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On Jacobi Sums in $\mathbb{Q}(\zeta_p)$

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Abstract

We study the p -adic behavior of Jacobi sums for $\mathbb{Q}(\zeta_p)$ and link this study to the p -Sylow subgroup of the class group of $\mathbb{Q}(\zeta_p)^+$ and to some properties of the jacobian of the Fermat curve $X^p + Y^p = 1$ over \mathbb{F}_ℓ where ℓ is a prime number distinct from p .

Let p be a prime number, $p \geq 5$. Iwasawa has shown that the p -adic properties of Jacobi sums for $\mathbb{Q}(\zeta_p)$ are linked to Vandiver's Conjecture (see [5]). In this paper, we follow Iwasawa's ideas and study the p -adic properties of the subgroup J of $\mathbb{Q}(\zeta_p)^*$ generated by Jacobi sums.

Let A be the p -Sylow subgroup of the class group of $\mathbb{Q}(\zeta_p)$. If E denotes the group of units of $\mathbb{Q}(\zeta_p)$, then if Vandiver's Conjecture is true for p , by Kummer Theory, we must have $\frac{A^-}{pA^-} \hookrightarrow \text{Gal}(\mathbb{Q}(\zeta_p)(\sqrt[p]{E})/\mathbb{Q}(\zeta_p))$. Note that J is analogous for the odd part to the group of cyclotomic units for the even part. We introduce a submodule W of $\mathbb{Q}(\zeta_p)^*$ which was already considered

by Iwasawa ([6]). This module can be thought as the analogue for the odd part of the group of units for the even part. We observe that $J \subset W$ and if the Iwasawa-Leopoldt Conjecture is true for p then $W(Q(\zeta_p)^*)^p = J(Q(\zeta_p)^*)^p$. We prove that if $pA^- = \{0\}$ then (Corollary 4.8):

$$\frac{A^+}{pA^+} \hookrightarrow \text{Gal}(\mathbb{Q}(\zeta_p)(\sqrt[p]{W})/\mathbb{Q}(\zeta_p)).$$

The last part of our paper is devoted to the study of the jacobian of the Fermat curve $X^p + Y^p = 1$ over \mathbb{F}_ℓ where ℓ is a prime number, $\ell \neq p$. It is well-known that Jacobi sums play an important role in the study of that jacobian. Following ideas developped by Greenberg ([4]), we prove that Vandiver's Conjecture is equivalent to some properties of that jacobian (for the precise statement see Corollary 5.3).

The authors thank Cornelius Greither for interesting discussions on the converse of Kummer's Lemma which led us to the study of the analogous statement for the odd part . The second author thanks the mathematicians of the Laboratoire de Mathématiques Nicolas Oresme for their hospitality during his stay at Caen.

1 Notations

Let p be a prime number, $p \geq 5$. Let $\zeta_p \in \mu_p \setminus \{1\}$, and let $L = \mathbb{Q}(\zeta_p)$. Set $\mathcal{O} = \mathbb{Z}[\zeta_p]$ and $E = \mathcal{O}^*$. Let $\Delta = \text{Gal}(L/\mathbb{Q})$ and let $\widehat{\Delta} = \text{Hom}(\Delta, \mathbb{Z}_p^*)$. Let \mathcal{I} be the group of fractional ideals of L which are prime to p , and let \mathcal{P} be the group of principal ideals in \mathcal{I} . Let A be the p -Sylow subgroup of the ideal class group of L .

Set $\pi = \zeta_p - 1$, $K = \mathbb{Q}_p(\zeta_p)$, $U = 1 + \pi^2 \mathbb{Z}_p[\zeta_p]$. Observe that if $\mathcal{A} \in \mathcal{P}$, then there exists $\alpha \in L^* \cap U$ such that $\mathcal{A} = \alpha \mathcal{O}$. If H is a subgroup of U , we will denote the closure of H in U by \overline{H} . Let $\omega \in \widehat{\Delta}$ be the Teichmüller character, i.e. :

$$\forall \sigma \in \Delta, \sigma(\zeta_p) = \zeta_p^{\omega(\sigma)}.$$

For $\rho \in \widehat{\Delta}$, we set:

$$e_\rho = \frac{1}{p-1} \sum_{\delta \in \Delta} \rho^{-1}(\delta) \delta \in \mathbb{Z}_p[\Delta].$$

If M is a $\mathbb{Z}_p[\Delta]$ -module, for $\rho \in \widehat{\Delta}$, we set:

$$M(\rho) = e_\rho M.$$

For $\psi \in \widehat{\Delta}$, ψ odd, recall that:

$$B_{1,\psi} = \frac{1}{p} \sum_{a=1}^{p-1} a\psi(a).$$

Set:

$$\theta = \frac{1}{p} \sum_{a=1}^{p-1} a\sigma_a^{-1} \in \mathbb{Q}[\Delta],$$

where $\sigma_a \in \Delta$ is such that $\sigma_a(\zeta_p) = \zeta_p^a$. Observe that we have the following equality in $\mathbb{C}[\Delta]$:

$$\theta = \frac{N}{2} + \sum_{\psi \in \widehat{\Delta}, \psi \text{ odd}} B_{1,\psi^{-1}} e_\psi,$$

where $N = \sum_{\delta \in \Delta} \delta$.

Let M be a $\mathbb{Z}[\Delta]$ -module, we set:

$$M^- = \{m \in M, \sigma_{-1}(m) = -m\},$$

$$M^+ = \{m \in M, \sigma_{-1}(m) = m\}.$$

If M is an abelian group of finite type, we set:

$$M[p] = \{m \in M, pm = 0\},$$

$$d_p M = \dim_{\mathbb{F}_p} \frac{M}{pM}.$$

2 Background on Jacobi Sums

Let $Cl(L)$ be the ideal class group of L , then:

$$Cl(L) \simeq \frac{\mathcal{I}}{\mathcal{P}}.$$

Note that we have a natural $\mathbb{Z}[\Delta]$ -morphism (see [6], pages 102-103):

$$\phi : (\text{Ann}_{\mathbb{Z}[\Delta]}Cl(L))^- \rightarrow \text{Hom}_{\mathbb{Z}[\Delta]}(Cl(L), \frac{E^+}{(E^+)^2}).$$

For the convenience of the reader, we recall the construction of ϕ . Let $x \in (\text{Ann}_{\mathbb{Z}[\Delta]}Cl(L))^-$. Let $\mathcal{A} \in \mathcal{I}$, we have:

$$\mathcal{A}^x = \gamma_a \mathcal{O},$$

where $\gamma_a \in L^* \cap U$. Now:

$$\overline{\gamma_a} = \varepsilon_a \gamma_a^{-1},$$

for some $\varepsilon_a \in E^+ \cap U$. One can prove that we obtain a well-defined morphism of $\mathbb{Z}[\Delta]$ -modules: $\phi(x) : Cl(L) \rightarrow \frac{E^+}{(E^+)^2}$, class of $\mathcal{A} \mapsto$ class of ε_a .

In this paragraph, we will study the kernel of the morphism ϕ .

Let \mathcal{W} be the set of elements $f \in \text{Hom}_{\mathbb{Z}[\Delta]}(\mathcal{I}, L^*)$ such that:

- $f(\mathcal{I}) \subset U$,
- there exists $\beta(f) \in \mathbb{Z}[\Delta]$ such that for all $\alpha \in L^* \cap U$, $f(\alpha \mathcal{O}) = \alpha^{\beta(f)}$.

One can prove that if $f \in \mathcal{W}$ then $\beta(f)$ is unique, the map $\beta : \mathcal{W} \rightarrow \mathbb{Z}[\Delta]$ is an injective $\mathbb{Z}[\Delta]$ -morphism and $\beta(\mathcal{W}) \subset \text{Ann}_{\mathbb{Z}[\Delta]}(Cl(L))$ (see [2]). If \mathcal{B} denotes the group of Hecke characters of type (A_0) that have values in $\mathbb{Q}(\zeta_p)$ (see [6]), then one can prove that \mathcal{B} is isomorphic to \mathcal{W} .

Lemma 2.1 $\text{Ker } \phi = \beta(\mathcal{W}^-)$.

Proof We just prove the inclusion $\text{Ker } \phi \subset \beta(\mathcal{W}^-)$. Let $x \in \text{Ker } \phi$. Let $\mathcal{A} \in \mathcal{I}$, then there exists a unique $\gamma_a \in L^* \cap U$ such that $\overline{\gamma_a} \gamma_a = 1$ and:

$$\mathcal{A}^x = \gamma_a \mathcal{O}.$$

Let $f : \mathcal{I} \rightarrow L^*$, $\mathcal{A} \mapsto \gamma_a$. It is not difficult to see that $f \in \text{Hom}_{\mathbb{Z}[\Delta]}(\mathcal{I}, L^*)$ and $f(\mathcal{I}) \subset U$. Now, let $\alpha \in L^* \cap U$, we have:

$$f(\alpha \mathcal{O}) = \alpha^x u,$$

for some $u \in E$. Since $x \in \mathbb{Z}[\Delta]^-$ and $\alpha, f(\alpha \mathcal{O}) \in U$, we must have $u = 1$. Therefore $f \in \mathcal{W}^-$ and $x = \beta(f)$. \diamond

Now, we recall some basic properties of Gauss and Jacobi sums (we refer the reader to [11], paragraph 6.1).

Let P be a prime ideal in \mathcal{I} and let ℓ be the prime number such that $\ell \in P$. We fix $\zeta_\ell \in \mu_\ell \setminus \{1\}$. Set $\mathbb{F}_P = \frac{\mathcal{O}}{P}$. Let $\chi_P : \mathbb{F}_P^* \rightarrow \mu_p$, such that:

$$\forall \alpha \in \mathbb{F}_P^*, \chi_P(\alpha) \equiv \alpha^{\frac{1-NP}{p}} \pmod{P},$$

where $NP = |\frac{\mathcal{O}}{P}|$. For $a \in \frac{\mathbb{Z}}{p\mathbb{Z}}$, we set:

$$\tau_a(P) = - \sum_{\alpha \in \mathbb{F}_P} \chi_P^a(\alpha) \zeta_\ell^{\text{Tr}_{\mathbb{F}_P/\mathbb{F}_\ell}(\alpha)}.$$

We also set $\tau(P) = \tau_1(P)$. For $a, b \in \frac{\mathbb{Z}}{p\mathbb{Z}}$, we set:

$$j_{a,b}(P) = - \sum_{\alpha \in \mathbb{F}_P} \chi_P^a(\alpha) \chi_P^b(1 - \alpha).$$

Then:

- if $a + b \equiv 0 \pmod{p}$,
 - i) if $a \not\equiv 0 \pmod{p}$, $j_{a,b}(P) = 1$,
 - ii) if $a \equiv 0 \pmod{p}$, $j_{a,b}(P) = 2 - NP$,
- if $a + b \not\equiv 0 \pmod{p}$, we have:

$$j_{a,b}(P) = \frac{\tau_a(P)\tau_b(P)}{\tau_{a+b}(P)}.$$

Observe that $\tau(P) \equiv 1 \pmod{\pi}$, and therefore (see [5], Theorem 1):

$$\forall a, b \in \frac{\mathbb{Z}}{p\mathbb{Z}}, j_{a,b}(P) \in U.$$

Let Ω be the compositum of the fields $\mathbb{Q}(\zeta_\ell)$ where ℓ runs through the prime numbers distinct from p . The map $P \mapsto \tau(P)$ induces by linearity a $\mathbb{Z}[\Delta]$ -morphism:

$$\tau : \mathcal{I} \rightarrow \Omega(\zeta_p)^*.$$

Let \mathcal{G} be the sub- $\mathbb{Z}[\Delta]$ -module of $\text{Hom}_{\mathbb{Z}[\Delta]}(\mathcal{I}, \Omega(\zeta_p)^*)$ generated by τ . We set:

$$\mathcal{J} = \mathcal{G} \cap \text{Hom}_{\mathbb{Z}[\Delta]}(\mathcal{I}, L^*).$$

Let \mathcal{S} be the Stickelberger ideal of L , i.e. : $\mathcal{S} = \mathbb{Z}[\Delta]\theta \cap \mathbb{Z}[\Delta]$. Then one can prove the following facts (see [2]):

- $\mathcal{J} \subset \mathcal{W}$,

- the map $\beta : \mathcal{W} \rightarrow \mathbb{Z}[\Delta]$ induces an isomorphism of $\mathbb{Z}[\Delta]$ -modules :

$$\mathcal{J} \simeq \mathcal{S}.$$

Lemma 2.2 *Let $\mathcal{N} \in \text{Hom}_{\mathbb{Z}[\Delta]}(I_L, L^*)$ be the ideal norm map. Then, as a \mathbb{Z} -module:*

$$\mathcal{J} = \mathcal{N}\mathbb{Z} \oplus \bigoplus_{n=1}^{(p-1)/2} j_{1,n}\mathbb{Z}.$$

Proof Recall that, for $1 \leq n \leq p-2$, for a prime P in \mathcal{I} , we have:

$$j_{1,n}(P) = - \sum_{\alpha \in \mathbb{F}_P} \chi_P(\alpha) \chi_P^n(1-\alpha) = \frac{\tau(P)\tau_n(P)}{\tau_{n+1}(P)}.$$

Thus, for $1 \leq n \leq p-2$, we have:

$$j_{1,n} = \tau^{1+\sigma_n-\sigma_{1+n}} = \frac{\tau\tau_n}{\tau_{n+1}},$$

where for $a \in \mathbb{F}_p^*$, $\tau^{\sigma^a} = \tau_a$. Observe that:

$$\forall a \in \mathbb{F}_p^*, \tau_a \tau_{-a} = N.$$

Thus $\mathcal{N} \in \mathcal{J}$. Since $\mathcal{J} \simeq \mathcal{S}$, \mathcal{J} is a \mathbb{Z} -module of rank $(p+1)/2$. It is not difficult to show that (see [5], Lemma 2):

$$\mathcal{J} = \tau^p \mathbb{Z} \bigoplus_{a=1}^{(p-1)/2} \tau_{-a} \tau^a \mathbb{Z}.$$

Observe also that, for $2 \leq n \leq p-2$, we have:

$$j_{1,p-n} = j_{1,n-1}.$$

Let V be the sub- \mathbb{Z} -module of \mathcal{J} generated by \mathcal{N} and the $j_{1,n}$, $1 \leq n \leq (p-1)/2$. Then for $1 \leq n \leq p-2$, $j_{1,n} \in V$. Furthermore:

$$\prod_{n=1}^{p-2} j_{1,n} = \frac{\tau^p}{\mathcal{N}}.$$

Therefore $\tau^p \in V$. Since $\tau_{-1}\tau^1 = \mathcal{N}$, $\tau_{-1}\tau^1 \in V$. Now, let $2 \leq r \leq (p-1)/2$ and assume that we have proved that $\tau_{-(r-1)}\tau^{r-1} \in V$. We have:

$$j_{1,r-1} = \frac{\tau\tau_{r-1}}{\tau_r} = \frac{\mathcal{N}\tau\tau_{1-r}^{-1}}{\mathcal{N}\tau_{-r}^{-1}}.$$

Thus:

$$\tau_{-r} = j_{1,r-1}^{-1}\tau_{1-r}\tau^{-1},$$

and

$$\tau_{-r}\tau^r = j_{1,r-1}^{-1}\tau_{-(r-1)}\tau^{r-1}.$$

Thus $\tau_{-r}\tau^r \in V$ and the Lemma follows. \diamond

Lemma 2.3 *Let ℓ be a prime number, $\ell \neq p$. Let P be a prime ideal of \mathcal{O} above ℓ and let $a \in \{1, \dots, p-2\}$. Then $\mathbb{Q}(j_{1,a}(P)) = L$ if and only if $\ell \equiv 1 \pmod{p}$ and $a^2 + a + 1 \not\equiv 0 \pmod{p}$ if $p \equiv 1 \pmod{3}$.*

Proof Since $j_{1,a}(P) \equiv 1 \pmod{\pi^2}$ and $j_{1,a}(P)j_{1,a}(P)^{\sigma^{-1}} = \ell^f$ where f is the order of ℓ in $(\mathbb{Z}/p\mathbb{Z})^*$, we have:

$$\forall \sigma \in \Delta, j_{1,a}(P)^\sigma = j_{1,a}(P) \Leftrightarrow j_{1,a}(P)^\sigma \mathcal{O} = j_{1,a}(P) \mathcal{O}.$$

Recall that:

$$\forall \sigma \in \Delta, j_{1,a}(P)^\sigma \mathcal{O} = j_{1,a}(P) \mathcal{O} \Leftrightarrow P^{(\sigma-1)(1+\sigma_a-\sigma_{1+a})\theta} = \mathcal{O}.$$

Since $j_{1,a}(P)^{\sigma^\ell} = j_{1,a}(P)$, we can assume $\ell \equiv 1 \pmod{p}$. Let $\sigma \in \Delta$, we have to consider the following equation in $\mathbb{C}[\Delta]$:

$$(\sigma - 1)(1 + \sigma_a - \sigma_{1+a})\theta = 0.$$

This is equivalent to:

$$\forall \psi \in \widehat{\Delta}, \psi \text{ odd}, (\psi(\sigma) - 1)(1 + \psi(a) - \psi(1+a)) = 0.$$

Assume that $\omega^3(\sigma) \neq 1$. Then:

$$1 + \omega^3(a) - \omega^3(1+a) = 0.$$

This implies:

$$a^2 + a \equiv 0 \pmod{p},$$

which is a contradiction. Thus $\omega^3(\sigma) = 1$. Let's suppose that $\sigma \neq 1$. We get:

$$1 + \omega(a) = \omega(1 + a),$$

which is equivalent to:

$$a^2 + a + 1 \equiv 0 \pmod{p}.$$

Conversely, one can see that if $p \equiv 1 \pmod{3}$, $a^2 + a + 1 \equiv 0 \pmod{p}$, $\omega^3(\sigma) = 1$, then:

$$\forall \psi \in \widehat{\Delta}, \psi \text{ odd}, (\psi(\sigma) - 1)(1 + \psi(a) - \psi(1 + a)) = 0.$$

The Lemma follows. \diamond

For $x \in \mathbb{Z}_p$, let $[x] \in \{0, \dots, p-1\}$ such that $x \equiv [x] \pmod{p}$. We set:

$$\eta = \left(\prod_{n=1}^{p-2} j_{1,n}^{[n^{-1}]} \right)^{1-\sigma_{-1}} \in \mathcal{J}^-.$$

Lemma 2.4

a) Let $\psi \in \widehat{\Delta}$, $\psi \neq \omega$, ψ odd. Then:

$$e_\psi \left(\sum_{n=1}^{p-2} (1 + \sigma_n - \sigma_{1+n}) [n^{-1}] \right) \in \mathbb{Z}_p^* e_\psi.$$

b) We have:

$$\frac{1}{p} e_\omega \left(\sum_{n=1}^{p-2} (1 + \sigma_n - \sigma_{1+n}) [n^{-1}] \right) \in \mathbb{Z}_p^* e_\omega.$$

Proof

a) Write $\psi = \omega^k$, k odd, $k \in \{3, \dots, p-2\}$. We have:

$$\sum_{n=2}^{p-2} (1 + \psi(n) - \psi(1+n)) [n^{-1}] \equiv \sum_{n=1}^{p-1} \frac{1 + n^k - (1+n)^k}{n} \equiv k \pmod{p}.$$

This implies a).

b) We have:

$$\forall a \in \mathbb{F}_p^*, \omega(a) \equiv a^p \pmod{p^2}.$$

Thus:

$$\frac{1}{p} \sum_{n=1}^{p-2} (1 + \omega(n) - \omega(1+n))[n^{-1}] \equiv - \sum_{n=1}^{p-1} \sum_{k=1}^{p-1} \frac{p!}{(p-k)! k! p} n^{k-1} \pmod{p}.$$

And we get:

$$\frac{1}{p} \sum_{n=1}^{p-2} (1 + \omega(n) - \omega(1+n))[n^{-1}] \equiv -1 \pmod{p}.$$

This implies b). \diamond

Lemma 2.5 *Let ℓ be a prime number, $\ell \neq p$. Let V_ℓ be the sub- $\mathbb{Z}[\Delta]$ -module of $L^*/(L^*)^p$ generated by $\{f(P), f \in \mathcal{J}\}$ where P is some prime of \mathcal{I} above ℓ . Let $\psi \in \widehat{\Delta}$, ψ odd and $\psi \neq \omega$. Then:*

$$V_\ell(\psi) = \mathbb{F}_p e_\psi \eta(P).$$

Proof Let $E = L(\zeta_\ell)$. Then:

$$\frac{L^*}{(L^*)^p}(\psi) \hookrightarrow \frac{E^*}{(E^*)^p}(\psi).$$

Now, in $\frac{E^*}{(E^*)^p}(\psi)$, we have:

$$V_\ell(\psi) = \mathbb{F}_p e_\psi \tau(P).$$

It remains to apply Lemma 2.4. \diamond

Finally, we mention the following Lemma:

Lemma 2.6 *We have:*

$$(\mathcal{J}^- : \mathbb{Z}[\Delta]\eta) = 2^{\frac{p-3}{2}} \frac{1}{p} \prod_{\psi \in \widehat{\Delta}, \psi \text{ odd}} \left(\sum_{n=1}^{p-2} (1 + \psi(n) - \psi(1+n))[n^{-1}] \right).$$

Furthermore:

$$(\mathcal{J}^- : \mathbb{Z}[\Delta]\eta) \not\equiv 0 \pmod{p}.$$

Proof Set $\widetilde{\mathcal{J}}^- = (1 - \sigma_{-1})\mathcal{J} \subset \mathcal{J}^-$. Ten (see [11], paragraph 6.4):

$$(\mathcal{J}^- : \widetilde{\mathcal{J}}^-) = 2^{\frac{p-3}{2}}.$$

Now, by the same kind of arguments as in [11], paragraph 6.4, we get:

$$(\widetilde{\mathcal{J}}^- : \mathbb{Z}[\Delta]\eta) = \frac{1}{p} \prod_{\psi \in \widehat{\Delta}, \psi \text{ odd}} \left(\sum_{n=1}^{p-2} (1 + \psi(n) - \psi(1+n))[n^{-1}] \right).$$

It remains to apply Lemma 2.4. to conclude the proof of this Lemma. \diamond

3 Jacobi Sums and the Ideal Class Group of $\mathbb{Q}(\zeta_p)$

Recall that the Iwasawa-Leopoldt Conjecture asserts that A is a cyclic $\mathbb{Z}_p[\Delta]$ -module. This Conjecture is equivalent to:

$$\forall \psi \in \widehat{\Delta}, \psi \text{ odd}, \psi \neq \omega, A(\psi) \simeq \frac{\mathbb{Z}_p}{B_{1,\psi^{-1}}\mathbb{Z}_p}.$$

It is well-known (see [11], Theorem 10.9) that:

$$\forall \psi \in \widehat{\Delta}, \psi \text{ odd}, \psi \neq \omega, A(\omega\psi^{-1}) = \{0\} \Rightarrow A(\psi) \simeq \frac{\mathbb{Z}_p}{B_{1,\psi^{-1}}\mathbb{Z}_p}.$$

In this paragraph, we will study the links between Jacobi sums and the structure of A^- .

We fix $\psi \in \widehat{\Delta}$, ψ odd and $\psi \neq \omega$. We set:

$$m(\psi) = v_p(B_{1,\psi^{-1}}).$$

Recall that, by [11], paragraph 13.6, we have:

$$|A(\psi)| = p^{m(\psi)}.$$

Let $p^{k(\psi)}$ be the exponent of the group $A(\psi)$. Then:

$$B_{1,\psi^{-1}} \equiv 0 \pmod{p^{k(\psi)}}.$$

Lemma 3.1 *Let P be a prime ideal in \mathcal{I} above a prime number ℓ . Then:*

$$e_\psi \eta(P)\mathcal{O} = 0 \text{ in } \frac{\mathcal{I}}{\mathcal{I}^p} \Leftrightarrow \psi(\ell) \neq 1 \text{ or } B_{1,\psi^{-1}} \equiv 0 \pmod{p}.$$

Proof First note that, if $\rho \in \widehat{\Delta}$, then $e_\rho P = 0$ in $\frac{\mathcal{I}}{\mathcal{I}^p}$ if and only if $\rho(\ell) \neq 1$. By the Stickelberger Theorem, we have:

$$\eta(P)\mathcal{O} = \left(\sum_{n=1}^{p-2} (1 + \sigma_n - \sigma_{1+n})[n^{-1}] \right) (1 - \sigma_{-1})\theta P.$$

Recall that:

$$e_\psi \theta = B_{1,\psi^{-1}} e_\psi.$$

The Lemma follows. \diamond

Lemma 3.2 *Let $f \in \mathcal{W}^-$. Then f lies in \mathcal{W}^p if and only if for all prime ideal $P \in \mathcal{I}$, $f(P) \in (L^*)^p$.*

Proof Let $f \in \mathcal{W}^-$ such that for all prime ideal $P \in \mathcal{I}$, $f(P) \in (L^*)^p$. Let $\mathcal{A} \in \mathcal{I}$. Then there exists $\gamma_a \in L^* \cap U$ such that $\gamma_a \bar{\gamma}_a = 1$ and:

$$f(\mathcal{A}) = \gamma_a^p.$$

Observe that $\beta(f) \in p(\mathbb{Z}[\Delta])^-$. Let $g : \mathcal{I} \rightarrow L^*$, $\mathcal{A} \mapsto \gamma_a$. Then one can verify that $f = g^p$ and $g \in \mathcal{W}^-$. \diamond

Let $m \geq 1$ such that $p^m > |A|$. Set $n = |Cl(L)| / |A|$. Let $e_m(\psi) \in \mathbb{Z}[\Delta]^-$ such that:

$$e_m(\psi) \equiv e_\psi \pmod{p^m}.$$

Set:

$$\beta_\psi = 2np^{k(\psi)}e_m(\psi) \in \mathbb{Z}[\Delta]^-.$$

Since $np^{k(\psi)}e_m(\psi) \in (\text{Ann}_{\mathbb{Z}[\Delta]}Cl(L))^-$, by Lemma 2.1, there exists a unique element $f_\psi \in \mathcal{W}^-$ such that $\beta(f_\psi) = \beta_\psi$. Recall that:

$$(\text{Ann}_{\mathbb{Z}_p[\Delta]}A)(\psi) = p^{k(\psi)}\mathbb{Z}_pe_\psi.$$

Therefore, for $0 \leq k \leq m$, $\frac{W^-}{(W^-)^{p^k}}(\psi)$ is cyclic of order p^k generated by the image of f_ψ . We set:

$$W = \{f(\mathcal{A}), \mathcal{A} \in \mathcal{I}, f \in \mathcal{W}\},$$

and:

$$J = \{f(\mathcal{A}), \mathcal{A} \in \mathcal{I}, f \in \mathcal{J}\}.$$

Observe that J is a sub- $\mathbb{Z}[\Delta]$ -module of W , and it is called the module of Jacobi sums of $\mathbb{Q}(\zeta_p)$. Note that, by Lemma 3.2, we have:

$$\frac{W(L^*)^p}{(L^*)^p}(\psi) \neq \{0\}.$$

Theorem 3.3 *The map f_ψ induces an isomorphism of groups:*

$$A(\psi) \simeq \frac{W(L^*)^{p^{k(\psi)}}}{(L^*)^{p^{k(\psi)}}}(\psi).$$

Proof First observe that $m \geq k(\psi) + 1$. Let P be a prime in \mathcal{I} . Then:

$$f_\psi(P)\mathcal{O} = P^{\beta_\psi}.$$

Let $\rho \in \widehat{\Delta}$, $\rho \neq \psi$. Then:

$$e_m(\rho)e_m(\psi) \equiv 0 \pmod{p^m}.$$

Therefore, there exists $\gamma \in L^* \cap U$ such that:

$$P^{(1-\sigma_{-1})ne_m(\rho)e_m(\psi)} = \left(\frac{\gamma}{\sigma_{-1}(\gamma)}\right)^p \mathcal{O}.$$

But $(1 - \sigma_{-1})e_m(\psi) = 2e_m(\psi)$. Thus, there exists $\alpha \in L^* \cap U$, $\alpha\sigma_{-1}(\alpha) = 1$, and:

$$f_\psi(P)^{e_m(\rho)} = \alpha^{p^{k(\psi)+1}}.$$

Therefore, $e_\rho f_\psi(\mathcal{I}) = 0$ in $L^*/(L^*)^{p^{k(\psi)+1}}$.

It is clear that f_ψ induces a morphism:

$$\frac{\mathcal{I}}{(\mathcal{I})^{p^m}\mathcal{P}}(\psi) \rightarrow \frac{L^*}{(L^*)^{p^{k(\psi)}}}(\psi).$$

Now, let P be a prime in \mathcal{I} such that $e_\psi f_\psi(P) = 0$ in $\frac{L^*}{(L^*)^{p^{k(\psi)}}}(\psi)$. Then, by the above remark, we get:

$$f_\psi(P) = 0 \text{ in } L^*/(L^*)^{p^{k(\psi)}}.$$

Thus, there exists $\gamma \in L^* \cap U$ such that:

$$P^{\beta_\psi} = \gamma^{p^{k(\psi)}} \mathcal{O}.$$

Thus:

$$P^{2ne_m(\psi)} = \gamma \mathcal{O}.$$

This implies:

$$e_\psi P = 0 \text{ in } \frac{\mathcal{I}}{(\mathcal{I})^{p^m}\mathcal{P}}(\psi).$$

Thus our map is injective. Now, observe that the image of the map induced by f_ψ is $\frac{W(L^*)^{p^{k(\psi)}}}{(L^*)^{p^{k(\psi)}}}(\psi)$ and that:

$$A(\psi) \simeq \frac{\mathcal{I}}{(\mathcal{I})^{p^m}\mathcal{P}}(\psi).$$

The Theorem follows. \diamond

Recall that:

$$\eta = \left(\prod_{n=1}^{p-2} j_{1,n}^{[n^{-1}]} \right)^{1-\sigma_{-1}} \in \mathcal{J}^-.$$

Set:

$$z = (1 - \sigma_{-1}) \sum_{n=1}^{p-2} (1 + \sigma_n - \sigma_{1+n}) [n^{-1}] \in \mathbb{Z}[\Delta]^+.$$

We have:

$$\beta(\eta) = z\theta.$$

Corollary 3.4

1) The map η induces an isomorphism of groups:

$$A(\psi) \simeq \frac{J(L^*)^{p^{m(\psi)}}}{(L^*)^{p^{m(\psi)}}}(\psi).$$

2) $\frac{J(L^*)^p}{(L^*)^p}(\psi) \neq \{0\}$ if and only if $A(\psi)$ is \mathbb{Z}_p -cyclic.

Proof

1) Let P be a prime in \mathcal{I} . Then one can show that:

$$f_\psi(P)^{z\theta} = \eta(P)^{2np^{k(\psi)}e_m(\psi)}.$$

The first assertion follows from Theorem 3.3.

2) Note that:

$$A(\psi) \text{ is } \mathbb{Z}_p\text{-cyclic} \Leftrightarrow m(\psi) = k(\psi).$$

Thus, if $A(\psi)$ is \mathbb{Z}_p -cyclic, then:

$$\frac{J(L^*)^p}{(L^*)^p}(\psi) = \frac{W(L^*)^p}{(L^*)^p}(\psi) \neq \{0\}.$$

By the proof of assertion 1), if $k(\psi) < m(\psi)$ and if P is a prime in \mathcal{I} , then:

$$\eta(P)^{e_m(\psi)} \in (L^*)^p.$$

Therefore, we get assertion 2). \diamond

4 The p -adic behavior of Jacobi Sums

Let M be a subgroup of $L^*/(L^*)^p$, we say that M is unramified if $L(^p\sqrt{M})/L$ is an unramified extension. Note that Kummer's Lemma asserts that ([11], Theorem 5.36):

$$\forall \rho \in \widehat{\Delta}, \rho \text{ even}, \rho \neq 1, \frac{E}{E^p}(\rho) \text{ is unramified} \Rightarrow B_{1,\rho\omega^{-1}} \equiv 0 \pmod{p}.$$

It is natural to ask if this implication is in fact an equivalence (see [1], [3]). We will say that the converse of Kummer's Lemma is true for the character ρ if we have:

$$\frac{E}{E^p}(\rho) \text{ is unramified} \Leftrightarrow B_{1,\rho\omega^{-1}} \equiv 0 \pmod{p}.$$

In this paragraph, we will study this question with the help of Jacobi sums.

Let F/L be the maximal abelian p -extension of L which is unramified outside p . Set $\mathcal{X} = \text{Gal}(F/L)$. We have an exact sequence of $\mathbb{Z}_p[\Delta]$ -modules ([11], Corollary 13.6):

$$0 \rightarrow \frac{U}{E} \rightarrow \mathcal{X} \rightarrow A \rightarrow 0.$$

Let $\rho \in \widehat{\Delta}$, observe that:

- if $\rho = 1, \omega$ then $\mathcal{X}(\rho) \simeq \mathbb{Z}_p$,
- if ρ is even, $\rho \neq 1$, $\mathcal{X}(\rho) \simeq \text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\rho)$,
- if ρ is odd, $\rho \neq \omega$, $\mathcal{X}(\rho) \simeq \mathbb{Z}_p \oplus \text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\rho)$.

Lemma 4.1 *Let $\psi \in \widehat{\Delta}$, ψ odd, $\psi \neq \omega$. Then:*

$$d_p \text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\psi) = d_p A(\omega\psi^{-1}).$$

Proof This is a consequence of the proof of Leopoldt's reflection Theorem ([11], Theorem 10.9). For the convenience of the reader, we give the proof of the above Lemma.

Let H be the Galois group of the maximal abelian extension of L which is unramified outside p and of exponent p . Then H is a $\mathbb{Z}_p[\Delta]$ -module and we have:

- $H(1) \simeq \mathbb{F}_p$ and corresponds to $L(\zeta_{p^2})/L$,
- $H(\omega) \simeq \mathbb{F}_p$ and corresponds to $L(^p\sqrt{p})/L$,

- if ρ is even, $\rho \neq 1$, $d_p H(\rho) = d_p \text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\rho)$,
- if ρ is odd, $\rho \neq \omega$, then $d_p H(\rho) = 1 + d_p \text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\rho)$.
Let V be the sub- $\mathbb{Z}[\Delta]$ -module of $L^*/(L^*)^p$ which corresponds to H , i.e. $H = \text{Gal}(L(p\sqrt{V})/L)$. Let M be the sub- $\mathbb{Z}[\Delta]$ -module of $L^*/(L^*)^p$ generated by E and $1 - \zeta_p$. We have an exact sequence:

$$0 \rightarrow M \rightarrow V \rightarrow A[p] \rightarrow 0.$$

Soit $\psi \in \widehat{\Delta}$, ψ odd, $\psi \neq \omega$. We have by Kummer theory :

$$1 + d_p \text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\psi) = d_p V(\omega\psi^{-1}).$$

And, by the above exact sequence, we have:

$$d_p V(\omega\psi^{-1}) = 1 + d_p A(\omega\psi^{-1}).$$

The Lemma follows. \diamond

Lemma 4.2 *Let $\rho \in \widehat{\Delta}$, ρ even and $\rho \neq 1$. If $\frac{E}{E^p}(\rho)$ is ramified then $d_p A(\rho) = d_p A(\omega\rho^{-1})$.*

Proof We keep the notations of the proof of Lemma 4.1. Let $V^{nr} \subset V$ which corresponds via Kummer theory to A/pA . Then:

$$V^{nr}(\rho) \simeq \frac{A}{pA}(\omega\rho^{-1}).$$

But we have:

$\frac{E}{E^p}(\rho)$ is ramified if and only if $V^{nr}(\rho) \hookrightarrow A[p](\rho)$.

Now recall that:

$$d_p A(\rho) \leq d_p A(\omega\rho^{-1}).$$

The Lemma follows. \diamond

Lemma 4.3 *There exists an unique $\mathbb{Z}[\Delta]$ -morphism $\varphi : K^* \rightarrow \mathbb{Z}_p[\Delta]$ such that:*

$$\forall x \in K^*, \varphi(x)\zeta_p = \text{Log}_p(x).$$

Furthermore, we have:

$$\text{Im } \varphi = \bigoplus_{\rho=1,\omega} p\mathbb{Z}_p e_\rho \bigoplus_{\rho \neq 1,\omega} \mathbb{Z}_p e_\rho.$$

Proof Let $\lambda \in K^*$ such that $\lambda^{p-1} = -p$. Then:

$$K^* = \lambda^{\mathbb{Z}} \times \mu_{p-1} \times \mu_p \times U.$$

Recall that:

- the kernel of Log_p on K^* is equal to $\lambda^{\mathbb{Z}} \times \mu_{p-1} \times \mu_p$,
- $\text{Log}_p U = \pi^2 \mathbb{Z}_p[\zeta_p]$.

For $\rho \in \widehat{\Delta}$, set:

$$\tau(\rho) = \sum_{a=1}^{p-1} \rho(a) \zeta_p^a \in \mathbb{Z}_p[\zeta_p].$$

Then:

$$e_\rho \zeta_p = \tau(\rho^{-1}).$$

But recall that that $\mathbb{Z}_p[\zeta_p] = \mathbb{Z}_p[\Delta]\zeta_p$. Thus:

$$e_\rho \mathbb{Z}_p[\zeta_p] = \mathbb{Z}_p \tau(\rho^{-1}).$$

If $\rho = \omega^k$, $k \in \{0, \dots, p-2\}$, we have:

$$v_p(\tau(\rho^{-1})) = \frac{k}{p-1}.$$

Therefore:

$$\pi^2 \mathbb{Z}_p[\zeta_p] = \bigoplus_{\rho=1, \omega} p \mathbb{Z}_p \tau(\rho^{-1}) \bigoplus_{\rho \neq 1, \omega} \mathbb{Z}_p \tau(\rho^{-1}).$$

The Lemma follows. \diamond

Let P be a prime in \mathcal{I} . We fix a generator $r_P \in \mathbb{F}_P^*$ such that:

$$\chi_P(r_P) = \zeta_p.$$

For $x \in \mathbb{F}_P^*$, let $\text{Ind}(P, x) \in \{0, \dots, NP-2\}$ such that:

$$x = r_P^{\text{Ind}(P, x)}.$$

We recall the following Theorem (see also [9] for a statement similar but weaker than part 2) of the following Theorem):

Theorem 4.4

- 1) $\varphi(1 - \zeta_p) = \sum_{\rho \in \widehat{\Delta}, \rho \neq 1, \rho \text{ even}} -(p-1)^{-1} L_p(1, \rho) e_\rho$.
- 2) Let $\psi \in \widehat{\Delta}$, ψ odd, $\psi \neq \omega$. Write $\psi = \omega^k$, $k \in \{2, \dots, p-2\}$. We have:

$$e_\psi \varphi(\eta(P)) \equiv 2k \text{Ind}(P, \prod_{a=1}^{p-1} \left(\frac{1 - \zeta_p^{-a}}{1 - \zeta_p^a} \right)^{a^{k-1}}) e_\psi \pmod{p}.$$

Proof

1) Let $\rho \in \widehat{\Delta}$, ρ even, $\rho \neq 1$. By [11], Theorem 5.18, we have:

$$L_p(1, \rho)\tau(\rho^{-1}) = -(p-1)e_\rho \text{Log}_p(1 - \zeta_p).$$

Thus the first assertion follows.

2) Let $\psi \in \widehat{\Delta}$, ψ odd, $\psi \neq \omega$. By a beautiful result of Uehara ([10], Theorem 1), we have:

$$e_\psi \text{Log}_p(\eta(P)) \equiv 2k \text{Ind}(P, \prod_{a=1}^{p-1} \left(\frac{1 - \zeta_p^{-a}}{1 - \zeta_p}\right)^{a^{k-1}}) \tau(\psi^{-1}) \pmod{p}.$$

This implies the second assertion. \diamond

Theorem 4.5 *Let $\psi \in \widehat{\Delta}$, $\psi \neq \omega$, ψ odd. We have the following exact sequences:*

$$0 \rightarrow \text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\psi) \rightarrow A(\psi) \rightarrow \frac{\overline{W}(\psi)}{U^{p^k(\psi)}(\psi)} \rightarrow 0,$$

$$0 \rightarrow \text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\psi) \rightarrow A(\psi) \rightarrow \frac{\overline{J}(\psi)}{U^{p^m(\psi)}(\psi)} \rightarrow 0.$$

Proof This Theorem is a consequence of the method developed by Iwasawa in [5]. Let's recall briefly this method.

Let $f \in \mathcal{W}$. Set, for $n \geq 2$, $\mathcal{P}_n = \{\alpha \mathcal{O}, \alpha \equiv 1 \pmod{\pi^n}\}$. Observe that:

$$f(\mathcal{P}_n) \subset 1 + \pi^n \mathbb{Z}_p[\zeta_p].$$

Let:

$$\tilde{\mathcal{X}} = \varprojlim \frac{\mathcal{I}}{\mathcal{P}_n}.$$

If \tilde{F} is the maximal abelian extension of L which is unramified outside p , then, we get by class field theory:

$$\tilde{\mathcal{X}} \simeq \text{Gal}(\tilde{F}/L).$$

By [11], Theorem 13.4, the natural surjective map $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ has a finite kernel of order prime to p . Thus f induces a map:

$$\overline{f} : \mathcal{X} \rightarrow U.$$

Furthermore:

$$\overline{f}(U) = U^{\beta(f)} \subset \overline{f}(\mathcal{X}).$$

Now let $\psi \in \widehat{\Delta}$, ψ odd, $\psi \neq \omega$. We have a map:

$$\overline{f} : \mathcal{X}(\psi) \rightarrow U(\psi).$$

But:

$$\mathcal{X}(\psi) \simeq \mathbb{Z}_p \bigoplus \text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\psi),$$

and:

$$U(\psi) \simeq \mathbb{Z}_p.$$

Thus, if $e_\psi \beta(f) \neq 0$, we get:

$$\text{Ker}(\overline{f} : \mathcal{X}(\psi) \rightarrow U(\psi)) = \text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\psi).$$

Therefore, if $e_\psi \beta(f) \neq 0$, we get the following exact sequence induced by f :

$$0 \rightarrow \text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\psi) \rightarrow A(\psi) \rightarrow \frac{\overline{f}(\mathcal{X})(\psi)}{U^{\beta(f)}(\psi)} \rightarrow 0.$$

It remains to apply this construction to f_ψ and η to get the desired exact sequences. \diamond

Corollary 4.6

1) Let $\psi \in \widehat{\Delta}$, ψ odd, $\psi \neq \omega$. Then:

$$d_p A(\psi) = 1 + d_p A(\omega\psi^{-1}) \Leftrightarrow B_{1,\psi^{-1}} \equiv 0 \pmod{p} \text{ and } \overline{W}(\psi) = U(\psi).$$

2) Let $\rho \in \widehat{\Delta}$, ρ even and $\rho \neq 1$. Assume that $B_{1,\rho\omega^{-1}} \equiv 0 \pmod{p}$ and that $\overline{W}(\omega\rho^{-1}) = U(\omega\rho^{-1})$ then the converse of Kummer's Lemma is true for the character ρ .

Proof

1) We apply Theorem 4.5. We identify $\text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\psi)$ with its image in $A(\psi)$. We can write:

$$A(\psi) = B \bigoplus C,$$

where C is cyclic of order $p^{k(\psi)}$ and $B \subset \text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\psi)$. Now:

$$(C : C \cap \text{Tor}_{\mathbb{Z}_p} \mathcal{X}(\psi)) = (\overline{W}(\psi) : U^{p^{k(\psi)}}(\psi)).$$

It remains to apply Lemma 4.1 to get the desired result.

2) We apply the first assertion and Lemma 4.1, we get:

$$d_p A(\rho) = d_p A(\omega \rho^{-1}) - 1.$$

It remains to apply Lemma 4.2. \diamond

We set:

$$W^{nr} = \{\alpha \in W, \alpha \in U^p\}.$$

Let $\psi \in \widehat{\Delta}$, ψ odd, $\psi \neq \omega$. We assume that $B_{1,\psi^{-1}} \equiv 0 \pmod{p}$. Write:

$$A(\psi) = \frac{\mathbb{Z}}{p^{e_1}\mathbb{Z}} \bigoplus \cdots \bigoplus \frac{\mathbb{Z}}{p^{e_t}\mathbb{Z}},$$

where $t = d_p A(\psi)$ and $1 \leq e_1 \leq \cdots \leq e_t = k(\psi)$. Set:

$$n(\psi) = |\{i \in \{1, \dots, t\}, e_i = k(\psi)\}|.$$

Corollary 4.7 *We have:*

$$n(\psi) - 1 \leq \dim_{\mathbb{F}_p} \frac{W^{nr}(L^*)^p}{(L^*)^p} \leq n(\psi).$$

Furthermore:

$$\dim_{\mathbb{F}_p} \frac{W^{nr}(L^*)^p}{(L^*)^p} = n(\psi) \Leftrightarrow \overline{W}(\psi) \neq U(\psi).$$

Proof By Theorem 4.5 and Theorem 3.3, we have:

$$\frac{W^{nr}(L^*)^{p^{k(\psi)}}}{(L^*)^{p^{k(\psi)}}} \simeq \text{Ker}(A(\psi) \rightarrow \frac{\overline{W}(\psi)}{Up^{k(\psi)}(\psi)}).$$

The Corollary follows. \diamond

Corollary 4.8 *Assume that $pA^- = \{0\}$. Then we have an isomorphism of groups:*

$$\text{Gal}(L(\sqrt[p]{W^{nr}})/L) \simeq \frac{A^+}{pA^+}.$$

Proof This result is a consequence of Kummer theory, Corollary 4.7 and Corollary 4.6. \diamond

Note that the above results lead us to ask the following problem (which is a restatement of the converse of Kummer's Lemma):

do we have $\varphi(\overline{W}^-) = (\text{Im } \varphi)^-$?

Observe that $e_\omega \varphi(\overline{W}^-) = e_\omega (\text{Im } \varphi)^-$, and since $K_4(\mathbb{Z}) = \{0\}$, we have $A(\omega^{-2}) = \{0\}$ (see [7]) and therefore $e_{\omega^3} \varphi(\overline{W}^-) = e_{\omega^3} (\text{Im } \varphi)^-$.

5 Remarks on the Jacobian of the Fermat Curve over a finite field

First we fix some notations and recall some basic facts about global function fields.

Let \mathbb{F}_q be a finite field having q elements. Let ℓ be the characteristic of \mathbb{F}_q , $\ell \neq p$. Let $\overline{\mathbb{F}_q}$ be a fixed algebraic closure of \mathbb{F}_q and let $\widetilde{\mathbb{F}_q} = \bigcup_{n \geq 1, n \not\equiv 0 \pmod{p}} \mathbb{F}_{q^n} \subset \overline{\mathbb{F}_q}$. Let k/\mathbb{F}_q be a global function field such that \mathbb{F}_q is algebraically closed in k . We set:

- D_k : the group of divisors of k ,
- D_k^0 : the group of divisors of degree zero of k ,
- P_k : the group of principal divisors of k ,
- J_k : the jacobian of k , note that we have:

$$\forall n \geq 1, J_k(\mathbb{F}_{q^n}) \simeq \frac{D_{\mathbb{F}_{q^n} k}^0}{P_{\mathbb{F}_{q^n} k}},$$

- g_k : the genus of k ,
- $L_k(Z) \in \mathbb{Z}[Z]$: the numerator of the zeta function of k , we recall that:

$$\frac{L_k(Z)}{(1-Z)(1-qZ)} = \prod_{v \text{ place of } k} (1 - Z^{\deg v})^{-1},$$

furthermore $\deg_Z L_k(Z) = 2g_k$ and $L_k(1) = |J_k(\mathbb{F}_q)|$,

- $C_k(\mathbb{F}_{q^n}) = J_k(\mathbb{F}_{q^n}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$,
- $\widetilde{d}_p J_k = d_p C_k(\widetilde{\mathbb{F}_q})$, observe that there exists an integer m , $m \not\equiv 0 \pmod{p}$, such that :

$$C_k(\widetilde{\mathbb{F}_q}) = C_k(\mathbb{F}_{q^m}).$$

Write:

$$L_k(Z) = \prod_{i=1}^{2g_k} (1 - \alpha_i Z).$$

For simplicity, we assume that $v_p(\alpha_i - 1) > 0$ for $i = 1, \dots, 2g_k$. In this case, we have:

$$C_k(\tilde{\mathbb{F}}_q) = C_k(\mathbb{F}_q).$$

Set $P_k(Z) = \prod_{i=1}^{2g_k} (Z - (\alpha_i - 1))$. Let γ be the Frobenius of \mathbb{F}_q , and set:

$$C_n(k) = C_k(\mathbb{F}_{q^{p^n}}).$$

Let $C_\infty(k) = \cup_{n \geq 0} C_n(k)$, and set:

$$M_k = \text{Hom}\left(\frac{\mathbb{Q}_p}{\mathbb{Z}_p}, C_\infty(k)\right).$$

Then M_k is isomorphic to the p -adic Tate-module of J_k . Set $\Lambda = \mathbb{Z}_p[[Z]]$ where Z corresponds to $\gamma - 1$. Then it is well-known that:

- M_k is a Λ -module of finite type and of torsion,
- as a \mathbb{Z}_p -module M_k is isomorphic to $\mathbb{Z}_p^{2g_k}$,
- $M_k/\omega_n M_k \simeq C_n(k)$, where $\omega_n = (1 + Z)^{p^n} - 1$,
- $\text{Char}_\Lambda M_k = P_k(Z)\Lambda$,
- the action of Z on M_k is semi-simple, i.e. the minimal polynomial of the action of Z on M_k has only simple roots.

Now, let ℓ be a prime number, $\ell \neq p$. We fix a prime P of \mathcal{O} above ℓ and we view \mathcal{O}/P as a subfield of $\overline{\mathbb{F}}_\ell$, thus $\mathbb{F}_q = \mathcal{O}/P \subset \tilde{\mathbb{F}}_\ell$. We identify ζ_p with its image in \mathbb{F}_q . Let X be an indeterminate over \mathbb{F}_q , we set $k = \mathbb{F}_\ell(X, Y)$ where $X^p + Y^p = 1$, and we set: $T = X^p$. For $a, b \in \mathbb{Z}$, let $\tau_{a,b} \in \text{Gal}(\overline{\mathbb{F}}_\ell k / \overline{\mathbb{F}}_\ell(T))$ such that:

$$\tau_{a,b}(X) = \zeta_p^a X \text{ and } \tau_{a,b}(Y) = \zeta_p^b Y.$$

Let $a \in \{1, \dots, p-2\}$. Let H_a be the subgroup of $\text{Gal}(\overline{\mathbb{F}}_\ell k / \overline{\mathbb{F}}_\ell(T))$ generated by $\tau_{1,[-a-1]}$. Set:

$$E_a = \mathbb{F}_\ell(T, XY^a).$$

We set:

$$E = \mathbb{F}_q E_a,$$

$$F = \mathbb{F}_q k,$$

and observe that:

$$\widetilde{\mathbb{F}}_\ell = \widetilde{\mathbb{F}}_q.$$

It is clear that:

$$F^{H_a} = E.$$

Finally, we set:

$$G = \text{Gal}(E/\mathbb{F}_q(T)).$$

Note that $g_E = (p-1)/2$.

Lemma 5.1 *We have:*

$$L_E(Z) = \prod_{\sigma \in \Delta} (1 - j_{1,a}(P)^\sigma Z).$$

Proof Let $\chi \in \widehat{G}$ such that:

$$\chi(g) = \zeta_p^{-1},$$

where $g \in G$ is such that $g(XY^a) = \zeta_p XY^a$. Note that:

$$L_E(Z) = \prod_{\sigma \in \Delta} L(Z, \chi^\sigma),$$

where:

$$L(Z, \chi) = \prod_{v \text{ place of } \mathbb{F}_q(T)} (1 - \chi(v) Z^{\deg v})^{-1}.$$

Since $2g_e = p-1$, we get:

$$\deg_Z L(Z, \chi) = 1.$$

For $b \in \mathbb{F}_q \setminus \{0, 1\}$, we denote the Frobenius of $T-b$ in $E/\mathbb{F}_q(T)$ by $Frob_b$.

We have:

$$Frob_b(XY^a) = (b(1-b)^a)^{(q-1)/p} XY^a.$$

But:

$$L(Z, \chi) \equiv 1 + \left(\sum_{b \in \mathbb{F}_q \setminus \{0,1\}} \chi(Frob_b) \right) X \pmod{X^2}.$$

Thus:

$$L(Z, \chi) = 1 + \left(\sum_{b \in \mathbb{F}_q \setminus \{0,1\}} \chi(Frob_b) \right) X.$$

But, we can write:

$$j_{1,a}(P) = - \sum_{i=0}^{p-1} N_i \zeta_p^{-i},$$

where $N_i = |\{ \alpha \in \mathbb{F}_q \setminus \{0, 1\}, (\alpha(1 - \alpha)^a)^{(q-1)/p} \equiv \zeta_p^{-i} \pmod{P} \}|$.
Therefore:

$$j_{1,a}(P) = - \sum_{b \in \mathbb{F}_q \setminus \{0,1\}} \chi(\overline{Frob}_b).$$

The Lemma follows. \diamond

Theorem 5.2 *Let n be the smallest integer (if it exists) such that $3 \leq n \leq p - 2$, n odd and $e_{\omega^n} j_{1,a}(P) \notin U^p$, then:*

$$J_k(\widetilde{\mathbb{F}}_\ell)^{H_a} \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^n.$$

If such an integer doesn't exist then:

- 1) $\widetilde{d}_p J_k^{H_a} = p - 1$,
- 2) *we have:*

$$J_k(\widetilde{\mathbb{F}}_\ell)^{H_a} \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{p-1} \Leftrightarrow \ell^{p-1} \not\equiv 1 \pmod{p^2}.$$

Proof The proof of this result is based on ideas developed by Greenberg in [4]. Set $H = H_a$. Let P_0 be the prime of E above T , P_1 the prime of E above $T - 1$ and P_∞ the prime of E above $\frac{1}{T}$. Recall that we have in D_E :

$$p(P_0 - P_\infty) = (T),$$

$$p(P_1 - P_\infty) = (T - 1),$$

$$P_0 - P_\infty + a(P_1 - P_\infty) = (XY^a).$$

Thus, by [4], paragraph 2, we get:

$$J_E(\mathbb{F}_q)^G \simeq \frac{\mathbb{Z}}{p\mathbb{Z}},$$

and $J_E(\mathbb{F}_q)^G$ is generated by the class of $P_0 - P_\infty$. Observe also that F/E is unramified and cyclic of order p . Let's start by the following exact sequence:

$$0 \rightarrow \mathbb{F}_q^* \rightarrow F^* \rightarrow P_F \rightarrow 0.$$

We get:

$$\frac{P_F^H}{P_E} \simeq \frac{\mathbb{Z}}{p\mathbb{Z}},$$

and $\frac{P_F^H}{P_E}$ is generated by the image of $P_0 - P_\infty$ in D_F . In particular:

$$\frac{P_F^H}{P_E} \simeq J_E(\mathbb{F}_q)^G.$$

Note that we also have:

$$0 \rightarrow H^1(H, P_F) \rightarrow H^2(H, \mathbb{F}_q^*) \rightarrow H^2(H, F^*).$$

But F/E is unramified and cyclic, therefore every element of \mathbb{F}_q^* is a norm in the extension F/E . Thus:

$$H^1(H, P_F) \simeq \frac{\mathbb{Z}}{p\mathbb{Z}}.$$

Now, we look at the exact sequence:

$$0 \rightarrow P_F \rightarrow D_F^0 \rightarrow J_F(\mathbb{F}_q) \rightarrow 0.$$

Since F/E is unramified:

$$H^1(H, D_F^0) = \{0\}.$$

Therefore, we have obtained the following exact sequence:

$$0 \rightarrow J_E(\mathbb{F}_q)^G \rightarrow J_E(\mathbb{F}_q) \rightarrow J_F(\mathbb{F}_q)^H \rightarrow \frac{\mathbb{Z}}{p\mathbb{Z}} \rightarrow 0.$$

Now, it is not difficult to deduce that, for all $n \geq 1$, we have the following exact sequence:

$$0 \rightarrow \frac{\mathbb{Z}}{p\mathbb{Z}} \rightarrow J_E(\mathbb{F}_{q^n}) \rightarrow J_F(\mathbb{F}_{q^n})^H \rightarrow \frac{\mathbb{Z}}{p\mathbb{Z}} \rightarrow 0.$$

From this, we get the following exact sequence of $\mathbb{Z}_p[G]$ -modules and Λ -modules:

$$0 \rightarrow M_E \rightarrow M_F^H \rightarrow \frac{\mathbb{Z}}{p\mathbb{Z}} \rightarrow 0.$$

Recall that in our situation, by Lemma 5.1, we have:

$$P_E(Z) = \prod_{\sigma \in \Delta} (Z - (j_{1,a}(P)^\sigma - 1)).$$

Furthermore the action of G and Z commute on M_F^H . Now, we have:

- $\text{Char}_\Lambda M_F^H = \text{Char}_\Lambda M_E = P_E(Z)\Lambda$,
- $M_F^H \simeq \mathbb{Z}_p^{p-1}$ as \mathbb{Z}_p -modules,
- $M_F^H/\omega_n \simeq C_n(F)^H$.

Observe that:

$$C_0(F)^H = J_k(\tilde{\mathbb{F}}_\ell)^{H_a} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

Note also that the minimal polynomial of the action of Z on M_F^H is:

$$\text{Irr}(j_{1,a}(P) - 1, \mathbb{Q}_p; Z) := G(Z).$$

Set $N = \sum_{\delta \in G} \delta$. Then one can see that:

$$NM_E = NM_F^H = \{0\}.$$

Thus M_F^H is a $\mathbb{Z}_p[G]/N\mathbb{Z}_p[G]$ -module. Now, we identify $\mathbb{Z}_p[G]/N\mathbb{Z}_p[G]$ with $\mathbb{Z}_p[\zeta_p]$. Since $M_F^H \simeq \mathbb{Z}_p^{p-1}$, there exists $m \in M_F^H$ such that:

$$M_F^H \simeq \mathbb{Z}_p[\zeta_p].m,$$

i.e. M_F^H is a free $\mathbb{Z}_p[\zeta_p]$ -module of rank one. Therefore there exists an element $x \in \mathbb{Z}_p[\zeta_p]$ such that:

$$Zm = xm.$$

Now set:

$$D(Z) = \prod_{\sigma \in \Delta} (Z - x^\sigma) \in \Lambda.$$

Then:

$$D(Z)M_F^H = \{0\}.$$

Therefore $G(Z)$ divides $D(Z)$ in Λ . Thus there exists $\sigma \in \Delta$ such that:

$$x^\sigma = j_{1,a}(P) - 1.$$

But:

$$C_0(F)^H \simeq \frac{M_F^H}{ZM_F^H} \simeq \frac{\mathbb{Z}_p[\zeta_p]}{x\mathbb{Z}_p[\zeta_p]}.$$

Therefore, we get:

$$J_k(\tilde{\mathbb{F}}_\ell)^{H_a} \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \frac{\mathbb{Z}_p[\zeta_p]}{(j_{1,a}(P) - 1)\mathbb{Z}_p[\zeta_p]}.$$

Recall that $j_{1,a}(P) \equiv 1 \pmod{\pi^2}$. Thus:

$$v_p(j_{1,a}(P) - 1) = v_p(\text{Log}_p(j_{1,a}(P))).$$

Now:

$$\text{Log}_p(j_{1,a}(P)) = \frac{1}{2}f\text{Log}_p(\ell) + \sum_{\psi \in \hat{\Delta}, \psi \text{ odd}} e_\psi \text{Log}_p(j_{1,a}(P)),$$

where f is the order of ℓ in $(\mathbb{Z}/p\mathbb{Z})^*$. Let $\psi \in \hat{\Delta}$, $\psi = \omega^n$, n odd. If $e_\psi \text{Log}_p(j_{1,a}(P)) \neq 0$, then:

$$v_p(e_\psi \text{Log}_p(j_{1,a}(P))) \equiv \frac{n}{p-1} \pmod{\mathbb{Z}},$$

furthermore:

$$v_p(e_\psi \text{Log}_p(j_{1,a}(P))) > \frac{n}{p-1} \Leftrightarrow e_\psi j_{1,a}(P) \in U^p.$$

Note also that:

$$v_p(e_\omega \text{Log}_p(j_{1,a}(P))) > \frac{1}{p-1}.$$

The Theorem follows. \diamond

Corollary 5.3 *Let $n \in \{3, \dots, p-2\}$, n odd. Let $a \in \{1, \dots, p-2\}$ such that $1+a^n - (1+a)^n \not\equiv 0 \pmod{p}$. The following assertions are equivalent:*

- 1) $A(\omega^{1-n}) = \{0\}$,
- 2) *there exists a prime number ℓ , $\ell \neq p$, such that $\tilde{d}_p J_k^{H_a} = n$.*

Proof Observe that 2) implies 1) by the above Theorem and Theorem 4.5.

Write $\psi = \omega^n$. Let ℓ be a prime number, $\ell \neq p$. Write:

$$\mathbb{F}_{(\ell)} = \frac{\mathcal{O}}{\ell\mathcal{O}},$$

and:

$$D_\ell = \frac{\mathbb{F}_{(\ell)}^*}{(\mathbb{F}_{(\ell)}^*)^p}.$$

Observe that D_ℓ is a $\mathbb{Z}_p[\Delta]$ -module. Let Cyc be the group of cyclotomic units of L . We denote the image of Cyc in D_ℓ by \overline{Cyc}^ℓ . Then Theorem 4.4 asserts that $e_\psi \overline{Cyc}^\ell = \{1\}$ in D_ℓ if and only if $e_\psi j_{1,a}(P) \in U^p$, where P is a prime of \mathcal{O} above ℓ . Let:

$$B = L(\sqrt[p]{\overline{Cyc}}).$$

We assume that 1) holds. We apply the Chebotarev density theorem to the extension B/L , then there exist infinitely many primes ℓ such that:

- $e_\rho \overline{Cyc}^\ell = \{1\}$ for $\rho \neq \psi$,
- $e_\psi \overline{Cyc}^\ell \neq \{1\}$.

It remains to apply Theorem 5.2 and the above remarks to get 2). \diamond

Now, let ℓ be a prime number. Let p be an odd prime number, $p \neq \ell$. Let T be an indeterminate over \mathbb{F}_ℓ and let $E_p/\mathbb{F}_\ell(T)$ be the imaginary quadratic extension defined by:

$$E_p = \mathbb{F}_\ell(T, X) \text{ where } X^2 - X + T^p = 0.$$

Let n be an odd integer, $n \geq 3$. Let $S_n(\ell)$ denote the set of primes p such that $\tilde{d}_p J_{E_p} = n$. By our above results, we remark that if $p \in S_n(\ell)$ then $A(\omega^{1-n}) = \{0\}$. Observe that if $\ell^n \not\equiv 1 \pmod{p}$ then $p \notin S_n(\ell)$, and therefore $S_n(\ell)$ is a finite set. Set $S(\ell) = \cup_n S_n(\ell)$, where n runs through the odd integers. Observe that if the order of ℓ modulo p is even then $p \notin S(\ell)$. Therefore, by a classical result of Hasse (see [8]) there exist infinitely many prime p not in $S(\ell)$ (in fact at least "2/3 of the prime numbers" are not in $S(\ell)$). Thus, we ask the following question:

is $S(\ell)$ infinite?

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