



**HAL**  
open science

## Discrete Riemann surfaces, linear and non-linear

Christian Mercat

► **To cite this version:**

Christian Mercat. Discrete Riemann surfaces, linear and non-linear. *Discrete Differential Geometry*, Mar 2006, Oberwolfach, Germany. pp.653–727, 10.4171/OWR/2015/13 . hal-00250245

**HAL Id: hal-00250245**

**<https://hal.science/hal-00250245v1>**

Submitted on 11 Feb 2008

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## Discrete Riemann surfaces, linear and non-linear

CHRISTIAN MERCAT

(joint work with A. Bobenko, Y. Suris)

To the first two orders, discrete holomorphic functions are whether complex linear or Möbius transformations. We discretize this notion to complex functions of vertices  $\diamond_0$  of an oriented quad-mesh  $\diamond$ . First, fix a direct embedding of this quad-mesh into the complex plane by a complex function of the vertices  $Z : \diamond_0 \rightarrow C$ , corresponding to the identity map. We will say that another function is *linear holomorphic* (with respect to  $Z$ ) if and only if the ratio along the diagonals are the same as  $Z$  on each quadrilateral:

$$\forall(x, y, x', y') \in \diamond_2, \frac{f(y') - f(y)}{f(x') - f(x)} = \frac{Z(y') - Z(y)}{Z(x') - Z(x)} = i \rho_{(x, x')}.$$

Likewise we will say that a function is *cross-ratio preserving* if the cross-ratio on each quadrilateral is the same as the one given by  $Z$ :

$$\frac{f(x) - f(y)}{f(y) - f(x')} \frac{f(x') - f(y')}{f(y') - f(x)} = \frac{Z(x) - Z(y)}{Z(y) - Z(x')} \frac{Z(x') - Z(y')}{Z(y') - Z(x)} = q_{(x, y, x', y')}.$$

An important class of cross-ratio preserving maps are given by circle patterns with prescribed intersection angles. In the form of a *Hirota system*, a connection between the two notions can be described: A discrete function  $F$  is cross-ratio preserving if its exterior differential can be written, on each edge  $(x, y) \in \diamond_1$  as  $F(y) - F(x) = f(x)f(y)(Z(y) - Z(x)) =: \int_{(x, y)} f dZ$  for a function  $f$ . The constraint on  $f$  is the Morera equation:  $\oint f dZ = 0$  around every quadrilateral. Looking at logarithmic derivatives  $f_\epsilon = f \times (1 + \epsilon g)$  of  $f$  that still form a Hirota system, one finds that  $g$  should be linear holomorphic with respect to  $F$ :  $\frac{g(y') - g(y)}{g(x') - g(x)} = \frac{F(y') - F(y)}{F(x') - F(x)}$ . Linear constraints can as well be reformulated as a Morera equation  $\oint f dZ = 0$  not for a multiplicative but for an additive coupling between functions and 1-forms:  $\int_{(x, y)} f dZ := \frac{f(x) + f(y)}{2} (Z(y) - Z(x))$ .

We will be interested in so called *critical* reference maps  $Z$  composed of rhombi, the length  $\delta = |Z(y) - Z(x)|$  is constant for all edges  $(x, y) \in \diamond_1$ . Then these two constraints are *integrable*, meaning that they give rise to a well-defined Bäcklund (or Darboux) transform: given a linear holomorphic resp. cross-ratio preserving map  $f$  and a starting point  $O \in \diamond_0$ , one can define a 2-parameters family  $f_{\lambda, \mu}$  of deformations of  $f$  that still fulfill the same (linear resp. cross-ratio) condition and starting value  $\mu$  at the point  $O$ . This is done by viewing the solution  $f$  lying on a ground level and building “vertically”, above each edge  $(x, y)$ , a quadrilateral on which the same kind of equation will be imposed:  $\frac{f_{\lambda, \mu}(x) - f(y)}{f_{\lambda, \mu}(y) - f(x)} = \frac{\lambda + Z(x) - Z(y)}{\lambda + Z(y) - Z(x)}$  for the linear constraint, resp.  $\frac{f_{\lambda, \mu}(x) - f(x)}{f(x) - f(y)} \frac{f(y) - f_{\lambda, \mu}(y)}{f_{\lambda, \mu}(y) - f_{\lambda, \mu}(x)} = \frac{\lambda^2}{(Z(y) - Z(x))^2}$  for the cross-ratio constraint.

In [5], based on the point of view of integrable systems, we define discrete holomorphicity in  $\mathbb{Z}^d$ , for  $d > 1$  finite, equipped with rapidities  $(\alpha_i)_{1 \leq i \leq d}$  and

the tools of integrable theory yields interesting results like a *Lax pair* governing a *moving frame*  $\Psi(\cdot, \lambda) : \mathbb{Z}^d \rightarrow GL_2(\mathbb{C})[\lambda]$  and *isomonodromic* solutions like the *Green function* found by Kenyon [4].

An important tool of the linear theory is the existence of an explicit basis of discrete holomorphic functions. In the rhombic case, for a discrete holomorphic function  $f$ , the 1-form  $f dZ$  is holomorphic (it is closed and its ratios on two dual diagonals are equal to the reference ratios) and can be integrated, yielding back a holomorphic function. Differential equations can be setup, producing explicit formulae for *exponentials and polynomials* which are shown to form a basis of discrete holomorphic functions.

Lots of results of the continuous theory can be extended to this discrete setting: There exists a *Hodge star*:  $*$  :  $C^k \rightarrow C^{2-k}$ , defined by  $\int_{(y,y')} * \alpha := \rho(x,x') \int_{(x,x')} \alpha$ ; the discrete *Laplacian* is written as usual  $\Delta := d d^* + d^* d$  with  $d^* := - * d *$ , which reads as weighted differences around neighbours  $\Delta f(x) = \sum \rho(x,x_k)(f(x) - f(x_k))$ . The weights are given by the usual cotan formula.

A wedge product defines an  $L^2$  norm for functions and forms by  $(\alpha, \beta) := \iint_{\diamond_2} \alpha \wedge * \beta$ . The norm of  $df$  is called the *Dirichlet energy* of the function  $f$ ,  $E_D(f) := \|df\|^2 = (df, df) = \frac{1}{2} \sum_{(x,x') \in \Lambda_1} \rho(x,x') |f(x') - f(x)|^2$ . The *conformal energy* of a map measures its conformality defect  $E_C(f) := \frac{1}{2} \|df - i * df\|^2$ . They are related through  $E_C(f) = E_D(f) - 2\mathcal{A}(f)$  just as in the continuous.

The Hodge star decomposes forms into exact, coexact and harmonic ones, the harmonic being the orthogonal sum of holomorphic and anti-holomorphic ones. A Weyl's lemma and a Green's identity are found.

Non closed 1-forms with prescribed diagonal ratios define *meromorphic* forms and the holonomy around a quadrilateral is called its *residue*. The compact case is covered with flat atlases of critical maps for a given euclidean metric with conic singularities. *Meromorphic* forms of prescribed holonomies and poles are defined and are used to form a basis of the space of holomorphic forms. It is  $2g$ -dimensional on a genus  $g$  surface, that is *twice* as large as the continuous case, defining two period matrices instead of one. This difference is explained by the doubling of degrees of freedom, and partially solved through continuous limit theorems: the two period matrices converge to the same limit when refinements of quad-meshes for a given Euclidean metric with conic singularities are taken. Every holomorphic function can be approximated by a converging sequence of discrete holomorphic functions on refinements of critical quad-meshes.

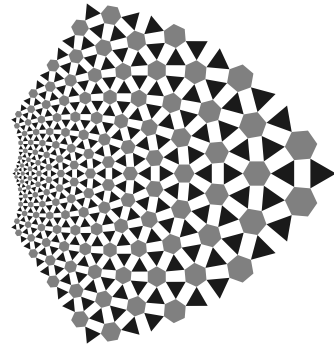
The Green function and potential allow one to setup a Cauchy integral formula giving the value at a point (in fact its average at two neighbours  $x, y$ ) as a contour integral:  $\oint_{\partial D} f dG_{x,y} = 2i\pi \frac{f(x)+f(y)}{2}$ .

We define a derivation with respect to  $Z$  by  $\partial : C^0(\diamond) \rightarrow C^2(\diamond)$  with  $\partial f = [(x, y, x', y') \mapsto -\frac{i}{2\mathcal{A}(x,y,x',y')} \oint_{(x,y,x',y')} f d\bar{Z}] = \frac{(f(x')-f(x))(\bar{y}'-\bar{y})-(\bar{x}'-\bar{x})(f(y')-f(y))}{(x'-x)(\bar{y}'-\bar{y})-(\bar{x}'-\bar{x})(y'-y)}$ ,

where  $Z(x)$  is simply written  $x$ ; and likewise  $\bar{\partial} f$  with  $f dZ$ . A holomorphic function  $f$  verifies  $\bar{\partial} f \equiv 0$  and  $\partial f(x, y, x', y') = \frac{f(y')-f(y)}{y'-y} = \frac{f(x')-f(x)}{x'-x}$ . The Jacobian

$$J = |\partial f|^2 - |\bar{\partial} f|^2 \text{ relates the areas } \iint_{(x,y,x',y')} df \wedge \bar{d}f = J \iint_{(x,y,x',y')} dZ \wedge \bar{d}Z.$$

Following Colin de Verdière and Kenyon, a geometrical interpretation of linear discrete holomorphicity is enlightening. As circle patterns with prescribed angles can be checked by eye, so can be a linear holomorphic map: The quad-mesh  $\diamond$ , when bipartite, decomposes into two dual graphs  $\Gamma$  and  $\Gamma^*$  whose edges are dual diagonals of each quadrilateral. Around each vertex  $x \in \Gamma_0$ , there is a polygon, image of the dual face  $x^* \in \Gamma_2^*$  by the reference map  $Z$ . Consider the identity map as a picture of all these polygons shrunk by a factor half. It represents both dual graphs at the same time as matching polygons. A map  $f : \diamond_0 \rightarrow \mathbb{C}$  is *discrete holomorphic* if and only if every polygon  $x^*$ , centered at  $f(x)$ , scaled and turned according to  $\partial f(x)$ , form into a polygonal pattern of the same combinatorics as the reference polygonal pattern, made of *similar* polygons.



The discrete exponential as a polygonal pattern on the triangular/hexagonal

The *dilatation* coefficient of a discrete map  $f$  is defined as  $D_f := \frac{\text{lattice } |\partial f| + |\bar{\partial} f|}{|\partial f| - |\bar{\partial} f|}$ . We will call  $f$  *quasi-conformal* when  $D_f \geq 1$ , that is  $|f_{\bar{z}}| \leq |\partial f|$ . It can be written in term of the *complex dilatation*:  $\mu_f = \frac{\bar{\partial} f}{\partial f} = \frac{(f(x')-f(x))(y'-y)-(x'-x)(f(y')-f(y))}{(f(x')-f(x))(\bar{y}'-\bar{y})-(\bar{x}'-\bar{x})(f(y')-f(y))}$ .

#### REFERENCES

- [1] Christian Mercat. *Holomorphic discrète et modèle d'Ising*. PhD thesis, Université Louis Pasteur, Strasbourg, France, 1998. under the direction of Daniel Bennequin, Prépublication de l'IRMA, available at <http://tel.archives-ouvertes.fr/tel-00001851/>.
- [2] Christian Mercat. Discrete Riemann surfaces and the Ising model. *Comm. Math. Phys.*, 218(1):177–216, 2001.
- [3] Christian Mercat. Exponentials form a basis of discrete holomorphic functions on a compact. *Bull. Soc. Math. France*, 132(2):305–326, 2004.
- [4] R. Kenyon. The Laplacian and Dirac operators on critical planar graphs. *Invent. Math.*, 150(2):409–439, 2002. [math-ph/0202018](https://arxiv.org/abs/math-ph/0202018).
- [5] Alexander I. Bobenko, Christian Mercat, and Yuri B. Suris. Linear and nonlinear theories of discrete analytic functions. Integrable structure and isomonodromic Green's function. *J. Reine Angew. Math.*, 583:117–161, 2005.
- [6] Ruben Costa-Santos and Barry M. McCoy. Dimers and the critical Ising model on lattices of genus  $> 1$ . *Nuclear Phys. B*, 623(3):439–473, 2002. [hep-th/0109167](https://arxiv.org/abs/hep-th/0109167).
- [7] R. J. Duffin. Basic properties of discrete analytic functions. *Duke Math. J.*, 23:335–363, 1956.
- [8] R. J. Duffin. Potential theory on a rhombic lattice. *J. Combinatorial Theory*, 5:258–272, 1968.
- [9] Jacqueline Lelong-Ferrand. *Représentation conforme et transformations à intégrale de Dirichlet bornée*. Gauthier-Villars, Paris, 1955.
- [10] Christian Mercat. Discrete Period Matrices and Related Topics. [math-ph/0111043](https://arxiv.org/abs/math-ph/0111043).