{p}\left[\zeta_{p}\right] .
$$

Let:

$$
\tilde{\mathcal{X}}=\lim _{\leftarrow} \frac{\mathcal{I}}{\mathcal{P}_{n}} .
$$

If $\widetilde{F}$ is the maximal abelian extension of $L$ which is unramified outside $p$, then, we get by class field theory:

$$
\widetilde{\mathcal{X}} \simeq \operatorname{Gal}(\widetilde{F} / L)
$$

By [11], Theorem 13.4, the natural surjective map $\widetilde{\mathcal{X}} \rightarrow \mathcal{X}$ has a finite kernel of order prime to $p$. Thus $f$ induces a map:

$$
\bar{f}: \mathcal{X} \rightarrow U .
$$

Furthermore:

$$
\bar{f}(U)=U^{\beta(f)} \subset \bar{f}(\mathcal{X})
$$

Now let $\psi \in \widehat{\Delta}, \psi$ odd, $\psi \neq \omega$. We have a map:

$$
\bar{f}: \mathcal{X}(\psi) \rightarrow U(\psi)
$$

But:

$$
\mathcal{X}(\psi) \simeq \mathbb{Z}_{p} \bigoplus \operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\psi)
$$

and:

$$
U(\psi) \simeq \mathbb{Z}_{p}
$$

Thus, if $e_{\psi} \beta(f) \neq 0$, we get:

$$
\operatorname{Ker}(\bar{f}: \mathcal{X}(\psi) \rightarrow U(\psi))=\operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\psi) .
$$

Therefore, if $e_{\psi} \beta(f) \neq 0$, we get the following exact sequence induced by $f$ :

$$
0 \rightarrow \operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\psi) \rightarrow A(\psi) \rightarrow \frac{\bar{f}(\mathcal{X})(\psi)}{U^{\beta(f)}(\psi)} \rightarrow 0
$$

It remains to apply this construction to $f_{\psi}$ and $\eta$ to get the desired exact sequences. $\diamond$

## Corollary 4.6

1) Let $\psi \in \widehat{\Delta}, \psi$ odd, $\psi \neq \omega$. Then:

$$
d_{p} A(\psi)=1+d_{p} A\left(\omega \psi^{-1}\right) \Leftrightarrow B_{1, \psi^{-1}} \equiv 0 \quad(\bmod p) \text { and } \bar{W}(\psi)=U(\psi)
$$

2) Let $\rho \in \widehat{\Delta}, \rho$ even and $\rho \neq 1$. Assume that $B_{1, \rho \omega^{-1}} \equiv 0(\bmod p)$ and that $\bar{W}\left(\omega \rho^{-1}\right)=U\left(\omega \rho^{-1}\right)$ then the converse of Kummer's Lemma is true for the character $\rho$.

## Proof

1) We apply Theorem 4.5. We identify $\operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\psi)$ with its image in $A(\psi)$. We can write:

$$
A(\psi)=B \bigoplus C
$$

where $C$ is cyclic of order $p^{k(\psi)}$ and $B \subset \operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\psi)$. Now:

$$
\left(C: C \cap \operatorname{Tor}_{\mathbb{Z}_{p}} \mathcal{X}(\psi)\right)=\left(\bar{W}(\psi): U^{U^{k(\psi)}}(\psi)\right) .
$$

It remains to apply Lemma 4.1 to get the desired result.
2) We apply the first assertion and Lemma 4.1, we get:

$$
d_{p} A(\rho)=d_{p} A\left(\omega \rho^{-1}\right)-1 .
$$

It remains to apply Lemma 4.2.
We set:

$$
W^{n r}=\left\{\alpha \in W, \alpha \in U^{p}\right\} .
$$

Let $\psi \in \widehat{\Delta}, \psi$ odd, $\psi \neq \omega$. We assume that $B_{1, \psi^{-1}} \equiv 0(\bmod p)$. Write:

$$
A(\psi)=\frac{\mathbb{Z}}{p^{e_{1} \mathbb{Z}}} \bigoplus \cdots \bigoplus \frac{\mathbb{Z}}{p^{e_{t}} \mathbb{Z}}
$$

where $t=d_{p} A(\psi)$ and $1 \leq e_{1} \leq \cdots \leq e_{t}=k(\psi)$. Set:

$$
n(\psi)=\left|\left\{i \in\{1, \cdots t\}, e_{i}=k(\psi)\right\}\right| .
$$

Corollary 4.7 We have:

$$
n(\psi)-1 \leq \operatorname{dim}_{\mathbb{F}_{p}} \frac{W^{n r}\left(L^{*}\right)^{p}}{\left(L^{*}\right)^{p}} \leq n(\psi)
$$

Furthermore:

$$
\operatorname{dim}_{\mathbb{F}_{p}} \frac{W^{n r}\left(L^{*}\right)^{p}}{\left(L^{*}\right)^{p}}=n(\psi) \Leftrightarrow \bar{W}(\psi) \neq U(\psi)
$$

Proof By Theorem4.5 and Theorem 3.3, we have:

$$
\frac{W^{n r}\left(L^{*}\right)^{p^{k(\psi)}}}{\left(L^{*}\right)^{p^{k(\psi)}}} \simeq \operatorname{Ker}\left(A(\psi) \rightarrow \frac{\bar{W}(\psi)}{U p^{k(\psi)}(\psi)}\right)
$$

The Corollary follows. $\diamond$
Corollary 4.8 Assume that $p A^{-}=\{0\}$. Then we have an isomorphism of groups:

$$
\operatorname{Gal}\left(L\left(\sqrt[p]{W^{n r}}\right) / L\right) \simeq \frac{A^{+}}{p A^{+}} .
$$

Proof This result is a consequence of Kummer theory, Corollary 4.7 and Corollary 4.6. $\diamond$

Note that the above results lead us to ask the following problem (which is a restatement of the converse of Kummer's Lemma):

$$
\text { do we have } \varphi\left(\bar{W}^{-}\right)=(\operatorname{Im} \varphi)^{-} \text {? }
$$

Observe that $e_{\omega} \varphi\left(\bar{W}^{-}\right)=e_{\omega}(\operatorname{Im} \varphi)^{-}$, and since $K_{4}(\mathbb{Z})=\{0\}$, we have $A\left(\omega^{-2}\right)=\{0\}$ (see [7]) and therefore $e_{\omega^{3}} \varphi\left(\bar{W}^{-}\right)=e_{\omega^{3}}(\operatorname{Im} \varphi)^{-}$.

## 5 Remarks on the Jacobian of the Fermat Curve over a finite field

First we fix some notations and recall some basic facts about global function fields.

Let $\mathbb{F}_{q}$ be a finite field having $q$ elements. Let $\ell$ be the charactersitic of $\mathbb{F}_{q}, \ell \neq p$. Let $\overline{\mathbb{F}_{q}}$ be a fixed algebraic closure of $\mathbb{F}_{q}$ and let $\widetilde{\mathbb{F}_{q}}=$ $\cup_{n \geq 1, n \neq 0}(\bmod p) \mathbb{F}_{q^{n}} \subset \overline{\mathbb{F}_{q}}$. Let $k / \mathbb{F}_{q}$ be a global function field such that $\mathbb{F}_{q}$ is algebraically closed in $k$. We set:

- $D_{k}$ : the group of divisors of $k$,
- $D_{k}^{0}$ : the group of divisors of degree zero of $k$,
- $P_{k}$ : the group of principal divisors of $k$,
- $J_{k}$ : the jacobian of $k$, note that we have:

$$
\forall n \geq 1, J_{k}\left(\mathbb{F}_{q^{n}}\right) \simeq \frac{D_{\mathbb{F}_{q^{n}} k}^{0}}{P_{\mathbb{F}_{q^{n}} k}}
$$

- $g_{k}$ : the genus of $k$,
- $L_{k}(Z) \in \mathbb{Z}[Z]$ : the numerator of the zeta function of $k$, we recall that:

$$
\frac{L_{k}(Z)}{(1-Z)(1-q Z)}=\prod_{v \text { placeof } k}\left(1-Z^{\operatorname{deg} v}\right)^{-1}
$$

furthermore $\operatorname{deg}_{Z} L_{k}(Z)=2 g_{k}$ and $L_{k}(1)=\left|J_{k}\left(\mathbb{F}_{q}\right)\right|$,

- $C_{k}\left(\mathbb{F}_{q^{n}}\right)=J_{k}\left(\mathbb{F}_{q^{n}}\right) \otimes_{\mathbb{Z}} \mathbb{Z}_{p}$,
- $\widetilde{d}_{p} J_{k}=d_{p} C_{k}\left(\widetilde{\mathbb{F}_{q}}\right)$, observe that there exists an integer $m, m \not \equiv 0(\bmod p)$, such that:

$$
C_{k}\left(\widetilde{\mathbb{F}_{q}}\right)=C_{k}\left(\mathbb{F}_{q^{m}}\right)
$$

Write:

$$
L_{k}(Z)=\prod_{i=1}^{2 g_{k}}\left(1-\alpha_{i} Z\right)
$$

For simplicity, we assume that $v_{p}\left(\alpha_{i}-1\right)>0$ for $i=1, \cdots, 2 g_{k}$. In this case, we have:

$$
C_{k}\left(\widetilde{\mathbb{F}_{q}}\right)=C_{k}\left(\mathbb{F}_{q}\right) .
$$

Set $P_{k}(Z)=\prod_{i=1}^{2 g_{k}}\left(Z-\left(\alpha_{i}-1\right)\right)$. Let $\gamma$ be the Frobenius of $\mathbb{F}_{q}$, and set:

$$
C_{n}(k)=C_{k}\left(\mathbb{F}_{q^{p^{n}}}\right) .
$$

Let $C_{\infty}(k)=\cup_{n \geq 0} C_{n}(k)$, and set:

$$
M_{k}=\operatorname{Hom}\left(\frac{\mathbb{Q}_{p}}{\mathbb{Z}_{p}}, C_{\infty}(k)\right)
$$

Then $M_{k}$ is isomorphic to the $p$-adic Tate-module of $J_{k}$. Set $\Lambda=\mathbb{Z}_{p}[[Z]]$ where $Z$ corresponds to $\gamma-1$. Then it is well-known that:

- $M_{k}$ is a $\Lambda$-module of finite type and of torsion,
- as a $\mathbb{Z}_{p}$-module $M_{k}$ is isomorphic to $\mathbb{Z}_{p}^{2 g_{k}}$,
- $M_{k} / \omega_{n} M_{k} \simeq C_{n}(k)$, where $\omega_{n}=(1+Z)^{p^{n}}-1$,
- $\operatorname{Char}_{\Lambda} M_{k}=P_{k}(Z) \Lambda$,
- the action of $Z$ on $M_{k}$ is semi-simple, i.e. the minimal polynomial of the action of $Z$ on $M_{k}$ has only simple roots.

Now, let $\ell$ be a prime number, $\ell \neq p$. We fix a prime $P$ of $\mathcal{O}$ above $\ell$ and we view $\mathcal{O} / P$ as a subfield of $\overline{\mathbb{F}_{\ell}}$, thus $\mathbb{F}_{q}=\mathcal{O} / \mathcal{P} \subset \widetilde{\mathbb{F}_{\ell}}$. We identify $\zeta_{p}$ with its image in $\mathbb{F}_{q}$. Let $X$ be an indeterminate over $\mathbb{F}_{q}$, we set $k=\mathbb{F}_{\ell}(X, Y)$ where $X^{p}+Y^{p}=1$, and we set: $T=X^{p}$. For $a, b \in \mathbb{Z}$, let $\tau_{a, b} \in \operatorname{Gal}\left(\overline{\mathbb{F}_{\ell}} k / \overline{\mathbb{F}_{\ell}}(T)\right)$ such that:

$$
\tau_{a, b}(X)=\zeta_{p}^{a} X \text { and } \tau_{a, b}(Y)=\zeta_{p}^{b} Y .
$$

Let $a \in\{1, \cdots, p-2\}$. Let $H_{a}$ be the subgroup of $\operatorname{Gal}\left(\overline{\mathbb{F}_{\ell}} k / \overline{\mathbb{F}_{\ell}}(T)\right)$ generated by $\tau_{1,\left[-a^{-1}\right]}$. Set:

$$
E_{a}=\mathbb{F}_{\ell}\left(T, X Y^{a}\right)
$$

We set:

$$
\begin{gathered}
E=\mathbb{F}_{q} E_{a}, \\
F=\mathbb{F}_{q} k,
\end{gathered}
$$

and observe that:

$$
\widetilde{\mathbb{F}_{\ell}}=\widetilde{\mathbb{F}_{q}} .
$$

It is clear that:

$$
F^{H_{a}}=E .
$$

Finally, we set:

$$
G=\operatorname{Gal}\left(E / \mathbb{F}_{q}(T)\right)
$$

Note that $g_{E}=(p-1) / 2$.
Lemma 5.1 We have:

$$
L_{E}(Z)=\prod_{\sigma \in \Delta}\left(1-j_{1, a}(P)^{\sigma} Z\right)
$$

Proof Let $\chi \in \widehat{G}$ such that:

$$
\chi(g)=\zeta_{p}^{-1}
$$

where $g \in G$ is such that $g\left(X Y^{a}\right)=\zeta_{p} X Y^{a}$. Note that:

$$
L_{E}(Z)=\prod_{\sigma \in \Delta} L\left(Z, \chi^{\sigma}\right)
$$

where:

$$
L(Z, \chi)=\prod_{v \text { place of } \mathbb{F}_{q}(T)}\left(1-\chi(v) Z^{\operatorname{deg} v}\right)^{-1}
$$

Since $2 g_{e}=p-1$, we get:

$$
\operatorname{deg}_{Z} L(Z, \chi)=1
$$

For $b \in \mathbb{F}_{q} \backslash\{0,1\}$, we denote the Frobenius of $T-b$ in $E / \mathbb{F}_{q}(T)$ by Frob $_{b}$. We have:

$$
\operatorname{Frob}_{b}\left(X Y^{a}\right)=\left(b(1-b)^{a}\right)^{(q-1) / p} X Y^{a} .
$$

But:

$$
L(Z, \chi) \equiv 1+\left(\sum_{b \in \mathbb{F}_{q} \backslash\{0,1\}} \chi\left(\text { Frob }_{b}\right)\right) X \quad\left(\bmod X^{2}\right) .
$$

Thus:

$$
L(Z, \chi)=1+\left(\sum_{b \in \mathbb{F}_{q} \backslash\{0,1\}} \chi\left(\text { Frob }_{b}\right)\right) X .
$$

But, we can write:

$$
j_{1, a}(P)=-\sum_{i=0}^{p-1} N_{i} \zeta_{p}^{-i}
$$

where $N_{i}=\left|\left\{\alpha \in \mathbb{F}_{q} \backslash\{0,1\},\left(\alpha(1-\alpha)^{a}\right)^{(q-1) / p} \equiv \zeta_{p}^{-i}(\bmod P)\right\}\right|$. Therefore:

$$
j_{1, a}(P)=-\sum_{b \in \mathbb{F}_{q} \backslash\{0,1\}} \chi\left(\text { Frob }_{b}\right) .
$$

The Lemma follows. $\diamond$
Theorem 5.2 Let $n$ be the smallest integer (if it exists) such that $3 \leq n \leq$ $p-2, n$ odd and $e_{\omega^{n}} j_{1, a}(P) \notin U^{p}$, then:

$$
J_{k}\left(\widetilde{\mathbb{F}_{\ell}}\right)^{H_{a}} \otimes_{\mathbb{Z}} \mathbb{Z}_{p} \simeq\left(\frac{\mathbb{Z}}{p \mathbb{Z}}\right)^{n}
$$

If such an integer doesn't exist then:

1) $\widetilde{d}_{p} J_{k}^{H_{a}}=p-1$,
2) we have:

$$
J_{k}\left(\widetilde{\mathbb{F}}_{\ell}\right)^{H_{a}} \otimes_{\mathbb{Z}} \mathbb{Z}_{p} \simeq\left(\frac{\mathbb{Z}}{p \mathbb{Z}}\right)^{p-1} \Leftrightarrow \ell^{p-1} \not \equiv 1 \quad\left(\bmod p^{2}\right)
$$

Proof The proof of this result is based on ideas developped by Greenberg in [4]. Set $H=H_{a}$. Let $P_{0}$ be the prime of $E$ above $T, P_{1}$ the prime of $E$ above $T-1$ and $P_{\infty}$ the prime of $E$ above $\frac{1}{T}$. Recall that we have in $D_{E}$ :

$$
\begin{gathered}
p\left(P_{0}-P_{\infty}\right)=(T), \\
p\left(P_{1}-P_{\infty}\right)=(T-1), \\
P_{0}-P_{\infty}+a\left(P_{1}-P_{\infty}\right)=\left(X Y^{a}\right) .
\end{gathered}
$$

Thus, by (4) paragraph 2 , we get:

$$
J_{E}\left(\mathbb{F}_{q}\right)^{G} \simeq \frac{\mathbb{Z}}{p \mathbb{Z}},
$$

and $J_{E}\left(\mathbb{F}_{q}\right)^{G}$ is generated by the class of $P_{0}-P_{\infty}$. Observe also that $F / E$ is unramified and cyclic of order $p$. Let's start by the folowing exact sequence:

$$
0 \rightarrow \mathbb{F}_{q}^{*} \rightarrow F^{*} \rightarrow P_{F} \rightarrow 0
$$

We get:

$$
\frac{P_{F}^{H}}{P_{E}} \simeq \frac{\mathbb{Z}}{p \mathbb{Z}},
$$

and $\frac{P_{F}^{H}}{P_{E}}$ is generated by the image of $P_{0}-P_{\infty}$ in $D_{F}$. In particular:

$$
\frac{P_{F}^{H}}{P_{E}} \simeq J_{E}\left(\mathbb{F}_{q}\right)^{G}
$$

Note that we also have:

$$
0 \rightarrow H^{1}\left(H, P_{F}\right) \rightarrow H^{2}\left(H, \mathbb{F}_{q}^{*}\right) \rightarrow H^{2}\left(H, F^{*}\right)
$$

But $F / E$ is unramified and cyclic, therefore every element of $\mathbb{F}_{q}^{*}$ is a norm in the extension $F / E$. Thus:

$$
H^{1}\left(H, P_{F}\right) \simeq \frac{\mathbb{Z}}{p \mathbb{Z}}
$$

Now, we look at the exact sequence:

$$
0 \rightarrow P_{F} \rightarrow D_{F}^{0} \rightarrow J_{F}\left(\mathbb{F}_{q}\right) \rightarrow 0
$$

Since $F / E$ is unramified:

$$
H^{1}\left(H, D_{F}^{0}\right)=\{0\} .
$$

Therefore, we have obtained the following exact sequence:

$$
0 \rightarrow J_{E}\left(\mathbb{F}_{q}\right)^{G} \rightarrow J_{E}\left(\mathbb{F}_{q}\right) \rightarrow J_{F}\left(\mathbb{F}_{q}\right)^{H} \rightarrow \frac{\mathbb{Z}}{p \mathbb{Z}} \rightarrow 0
$$

Now, it is not difficult to deduce that, for all $n \geq 1$, we have the following exact sequence:

$$
0 \rightarrow \frac{\mathbb{Z}}{p \mathbb{Z}} \rightarrow J_{E}\left(\mathbb{F}_{q^{n}}\right) \rightarrow J_{F}\left(\mathbb{F}_{q^{n}}\right)^{H} \rightarrow \frac{\mathbb{Z}}{p \mathbb{Z}} \rightarrow 0
$$

From this, we get the following exact sequence of $\mathbb{Z}_{p}[G]$-modules and $\Lambda$ modules:

$$
0 \rightarrow M_{E} \rightarrow M_{F}^{H} \rightarrow \frac{\mathbb{Z}}{p \mathbb{Z}} \rightarrow 0
$$

Recall that in our situation, by Lemma 5.1, we have:

$$
P_{E}(Z)=\prod_{\sigma \in \Delta}\left(Z-\left(j_{1, a}(P)^{\sigma}-1\right)\right) .
$$

Furthermore the action of $G$ and $Z$ commute on $M_{F}^{H}$. Now, we have:

- $\operatorname{Char}_{\Lambda} M_{F}^{H}=\operatorname{Char}_{\Lambda} M_{E}=P_{E}(Z) \Lambda$,
- $M_{F}^{H} \simeq \mathbb{Z}_{p}^{p-1}$ as $\mathbb{Z}_{p}$-modules,
- $M_{F}^{H} / \omega_{n} \simeq C_{n}(F)^{H}$.

Observe that:

$$
C_{0}(F)^{H}=J_{k}\left(\widetilde{\mathbb{F}}_{\ell}\right)^{H_{a}} \otimes_{\mathbb{Z}} \mathbb{Z}_{p}
$$

Note also that the minimal polynomial of the action of $Z$ on $M_{F}^{H}$ is:

$$
\operatorname{Irr}\left(j_{1, a}(P)-1, \mathbb{Q}_{p} ; Z\right):=G(Z)
$$

Set $N=\sum_{\delta \in G} \delta$. Then one can see that:

$$
N M_{E}=N M_{F}^{H}=\{0\}
$$

Thus $M_{F}^{H}$ is a $\mathbb{Z}_{p}[G] / N \mathbb{Z}_{p}[G]$-module. Now, we identify $\mathbb{Z}_{p}[G] / N \mathbb{Z}_{p}[G]$ with $\mathbb{Z}_{p}\left[\zeta_{p}\right]$. Since $M_{F}^{H} \simeq \mathbb{Z}_{p}^{p-1}$, there exists $m \in M_{F}^{H}$ such that:

$$
M_{F}^{H} \simeq \mathbb{Z}_{p}\left[\zeta_{p}\right] \cdot m
$$

i.e. $\quad M_{F}^{H}$ is a free $\mathbb{Z}_{p}\left[\zeta_{p}\right]$-module of rank one. Therefore there exists an element $x \in \mathbb{Z}_{p}\left[\zeta_{p}\right]$ such that:

$$
Z m=x m
$$

Now set:

$$
D(Z)=\prod_{\sigma \in \Delta}\left(Z-x^{\sigma}\right) \in \Lambda
$$

Then:

$$
D(Z) M_{F}^{H}=\{0\}
$$

Therefore $G(Z)$ divides $D(Z)$ in $\Lambda$. Thus there exists $\sigma \in \Delta$ such that:

$$
x^{\sigma}=j_{1, a}(P)-1
$$

But:

$$
C_{0}(F)^{H} \simeq \frac{M_{F}^{H}}{Z M_{F}^{H}} \simeq \frac{\mathbb{Z}_{p}\left[\zeta_{p}\right]}{x \mathbb{Z}_{p}\left[\zeta_{p}\right]}
$$

Therefore, we get:

$$
J_{k}\left(\widetilde{\mathbb{F}}_{\ell}\right)^{H_{a}} \otimes_{\mathbb{Z}} \mathbb{Z}_{p} \simeq \frac{\mathbb{Z}_{p}\left[\zeta_{p}\right]}{\left(j_{1, a}(P)-1\right) \mathbb{Z}_{p}\left[\zeta_{p}\right]}
$$

Recall that $j_{1, a}(P) \equiv 1 \quad\left(\bmod \pi^{2}\right)$. Thus:

$$
v_{p}\left(j_{1, a}(P)-1\right)=v_{p}\left(\log _{p}\left(j_{1, a}(P)\right)\right)
$$

Now:

$$
\log _{p}\left(j_{1, a}(P)\right)=\frac{1}{2} f \log _{p}(\ell)+\sum_{\psi \in \widehat{\Delta}, \psi \text { odd }} e_{\psi} \log _{p}\left(j_{1, a}(P)\right)
$$

where $f$ is the order of $\ell$ in $(\mathbb{Z} / p \mathbb{Z})^{*}$. Let $\psi \in \widehat{\Delta}, \psi=\omega^{n}$, $n$ odd. If $e_{\psi} \log _{p}\left(j_{1, a}(P)\right) \neq 0$, then:

$$
v_{p}\left(e_{\psi} \log _{p}\left(j_{1, a}(P)\right)\right) \equiv \frac{n}{p-1} \quad(\bmod \mathbb{Z})
$$

furthermore:

$$
v_{p}\left(e_{\psi} \log _{p}\left(j_{1, a}(P)\right)\right)>\frac{n}{p-1} \Leftrightarrow e_{\psi} j_{1, a}(P) \in U^{p}
$$

Note also that:

$$
v_{p}\left(e_{\omega} \log _{p}\left(j_{1, a}(P)\right)\right)>\frac{1}{p-1}
$$

The Theorem follows. $\diamond$
Corollary 5.3 Let $n \in\{3, \cdots, p-2\}$, $n$ odd. Let $a \in\{1, \cdots, p-2\}$ such that $1+a^{n}-(1+a)^{n} \not \equiv 0 \quad(\bmod p)$. The following assertions are equivalent: 1) $A\left(\omega^{1-n}\right)=\{0\}$,
2) there exists a prime number $\ell, \ell \neq p$, such that $\widetilde{d}_{p} J_{k}^{H_{a}}=n$.

Proof Observe that 2) implies 1) by the above Theorem and Theorem 4.5. Write $\psi=\omega^{n}$. Let $\ell$ be a prime number, $\ell \neq p$. Write:

$$
\mathbb{F}_{(\ell)}=\frac{\mathcal{O}}{\ell \mathcal{O}}
$$

and:

$$
D_{\ell}=\frac{\mathbb{F}_{(\ell)}^{*}}{\left(\mathbb{F}_{(\ell)}^{*}\right)^{p}}
$$

Observe that $D_{\ell}$ is a $\mathbb{Z}_{p}[\Delta]$-module. Let $C y c$ be the group of cyclotomic units of $L$. We denote the image of $C y c$ in $D_{\ell}$ by $\overline{C y c}{ }^{\ell}$. Then Theorem 4.4 asserts that $e_{\psi} \overline{C y c}^{\ell}=\{1\}$ in $D_{\ell}$ if and only if $e_{\psi} j_{1, a}(P) \in U^{p}$, where $P$ is a prime of $\mathcal{O}$ above $\ell$. Let:

$$
B=L(\sqrt[p]{C y c}) .
$$

We assume that 1) holds. We apply the Chebotarev density theorem to the extension $B / L$, then there exist infinitely many primes $\ell$ such that:
$-e_{\rho} \overline{C y c}^{\ell}=\{1\}$ for $\rho \neq \psi$,
$-e_{\psi} \overline{C y c}^{\ell} \neq\{1\}$.
It remains to apply Theorem 5.2 and the above remarks to get 2 ). $\diamond$
Now, let $\ell$ be a prime number. Let $p$ be an odd prime number, $p \neq \ell$. Let $T$ be an indeterminate over $\mathbb{F}_{\ell}$ and let $E_{p} / \mathbb{F}_{\ell}(T)$ be the imaginary quadratic extension defined by:

$$
E_{p}=\mathbb{F}_{\ell}(T, X) \text { where } X^{2}-X+T^{p}=0 .
$$

Let $n$ be an odd integer, $n \geq 3$. Let $S_{n}(\ell)$ denote the set of primes $p$ such that $\widetilde{d}_{p} J_{E_{p}}=n$. By our above results, we remark that if $p \in S_{n}(\ell)$ then $A\left(\omega^{1-n}\right)=\{0\}$. Observe that if $\ell^{n} \not \equiv 1(\bmod p)$ then $p \notin S_{n}(\ell)$, and therefore $S_{n}(\ell)$ is a finite set. Set $S(\ell)=\cup_{n} S_{n}(\ell)$, where $n$ runs through the odd integers. Observe that if the order of $\ell$ modulo $p$ is even then $p \notin S(\ell$. Therefore, by a classical result of Hasse (see [8]) there exist infinitely many prime $p$ not in $S(\ell)$ (in fact at least " $2 / 3$ of the prime numbers "are not in $S(\ell)$ ). Thus, we ask the following question:

$$
\text { is } S(\ell) \text { infinite? }
$$

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