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# A semi-classical inverse problem II: reconstruction of the potential

Yves Colin de Verdière\*

February 12, 2008

## 1 Introduction

This paper is the continuation of [4], where Victor Guillemin and I proved the following result: the Taylor expansion of the potential  $V(x)$  ( $x \in \mathbb{R}$ ) at a non degenerate critical point  $x_0$  of  $V$ , satisfying  $V'''(x_0) \neq 0$ , is determined by the semi-classical spectrum of the associated Schrödinger operator near the corresponding critical value  $V(x_0)$ . Here, I prove results which are stronger in some aspects: the potential itself, without any analyticity assumption, but with some genericity conditions, is determined from the semi-classical spectrum. Moreover, our method gives an explicit way to reconstruct the potential.

Inverse spectral results for Sturm-Liouville operators are due to Borg, Gelfand, Levitan, Marchenko and others (see for example [8]). They need the spectra of the differential operator with two different boundary conditions in order to recover the potential. Our results are different in several aspects:

- They are local using only the part of the spectrum included in some interval  $]-\infty, E[$  in order to get  $V$  in the inverse image  $\{x|V(x) < E\}$  of this interval.
- They need only approximate spectra.
- They still apply if the operator is essentially self-adjoint.

After having completed the present work, I founded that similar methods were already used by David Gurarie [7] in order to recover a surface of revolution from the joint spectrum of the Laplace operator and the momentum operator  $L_z$ . Our genericity assumptions are weaker and more explicit:

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- David Gurarie assumes that the potential is a Morse function with pairwise different critical values, while we assume only a weak non degeneracy condition (see Section 10.1.1).
- His argument for the separation of spectra associated to the different wells is less explicit than ours which uses the semi-classical trace formula (see Section 11.3).
- He does not say a word about the problem of a non generic symmetry defect and explicit non isomorphic potentials with the same semi-classical spectra (Section 7 and Assumption 3 in Theorem 5.1).

For a recent review on the use of semi-classics in inverse spectral problems, the reader could look at [9].

## 2 Motivation I: surfaces of revolution

Let us consider a surface of revolution with a metric

$$ds^2 = dx^2 + a^4(x)dy^2$$

with  $x \in [0, L]$  and  $y \in \mathbb{R}/2\pi\mathbb{Z}$ . We assume that  $a(0) = a(L) = 0$ ,  $a(x) > 0$  for  $0 < x < L$  and  $a$  is smooth. The volume element is given by  $dv = a^2(x)|dxdy|$ . The Laplace operator is:

$$\Delta = -\frac{\partial^2}{\partial x^2} - \frac{2a'}{a} \frac{\partial}{\partial x} - \frac{1}{a^4} \frac{\partial^2}{\partial y^2} .$$

Using the change of function  $f = Fa$ , we get the operator  $P = a\Delta a^{-1}$  which is formally symmetric w.r. to  $|dxdy|$ :

$$P = -\frac{\partial^2}{\partial x^2} + \frac{a''}{a} - \frac{1}{a^4} \frac{\partial^2}{\partial y^2} .$$

If  $F(x, y) = \varphi(x)\exp(ily)$  with  $l \in \mathbb{Z}$ , we define  $Q_l$  as follows

$$PF = l^2(Q_l\varphi)e^{ily} ,$$

and putting  $\hbar = l^{-1}$ , we get

$$Q_\hbar\varphi = -\hbar^2\varphi'' + (a^{-4} + \hbar^2W)\varphi$$

with  $W = \frac{a''}{a}$ . It implies that the knowledge of the joint spectrum of  $\Delta$  and  $\partial_y$  is closely related to the spectra of  $Q_\hbar$  for  $\hbar = 1/l$  with  $l \in \mathbb{Z} \setminus 0$ . This relates our paper to Gurarie's result [7].

### 3 Motivation II: effective surface waves Hamiltonian

In our paper [2], we started with the following acoustic wave equation<sup>1</sup>

$$\begin{cases} u_{tt} - \operatorname{div}(n \operatorname{grad} u) = 0 \\ u(\mathbf{x}, 0, t) = 0 \end{cases} \quad (1)$$

in the half space  $X = \mathbb{R}_{\mathbf{x}}^{d-1} \times ]-\infty, 0]_z$  where  $n(z) : \mathbb{R}_- \rightarrow \mathbb{R}_+$  is a non negative function which satisfies

$$0 < n_0 := \inf n(z) < n_\infty := \liminf_{z \rightarrow -\infty} n(z) .$$

This equation describes the propagation of acoustic waves in a medium which is stratified: the variations of the density are on much smaller scales vertically than horizontally<sup>2</sup>. This equation admits solutions of the form  $\exp(i(\omega t - \mathbf{x}\xi))v(z)$  provided that  $v$  is an eigenfunction of the operator  $L_\xi$  on the half line  $z \leq 0$  defined as follows:

$$L_\xi v := -\frac{d}{dz} \left( n(z) \frac{dv}{dz} \right) + n(z) |\xi|^2 v \quad (2)$$

with Dirichlet boundary conditions and eigenvalue  $\omega^2$ . These solutions are exponentially localized near the boundary provided that  $\omega^2$  is in the discrete spectrum of  $L_\xi$  contained in  $J := ]n_0|\xi|^2, n_\infty|\xi|^2[$ .

Let us denote by  $\lambda_1(\xi) < \lambda_2(\xi) < \dots < \lambda_j(\xi) < \dots$  the spectrum of  $L_\xi$  in the interval  $J$  and  $v_j(\xi, z)$  the associated normalized eigenfunctions. The unitary map from  $L^2(\partial X)$  into  $L^2(X)$  defined by

$$T_j(a) := (2\pi)^{-(d-1)} \int_{\mathbb{R}^{d-1}} \hat{a}(\xi) v_j(\xi, z) e^{i\mathbf{x}\xi} d\xi ,$$

with  $\hat{a}(\xi) := \int_{\mathbb{R}^{d-1}} a(\mathbf{x}) e^{-i\mathbf{x}\xi} d\mathbf{x}$ , satisfies:

$$PT_j = T_j \operatorname{Op}(\lambda_j) ,$$

where  $P = -\operatorname{div}(n \operatorname{grad} u)$  with Dirichlet boundary conditions and  $\operatorname{Op}(\lambda_j)$  is an elliptic pseudo-differential operator of degree 2 and of symbol  $\lambda_j$ . So that, for each  $j = 1, \dots$ , we get an effective surface wave Hamiltonian with the Hamiltonian

<sup>1</sup> $u = u(\mathbf{x}, z, t)$  is the pressure,  $n = K/\rho$  with  $\rho$  the density and  $K > 0$  the incompressibility assumed to be a constant. The acoustic wave equation is a simplification of the elastic wave equation which holds if the medium is fluid.

<sup>2</sup>In [2], we took a more complicated function  $n(\mathbf{x}, z) = N(\mathbf{x}, z/\varepsilon, z)$  with  $N$  smooth and  $\varepsilon$  small

$\lambda_j$ . The map  $T : \oplus_{j=1}^{\infty} L^2(\partial X) \rightarrow L^2(X)$  given by  $T = \oplus_{j=1}^{\infty} T_j$  is an injective isometry.

We see that the high frequency surface waves are associated to the semi-classical spectrum of a Schrödinger type operator

$$\mathcal{L}_{\hbar} = -\hbar^2 \frac{d}{dz} \left( n(z) \frac{d}{dz} \right) + n(z) ,$$

with  $\hbar = \|\xi\|^{-1}$ . One can try to recover  $n(z)$  from the propagation of surface waves: this is equivalent to get the operator  $\mathcal{L}_{\hbar}$  from its semi-classical spectrum.

## 4 Some notations

The following notations will be used everywhere in this paper. The *interval*  $I$  is defined by  $I = ]a, b[$  with  $-\infty \leq a < b \leq +\infty$ . The *potential*  $V : I = ]a, b[ \rightarrow \mathbb{R}$  is a smooth function with  $-\infty < E_0 := \inf V < E_{\infty} = \liminf_{x \rightarrow \partial I} V(x)$ . We will denote by  $\hat{H}$  any self-adjoint extension of the operator  $-\hbar^2 \frac{d^2}{dx^2} + V(x)$  defined on  $C_0^{\infty}(I)$ . The discrete spectrum of  $\hat{H}_{\hbar}$  will be denoted by

$$(E_0 <) \lambda_1(\hbar) < \lambda_2(\hbar) < \dots < \lambda_l(\hbar) < \dots .$$

The semi-classical limit is associated to the classical Hamiltonian  $H = \xi^2 + V(x)$  and the dynamics  $dx/dt = \xi$ ,  $d\xi/dt = -V'(x)$ .

**Definition 4.1** *We say that  $\mu_l(\hbar)$  is a semi-classical spectrum of  $\hat{H}$  mod  $o(\hbar^N)$  in  $[E_0, E]$  if, for any  $F < E$ ,*

$$\left( \sum_{\lambda_l(\hbar) \leq F} |\lambda_l(\hbar) - \mu_l(\hbar)|^2 \right)^{\frac{1}{2}} = o(\hbar^{N-\frac{1}{2}}) .$$

If we have a uniform approximation of the eigenvalues up to  $o(\hbar^N)$ , it is also a semi-classical spectrum of  $\hat{H}$  mod  $o(\hbar^N)$  in the previous  $l^2$  sense because the number of eigenvalues in  $] -\infty, F]$  is  $O(\hbar^{-1})$ .

## 5 A Theorem for one well potentials

**Theorem 5.1** *Let us assume that the potential  $V : I \rightarrow \mathbb{R}$  satisfies:*

1. **A single well below  $E$ :** *there exists  $E \leq E_{\infty}$  so that, for any  $y \leq E$ , the sets  $I_y := \{x | V(x) \leq y\}$  are connected. The intervals  $I_y$  are compact for  $y < E$ . There exists a unique  $x_0$  so that  $V(x_0) = E_0 (= \inf_{x \in I} V(x))$ . For any  $y$  with  $E_0 < y \leq E$ , if the interval  $I_y$  is defined by  $I_y = [f_-(y), f_+(y)]$ , we have  $V'(x_0) = 0$ ,  $V'(x) < 0$  for  $f_-(E) < x < x_0$  and  $V'(x) > 0$  for  $x_0 < x < f_+(E)$ .*

2. **A genericity hypothesis at the minimum:** *there exists  $N \geq 2$  so that the  $N$ -th derivative  $V^{(N)}(x_0)$  does not vanish.*
3. **A generic symmetry defect:** *if there exists  $x_{\pm}$ , satisfying  $f_-(E) < x_- < x_+ < f_+(E)$  and  $\forall n \in \mathbb{N}$ ,  $V^{(n)}(x_-) = (-1)^n V^{(n)}(x_+)$ , then  $V$  is globally even w.r. to  $x_0 = (x_- + x_+)/2$  on the interval  $I_E$ . This is true for example if  $V$  is real analytic.*

Then the spectra modulo  $o(\hbar^2)$  in the interval  $] - \infty, E[$  of the Schrödinger operators  $\hat{H}_{\hbar}$ , for a sequence  $\hbar_j \rightarrow 0^+$ , determine  $V$  in the interval  $I_E$  up to a symmetry-translation  $V(x) \rightarrow V(c \pm x)$ .

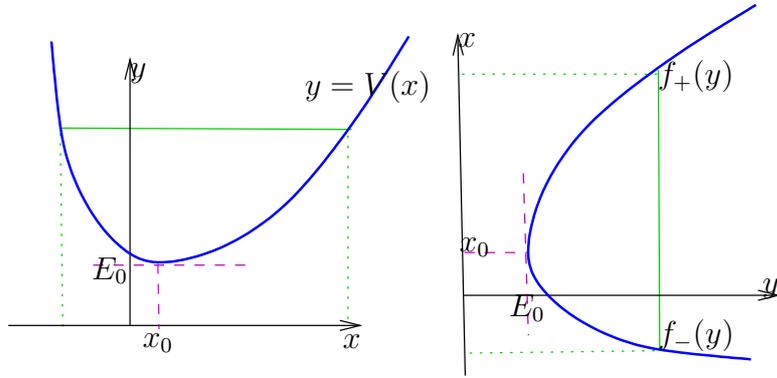


Figure 1: The potential  $V$  and the functions  $f_+$  and  $f_-$

## 6 One well potentials : Bohr-Sommerfeld rules and a $\Psi DO$ trace formula

From [3], we know that the semi-classical spectrum (i.e. the spectrum up to  $O(\hbar^\infty)$ ) of  $\hat{H}_{\hbar}$  in the interval  $]E_0, E[$  is given by

$$\Sigma(\hbar) = \{y \mid E_0 < y < E \text{ and } S(y) \in 2\pi\hbar\mathbb{Z}\}$$

where, for  $E_0 < y < E$ , the function  $S$  admits the formal series expansion  $S(y) \equiv S_0(y) + \hbar\pi + \hbar^2 S_2(y) + \hbar^4 S_4(y) + \dots$  (the formal series  $S$  will be called the *semi-classical action* and the remainder term in the expansion is uniform in every compact sub-interval of  $]E_0, E[$ ) with

- $S_0(y) = \int_{\gamma_y} \xi dx$  with  $\gamma_y = \{(x, \xi) \mid H(x, \xi) = y\}$  oriented according to the classical dynamics and

$$\frac{dS_0}{dy}(y) = \int_{f_-(y)}^{f_+(y)} \frac{dx}{\sqrt{y - V(x)}}$$

is the *period*  $T(y)$  of the trajectory of energy  $y$  for the classical Hamiltonian  $H$ ,

- If  $t$  is the time parametrization of  $\gamma_y$ ,

$$S_2(y) = -\frac{1}{12} \frac{d}{dy} \int_{\gamma_y} V''(x) dt ,$$

which can be rewritten as:

$$S_2(y) = -\frac{1}{12} \frac{d}{dy} \left( \int_{f_-(y)}^{f_+(y)} \frac{V''(x) dx}{\sqrt{y - V(x)}} \right) .$$

- For  $j \geq 1$ ,  $S_{2j}(y)$  is a linear combination of expressions of the form

$$\left( \frac{d}{dy} \right)^n \int_{\gamma_y} P(V', V'', \dots) dt ,$$

where  $dt$  is the differential of the time on  $\gamma_y$ : outside the caustic set  $dt = dx/2\xi$ .

In what follows, we will use only  $S_0$  and  $S_2$ . It will be convenient to relate the semi-classical action to the spectra by using the following trace formula:

**Theorem 6.1** ( *$\Psi$ DO trace formula*) *Let  $f \in C_o^\infty(]E_0, E[)$  and  $F(y) := -\int_y^\infty f(u) du$ , we have, with  $Z = T^*I$ :*

$$\text{Trace} F(\hat{H}) = \frac{1}{2\pi\hbar} \left( \int_Z F(H) dx d\xi + \hbar^2 \int_{E_0}^E f(y) (S_2(y) + \hbar^2 S_4(y) + \dots) dy \right) + O(\hbar^\infty) .$$

*This formula implies that  $S_0$  and  $S_2$  are determined by the semi-classical spectrum mod  $o(\hbar^2)$  in  $] -\infty, E[$ .*

This Theorem is closely related to (but a bit stronger) than what is proved in my paper [3]. The trace formula contains implicitly the Maslov index.

## 7 Two potentials with the same semi-classical spectra

We introduced a genericity Assumption 3 on symmetry defects in Theorem 5.1. The Figure 2 shows two one well potentials with the same semi-classical spectra mod  $O(\hbar^\infty)$ . The fact that they have the same semi-classical spectra comes from the description of Bohr-Sommerfeld rules in Section 6.

It would be nice to prove that they do NOT have the same spectra!

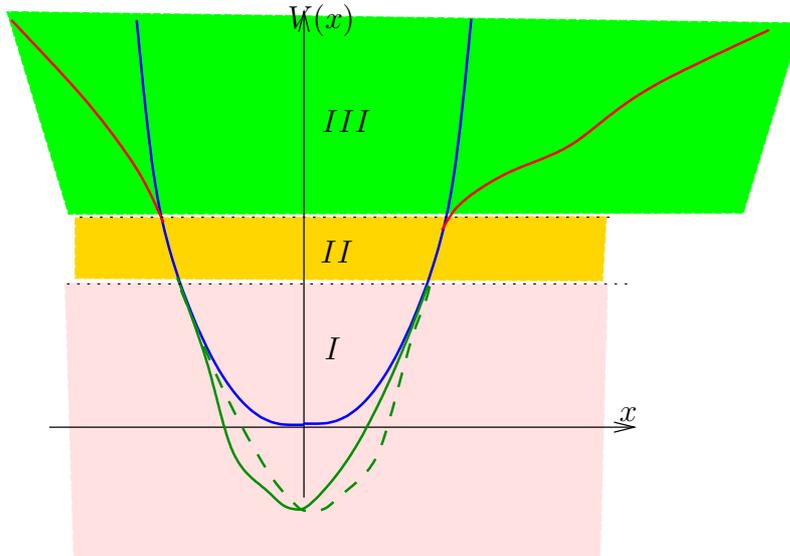


Figure 2: The (graphs of the) two potentials are the same in the sets  $II$  and  $III$ , they are mirror image of each other in  $I$  (green curve and dotted green curve), the potential is even in the set  $II$ .

## 8 One well potentials : the proof of Theorem 5.1

### 8.1 Some useful Lemmas

**Lemma 8.1** *The semi-classical spectra modulo  $o(\hbar^2)$  in  $]E_0, E[$  determine the actions  $S_0(y)$  and  $S_2(y)$  for  $y \in ]E_0, E[$ .*

It is a consequence of Theorem 6.1.

**Lemma 8.2** *If  $V$  satisfies Assumption 2 in Theorem 5.1, we have:*

$$\lim_{y \rightarrow E_0} \int_{\gamma_y} V''(x) dt = \pi \sqrt{2V''(x_0)} .$$

*This holds even if the minimum is degenerate<sup>3</sup>.*

The Lemma is clear if  $V''(x_0) > 0$ : the limit is then  $V''(x_0)$  times the period of small oscillations of a pendulum which is  $\pi/\sqrt{2/V''(x_0)}$ .

Let us consider the case of an isolated degenerate minimum with  $V(x) = E_0 + a(x - x_0)^N(1 + o(1))$  ( $a > 0$ ,  $N > 2$ ), we can check that the integral to be evaluated is  $O\left((y - E_0)^{\frac{3}{2} - \frac{3}{N}}\right) = o(1)$ .

<sup>3</sup>I do not know if this is still true without the genericity Assumption 2 in Theorem 5.1; it is the only place where I use it

**Lemma 8.3** *We have*

$$\lim_{y \rightarrow 0} \left( \frac{1}{f'_+(y)} - \frac{1}{f'_-(y)} \right) = 0 .$$

**Lemma 8.4** *If  $x_0$  is the unique point where  $V(x_0) = \inf V = E_0$ , the first eigenvalue of  $\hat{H}_\hbar$  satisfies  $\lambda_1(\hbar) = E_0 + \hbar\sqrt{V''(x_0)}/2 + o(\hbar)$*

This is well known if  $V''(x_0) > 0$  and is still true otherwise by comparison: if  $E_0 \leq V(x) \leq A(x - x_0)^2$  with  $A > 0$ , near  $x_0$  then  $E_0 < \lambda_1(\hbar) \leq 2\pi\hbar\sqrt{A}$ .

## 8.2 Rewriting $V$ using $F$ and $G$

We will denote by  $F = \frac{1}{2}(f_+ + f_-)$  and  $G = \frac{1}{2}(f_+ - f_-)$ .

- The function  $F$  is smooth on  $]E_0, E[$ , continuous on  $[E_0, E[$  (smooth in the non degenerate case  $V''(x_0) > 0$  as a consequence of the Morse Lemma), with  $F(E_0) = x_0$ , and is constant if and only if  $V$  is even w.r. to  $x_0$ . More generally, if  $F$  is constant on some interval,  $V$  is even on the inverse image of that interval. We call  $F$  the *parity defect*.

**Lemma 8.5** *Under the Assumption 3 in Theorem 5.1, the function  $F'$  is determined up to  $\pm$  by its square.*

- The function  $G$  is smooth on  $]E_0, E[$ , continuous at  $y = E_0$ . We have  $G(E_0) = 0$ . It is clear that, from  $F$  and  $G$ , we can recover the restriction of  $V$  to  $I_E$ .

## 8.3 How to get $V$ from $S_0$ and $S_2$

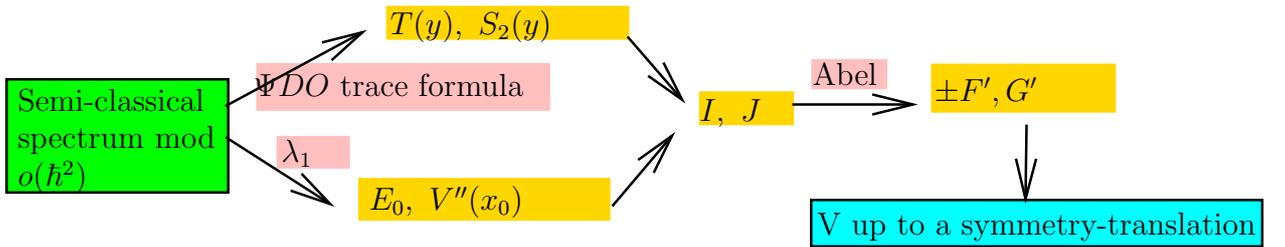


Figure 3: The scheme of the proof

Let us consider, for  $E_0 < y < E$ ,

$$I(y) := \int_{f_-(y)}^{f_+(y)} \frac{dx}{\sqrt{y - V(x)}}$$

and

$$J(y) = \int_{f_-(y)}^{f_+(y)} \frac{V''(x)dx}{\sqrt{y - V(x)}}.$$

We have  $I(y) = dS_0(y)/dy$  and  $S_2(y) = -(1/12)dJ(y)/dy$ . This implies that  $S_0$ ,  $S_2$  and the limit  $J(E_0)$  determine  $I$  and  $J$ . The limit  $J(E_0)$  is determined by  $V''(x_0)$  (Lemma 8.2) which is determined by the first semi-classical eigenvalue (Lemma 8.4). We can express  $I$  and  $J$  using  $F$  and  $G$ . Using the change of variables  $x = f_+(u)$  for  $x > x_0$  and  $x = f_-(u)$  for  $x < x_0$ , we get:

$$I(y) = 4 \int_{E_0}^y \frac{G'(u)du}{\sqrt{y - u}}$$

$$J(y) = \int_{E_0}^y \frac{d}{du} \left( \frac{1}{f'_+(u)} - \frac{1}{f'_-(u)} \right) \frac{du}{\sqrt{y - u}}.$$

Using Abel's result [1] (and Appendix A), we can recover  $G'$  and

$$\frac{d}{dy} \left( \frac{1}{f'_+(y)} - \frac{1}{f'_-(y)} \right) = \frac{d}{dy} \left( \frac{2G'}{G'^2 - F'^2} \right).$$

Using Lemma 8.3, we recover  $F'^2$ . The Assumption 3 implies that there exists a unique square root to  $F'^2$  up to signs. From that we recover  $G'$  and  $\pm F'$  and hence  $\pm F$  and  $G$  modulo constants. This gives  $V$  up to change of  $x$  into  $c \pm x$ .

## 9 Taylor expansions

From the previous section, we see that the semi-classical spectra determine  $F'^2$  and  $G$  even without assuming the hypothesis 3 of Theorem 5.1 on symmetry defect. It is not difficult to see that, if  $V$  satisfies the hypothesis 2 of Theorem 5.1, the parity defect  $F$  is a smooth function of  $y^{2/N}$ . We have the following:

**Lemma 9.1** *Let us give two formal powers series  $a = \sum_{j=0}^{\infty} a_j t^j$  and  $b = \sum_{j=0}^{\infty} b_j t^j$  which satisfy  $a^2 = b$ . The equation  $f^2 = b$  has exactly two solutions as formal powers series:  $f = \pm a$ .*

From this Lemma, we deduce the:

**Theorem 9.1** *Under the Assumptions 1 and 2 of Theorem 5.1, but without Assumption 3, the Taylor expansion of  $V$  at a local minimum  $x_0$  is determined (up to mirror symmetry) by the semi-classical spectrum modulo  $o(\hbar^2)$  in a fixed neighbourhood of  $E_0$ .*

In some aspects, this result is stronger than the one obtained in [4], but it requires the knowledge of the semi-classical spectrum in a fixed neighbourhood of  $E_0$ , while, in [4], we need only  $N$  semi-classical eigenvalues in order to get  $2N$  terms in the Taylor expansion.

## 10 A Theorem for a potential with several wells

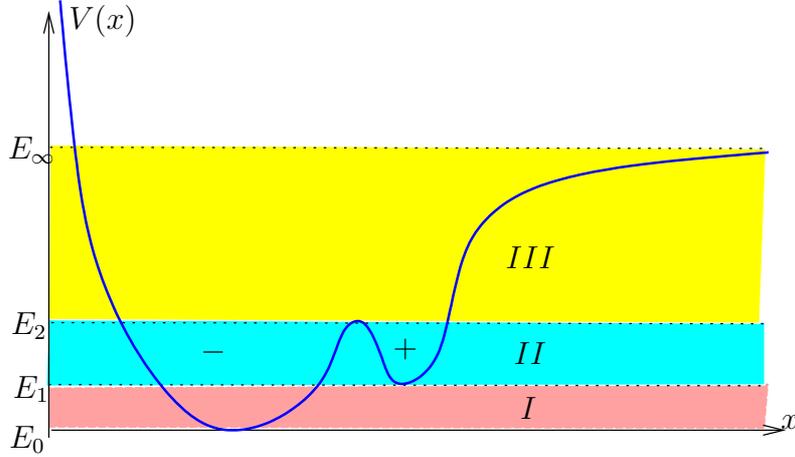


Figure 4: a 2 wells potential  $V$

We will extend our main result to cases including that of Figure 4: a two wells potential with three critical values,  $E_0 = 0$ ,  $E_1$  and  $E_2$ . We can take any boundary condition at  $x = 0$ .

### 10.1 The genericity Assumptions

In what follows, we choose  $E$  so that  $E_0 < E \leq E_\infty$  and define  $I_E = \{x | V(x) < E\}$ . The goal is to determine the restriction of  $V$  to  $I_E$  from the semi-classical spectrum in  $] -\infty, E]$ .

We need the following Assumptions which are generically satisfied. We introduce a:

**Definition 10.1** *Two smooth functions  $f, g : J \rightarrow \mathbb{R}$  are weakly transverse if, for every  $x_0$  so that  $f(x_0) = g(x_0)$ , there exists an integer  $N$  such that the  $N$ th derivative  $(f - g)^{(N)}(x_0)$  does not vanish.*

#### 10.1.1 Assumption on critical points

- for any point  $x_0$  so that  $V'(x_0) = 0$  and  $V(x_0) < E$ , there exists  $N \geq 2$  so that, the  $N$ -th derivative  $V^{(N)}(x_0)$  does not vanish.
- The *critical values* associated to different critical points are *distinct*.

**The wells:** *Let us label the critical values of  $V$  below  $E_\infty$  as  $E_0 < E_1 < \dots < E_k < \dots < E_\infty$  and the corresponding critical points by  $x_0, x_1, \dots$ . The critical values can only accumulate at  $E_\infty$  because the critical points are isolated*

and hence only a finite number of them lies in  $\{x|V(x) < E_\infty - c\}$  for any  $c > 0$ . Let us denote, for  $k = 1, 2, \dots$  by  $J_k = ]E_{k-1}, E_k[$ .

**Definition 10.2** A well of order  $k$  is a connected component of  $\{x|V(x) < E_k\}$ .

Let us denote by  $N_k$  the number of wells of order  $k$ .

For any  $k$ ,  $H^{-1}(J_k)$  is a union of  $N_k$  topological annuli  $A_j^k$  and the map  $H : A_j^k \rightarrow J_k$  is a submersion whose fibers  $H^{-1}(y) \cap A_j^k$  are topological circles  $\gamma_j^k(y)$  which are periodic trajectories of the classical dynamics: if  $y \in J_k$ ,  $H^{-1}(y) = \cup_{j=1}^{N_k} \gamma_j^k(y)$ . We will denote by  $T_j^k(y) = \int_{\gamma_j^k} dt$ , the corresponding classical periods. We will often remove the index  $k$  in what follows.

The semi-classical spectrum in  $J_k$  is the union of  $N_k$  spectra which are given by Bohr-Sommerfeld rules associated to actions  $S_j^k(y)$  given as in Section 6.

### 10.1.2 A generic symmetry defect

If there exists  $x_- < x_+$ , satisfying  $V(x_-) = V(x_+) < E$  and,  $\forall n \in \mathbb{N}$ ,  $V^{(n)}(x_-) = (-1)^n V^{(n)}(x_+)$ , then  $V$  is globally even on  $I_E$ .

### 10.1.3 Separation of the wells

For any  $k = 1, 2, \dots$  and any  $j$  with  $1 \leq j < l \leq N_k$ , the classical periods  $T_j(y)$  and  $T_l(y)$  are weakly transverse in  $J_k$ . This is assumed to hold also at  $E_{k-1}$  if  $x_k$  is a local non degenerate minimum of  $V$  (in this case, the period of the new periodic orbit is smooth at  $(E_{k-1})_+$ ).

## 10.2 Quartic potentials

If  $V$  is a polynomial of degree four with two wells like  $V(x) = x^4 + ax^3 + bx^2$  with  $b < 0$ , the periods of the two wells (between  $E_1$  and  $E_2(=0)$ ) are identical. This is because, on the complex projective compactification  $X_E$  (with  $E < 0$ ) of  $\xi^2 + V(x) = E$ , the differential  $dx/\xi$  is holomorphic and the real part of  $X$  consists of 2 homotopic curves in  $X_E$ . One can check directly that all other actions  $S_{2j}$ ,  $j \geq 1$  coincide; this is also proved for example in [5] p. 191.

## 10.3 The statement of the result

Our result is:

**Theorem 10.1** Under the three Assumptions in Sections 10.1.1, 10.1.2 and 10.1.3,  $V$  is determined in the domain  $I_E := \{x|V(x) < E\}$  by the semi-classical spectrum in  $] - \infty, E[$  modulo  $o(\hbar^4)$  up to the following moves:  $I_E$  is an union of open intervals  $I_{E,m}$ , each interval  $I_{E,m}$  is defined up to translation and the restriction of  $V$  to each  $I_{E,m}$  is defined up to  $V(x) \rightarrow V(c - x)$ .

**Remark 10.1** We need  $o(\hbar^4)$  in the previous Theorem while we needed only  $o(\hbar^2)$  in the one well case. This is due to the way we are able to separate the spectra associated to the different wells.

## 11 The case of several wells: the proof of Theorem 10.1

### 11.1 What can be read from the Weyl's asymptotics?

**Lemma 11.1** Under the Assumption 10.1.1, the singular (non smooth) points of the function  $y \rightarrow A(y) = \int_{H(x,\xi) \leq y} dx d\xi$  are exactly the critical values  $E_0, E_1, \dots$  of  $V$ . Moreover,

- the function  $A(y)$  is smooth on  $]E_k - c, E_k]$ , with  $c > 0$ , if and only if  $x_k$  is a local minimum of  $V$ ,
- From the singularity of  $A(y)$  at  $E_k$ , one can read the value of  $V''(x_k)$ .

The function  $A(y)$  is determined by the semi-classical spectrum, this is a consequence of the Weyl asymptotics:

$$\#\{\lambda_l(\hbar) \leq y\} \sim \frac{A(y)}{2\pi\hbar}.$$

This implies that the critical values  $E_k$  of  $V$  are determined by the semi-classical spectrum.

### 11.2 The scheme of the reconstruction

The proof is by “induction” on  $E$ .

We start by constructing the piece of  $V$  where  $V(x) \leq E_1$  using Theorem 5.1.

We want then to construct  $V$  where  $E_1 \leq V(x) \leq E_2$ .

There are two cases:

1.  $x_1$  is not an extremum: then we are able to extend the proof of Theorem 5.1 using the fact that we know, using Section 11.4, the limits of  $\int_{\gamma_y} V''(x) dt$  and  $f'_\pm(y)$  as  $y \rightarrow E_1^+$ . We can reduce to an Abel transform starting from  $E_1$  using

$$\int_{V(x) \leq y} = \int_{V(x) \leq E_1} + \int_{E_1 \leq V(x) \leq y}$$

where the first part is known from the knowledge of  $V(x)$  in  $\{x | V(x) \leq E_1\}$ .

2.  $x_1$  is a local minimum: using the separation of spectra (Section 11.3) and Theorem 5.1, we can construct the 2 wells of order 2 if we know  $V''(x_1)$ . But the estimate

$$A(y) = A(E_1) + \pi\sqrt{2/V''(x_1)}(y - E_1)_+ + a(y - E_1) + o(y - E_1)$$

shows that the singularity of  $A(y)$  at  $y = E_1$  determines  $V''(x_1)$ .

We then proceed to the interval  $[E_2, E_3]$ . A new case arises when  $x_2$  is a local maximum. Then we need to glue together the wells of order 2. This case works then as before.

### 11.3 Separation of spectra

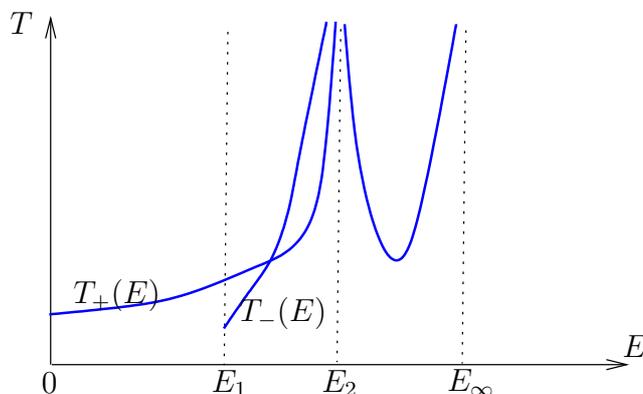


Figure 5: The primitive periods as functions of  $y$  for the Example of Figure 4

Let us start with a:

**Lemma 11.2** *Let us give some open interval  $J$  and assume that we have a function*

$$F(x) = \sum_{j=1}^N a_j(x) e^{iS_j(x)/\hbar}$$

*with the functions  $S_j$  and  $S_k$  weakly transverse for any  $j \neq k$ . If for any compact interval  $K \subset J$ , we have*

$$\int_K |F|^2(x) dx = o(1)$$

*then all  $a_j$ 's vanish identically.*

If  $P = \text{Op}(p)$  with  $p \in C_o^\infty(T^*J)$ , using the  $L^2$   $\hbar$ -uniform continuity of  $P$ , we have

$$PF(x) = \sum_{j=1}^N p(x, S'_j(x)) a_j(x) e^{iS_j(x)/\hbar} = o(1) .$$

One sees that the  $a_j$ 's vanish by choosing  $p$  in an appropriate way, i.e. supported near a point  $(x_0, S'_{j_0}(x_0))$ .

**Lemma 11.3** *Let us consider the distributions  $D_k(\hbar)$  on  $J_k$  defined by  $D_k(\hbar) = \sum_{\lambda_l(\hbar) \in J_k} \delta(\lambda_l(\hbar))$ , then  $D_k$  is microlocally in  $T^*J_k$  a locally finite sum of WKB functions  $D_{j,l}$  associated to the Lagrangian manifolds  $t = lS'_j(y)$  with  $j = 1, \dots, N_k$  and  $l \in \mathbb{Z}$ . We have*

$$D_{j,l} = \frac{1}{2\pi\hbar} e^{ilS_{j,h}(y)/\hbar} S'_{j,h}(y) ,$$

and

$$D_{j,l} = \frac{(-1)^l}{2\pi\hbar} e^{ilS_{j,o}(y)/\hbar} T_j(y) (1 + il\hbar S_{j,2}(y) + O(\hbar^2)) ,$$

with  $S_{j,h} \equiv \sum_{k=0}^{\infty} \hbar^k S_{j,k}$  the semi-classical actions associated to the  $j$ -th well.

This is a formulation of the semi-classical trace formula (see Appendix C).

**Lemma 11.4** *If  $\mu_l(\hbar)$  is a semi-classical spectrum modulo  $o(\hbar^4)$  and  $\tilde{D}_k(\hbar) = \sum_{\mu_l(\hbar) \in J_k} \delta(\mu_l(\hbar))$ , then, for any pseudo-differential operator  $P = \text{Op}_{\hbar}(p)$ , with  $p \in C_{\infty}^0(T^*J_k)$ , we have*

$$\|P(D_k - \tilde{D}_k)\|_{L^2(J_k)} = o(\hbar) .$$

It is enough to prove it for  $p = \chi(E)\hat{\rho}(t)$  and then it is elementary because  $P\delta(\lambda) = \hbar^{-1}\chi(\lambda)\rho((y-\lambda)/\hbar)$ .

From the three previous Lemmas, it follows that, with Assumption 10.1.3, the spectrum in  $J_k$  modulo  $o(\hbar^4)$  determine the periods  $T_j(y)$  and the actions  $S_{j,2}(y)$ .

## 11.4 Limit values of some integrals

Using the trick of Section 8.3, we can use Abel's result (Section 12.3) once we know the following limits (or asymptotic behaviours) as  $y \rightarrow E_j^+$  ( $j = 0, 1, \dots$ ):

- $f_{\pm}^j(y)$
- $\int_{H^{-1}(y)} V''(x) dt$  where  $H = \xi^2 + V(x)$  is the classical Hamiltonian. Here  $H^{-1}(y)$  is oriented so that  $dt > 0$ .
- $f_{\pm}^{\prime j}(y)$

All of them are determined by the knowledge of  $V$  in the set  $\{x | V(x) \leq E_j\}$ .

It is clear, except for the second one; we have:

**Lemma 11.5** *Let us assume that  $V$  satisfies Assumption 1 of Section 10.1. If  $E_j$  is a critical value of  $V$  which is not a local minimum and  $\tau(z) := \int_{H^{-1}(E_j+z)} V''(x) dt - \int_{H^{-1}(E_j-z)} V''(x) dt$ , then  $\lim_{z \rightarrow 0^+} \tau(z) = 0$ .*

*Proof.*–

We cut the integrals into pieces. One piece near each critical point and another piece far from them. Far from the critical points, the convergence is clear.

- *Local maximum:* let us take a critical point where  $V(x) = E_j - A(x - x_0)^{2N}(1 + o(1))$  with  $N \geq 1$  and  $A > 0$ . We use a smooth change of variable  $x = \psi(y)$  with  $\psi(0) = x_0$  so that  $V(\psi(y)) = E_j - y^{2N}$ . We are reduced to check that

$$\lim_{\varepsilon \rightarrow 0^+} \left( \int_0^1 \frac{W(y)dy}{\sqrt{\varepsilon + y^{2N}}} - \int_{\varepsilon^{1/2N}}^1 \frac{W(y)dy}{\sqrt{y^{2N} - \varepsilon}} \right) = 0 ,$$

assuming that  $W(y) = O(y^{2N-2})$ .

- *Other critical points:* let us take a critical point where  $V(x) = E_j + A(x - x_0)^{2N+1}(1 + o(1))$  with  $N \geq 1$  and  $A > 0$ . We use the same method.

□

## 12 Extensions to other operators

### 12.1 The statement

Let us indicate in this Section how to extend the previous results to the operator

$$L_{\hbar} = -\hbar^2 \frac{d}{dx} \left( n(x) \frac{d}{dx} \right) + n(x)$$

which was found in Section 3. We want to recover the function  $n(x)$ . Let us sketch the one well case for which we will get:

**Theorem 12.1** *Assuming that*

- *the function  $n(x)$  admits a non degenerate minimum  $n(x_0) = E_0 > 0$ ,*
- *the function  $n(x)$  has no critical values in  $]E_0, E_1]$  with  $E_1 \leq \liminf_{x \rightarrow \partial I} n(x)$ ,*
- *the function  $n(x)$  has a generic symmetry defect as in Theorem 5.1,*

*then the function  $n$  is determined in  $\{x | n(x) \leq E_1\}$  by the semi-classical spectrum of  $L_{\hbar}$  modulo  $o(\hbar^2)$ .*

The proof works along the same lines as that of Theorem 5.1 except that we get an integral transform which is not exactly Abel's transform.

## 12.2 The Weyl symbol and the actions

The Weyl symbol  $l$  of  $L$  can be computed, using the Moyal product, as  $l = \xi \star n \star \xi + n$ . We get:

$$l(x, \xi) = n(x)(1 + \xi^2) + \frac{\hbar^2}{4}n''(x) .$$

The action  $S_0$  satisfies:

$$\frac{dS_0}{dy}(y) = T(y) = \int_{n(x) \leq y} \frac{dx}{\sqrt{n(x)(y - n(x))}} .$$

The action  $S_2$  is given from [3] by

$$S_2(y) = -\frac{1}{12} \frac{d}{dy} \int_{\gamma_y} \left( yn'' - 2 \left( \frac{y}{n} - 1 \right) n'^2 \right) dt - \frac{1}{4} \int_{\gamma_y} n'' dt ,$$

which we rewrite:

$$S_2(y) = -\frac{1}{12} \frac{d}{dy} J(y) - \frac{1}{4} K(y) .$$

- **The integral J:**

$$J(y) = \int_{x_-(y)}^{x_+(y)} \left( yn'' - 2 \left( \frac{y}{n} - 1 \right) n'^2 \right) \frac{dx}{\sqrt{n(y - n)}} .$$

Using  $x = f_{\pm}(y)$  as in Section 5 and

$$\Phi(y) = \frac{1}{f'_+(y)} - \frac{1}{f'_-(y)} ,$$

we get  $J(y) = (\mathcal{J}\Phi)(y)$ , with

$$(\mathcal{J}\Phi)(y) = \int_{E_0}^y \left( y\Phi'(u) - 2 \left( \frac{y}{u} - 1 \right) \Phi(u) \right) \frac{du}{\sqrt{u(y - u)}} .$$

- **The integral K:**

$$K(y) = \int_{E_0}^y \Phi'(u) \frac{du}{\sqrt{u(y - u)}}$$

and

$$K(y) = 2 \frac{d}{dy} \int_{E_0}^y \Phi'(u) \frac{\sqrt{y - u} du}{\sqrt{u}}$$

which is rewritten as:

$$K(y) = 2 \frac{d}{dy} (\mathcal{K}\Phi)(y) .$$

## 12.3 An integral transform

**Lemma 12.1** *If  $0 < E_0 < E_1$ , the kernel of  $A := \mathcal{J} + 6\mathcal{K}$  on the space of continuous function on  $[E_0, E_1]$  at most two dimensional and all functions in this kernel are smooth.*

*Proof.*–

we have

$$A\Phi(y) = \int_{E_0}^y \left( (7y - 6u)\Phi'(u) - 2\left(\frac{y}{u} - 1\right)\Phi(u) \right) \frac{du}{\sqrt{u(y-u)}}. \quad (3)$$

We compute  $T \circ A$  with the operator  $T$  defined by  $T\psi(y) = \int_{E_0}^y \frac{\psi(u)du}{\sqrt{y-u}}$ . We will need the easy:

**Lemma 12.2** *We have:*

$$\int_{E_0}^y \frac{udu}{\sqrt{y-u}} \int_{E_0}^u f(t) \frac{dt}{\sqrt{u-t}} = \frac{\pi}{2} \int_{E_0}^y (t+y)f(t)dt,$$

and

$$\int_{E_0}^y \frac{du}{\sqrt{y-u}} \int_{E_0}^u f(t) \frac{dt}{\sqrt{u-t}} = \pi \int_{E_0}^y f(t)dt,$$

Applying the previous formulae, we get:

$$T \circ A(\Phi)(y) = \frac{\pi}{2} \int_{E_0}^y \left[ (t+y)(7\Phi'(t) - 2\frac{\Phi(t)}{t}) + 2(-6t\Phi'(t) + 2\Phi(t)) \right] \frac{dt}{\sqrt{t}}.$$

Taking two derivatives:

$$\frac{\pi}{y^{3/2}} \frac{d^2}{dy^2} ((T \circ A)\Phi)(y) = y^2\Phi''(y) + 4y\Phi'(y) - \Phi(y).$$

From  $S_2$  and  $A\Phi(E_0)$ , we get  $A\Phi$ , then we get  $P(\Phi)$  where  $P\phi = y^2\phi'' + 4y\phi' - \phi$  is a non singular linear differential equation (remind that  $E_0 > 0$ ). So, if we know also  $\Phi(E_0)$  and the asymptotic behaviour of  $\Phi'(E_0)$ , we can get  $\Phi$ . Let us assume  $n''(x_0) = a > 0$ . Then we have:

- $A\Phi(E_0) = 2\pi\sqrt{aE_0}$
- $\Phi(E_0) = 0$
- $\Phi'(y) \sim 4\sqrt{a}/\sqrt{y - E_0}$ .

□

## Appendix A: Abel's result

Let us consider the linear operator  $T$  which acts on continuous functions on  $[E_0, E[$  defined by:

$$Tf(x) = \int_{E_0}^x \frac{f(y)dy}{\sqrt{x-y}}.$$

Then  $T^2f(x) = \pi \int_{E_0}^x f(y)dy$ . This implies that  $T$  is injective! This is the content of [1].

## Appendix B: a proof of the $\Psi DO$ trace formula of Section 6

For this Section, one can read [6]. This can be seen as a complement and a partial rewriting of my paper [3] with a better trace formula. The formula we will prove is more general than that in Section 6. It is valid even for several wells. Let us state it:

**Theorem 12.2** *Let  $f \in C_o^\infty(J_k)$  and  $F(y) := -\int_y^\infty f(u)du$ , we have, with  $Z = T^*I$ , modulo  $O(\hbar^\infty)$ :*

$$\text{Trace}F(\hat{H}) \equiv \frac{1}{2\pi\hbar} \left( \int_Z F(H)dx d\xi + \hbar^2 \int_{J_k} f(y) \left( \sum_{j=1}^{N_k} (S_{2,j}^k(y) + \hbar^2 S_{4,j}^k(y) + \dots) \right) dy \right).$$

*Proof.*–

1. *Reduction to  $N_k = 1$ :* we can decompose both the lefthandside and the righthandside according to the  $N_k$  wells: for the lhs, it uses the fact that the classical spectrum splits into  $N_k$  parts; for the rhs, it is enough to decompose the first integral terms according to the connected component of  $H < E_k$ .
2. *Reduction from  $N_k = 1$  to one well:* the whole Moyal symbol of  $F(\hat{H})$  is  $\equiv F(E_0)$  in  $\{H \leq E_{k-1}\}$ .
3. *The harmonic oscillator case ( $\hat{H} = \Omega$ ):*

$$\text{Trace}F(\Omega) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{F} \left( \left( n + \frac{1}{2} \right) \hbar \right)$$

with  $\tilde{F}$  even and coinciding with  $F$  on the positive axis. We get with Poisson summation formula:

$$\text{Trace}F(\Omega) = \frac{1}{2\pi\hbar} \iint F \left( \frac{x^2 + \xi^2}{2} \right) dx d\xi + O(\hbar^\infty).$$

4. *The case where  $F$  is compactly supported:* using Poisson summation formula as in [3], we get

$$\text{Trace}F(\hat{H}) = \frac{1}{2\pi\hbar} \int F(y)S'(y)dy$$

and we get this case by integration by part.

5. *The final step:* we can assume that  $H = \frac{(x-x_0)^2 + \xi^2}{2} + E_0$  near  $(x_0, 0)$  and we split  $F = F_0 + F_1$  where

$$F_0(H) \equiv F_1 \left( \frac{(x-x_0)^2 + \xi^2}{2} + E_0 \right) .$$

The formula then follows from the two particular cases computed before.

□

*For the convenience of the reader, we regive also the way to get  $S_2$  from the Moyal formula.*

Defining  $F^*(H)$  by  $F(\hat{H}) = \text{Op}_{\text{Weyl}}(F^*(H))$  we know that, with  $z_0 = (x_0, \xi_0)$  and  $H_0 = H(z_0)$ ,

$$F^*(H)(z_0) = F(H_0) + \frac{1}{2}F''(H_0)(H-H_0)^{*2}(z_0) + \frac{1}{6}F'''(H_0)(H-H_0)^{*3}(z_0) + O(\hbar^4) .$$

Computing the Moyal powers of  $H - H_0$  at the point  $z_0 \bmod O(\hbar^4)$ , gives

$$F^*(H) = F(H) - \hbar^2 \left( \frac{1}{8}f'(H)\det(H'') + \frac{1}{24}f''(H)H''(X_H, X_H) \right) + O(\hbar^4) .$$

If  $\alpha = \iota(X_H)H''$ , we have  $d\alpha = 2\det(H'')d\xi \wedge dx$ , and we get, by Stokes and with  $\gamma_y$  oriented according to the dynamics:

$$\int_{\gamma_y} \alpha = 2 \int_{H \leq y} \det(H'')dx d\xi$$

and the final result for  $S_2(y)$  using an integration by part and the formula  $dt dy = dx d\xi$ :

$$S_2(y) = -1/24 \int_{\gamma_y} \det(H'')dt .$$

## Appendix C: the semi-classical trace formula

In this Section, we want to give a proof of Lemma 11.3.

We want to evaluate mod  $O(\hbar^\infty)$  the sums:

$$D(y) := \frac{1}{\hbar} \sum_{l \in \mathbb{Z}} \rho \left( \frac{y - S^{-1}(2\pi l \hbar)}{\hbar} \right) ,$$

where  $S : \mathbb{R} \rightarrow \mathbb{R}$  is an extension to  $\mathbb{R}$  of the given function  $S_j$  on  $\Delta$  which is  $\equiv \text{Id}$  near infinity. This is equal to  $D_{\Delta, \rho}^j(y)$  up to  $O(\hbar^\infty)$ . Using the Poisson summation formula and defining

$$F_y(x) = \int_{\mathbb{R}} \rho \left( \frac{y - S^{-1}(\hbar y)}{\hbar} \right) e^{-ixy} dy ,$$

we get

$$D(y) = \frac{1}{2\pi\hbar} \sum_{m \in \mathbb{Z}} F_y(m) . \quad (4)$$

Using the change of variable,  $y - S^{-1}(\hbar y) = \hbar z$  or  $y = S(y - \hbar z)/\hbar$ , we get:

$$F_y(x) = \int \rho(z) e^{-ixS(y-\hbar z)/\hbar} S'(y - \hbar z) dz .$$

Using the fact that all moments of  $\rho$  vanish and Taylor expanding  $S(y - \hbar z)$  w.r. to  $\hbar$ , we get

$$F_y(x) = e^{-ixS(y)/\hbar} S'(y) \hat{\rho}(-xS'(y)) + O(\hbar^\infty) .$$

If the support of  $\hat{\rho}$  is close enough to  $S'(y)$ , we get the final answer taking the contribution of  $m = -1$  to Equation (4). This way, we get the formula of Lemma 11.3.

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