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JINQIAO DUAN AND ANNIE MILLET

Abstract. A Boussinesq model for the Bénard convection under random influences is considered as a system of stochastic partial differential equations. This is a coupled system of stochastic Navier-Stokes equations and the transport equation for temperature. Large deviations are proved, using a weak convergence approach based on a variational representation of functionals of infinite dimensional Brownian motion.

1. Introduction

The need to take stochastic effects into account for modeling complex systems has now become widely recognized. Stochastic partial differential equations arise naturally as mathematical models for nonlinear macroscopic dynamics under random influences. It is thus desirable to understand the impact of such random influences on the system evolution [24, 8, 20].

The Navier-Stokes equations are often coupled with other equations, especially, with the scalar transport equations for fluid density, salinity, or temperature. These coupled equations (often with the Boussinesq approximation) model a variety of phenomena in environmental, geophysical, and climate systems [9, 10, 17]. We consider the Boussinesq equations in which the scalar quantity is temperature, under different boundary conditions for the temperature at different parts (top and bottom) of the boundary. This is a Bénard convection problem. With other boundary conditions, the Boussinesq equations model various phenomena in weather and climate dynamics, for example. We take random forcings into account and formulate the Bénard convection problem as a system of stochastic partial differential equations (SPDEs). This is a coupled system of the stochastic Navier-Stokes equations and the stochastic transport equation for temperature.

In various papers about large deviation principle (LDP) for solutions u^ε to SPDEs or to evolution equations in a semi-linear framework [3, 5, 4, 6, 14, 15, 18, 21, 26], the strategy used is similar to the classical one for diffusion processes. A very general version of Schilder's theorem yields the LDP for the Gaussian noise $\sqrt{\varepsilon}W$ driving the

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stochastic forcing term, with a good rate function $\tilde{I}$ written in terms of its reproducing kernel Hilbert space (RKHS). However, since the noise is not additive, the process u^ε is not a continuous function of the noise, which creates technical difficulties. As if the contraction principle were true, one defines deterministic controlled equations u_h which are similar to the stochastic one, replacing the stochastic integral with respect to the noise $\sqrt{\varepsilon}W$ by deterministic integrals in terms of elements h of its RKHS. Once well-posedness of this controlled equation is achieved, one proves that solution u^ε to the stochastic evolution equation satisfies a LDP with a rate function I defined in terms of $\tilde{I}$ and of u_h by means of an energy minimization problem. In order to transfer the LDP from the noise to the process, there are two classical proofs, each of which contains two main steps. One way consists in proving a continuity property of the map $h \mapsto u_h$ on level sets of the rate function $\tilde{I}$ and then some Freidlin-Wentzell inequality, which states continuity of the process with respect to the noise except on an exponentially small set. Another classical method in proving LDP for evolution equations is to establish both some exponential tightness and exponentially good approximations for some approximating sequence where the diffusion coefficient is stepwise constant. These methods require some time Hölder regularity that one can obtain when the diffusion coefficient is controlled in term of the L^2 norm of the solution, but not in the framework we will use here, where the bilinear term creates technical problems. An alternative approach [11] for large deviations is based on nonlinear semi-group theory and infinite dimensional Hamilton-Jacobi equations, and it also requires to establish exponential tightness.

The method used in the present paper is related to the Laplace principle. One proves directly that the level sets of the rate function I are compact and then establishes weak convergence of solutions to stochastic controlled equations written in terms of the noise $\sqrt{\varepsilon}W$ shifted by a random element h_ε of its RKHS. This is again some kind of continuity property written in terms of the distributions. Unlike [22], well-posedness and apriori estimates are proved directly for very general stochastic controlled equations with a forcing term including a stochastic integral and a deterministic integral with respect to a random element h_ε of the RKHS of the noise, and for diffusion coefficients which may depend on the gradient. Indeed, if the well-posedness for the stochastic controlled equation can be deduced from that of the stochastic equation by means of a Girsanov transformation, the apriori estimates uniform in $\varepsilon > 0$, which are a key ingredient of the proof of the weak convergence result, cannot be deduced from the corresponding ones for the stochastic Bénard equation since as $\varepsilon \to 0$, the $p > 1$ moments of the Girsanov density go to infinity exponentially fast. Well-posedness has been proved in [12] for the stochastic Boussinesq equation only in the particular case of an additive noise on the velocity component. This weak convergence approach has been introduced in [1, 2]. This method has been recently applied to SPDEs [22, 25] or SDEs in infinite dimensions [19]. Finally note that the proofs of the weak convergence and compactness property require more assumptions on the diffusion coefficient σ which may not depend on the gradient. Indeed, in order to prove convergence of integrals defined in terms of

elements h_ε of the RKHS of the noise only using weak convergence of h_ε, we also need to deal with localized integral estimates of time increments. With additional assumptions on the diffusion coefficient we are able to provide complete details of the proof of this statement which was missing in [22].

This paper is organized as follows. The mathematical formulation for the stochastic Bénard model is in §2. Then the well-posedness and general apriori estimates for the model are proved in §3. Finally, a large deviation principle is shown in §4.

2. Mathematical formulation

Let $D = (0, l) \times (0, 1)$ be a rectangular domain in the vertical plane. Denote by $x = (x_1, x_2)$ the spatial variable, $u = (u_1, u_2)$ the velocity field, p the pressure field, θ the temperature field, and (e_1, e_2) the standard basis in $\mathbb{R}^2$.

We consider the following stochastic coupled Navier-Stokes and heat transport equations for the Bénard convection problem [13]:

$$\frac{\partial}{\partial t}u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon - \nu\Delta u^\varepsilon + \nabla p = \theta e_2^\varepsilon + \sqrt{\varepsilon}\, n_1(t), \qquad \nabla\cdot u^\varepsilon = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}\theta^\varepsilon + u^\varepsilon\cdot\nabla\theta^\varepsilon - u_2^\varepsilon - \kappa\Delta\theta^\varepsilon = \sqrt{\varepsilon}\, n_2(t), \tag{2.2}$$

with boundary conditions

$$u^\varepsilon = 0 \ \ \& \ \ \theta^\varepsilon = 0 \ \text{ on } \ x_2 = 0 \text{ and } x_2 = 1, \tag{2.3}$$

$$u^\varepsilon, p, \theta^\varepsilon, u^\varepsilon_{x_1}, \theta^\varepsilon_{x_1} \text{ are periodic in } x_1 \text{ with period } l, \tag{2.4}$$

where n_1, n_2 are noise forcing terms and $\varepsilon > 0$ is a small parameter.

We consider the abstract functional setting for this system as in [13, 12]; see also [7, 23]. Let $L^2(D)$ be endowed with the usual scalar product and the induced norm. Consider another Hilbert space of vector-valued functions:

$$\dot{\mathbf{L}}^2(D) = \{u \in L^2(D)^2,\ \nabla\cdot u = 0,\ u|_{x_2=0} = u|_{x_2=1} = 0,\ u \text{ is periodic in } x_1 \text{ with period } l\}$$

$$\dot{L}^2(D) = \{\theta \in L^2(D),\ \ \theta|_{x_2=0} = \theta|_{x_2=1} = 0,\ \theta \text{ is periodic in } x_1 \text{ with period } l\}$$

Let $H = \dot{\mathbf{L}}^2(D) \times \dot{L}^2(D)$ be the product Hilbert space. We denote by the same notations, $(\cdot,\cdot)$ and $|\cdot|$, the scalar product and the induced norm, in $\dot{\mathbf{L}}^2(D)$, $\dot{L}^2(D)$ and H,

$$(\phi, \psi) = \int_D \phi(x)\psi(x)dx, \ \ |\phi| = \sqrt{(\phi,\phi)} = \sqrt{|\phi_1|^2 + |\phi_2|^2}.$$

Define $V = V_1 \times V_2$, where

$$V_1 = \{v \in H^1(D)^2 : \nabla\cdot v = 0,\ v|_{x_2=0} = v|_{x_2=1} = 0;\ v \text{ is periodic in } x_1 \text{ with period } l\},$$

$$V_2 = \{f \in H^1(D) : f|_{x_2=0} = f|_{x_2=1} = 0;\ f \text{ is periodic in } x_1 \text{ with period } l\}.$$

Then V is a product Hilbert space with the scalar product and the induced norm,

$$((\phi, \psi)) = \int_D \nabla\phi\cdot\nabla\psi dx, \ \ \|\phi\| = \sqrt{((\phi,\phi))} = \sqrt{\|\phi_1\|^2 + \|\phi_2\|^2},$$

where to ease the notation the space variable x is omitted when writing integrals on D. Again, we also use the same notations for the scalar product and the induced norm in V_1 and V_2. Let V' be the dual space of V. We have the dense and continuous embeddings $V \hookrightarrow H = H' \hookrightarrow V'$ and denote by $\langle \phi, \psi \rangle$ the duality between $\phi \in V$ (resp. V_i) and $\phi \in V'$ (resp. V_i').

Consider an unbounded linear operator $A = (\nu A_1, \kappa A_2) : H \to H$ with $D(A) = D(A_1) \times D(A_2)$ where $D(A_1) = V_1 \cap H^2(D)^2$, $D(A_2) = V_2 \cap H^2(D)$ and define

$$\langle A_1 u, v \rangle = ((u, v)),\ \langle A_2 \theta, \eta \rangle = ((\theta, \eta)),\ \forall u, v \in D(A_1),\ \forall \theta, \eta \in D(A_2).$$

Both the Stokes operator A_1 and the Laplace operator A_2 are self-adjoint, positive, with compact self-adjoint inverses. They map V to V'. We also introduce the bilinear operators B_1 and B_2 as follows: for $u, v, w \in V_1$ and $\theta, \eta \in V_2$,

$$\begin{aligned} \langle B_1(u, v), w \rangle &= \int_D [u \cdot \nabla v] w dx \quad := \sum_{i,j=1,2} \int_D u_i\, \partial_i v_j\, w_j dx, \\ \langle B_2(u, \theta), \eta \rangle &= \int_D [u \cdot \nabla \theta] \eta dx \quad := \sum_{i=1,2} \int_D u_i\, \partial_i \theta\, \eta\, dx. \end{aligned}$$

With the notation $\phi^\varepsilon = (u^\varepsilon, \theta^\varepsilon)$ and under the above formulation, we assume that the noise terms n_1 and n_2 are taken as, respectively $\sigma_1(t, \phi) \frac{\partial}{\partial t} W^1(t)$, $\sigma_2(t, \phi) \frac{\partial}{\partial t} W^2(t)$, where $W^1(t), W^2(t)$ are independent Wiener processes defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, taking values in $\dot{\mathbf{L}}^2(D)$ and $\dot{L}^2(D)$, with linear symmetric positive covariant operators Q_1 and Q_2, respectively. We denote $Q = (Q_1, Q_2)$. It is a linear symmetric positive covariant operator in the Hilbert space H. We assume that Q_1, Q_2 and thus Q are trace class (and hence compact [8]), i.e., $tr(Q) < \infty$.

As in [22], let $H_0 = Q^{\frac{1}{2}} H$. Then H_0 is a Hilbert space with the scalar product

$$(\phi, \psi)_0 = (Q^{-\frac{1}{2}} \phi, Q^{-\frac{1}{2}} \psi),\ \forall \phi, \psi \in H_0$$

together with the induced norm $|\cdot|_0 = \sqrt{(\cdot, \cdot)_0}$. The embedding $i : H_0 \to H$ is Hilbert-Schmidt and hence compact, and moreover, $i\, i^* = Q$.

Let L_Q be the space of linear operators S such that $SQ^{\frac{1}{2}}$ is a Hilbert-Schmidt operator (and thus a compact operator [8]) from H to H. The norm in the space L_Q is defined by $|S|^2_{L_Q} = tr(SQS^*)$, where S^* is the adjoint operator of S.

Note that the above formulation is equivalent to projecting the first governing equation from $\dot{L}^2(D)^2$ into the "divergence-free" space and thus the pressure term is absent. With these notation, the above Boussinesq system (2.1)-(2.2) becomes

$$du^\varepsilon + [\nu A_1 u^\varepsilon + B_1(u^\varepsilon, u^\varepsilon) - \theta^\varepsilon e_2] dt = \sqrt{\varepsilon}\, \sigma_1(t, \phi^\varepsilon)\, dW^1(t), \tag{2.5}$$

$$d\theta^\varepsilon + [\kappa A_2 \theta^\varepsilon + B_2(u^\varepsilon, \theta^\varepsilon) - u_2^\varepsilon] dt = \sqrt{\varepsilon}\, \sigma_2(t, \phi^\varepsilon)\, dW^2(t). \tag{2.6}$$

Thus, we write this system for $\phi^\varepsilon = (u^\varepsilon, \theta^\varepsilon)$ as

$$d\phi^\varepsilon + [A\phi^\varepsilon + B(\phi^\varepsilon) + R\phi^\varepsilon] dt = \sqrt{\varepsilon}\, \sigma(\phi^\varepsilon) dW(t), \quad \phi^\varepsilon(0) = \xi := (u_0^\varepsilon, \theta_0^\varepsilon), \tag{2.7}$$

where $W(t) = (W^1(t), W^2(t))$ and

$$\begin{aligned} A\phi &= (\nu A_1 u, \kappa A_2 \theta), && (2.8)\\ B(\phi) &= (B_1(u,u), B_2(u,\theta)), && (2.9)\\ R\phi &= (-\theta e_2, -u_2), && (2.10)\\ \sigma(t,\phi) &= (\sigma_1(t,\phi), \sigma_2(t,\phi)). && (2.11) \end{aligned}$$

The noise intensity $\sigma : [0,T] \times V \to L_Q(H_0, H)$ is assumed to satisfy the following:

Assumption A: There exist positive constants K and L such that
(A.1) $\sigma \in C\big([0,T] \times H; L_Q(H_0, H)\big)$
(A.2) $|\sigma(t,\phi)|^2_{L_Q} \leq K(1 + \|\phi\|^2), \quad \forall t \in [0,T]\ , \forall \phi \in V.$
(A.3) $|\sigma(t,\phi) - \sigma(t,\psi)|^2_{L_Q} \leq L\|\phi - \psi\|^2, \quad \forall t \in [0,T], \forall \phi, \psi \in V.$

In the sequel, to ease the notation, we will suppose that $\sigma(t,\phi) = \sigma(\phi)$; however, all the results have a straightforward extension to time-dependent noise intensity under the assumption A. When no confusion arises, we set $L^p := L^p(D)$ for $1 \leq p < +\infty$ and denote by C a constant which may change from one line to the next one.

3. Well-posedness

The solution for the Bénard problem under random influences (2.7) is denoted as ϕ^ε. The goal for this paper is to show the large deviation principle for $(\phi^\varepsilon, \varepsilon > 0)$ as $\varepsilon \to 0$.

Let $\mathcal{A}$ be the class of H_0-valued $(\mathcal{F}_t)-$predictable stochastic processes ϕ with the property $\int_0^T |\phi(s)|^2_0 ds < \infty$, a.s. Let

$$S_M = \Big\{ h \in L^2(0,T;H_0) : \int_0^T |h(s)|^2_0 ds \leq M \Big\}.$$

The set S_M endowed with the following weak topology is a Polish space (complete separable metric space) [2]: $d_1(h,k) = \sum_{i=1}^\infty \frac{1}{2^i} \big| \int_0^T \big(h(s) - k(s), \tilde{e}_i(s)\big)_0 ds \big|$, where $\{\tilde{e}_i(s)\}_{i=1}^\infty$ is a complete orthonormal basis for $L^2(0,T;H_0)$. Define

$$\mathcal{A}_M = \{\phi \in \mathcal{A} : \phi(\omega) \in S_M,\ a.s.\}. \tag{3.1}$$

As in [22], we prove existence and uniqueness of the solution to the Bénard equation. However, in the sequel, we will need some precise bounds on the norm of the solution to a more general equation, which contains an extra forcing (or control) term driven by an element of $\mathcal{A}_M$. These required estimates cannot be deduced from the corresponding ones by means of a Girsanov transformation. More precisely, let $h \in \mathcal{A}$, $\varepsilon \geq 0$ and consider the following generalized Bénard equation with initial condition $\phi^\varepsilon_h(0) = \xi$, and for σ , $\tilde{\sigma} \in C(H; L_Q(H_0, H))$ which satisfy Assumption A

$$d\phi^\varepsilon_h(t) + \big[A\phi^\varepsilon_h(t) + B(\phi^\varepsilon_h(t)) + R\phi^\varepsilon_h(t)\big]dt = \sqrt{\varepsilon}\sigma(\phi^\varepsilon_h(t))dW(t) + \tilde{\sigma}(\phi^\varepsilon_h(t))h(t)dt. \tag{3.2}$$

Recall that a stochastic process $\phi^\varepsilon_h(t,\omega)$ is called the weak solution for the generalized stochastic Bénard problem (3.2) on $[0,T]$ with initial condition ξ if ϕ^ε_h is in

$C([0,T];H)\cap L^2((0,T);V)$, a.s., and satisfies

$$(\phi_h^\varepsilon(t),\psi)-(\xi,\psi)+\int_0^t[\langle\phi_h^\varepsilon(s),A\psi\rangle+\big\langle B(\phi_h^\varepsilon(s)),\psi\big\rangle+(R\phi_h^\varepsilon(s),\psi)]ds$$
$$=\sqrt{\varepsilon}\int_0^t\big(\sigma(\phi_h^\varepsilon(s))dW(s),\psi\big)+\int_0^t\big(\tilde\sigma(\phi_h^\varepsilon(s))h(s)\,,\,\psi\big)\,ds,\ \ a.s., \tag{3.3}$$

for all $\psi\in D(A)$ and all $t\in[0,T]$. In most of the analysis here, we work in the Banach space $X:=C\big([0,T];H\big)\cap L^2\big((0,T);V\big)$ with the norm

$$\|\phi\|_X=\Big\{\sup_{0\le s\le T}|\phi(s)|^2+\int_0^T\|\phi(s)\|^2ds\Big\}^{\frac12}. \tag{3.4}$$

Theorem 3.1. *(Well-posedness and apriori bounds)*
Fix $M>0$; then there exists $\varepsilon_0:=\varepsilon_0(\nu,\kappa,K,L,T,M)>0$, such that the following existence and uniqueness result is true for $0\le\varepsilon\le\varepsilon_0$. Let the initial datum satisfy $\mathbb{E}|\xi|^4<\infty$, let $h\in\mathcal{A}_M$ and $\varepsilon\in[0,\varepsilon_0]$; then there exists a pathwise unique weak solution ϕ_h^ε of the generalized stochastic Bénard problem (3.2) *with initial condition $\phi_h^\varepsilon(0)=\xi\in H$ and such that $\phi_h^\varepsilon\in X$ a.s. Furthermore, there exists a constant $C_1:=C_1(\nu,\kappa,K,L,T,M)$ such that for $\varepsilon\in[0,\varepsilon_0]$ and $h\in\mathcal{A}_M$,*

$$E\|\phi_h^\varepsilon\|_X^2\le1+E\Big(\sup_{0\le t\le T}|\phi_h^\varepsilon(t)|^4+\int_0^T\|\phi_h^\varepsilon(t)\|^2\,dt\Big)\le C_1\left(1+E|\xi|^4\right). \tag{3.5}$$

Remark 3.2. *Note that if $\sigma=0$, i.e., when the noise term is absent, we deduce the existence and uniqueness of the solution to the "deterministic" control equation defined in terms of an element $h\in L^2((0,T);H_0)$ and an initial condition $\xi\in H$*

$$d\phi(t)+\big[A\phi(t)+B(\phi(t))+R\phi(t)\big]dt=\tilde\sigma(\phi(t))h(t)dt,\ \ \phi(0)=\xi. \tag{3.6}$$

If $h\in S_M$, the solution ϕ to (3.6) *satisfies*

$$\sup_{0\le s\le T}|\phi(s)|^2+\int_0^T\|\phi(s)\|^2ds\le\tilde C_1(\nu,\kappa,K,L,T,M,|\xi|). \tag{3.7}$$

Remark 3.3. *Finally, note that when ϕ_h^ε is a solution to the stochastic Boussinesq equation* (2.7)*, a similar argument shows that Theorem 3.1 holds for any $\varepsilon\ge0$ if the coefficients σ and $\tilde\sigma$ belong to $C([0,T]\times H;L_Q(H_0,H))$ and are such that in the upper estimates of the L_Q-norm appearing in the right hand sides of conditions (A.2) and (A.3), one replaces the V norms of ϕ and $\phi-\psi$ by their H-norms.*

The proof of this theorem will require several steps. The following lemmas gather some properties of B_1 and B_2. We send the reader to [7] or [23] for the results on B_1 which are classical and sketch some proofs of the corresponding results on B_2.

Lemma 3.4. *For $u,v,w\in V_1$ and $\theta,\eta\in V_2$,*

$$\begin{aligned}\langle B_1(u,v),v\rangle&=0,\quad\langle B_2(u,\theta),\theta\rangle=0,\\ \langle B_1(u,v),w\rangle&=-\langle B_1(u,w),v\rangle,\quad\langle B_2(u,\theta),\eta\rangle=-\langle B_2(u,\eta),\theta\rangle.\end{aligned}$$

Let $u \in V_1$, $\theta \in V_2$ and $\phi = (u, \theta) \in V$; note that $|\phi|^2 = |u|^2 + |\theta|^2$ and $\|\phi\|^2 = \|u\|^2 + \|\theta\|^2$. The following lemma provides upper bound estimates of B_1 and B_2.

Lemma 3.5. *For some positive constant c and any $u \in V_1$, $\theta, \eta \in V_2$ and $\phi = (u, \theta)$, one has*

$$\begin{aligned} |B_1(u,u)|_{V_1'} &\leq |u|_{L^4}^2 \leq c\,|u|\,\|u\|, && (3.8)\\ |\langle B_2(u,\theta),\eta\rangle| &\leq |u|_{L^4}\,|\theta|_{L^4}\,\|\eta\| \leq c\,|\phi|\,\|\phi\|\|\eta\|. && (3.9) \end{aligned}$$

Proof. We only check the properties on B_2. For $\phi = (u, \theta) \in V$ and $\eta \in V_2$, Lemma 3.4, Hölder's inequality, and the second part of (3.8) imply

$$\big|\big\langle B_2(u,\theta),\eta\big\rangle\big| = \big|\big\langle B_2(u,\eta),\theta\big\rangle\big| \leq \|\eta\|\,|u|_{L^4}\,|\theta|_{L^4} \leq c\|\eta\|\,|u|^{\frac12}\|u\|^{\frac12}|\theta|^{\frac12}\|\theta\|^{\frac12}.$$

This yields (3.9). □

Lemma 3.6. *Let $\phi = (u, \theta) \in V$, and let v and η be in $L^4(D)$. For any constant $\alpha > 0$, the following estimates hold:*

$$\begin{aligned} |\langle B_1(u,u),v\rangle| &\leq \alpha\|u\|^2 + \frac{3^3}{4^4\alpha^3}\,|u|^2\,|v|_{L^4}^4, && (3.10)\\ |\langle B_2(\phi),\eta\rangle| &\leq \alpha\,\|\phi\|^2 + \frac{3^3}{4^4\alpha^3}\,|u|^2\,|\eta|_{L^4}^4. && (3.11) \end{aligned}$$

Proof. We only check (3.11). The second half of (3.8) and Young's inequality yield

$$\begin{aligned} |\langle B_2(\phi),\eta\rangle| = |\langle B_2(u,\eta)\,,\theta\rangle| &\leq |\eta|_{L^4}\,|u|_{L^4}\,|\nabla\theta|_{L^2} \leq |\eta|_{L^4}\,|u|_{L^2}^{\frac12}\,|\nabla u|_{L^2}^{\frac12}\,|\nabla\theta|_{L^2}\\ &\leq |\eta|_{L^4}\,|u|_{L^2}^{\frac12}\,\|\phi\|^{\frac32} \leq \alpha\|\phi\|^2 + \frac{3^3}{4^4\alpha^3}\,|u|^2\,|\eta|_{L^4}^4. \end{aligned}$$

□

The following lemma allows to rewrite differences of B_i for $i = 1, 2$.

Lemma 3.7. *Let $\phi = (u, \theta)$ and $\psi = (v, \eta)$ belong to V. Then*

$$\begin{aligned} \big\langle B_1(u,u) - B_1(v,v), u - v\big\rangle &= -\big\langle B_1(u-v,u-v),v\big\rangle,\\ \big\langle B_2(\phi) - B_2(\psi),\theta-\eta\big\rangle &= -\big\langle B_2(\phi-\psi),\eta\big\rangle. \end{aligned}$$

Proof. Integration by parts, the boundary conditions and $div(u) = \nabla\cdot u = 0$ yield

$$\begin{aligned} \big\langle B_2(\phi) - B_2(\psi),\theta-\eta\big\rangle &= \int_D \big(u.\nabla\theta\big)(\theta-\eta)dx - \int_D \big(v.\nabla\eta\big)(\theta-\eta)dx\\ &= -\int_D \big(u.\nabla(\theta-\eta)\big)\theta dx + \int_D \big(v.\nabla(\theta-\eta)\big)\eta dx \end{aligned}$$

Since $\langle B_2(u,w)\,,\,w\rangle = \int_D \big(u.\nabla w\big)w dx = 0$ for any $w \in V_2$, we deduce that

$$\big\langle B_2(\phi) - B_2(\psi),\theta-\eta\big\rangle = -\int_D \big(u.\nabla(\theta-\eta)\big)\eta dx + \int_D \big(v.\nabla(\theta-\eta)\big)\eta dx,$$

which completes the proof. □

For $\phi = (u, \theta) \in V$, define

$$F(\phi) = -A\phi - B(\phi) - R\phi. \tag{3.12}$$

The following lemma gives a crucial monotonicity property of F. Let $\nu \wedge \kappa := \min(\nu, \kappa)$.

Lemma 3.8. *Let $r > 0$. Assume that $\phi = (u, \theta) \in V$ and $\psi = (v, \eta) \in V$ are such that $|v|_{L^4} \leq r$ and $|\eta|_{L^4} \leq r$. Then*

$$\big\langle F(\phi) - F(\psi), \phi - \psi \big\rangle + \frac{\nu \wedge \kappa}{2} \|\phi - \psi\|^2 \leq \Big(\frac{3^3 r^4}{2^4 (\nu \wedge \kappa)^3} + 1 \Big) \ |\phi - \psi|^2. \tag{3.13}$$

Proof. Set $U = u - v$, $\Theta = \theta - \eta$ and $\Phi = \phi - \psi := (U, \Theta)$. Integrating by parts and using Lemmas 3.6 and 3.7, we obtain for $\alpha > 0$,

$$\begin{aligned} \big\langle F(\phi) - F(\psi), \Phi \big\rangle \ &= \ -\nu \|U\|^2 - \kappa \|\Theta\|^2 - \langle B_1(U, U), v \rangle - \langle B_2(\Phi), \eta \rangle + 2(U_2, \Theta) \\ &\leq \ -\nu \|U\|^2 - \kappa \|\Theta\|^2 + \alpha \big(\|U\|^2 + \|\Phi\|^2 \big) + \frac{3^3}{4^4 \alpha^3} \, |U|^2 \, \big(|v|_{L^4}^4 + |\eta|_{L^4}^4 \big) + |U|^2 + |\Theta|^2. \end{aligned}$$

Choosing $\alpha = \frac{\nu \wedge \kappa}{2}$, we deduce (3.13). □

The proof of Theorem 3.1 involves Galerkin approximations. Let $\{\varphi_n\}_{n \geq 1}$ be a complete orthonormal basis of the Hilbert space H such that $\varphi_n \in Dom(A)$, domain of definition of the operator A. For any $n \geq 1$, let $H_n = span(\varphi_1, \cdots, \varphi_n) \in Dom(A)$ and $P_n : H \to H_n$ denotes the orthogonal projection onto H_n. Note that P_n contracts the H and V norms. Suppose that the $H-$valued Wiener process W with covariance operator Q is such that

$$P_n Q^{\frac{1}{2}} = Q^{\frac{1}{2}} P_n, \ \ n \geq 1,$$

which is true if $Qh = \sum_{n \geq 1} \lambda_n \varphi_n$ with trace $\sum_{n \geq 1} \lambda_n < \infty$. Then for $H_0 = Q^{\frac{1}{2}} H$ and $(\phi, \psi)_0 = (Q^{-\frac{1}{2}} \phi, Q^{-\frac{1}{2}} \psi)$, for $\phi, \psi \in H_0$, we see that $P_n : H_0 \to H_0 \cap H_n$ is a contraction both of the H and H_0 norms. Let $W_n = P_n W$, $\sigma_n = P_n \sigma$ and $\tilde{\sigma}_n = P_n \tilde{\sigma}$.

For $h \in \mathcal{A}_M$, consider the following stochastic ordinary differential equation on the n-dimensional space H_n defined by

$$d(\phi_{n,h}^\varepsilon, \psi) = \big[\langle F(\phi_{n,h}^\varepsilon), \psi \rangle + (\tilde{\sigma}_n(\phi_{n,h}^\varepsilon) h, \psi) \big] dt + \sqrt{\varepsilon} \, (\sigma_n(\phi_{n,h}^\varepsilon) dW_n, \psi), \tag{3.14}$$

for $\psi = (v, \eta) \in H_n$ and $\phi_{n,h}^\varepsilon(0) = P_n \xi$.

Note that for $\psi = (v, \eta) \in V$, the map $\phi \in H_n \mapsto \langle (A + R)(\phi), \psi \rangle$ is globally Lipschitz, while using Lemma 3.5 the map $\phi = (u, \theta) \in H_n \mapsto \sum_{i,j=1,2} \int_D u_i \, v_j \, \partial_i u_j \, dx + \sum_{i=1,2} \int_D u_i \, \eta \, \partial_i \, \theta \, dx$ is locally Lipschitz. Furthermore, condition (A.3) implies that the map $\phi \in H_n \to \sigma_n(\phi)$ is globally Lipschitz from H_n to $n \times n$ matrices. Hence by a well-posedness result for stochastic ordinary differential equations [16], there exists a maximal solution to (3.14), i.e., a stopping time $\tau_{n,h}^\varepsilon \leq T$ such that (3.14) holds for $t < \tau_{n,h}^\varepsilon$ and as $t \uparrow \tau_{n,h}^\varepsilon < T$, $|\phi_{n,h}^\varepsilon(t)| \to \infty$. For every $N > 0$, set

$$\tau_N = \inf\{t : \ |\phi_{n,h}^\varepsilon(t)| \geq N\} \wedge T. \tag{3.15}$$

Almost surely, $\phi^{\varepsilon}_{n,h} \in C([0,T],H_n)$ on $\{\tau_N = T\}$. The following proposition shows that $\tau^{\varepsilon}_{n,h} = T$ a.s. and gives estimates on $\phi^{\varepsilon}_{n,h}$ depending only on the physical constants ν and κ, T, M, $\mathbb{E}|\xi|^{2p}$ which are valid for all n and all $\varepsilon \in [0,\varepsilon_0]$ for some $\varepsilon_0 > 0$. Its proof depends on the following version of Gronwall's lemma.

Lemma 3.9. *Let X, Y and I be non decreasing, non-negative processes, φ be a non negative process and Z be a non-negative integrable random variable. Assume that $\int_0^T \varphi(s)\,ds \le C$ almost surely and that there exist positive constants $\alpha, \beta \le \frac{1}{2(1+Ce^C)}$, $\gamma \le \frac{\alpha}{2(1+Ce^C)}$ and $\tilde{C} > 0$ such that for $0 \le t \le T$,*

$$X(t) + \alpha Y(t) \;\le\; Z + \int_0^t \varphi(r)\, X(r)\, dr + I(t), \text{ a.s.} \tag{3.16}$$

$$\mathbb{E}(I(t)) \;\le\; \beta\, \mathbb{E}(X(t)) + \gamma\, \mathbb{E}(Y(t)) + \tilde{C}. \tag{3.17}$$

Then if $X \in L^\infty([0,T]\times\Omega)$, we have for $t \in [0,T]$

$$\mathbb{E}\big[X(t) + \alpha Y(t)\big] \le 2(1+Ce^C)\big(\mathbb{E}(Z) + \tilde{C}\big). \tag{3.18}$$

Proof. Iterating the inequality (3.16) and ignoring Y, an induction argument on n yields for $t \in [0,T]$, $n \ge 1$

$$\begin{aligned} X(t) \;&\le\; Z + \int_0^t \varphi(s_1)\Big[Z + \int_0^{s_1} \varphi(s_2)X(s_2)ds_2 + I(s_1)\Big]ds_1 + I(t) \\ &\le Z + I(t) + \sum_{1\le k\le n} \int_0^t \varphi(s_1) \int_0^{s_1} \varphi(s_2) \cdots \int_0^{s_{k-1}} \varphi(s_k)\,[Z + I(s_k)]\,ds_k \cdots ds_1 \\ &\quad + \int_0^t \varphi(s_1) \int_0^{s_1} \varphi(s_2) \cdots \int_0^{s_n} \varphi(s_{n+1})\, X(s_{n+1})\, ds_{n+1} ds_n \cdots ds_1. \end{aligned}$$

Recall that $X(s,\omega)$ is a.e. bounded, I is non decreasing and $\varphi(s) \le C$; thus $X(t) \le e^C\,[Z + I(t)]$. Using this inequality in (3.16), we deduce that $X(t) + \alpha Y(t) \le \big[Z + I(t)\big]\big(1 + Ce^C\big)$. Taking expected values and using (3.17), we conclude the proof. □

Proposition 3.10. *There exists $\varepsilon_{p,0} := \varepsilon_{p,0}(\nu,\kappa,K,T,M)$ such that for $0 \le \varepsilon \le \varepsilon_{p,0}$ the following result holds for an integer $p \ge 1$ (with the convention $x^0 = 1$). Let $h \in \mathcal{A}_M$ and $\xi \in L^{2p}(\Omega,H)$. Then $\tau_{n,h} = T$ a.s. and the equation (3.14) has a unique solution with a modification $\phi^{\varepsilon}_{n,h} \in C([0,T],H_n)$ and satisfying*

$$\begin{aligned} \sup_n \mathbb{E}\Big(\sup_{0\le t\le T} |\phi^{\varepsilon}_{n,h}(t)|^{2p} + \int_0^T \|\phi^{\varepsilon}_{n,h}(s)\|^2\, |\phi^{\varepsilon}_{n,h}(s)|^{2(p-1)} ds \Big) \\ \le C(p,\nu,\kappa,K,T,M)\big(\mathbb{E}|\xi|^{2p} + 1\big). \end{aligned} \tag{3.19}$$

Proof. Itô's formula yields that for $t \in [0,T]$ and τ_N defined by (3.15),

$$|\phi^{\varepsilon}_{n,h}(t\wedge\tau_N)|^2 = |P_n\xi|^2 + 2\sqrt{\varepsilon} \int_0^{t\wedge\tau_N} \big(\sigma_n(\phi^{\varepsilon}_{n,h}(s))dW_n(s), \phi^{\varepsilon}_{n,h}(s)\big) \tag{3.20}$$

$$+ 2\int_0^{t\wedge\tau_N} \big\langle F(\phi^\varepsilon_{n,h}(s)), \phi^\varepsilon_{n,h}(s)\rangle ds + 2\int_0^{t\wedge\tau_N} \big(\tilde\sigma_n(\phi^\varepsilon_{n,h}(s))h(s), \phi^\varepsilon_{n,h}(s)\big)\, ds$$
$$+ \varepsilon\int_0^{t\wedge\tau_N} |\sigma_n(\phi^\varepsilon_{n,h}(s))\, P_n|^2_{L_Q}\, ds. \tag{3.21}$$

Apply again Itô's formula for x^p when $p \geq 2$ and then use Lemma 3.4. With the convention $p(p-1)x^{p-2} = 0$ for $p = 1$, this yields for $t \in [0, T]$,

$$|\phi^\varepsilon_{n,h}(t\wedge\tau_N)|^{2p} + 2p\int_0^{t\wedge\tau_N} |\phi^\varepsilon_{n,h}(r)|^{2(p-1)} \left[\nu\|u^\varepsilon_{n,h}(r)\|^2 + \kappa\|\theta^\varepsilon_{n,h}(r)\|^2\right] dr$$
$$\leq\ |P_n\xi|^{2p} + \sum_{1\leq j\leq 5} T_j(t), \tag{3.22}$$

where

$$\begin{aligned}
T_1(t) &= 4p\int_0^{t\wedge\tau_N} |(\theta^\varepsilon_{n,h}(r), u^\varepsilon_{n,h,2}(r))||\phi^\varepsilon_{n,h}(r)|^{2(p-1)} dr,\\
T_2(t) &= 2p\sqrt{\varepsilon}\,\Big|\int_0^{t\wedge\tau_N} \big(\sigma_n(\phi^\varepsilon_{n,h}(r))\, dW_n(r), \phi^\varepsilon_{n,h}(r)\big)\, |\phi^\varepsilon_{n,h}(r)|^{2(p-1)}\Big|,\\
T_3(t) &= 2p\int_0^{t\wedge\tau_N} |(\tilde\sigma_n(\phi^\varepsilon_{n,h}(r))\, h(r), \phi^\varepsilon_{n,h}(r))|\, |\phi^\varepsilon_{n,h}(r)|^{2(p-1)} dr,\\
T_4(t) &= p\,\varepsilon\int_0^{t\wedge\tau_N} |\sigma_n(\phi^\varepsilon_{n,h}(r))\, P_n|^2_{L_Q}\, |\phi^\varepsilon_{n,h}(r)|^{2(p-1)} dr,\\
T_5(t) &= 2p(p-1)\varepsilon\int_0^{t\wedge\tau_N} |\sigma^*_n(\phi^\varepsilon_{n,h}(r))\, \phi^\varepsilon_{n,h}(r)|^2_{H_0}\, |\phi^\varepsilon_{n,h}(r)|^{2(p-2)} dr.
\end{aligned}$$

The Cauchy-Schwarz inequality implies that $2|(\theta^\varepsilon_{n,h}(r), u_{n,h,2}(r))| \leq |\phi^\varepsilon_{n,h}(r)|^2$. Hence

$$T_1(t) \leq 2p\int_0^{t\wedge\tau_N} |\phi^\varepsilon_{n,h}(r)|^{2p}\, dr. \tag{3.23}$$

Since $h \in \mathcal{A}_M$, the Cauchy-Schwarz inequality and (A.2) imply that for $\delta_1 > 0$,

$$\begin{aligned}
T_3(t) &\leq 2p\int_0^{t\wedge\tau_N} \left[K(1 + \|\phi^\varepsilon_{n,h}(r)\|^2)\right]^{\frac12} |h(r)|_0\, |\phi^\varepsilon_{n,h}(r)|^{2p-1} dr\\
&\leq \delta_1\int_0^{t\wedge\tau_N} \|\phi^\varepsilon_{n,h}(r)\|^2\, |\phi^\varepsilon_{n,h}(r)|^{2(p-1)}\, dr + \frac{p^2K}{\delta_1}\int_0^{t\wedge\tau_N} |h(r)|^2_0\, |\phi^\varepsilon_{n,h}(r)|^{2p}\, dr\\
&\quad + \delta_1\int_0^{t\wedge\tau_N} |\phi^\varepsilon_{n,h}(r)|^{2(p-1)} dr.
\end{aligned} \tag{3.24}$$

Using again (A.2), we deduce that

$$\begin{aligned}
T_4(t) + T_5(t) &\leq 2p^2\, K\,\varepsilon\int_0^{t\wedge\tau_N} \|\phi^\varepsilon_{n,h}(r)\|^2\, |\phi^\varepsilon_{n,h}(r)|^{2(p-1)}\, dr\\
&\quad + 2p^2\, K\,\varepsilon\int_0^{t\wedge\tau_N} |\phi^\varepsilon_{n,h}(r)|^{2(p-1)}\, dr.
\end{aligned} \tag{3.25}$$

Finally, the Burkholder-Davies-Gundy inequality, (A.2) and Schwarz's inequality yield that for $t \in [0,T]$ and $\delta_2 > 0$,

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le s\le t}|T_2(s)|\Big) &\le 6p\sqrt{\varepsilon}\,\mathbb{E}\Big\{\int_0^{t\wedge\tau_N}|\phi^\varepsilon_{n,h}(r)|^{2(2p-1)}\,|\sigma_{n,h}(\phi^\varepsilon_{n,h}(r))\,P_n|^2_{L_Q}\,dr\Big\}^{\frac12}\\ &\le \delta_2\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_N}|\phi^\varepsilon_{n,h}(s)|^{2p}\Big)+\frac{9p^2K\varepsilon}{\delta_2}\,\mathbb{E}\int_0^{t\wedge\tau_N}|\phi^\varepsilon_{n,h}(r)|^{2(p-1)}dr\\ &\quad+\frac{9p^2K\varepsilon}{\delta_2}\,\mathbb{E}\int_0^{t\wedge\tau_N}\|\phi^\varepsilon_{n,h}(r)\|^2\,|\phi^\varepsilon_{n,h}(r)|^{2(p-1)}dr. \qquad (3.26)\end{aligned}$$

Consider the following property $I(i)$ for an integer $i \ge 0$:
I(i) There exists $\varepsilon_{0,i} := \varepsilon_{0,i}(\nu,\kappa,K,T,M) > 0$ such that for $0 \le \varepsilon \le \varepsilon_{0,i}$

$$\sup_n \mathbb{E}\int_0^{t\wedge\tau_N}|\phi^\varepsilon_{n,h}(r)|^{2i}dr \le C(i) := C(i,\nu,\kappa,K,T,M) < +\infty.$$

The property $I(0)$ obviously holds with $\varepsilon_{0,0} = 1$ and $C(0) = T$. Assume that for some integer i with $1 \le i \le p$, the property I(i-1) holds; we prove that I(i) holds.

Set $\delta_1 = \frac{(\nu\wedge\kappa)\,i}{2}$, $\varphi_i(r) = 2i + \frac{i^2K}{\delta_1}|h(r)|^2_0$, $Z = \delta_1\int_0^{\tau_N}|\phi^\varepsilon_{n,h}(r)|^{2(i-1)}dr + |\xi|^{2i}$, $X(t) = \sup_{0\le s\le t}|\phi^\varepsilon_{n,h}(s\wedge\tau_N)|^{2i}$, $Y(t) = \int_0^{t\wedge\tau_N}\|\phi^\varepsilon_{n,h}(s)\|^2\,|\phi^\varepsilon_{n,h}(s)|^{2(i-1)}\,ds$ and $I(t) = \sup_{0\le s\le t}|2i\sqrt{\varepsilon}\,\Big|\int_0^{t\wedge\tau_N}\big(\sigma_n(\phi^\varepsilon_{n,h}(r))\,dW_n(r),\phi^\varepsilon_{n,h}(r)\big)\,|\phi^\varepsilon_{n,h}(r)|^{2(i-1)}\Big|$.
Then $\int_0^T\varphi_i(s)ds \le C_i(M) := 2iT + \frac{i^2K}{\delta_1}M$. Let $\alpha = i\,(\nu\wedge\kappa)$, $\beta = \delta_2 = \frac{1}{2\left[1+C_i(M)e^{C_i(M)}\right]}$ and $\tilde{C} = \frac{9i^2K}{\delta_2}\mathbb{E}\int_0^{\tau_N}|\phi^\varepsilon_{n,h}(s)|^{2(i-1)}ds$. Let

$$\varepsilon_{0,i} = 1 \wedge \frac{\nu\wedge\kappa}{8iK} \wedge \frac{\nu\wedge\kappa}{144\,i\,K\,[1+C_i(M)e^{C_i(M)}]^2} \wedge \varepsilon_{i-1,0}.$$

Then for $0 \le \varepsilon \le \varepsilon_{i,0}$ the inequalities (3.23)-(3.26) show that the assumptions of Lemma 3.9 hold with $\gamma = \frac{9i^2K\varepsilon}{\delta_2} \le \alpha\beta$, which yields I(i).

An induction argument shows that $I(p-1)$ holds, and hence the previous computations with $i = p$ and Lemma 3.9 yield that for $t = T$ and $0 \le \varepsilon \le \varepsilon_{p,0}$,

$$\sup_n \mathbb{E}\Big(\sup_{0\le s\le\tau_N}|\phi^\varepsilon_{n,h}(s)|^{2p} + \int_0^{\tau_N}\|\phi^\varepsilon_{n,h}(s)\|^2\,\phi^\varepsilon_{n,h}(s)|^{2(p-1)}\,ds\Big) \le C(p,\nu,\kappa,K,T,M).$$

As $N \to \infty$, $\tau_N \uparrow \tau_{n,h}$ and on $\{\tau_{n,h} < T\}$, $\sup_{0\le s\le t\wedge\tau_N}|\phi_{n,h}(s)| \to \infty$. Hence $\mathbb{P}(\tau_{n,h} < T) = 0$ and for almost all ω, for $N(\omega)$ large enough, $\tau_{N(\omega)}(\omega) = T$ and $\phi_{n,h}(.)(\omega) \in C([0,T],H_n)$. By the Lebesgue monotone convergence theorem, we complete the proof of the proposition. □

We now have the following bound in L^4.

Proposition 3.11. *Let $h \in \mathcal{A}_M$ and $\xi \in L^4(\Omega,H)$. Let $\varepsilon_{2,0}$ be defined as in Proposition 3.10 with $p = 2$. Then there exists a constant $C_2 := C_2(\nu,\kappa,K,T,M)$*

such that

$$\sup_n \mathbb{E}\int_0^T |\phi^\varepsilon_{n,h}(s)|^4_{L^4} ds \quad \leq \quad C_2(1+\mathbb{E}|\xi|^4). \tag{3.27}$$

Proof. Let $f_{n,h}(t) = u_{n,h,i}(t)$ or $\theta^\varepsilon_{n,h}(t)$, with $i = 1, 2$. Then (3.19) with $p = 2$ implies that

$$\sup_n \mathbb{E}\int_0^T \|f_{n,h}(s)\|^2 |f_{n,h}(s)|^2 ds \leq C_2(\nu,\kappa,K,T,M)(1+\mathbb{E}|\xi|^4).$$

Hence by the second part of (3.8), we finish the proof of (3.27). □

We are now ready to prove the main result of this section.

Proof of Theorem 3.1:
Let $\Omega_T = [0,T]\times\Omega$ be endowed with the product measure $ds\otimes d\mathbb{P}$ on $\mathcal{B}([0,T])\otimes\mathcal{F}$. Let $\varepsilon_{2,0}$ be defined by Proposition 3.10 with $p = 2$ and set $\varepsilon_0 := \varepsilon_{2,0} \wedge \frac{\nu\wedge\kappa}{2L}$. The proof consists of several steps.

Step 1: The inequalities (3.19) and (3.27) imply the existence of a subsequence of $\{\phi^\varepsilon_{n,h}\}_{n\geq 0}$ (still denoted by the same notation), of processes $\phi^\varepsilon_h \in L^2(\Omega_T, V)\cap L^4(\Omega_T, L^4(D))\cap L^4(\Omega, L^\infty([0,T],H))$, $F^\varepsilon_h \in L^2(\Omega_T, V')$ and $S^\varepsilon_h, \tilde{S}^\varepsilon_h \in L^2(\Omega_T, L_Q)$, and of random variables $\tilde{\phi}^\varepsilon_h(T) \in L^2(\Omega, H)$, for which the following properties hold:
(i) $\phi^\varepsilon_{n,h} \to \phi^\varepsilon_h$ weakly in $L^2(\Omega_T, V)$,
(ii) $\phi^\varepsilon_{n,h} \to \phi^\varepsilon_h$ weakly in $L^4(\Omega_T, L^4(D))$,
(iii) $\phi^\varepsilon_{n,h}$ is weak star converging to ϕ^ε_h in $L^4(\Omega, L^\infty([0,T],H))$,
(iv) $\phi^\varepsilon_{n,h}(T) \to \tilde{\phi}^\varepsilon_h(T)$ weakly in $L^2(\Omega, H)$,
(v) $F(\phi^\varepsilon_{n,h}) \to F^\varepsilon_h$ weakly in $L^2(\Omega_T, V')$,
(vi) $\sigma_n(\phi^\varepsilon_{n,h})P_n \to S^\varepsilon_h$ weakly in $L^2(\Omega_T, L_Q)$,
(vii) $\tilde{\sigma}_n(\phi^\varepsilon_{n,h})h \to \tilde{S}^\varepsilon_h$ in the $\sigma(L^1(\Omega_T,H), L^\infty(\Omega_T,H))$ topology.

Indeed, (i)-(iv) are straightforward consequences of Propositions 3.10 and 3.11, and of uniqueness of the limit of $\mathbb{E}\int_0^T \phi^\varepsilon_{n,h}(t)\psi(t)dt$ for appropriate ψ.

Furthermore, given $\psi = (v,\eta) \in L^2(\Omega_T, V)$, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \Big[\nu\langle A_1(u^\varepsilon_{n,h}(t), v(t)\rangle + \kappa\langle A_2(\theta^\varepsilon_{n,h}(t)), \eta(t)\rangle\Big]\, dt \\
&\qquad = \ -\nu\mathbb{E}\int_0^T (\nabla u^\varepsilon_{n,h}(t), \nabla v(t))dt - \kappa\mathbb{E}\int_0^T (\nabla\theta^\varepsilon_{n,h}(t), \nabla\eta(t))dt \\
&\qquad \to \ -\nu\mathbb{E}\int_0^T (\nabla u^\varepsilon_{h}(t), \nabla v(t))dt - \kappa\mathbb{E}\int_0^T (\nabla\theta^\varepsilon_{h}(t), \nabla\eta(t))dt.
\end{aligned} \tag{3.28}$$

Using (3.19) with $p = 2$, (3.8), (3.9), the Cauchy-Schwarz and Poincaré inequalities, we deduce

$$\sup_n \mathbb{E}\int_0^T \Big|\langle B_1(u^\varepsilon_{n,h}(t), u^\varepsilon_{n,h}(t)), v(t)\rangle + \langle B_2(\phi^\varepsilon_{n,h}(t))\,,\, \eta(t)\rangle + (R\phi^\varepsilon_{n,h}(t), \psi(t))\Big|\, dt$$

$$\begin{aligned}
&\leq C \sup_n \mathbb{E} \int_0^T \big\{ \|u^\varepsilon_{n,h}(t)\| \, |u^\varepsilon_{n,h}(t)| \|v(t)\| + \|\phi^\varepsilon_{n,h}(t)\| \, |\phi^\varepsilon_{n,h}(t)| \, \|\eta(t)\| \\
&\qquad + |\theta^\varepsilon_{n,h}(t)| \, |v_2(t)| + |u^\varepsilon_{n,h,2}(t)| \, |\eta(t))| \big\} dt \\
&\leq C_3(\nu, \kappa, K, T, M)\big(1 + E|\xi|^4\big) + \mathbb{E} \int_0^T \|\psi(t)\|^2 dt.
\end{aligned}$$

Hence $\{B(\phi^\varepsilon_{n,h}(t)) + R\phi^\varepsilon_{n,h}(t)\,,\, n \geq 1\}$ has a subsequence converging weakly in $L^2(\Omega_T, V')$, which proves (v).

Since P_n contracts the $|\cdot|_0$ and $|\cdot|$ norms, (A.2) and (3.19) imply that

$$\sup_n \mathbb{E} \int_0^T |\sigma_n(\phi^\varepsilon_{n,h}(t))P_n|^2_{L_Q} dt \leq K \sup_n \mathbb{E} \int_0^T (1 + \|\phi^\varepsilon_{n,h}(t)\|^2) dt \, < \, \infty,$$

which proves (vi).

Let λ denote the Lebesgue measure on $[0,T]$. We finally check that for $h \in \mathcal{A}_M$, the sequence $(\big|\tilde{\sigma}_n(\phi^\varepsilon_{n,h})\big|_{L_Q} |h|_0\,,\, n \geq 1)$ is uniformly integrable on Ω_T with respect to the measure $\lambda \otimes P$, and hence relatively compact for the weak topology on $L^1(\Omega_T)$. Let $\alpha \in]0,1[$ and for any $N > 0$ set $C(N) = N^{\frac{1}{4}}$. Then if $A(n,N) = \{(s,\omega) \in \Omega_T : |h(s)|_0 \, \|\phi^\varepsilon_{n,h}(s)\| \geq N\}$, we deduce that

$$T(n,N) := \mathbb{E} \int_0^T 1_{A(n,N)} |h(s)|_0 \, \|\phi^\varepsilon_{n,h}(s)\| \, ds \leq T_1(n,N) + T_2(n,N),$$

where

$$\begin{aligned}
T_1(n,N) &= \mathbb{E} \int_0^T 1_{\{|h(s)|_0 \geq C(N)\}} |h(s)|_0 \, \|\phi^\varepsilon_{n,h}(s)\| \, ds, \\
T_2(n,N) &= C(N)^{1-\alpha} \mathbb{E} \int_0^T 1_{A(n,N)} |h(s)|_0^\alpha \, \|\phi^\varepsilon_{n,h}(s)\| \, ds.
\end{aligned}$$

Since $|h(.)|_0^2 \in L^1(\Omega_T)$, it is uniformly integrable. Thus, Schwarz's inequality and (3.19) with $p = 1$ yield

$$T_1(n,N) \leq \Big(\mathbb{E} \int_0^T 1_{\{|h(s)|_0 \geq C(N)\}} |h(s)|_0^2 ds\Big)^{\frac{1}{2}} \Big(\mathbb{E} \int_0^T \|\phi^\varepsilon_{n,h}(s)\|^2 ds\Big)^{\frac{1}{2}},$$

so that $\lim_{N\to\infty} \sup_{n\geq 1} T_1(n,N) \to 0$. Using twice Hölder's inequality with conjugate exponents $\frac{2}{1+\alpha}$ and $\frac{2}{1-\alpha}$, then with $\frac{1+\alpha}{2\alpha}$ and $\frac{1+\alpha}{1-\alpha}$, we deduce

$$\begin{aligned}
T_2(n,N) \leq & C(N)^{1-\alpha} \Big(\mathbb{E} \int_0^T \|\phi^\varepsilon_{n,h}(s)\|^2 \, ds\Big)^{\frac{1-\alpha}{2}} \Big(\mathbb{E} \int_0^T 1_{A(n,N)} |h(s)|_0^{\frac{2\alpha}{1+\alpha}} \, \|\phi^\varepsilon_{n,h}\|^{\frac{2\alpha}{1+\alpha}} ds\Big)^{\frac{1+\alpha}{2}} \\
\leq & C \, C(N)^{1-\alpha} \Big(\mathbb{E} \int_0^T |h(s)|_0 \, \|\phi^\varepsilon_{n,h}(s)\| \, ds\Big)^{\alpha} \Big((\lambda \otimes P) A(n,N)\Big)^{\frac{1-\alpha}{2}}.
\end{aligned}$$

Using Markov's and Schwarz's inequalities, we deduce that

$$(\lambda \otimes P) A(n,N) \leq N^{-1} \mathbb{E} \int_0^T |h(s)|_0 \, \|\phi^\varepsilon_{n,h}(s)\| \, ds \leq \Big(M \, \mathbb{E} \int_0^T \|\phi^\varepsilon_{n,h}(s)\|^2 ds\Big)^{\frac{1}{2}} N^{-1}.$$

Hence using (3.19) with $p = 1$ and

$$T_2(n,N) \leq C\, N^{-\frac{1-\alpha}{4}} \left(M\, \mathbb{E} \int_0^T \|\phi^\varepsilon_{n,h}(s)\|^2 ds \right)^{\frac{1+\alpha}{4}},$$

we obtain $\lim_{N\to\infty} \sup_{n\geq 1} T_2(n,N) \to 0$. This completes the proof of the uniformly integrability of the sequence $(|h|_0 \, \|\phi^\varepsilon_{n,h}\|\,,\, n \geq 1)$. Finally, using (A2) and the uniform integrability of $|h|_0$, we deduce that the sequence $(\left|\tilde{\sigma}(\phi^\varepsilon_{n,h})\right|_{L_Q} |h|_0\,,\, n \geq 1)$ is uniformly integrable with respect to $\lambda \otimes P$. Since H is a Hilbert space, this immediately proves (vii).

Step 2: For $\delta > 0$, let $f \in H^1(-\delta, T+\delta)$ be such that $\|f\|_\infty = 1$, $f(0) = 1$ and for any integer $j \geq 1$ set $g_j(t) = f(t)\varphi_j$, where $\{\varphi_j\}_{j\geq 1}$ is the previously chosen orthonormal basis for H. The Itô formula implies that for any $j \geq 1$, and for $0 \leq t \leq T$,

$$\left(\phi^\varepsilon_{n,h}(T)\,,\, g_j(T)\right) = \left(\phi^\varepsilon_{n,h}(0)\,,\, g_j(0)\right) + \sum_{i=1}^4 I^i_{n,k}, \tag{3.29}$$

where

$$\begin{aligned}
I^1_{n,k} &= \int_0^T (\phi^\varepsilon_{n,h}(s), \varphi_j) f'(s) ds, \\
I^2_{n,k} &= \sqrt{\varepsilon} \int_0^T \left(\sigma_n(\phi^\varepsilon_{n,h}(s)) P_n dW_n(s), g_j(s)\right), \\
I^3_{n,k} &= \int_0^T \langle F(\phi^\varepsilon_{n,h}(s)), g_j(s)\rangle ds, \\
I^4_{n,k} &= \int_0^T \left(\tilde{\sigma}_n(\phi^\varepsilon_{n,h}(s)) h(s), g_j(s)\right) ds.
\end{aligned}$$

Since $f' \in L^2([0,T])$ and for every $X \in L^2(\Omega)$, $(t,\omega) \mapsto \varphi_j X(\omega)\, f'(t) \in L^2(\Omega, H)$, (i) above implies that as $n \to \infty$, $I^1_{n,k} \to \int_0^T (\phi^\varepsilon_h(s), \varphi_j) f'(s) ds$ weakly in $L^2(\Omega)$. Similarly, (v) implies that as $n \to \infty$, $I^3_{n,k} \to \int_0^T \langle F^\varepsilon_h(s), g_j(s)\rangle ds$ weakly in $L^2(\Omega)$, while (vii) implies that $I^4_{n,k} \to \int_0^T \left(\tilde{S}^\varepsilon_h(s), g_j(s)\right) ds$ in the $\sigma(L^1(\Omega), L^\infty(\Omega))$ topology.

To prove the convergence of $I^2_{n,k}$, as in [22], let $\mathcal{P}_T$ denote the class of predictable processes in $L^2(\Omega_T, L_Q(H_0, H))$ with the inner product

$$(G,J)_{\mathcal{P}_T} = \mathbb{E} \int_0^T \left(G(s), J(s)\right)_{\mathcal{P}_T} ds = \mathbb{E} \int_0^T trace(G(s) Q J(s)^*) ds.$$

The map $\mathcal{T} : \mathcal{P}_T \to L^2(\Omega)$ defined by $\mathcal{T}(G)(t) = \int_0^T \left(G(s) dW(s), g_j(s)\right)$ is linear and continuous because of the Itô isometry. Furthermore, (vi) shows that for every $G \in \mathcal{P}_T$, as $n \to \infty$, $\left(\sigma_n(\phi^\varepsilon_{n,h}) P_n, G\right)_{\mathcal{P}_T} \to (S^\varepsilon_h, G)_{\mathcal{P}_T}$ weakly in $L^2(\Omega)$.

Finally, as $n \to \infty$, $P_n\xi = \phi^\varepsilon_{n,h}(0) \to \xi$ in H and by (iv), $(\phi^\varepsilon_{n,h}(T), g_j(T)) \to (\tilde\phi^\varepsilon_h(T), g_j(T))$ weakly in $L^2(\Omega)$. Therefore, (3.29) leads to, as $n \to \infty$,

$$(\tilde\phi^\varepsilon_h(T), \varphi_j)\, f(T) = \big(\xi, \varphi_j\big) + \int_0^T \big(\phi^\varepsilon_h(s), \varphi_j\big) f'(s) ds + \sqrt{\varepsilon} \int_0^T \big(S^\varepsilon_h(s) dW(s), g_j(s)\big) + \int_0^T \langle F^\varepsilon_h(s), g_j(s)\rangle ds + \int_0^T \big(\tilde S^\varepsilon_h(s), g_j(s)\big) ds. \tag{3.30}$$

For $\delta > 0$, $k > \frac{1}{\delta}$, $t \in [0,T]$, let $f_k \in H^1(-\delta, T+\delta)$ be such that $\|f_k\|_\infty = 1$, $f_k = 1$ on $(-\delta, t - \frac{1}{k})$ and $f_k = 0$ on $\big(t, T+\delta\big)$. Then $f_k \to 1_{(-\delta,t)}$ in L^2, and $f_k' \to -\delta_t$ in the sense of distributions. Hence as $k \to \infty$, (3.30) written with $f := f_k$ yields

$$\begin{aligned} 0 &= \big(\xi, \varphi_j\big) - \big(\phi^\varepsilon_h(t), \varphi_j\big) + \sqrt{\varepsilon} \int_0^t \big(S^\varepsilon_h(s) dW(s), g_j(s)\big) \\ &\quad + \int_0^t \langle F^\varepsilon_h(s), g_j(s)\rangle ds + \int_0^t \big(\tilde S^\varepsilon_h(s), g_j(s)\big) ds. \end{aligned}$$

Note that j is arbitrary and $\mathbb{E}\int_0^T |S^\varepsilon_h(s)|^2_{L_Q} ds < \infty$; we deduce that for $0 \le t \le T$,

$$\phi^\varepsilon_h(t) = \xi + \sqrt{\varepsilon} \int_0^t S^\varepsilon_h(s) dW(s) + \int_0^t F^\varepsilon_h(s) ds + \int_0^t \tilde S^\varepsilon_h(s) ds \in H. \tag{3.31}$$

Indeed, $\int_0^t F^\varepsilon_h(s) ds$, as linear combination of $H-$valued terms, also belongs to H. Moreover, let $f = 1_{(-\delta, T+\delta)}$. Using (3.30) again, we obtain

$$\tilde\phi^\varepsilon_h(T) = \xi + \sqrt{\varepsilon} \int_0^T S^\varepsilon_h(s) dW(s) + \int_0^T F^\varepsilon_h(s) ds + \int_0^T \tilde S^\varepsilon_h(s) ds.$$

This equation and (3.31) yield that $\tilde\phi^\varepsilon_h(T) = \phi^\varepsilon_h(T)$ a.s.

Step 3: In (3.31) we still have to prove that $ds \otimes d\mathbb{P}$ a.s. on Ω_T, one has

$$S^\varepsilon_h(s) = \sigma(\phi^\varepsilon_h(s)),\ F^\varepsilon_h(s) = F(\phi^\varepsilon_h(s)) \text{ and } \tilde S^\varepsilon_h(s) = \tilde\sigma(\phi^\varepsilon_h(s))\, h(s).$$

Let

$$\begin{aligned} \mathcal{X} &:= \{\psi \in L^4(\Omega_T, L^4(D)) \cap L^4\big(\Omega, L^\infty([0,T], H)\big) \cap L^2(\Omega_T, V)\ : \\ &\qquad \int_0^T \big(\|\psi(t)\|^2 + \|\phi^\varepsilon_h(t)\|^2\big) |\psi(t) - \phi^\varepsilon_h(t)|^2\, dt < +\infty\}. \end{aligned}$$

Then (i)-(iii) yield $\phi^\varepsilon_h \in \mathcal{X}$ and since $\|u\| \le C(m)|u|$ for every $u \in H_m$, using (3.8) and the fact that $\phi^\varepsilon_h \in L^2(\Omega_T, V)$, we deduce that for any $m \ge 1$, $L^\infty(\Omega_T, H_m) \subset \mathcal{X}$. Let $\psi = (v, \eta) \in L^\infty(\Omega_T, H_m)$. For every $t \in [0,T]$, if $a \wedge b := \inf(a,b)$, set

$$r(t) = \int_0^t \Big[\frac{3^3}{2^3(\nu \wedge \kappa)^3} \big(|v(s)|^4_{L^4} + |\eta(s)|^4_{L^4}\big) + 2 + \frac{4L^2}{\nu \wedge \kappa} |h(s)|^2_0 \Big] ds. \tag{3.32}$$

Then $r(t) < \infty$ for all $t \in [0,T]$ and Fatou's lemma implies

$$\mathbb{E}\big(|\phi^\varepsilon_h(T)|^2\, e^{-r(T)}\big) \le \mathbb{E}\big(\liminf_n |\phi^\varepsilon_{n,h}(T)|^2\, e^{-r(T)}\big) \le \liminf_n \mathbb{E}\big(|\phi^\varepsilon_{n,h}(T)|^2\, e^{-r(T)}\big).$$

Apply Itô's formula to (3.31) and (3.14), and for $\phi = \phi_h^\varepsilon$ or $\phi = \phi_{n,h}^\varepsilon$, let $\phi = \psi + (\phi - \psi)$. After simplification, this yields

$$\mathbb{E}|\xi|^2 + \mathbb{E}\int_0^T e^{-r(s)}\big[-r'(s)\big\{\big|\phi_h^\varepsilon(s)-\psi(s)\big|^2 + 2\big(\phi_h^\varepsilon(s)-\psi(s)\,,\,\psi(s)\big)\big\} + 2\langle F_h^\varepsilon(s),\phi_h^\varepsilon(s)\rangle$$
$$+\varepsilon|S_h^\varepsilon(s)|^2_{L_Q^2} + 2\big(\tilde S_h^\varepsilon(s)\,,\,\phi_h^\varepsilon(s)\big)\big]\,ds \le \liminf_n\big(\mathbb{E}|P_n(\xi)|^2 + X_n\big), \tag{3.33}$$

where

$$X_n = \mathbb{E}\int_0^T e^{-r(s)}\big[-r'(s)\big\{\big|\phi_{n,h}^\varepsilon(s)-\psi(s)\big|^2 + 2\big(\phi_{n,h}^\varepsilon(s)-\psi(s)\,,\,\psi(s)\big)\big\}$$
$$+2\langle F(\phi_{n,h}^\varepsilon(s)),\phi_{n,h}^\varepsilon(s)\rangle + \varepsilon|\sigma_n(\phi_{n,h}^\varepsilon(s))P_n|^2_{L_Q^2} + 2\big(\tilde\sigma(\phi_{n,h}^\varepsilon(s))h(s)\,,\,\phi_{n,h}^\varepsilon(s)\big)\big]\,ds.$$

Set $a\vee b := \max(a,b)$. The inequalities (3.13), (A.3), (3.32) and Schwarz's inequality imply that for $0\le\varepsilon\le\varepsilon_0\le\frac{\nu\wedge\kappa}{2L}$,

$$\begin{aligned}
Y_n \;:=\; & \mathbb{E}\int_0^T e^{-r(s)}\big[-r'(s)|\phi_{n,h}^\varepsilon(s)-\psi(s)|^2\\
&+\Big[2\langle F(\phi_{n,h}^\varepsilon(s))-F(\psi(s)),\phi_{n,h}^\varepsilon(s)-\psi(s)\rangle \;+\varepsilon|\sigma_n(\phi_{n,h}^\varepsilon(s))\;P_n-\sigma_n(\psi(s))\;P_n|^2_{L_Q}\\
&+2\big(\big\{\tilde\sigma_n(\phi_{n,h}^\varepsilon(s))-\tilde\sigma_n(\psi(s))\big\}\;h(s),\phi_{n,h}^\varepsilon(s)-\psi(s)\big)\Big]ds\\
\le\;& \mathbb{E}\int_0^T e^{-r(s)}\;|\phi_{n,h}^\varepsilon(s)-\psi(s)|^2\Big\{-r'(s)\\
&\qquad+\frac{3^3}{2^3(\nu\wedge\kappa)^3}\big[|v(s)|^4_{L^4}\vee|\eta(s)|^4_{L^4}\big]+2+\frac{2L}{\nu\wedge\kappa}|h(s)|_0^2\Big\}ds\le 0.
\end{aligned}\tag{3.34}$$

Furthermore, $X_n = Y_n + \sum_{i=1}^2 Z_n^i$, with

$$\begin{aligned}
Z_n^1 \;=\;& \mathbb{E}\int_0^T e^{-r(s)}\Big[-2r'(s)(\phi_{n,h}^\varepsilon(s))-\psi(s),\psi(s))+2\langle F(\phi_{n,h}^\varepsilon(s)),\psi(s)\rangle\\
&+2\langle F(\psi(s)),\phi_{n,h}^\varepsilon(s)\rangle-2\langle F(\psi(s)),\psi(s)\rangle+2\varepsilon\big(\sigma_n(\phi_{n,h}^\varepsilon(s))P_n\,,\,\sigma(\psi(s))\big)_{L_Q}\\
&+2\big(\tilde\sigma_n(\phi_{n,h}^\varepsilon(s))h(s),\psi(s)\big)+2\big(\tilde\sigma(\psi(s))\;h(s),\phi_{n,h}^\varepsilon(s)\big)-2(P_n\tilde\sigma(\psi(s))h(s),\psi(s))\;\Big]\,ds,\\
Z_n^2 \;=\;& \mathbb{E}\int_0^T e^{-r(s)}\Big[2\big(\sigma_n(\phi_{n,h}^\varepsilon(s))P_n,[\sigma(\psi(s))P_n-\sigma(\psi(s))]\big)_{L_Q}-\varepsilon|P_n\sigma(\psi(s))P_n|^2_{L_Q}\Big]ds.
\end{aligned}$$

The weak convergence properties (i)-(vii) imply that, as $n\to\infty$, $Z_n^1\to Z^1$ where

$$\begin{aligned}
Z^1 \;=\;& \mathbb{E}\int_0^T e^{-r(s)}\big[-2r'(s)\big(\phi_h^\varepsilon(s)-\psi(s),\psi(s)\big)+2\langle F_h^\varepsilon(s),\psi(s)\rangle+2\langle F(\psi(s)),\phi_h^\varepsilon(s)\rangle\\
&-2\langle F(\psi(s)),\psi(s)\rangle+2\varepsilon\;\big(S_h^\varepsilon(s)\,,\,\sigma(\psi(s))\big)_{L_Q}+2(\tilde S_h^\varepsilon(s),\psi(s))\\
&+2\big(\tilde\sigma(\psi(s))\;h(s),\phi_h^\varepsilon(s)\big)-2\big(\tilde\sigma(\psi(s))h(s),\psi(s)\big)\;\big]ds.
\end{aligned}\tag{3.35}$$

Now we study (Z_n^2); when $n \to \infty$, $|\sigma(\psi(s))(P_n - Id_{H_0})|_{L_Q} \to 0$ a.s., and by (A.2),

$$\mathbb{E}\int_0^T e^{-r(s)} \sup_n |\sigma(\psi(s))(P_n - Id_{H_0})|_{L_Q}^2 ds < \infty.$$

Hence the Lebesgue dominated convergence theorem implies that, as $n \to \infty$,

$$\mathbb{E}\int_0^T e^{-r(s)} |\sigma(\psi(s))(P_n - Id_{H_0})|_{L_Q}^2 ds \to 0.$$

Since $\sup_n \mathbb{E}\int_0^T e^{-r(s)}|\sigma_n(\phi^\varepsilon_{n,h}(s))P_n|_{L_Q}^2 ds < \infty$, by the Cauchy-Schwarz inequality, we see that $Z_n^2 \to -\varepsilon\mathbb{E}\int_0^T e^{-r(s)}|\sigma(\psi(s))|_{L_Q}^2 ds$.

Thus, (3.33)-(3.35) imply that for any $m \geq 1$ and any $\psi \in L^\infty(\Omega_T, H_m)$,

$$\begin{aligned}\mathbb{E}\int_0^T e^{-r(s)}\Big\{ &- r'(s)|\phi^\varepsilon_h(s) - \psi(s)|^2 + 2\langle F^\varepsilon_h(s) - F(\psi(s)), \phi^\varepsilon_h(s) - \psi(s)\rangle \\ &+ \varepsilon|S^\varepsilon_h(s) - \sigma(\psi(s))|_{L_Q}^2 + 2\Big(\tilde{S}^\varepsilon_h(s) - \tilde{\sigma}(\psi(s))h(s)\,,\, \phi^\varepsilon_h(s) - \psi(s)\Big)\Big\}ds \leq 0. \quad (3.36)\end{aligned}$$

By a density argument, this inequality extends to all $\psi \in \mathcal{X}$. Taking $\psi = \phi^\varepsilon_h \in \mathcal{X}$, we conclude that $S^\varepsilon_h(s) = \sigma(\phi^\varepsilon_h(s))$, $ds \otimes d\mathbb{P}$ *a.e.* For a real number λ, $\tilde{\psi} = (v, \eta) \in L^\infty(\Omega_T, H_m)$ for some m, set $\psi_\lambda = \phi^\varepsilon_h - \lambda\tilde{\psi} \in \mathcal{X}$. Thus applying (3.36) to ψ_λ and neglecting $\varepsilon|\sigma(\phi^\varepsilon_h(s)) - \sigma(\psi_\lambda(s))|_{L_Q}^2$, we obtain

$$\begin{aligned}\mathbb{E}\int_0^T e^{-r(s)}\Big[&- \lambda^2 r'(s)|\tilde{\psi}(s)|^2 + 2\lambda\Big\{\langle F^\varepsilon_h(s) - F(\psi_\lambda(s)), \tilde{\psi}(s)\rangle \\ &+ \Big(\tilde{S}^\varepsilon_h(s) - \tilde{\sigma}(\psi_\lambda(s))h(s), \tilde{\psi}(s)\Big)\Big\}\Big]\, ds \leq 0. \quad (3.37)\end{aligned}$$

Using (A.3), we have for almost every $(s, \omega) \in \Omega_T$ as $\lambda \to 0$,

$$\Big|\Big(\big[\tilde{\sigma}(\psi_\lambda(s)) - \tilde{\sigma}(\phi^\varepsilon_h(s))\big]\, h(s)\,,\, \tilde{\psi}(s)\Big)\Big| \leq \sqrt{L}\lambda\, \|\tilde{\psi}(s)\|\, |h(s)|_0\, |\tilde{\psi}(s)| \to 0.$$

Furthermore, (A.2) implies that

$$\begin{aligned}&\mathbb{E}\int_0^T \sup_{|\lambda|\leq 1}\Big|\Big(\tilde{\sigma}(\psi_\lambda(s))\; h(s), \tilde{\psi}(s)\Big)\Big| ds \\ &\quad\leq\; \sqrt{K}\,\mathbb{E}\int_0^T \big(1 + 2\|\phi^\varepsilon_h(s)\|^2 + 2\|\tilde{\psi}(s))\|^2\big)^{\frac{1}{2}}\; |h(s)|_0\; |\tilde{\psi}(s)| ds \\ &\quad\leq\; KM + \mathbb{E}\int_0^T \Big[\big\{1 + 6\|\phi^\varepsilon_h(s)\|^2 + 4\|\psi_1(s)\|^2\big\}\, |\psi_1(s) - \phi^\varepsilon_h(s)|^2\Big] ds < \infty.\end{aligned}$$

Hence, the Lebesgue dominated convergence theorem yields, as $\lambda \to 0$,

$$\mathbb{E}\int_0^T \Big(\big\{\tilde{S}^\varepsilon_h(s) - \tilde{\sigma}(\psi_\lambda(s))\big\}h(s), \tilde{\psi}(s)\Big) ds \to \mathbb{E}\int_0^T \Big(\big\{\tilde{S}^\varepsilon_h(s) - \tilde{\sigma}(\phi_h(s))\big\}h(s), \tilde{\psi}(s)\Big) ds.$$

Furthermore, Lemmas 3.5 and 3.7 yield for $\lambda \neq 0$

$$\big|\langle F(\psi_\lambda(s)) - F(\phi^\varepsilon_h(s)), \tilde{\psi}(s)\rangle\big| \leq \nu\lambda\|v(s)\|^2 + \kappa\lambda\|\eta(s)\|^2$$

$$+ |B_1(\lambda v(s), \lambda v(s))|_{V_1'} \, \|v(s)\| + |B_2(\lambda\tilde{\psi}(s))|_{V_2'} \, \|\eta(s)\| + 2\lambda|\tilde{\psi}(s)|^2$$
$$\leq c\lambda\big[|\tilde{\psi}(s)|^2 + \|\tilde{\psi}(s)\|^2 + \lambda\|v(s)\|^2|v(s)| + \lambda\|\eta(s)\|^2|\eta(s)|\big].$$

Using again the dominated convergence theorem, we deduce as $\lambda \to 0$,

$$\mathbb{E}\int_0^T \langle F_h^\varepsilon(s) - F(\psi_\lambda(s)), \tilde{\psi}(s)\rangle ds \to \mathbb{E}\int_0^T \langle F_h^\varepsilon(s) - F(\phi_h^\varepsilon(s)), \tilde{\psi}(s)\rangle ds.$$

Thus, dividing (3.37) by $\lambda > 0$ and letting $\lambda \to 0$ we obtain that for every m and $\tilde{\psi} \in L^\infty(\Omega_T, H_m)$,

$$\mathbb{E}\int_0^T \Big[\langle F_h^\varepsilon(s) - F(\phi_h^\varepsilon(s)), \tilde{\psi}(s)\rangle + \big(\big\{\tilde{S}_h^\varepsilon(s) - \tilde{\sigma}(\phi_h^\varepsilon(s))\big\}h(s), \tilde{\psi}(s)\big)\Big] ds \leq 0,$$

while a similar calculation for $\lambda < 0$ yields the opposite inequality. Therefore for almost every $(s,\omega) \in \Omega_T$,, for every $\tilde{\psi}$ in a dense subset of $L^2(\Omega_T, V)$,

$$\mathbb{E}\int_0^T \Big[\big\langle F_h^\varepsilon(s) - F(\phi_h^\varepsilon(s))\,,\, \tilde{\psi}(s)\big\rangle + \big(\big\{\tilde{S}_h^\varepsilon(s) - \tilde{\sigma}(\phi_h^\varepsilon(s))\big\}h(s), \tilde{\psi}(s)\big)\Big] ds = 0. \quad (3.38)$$

Hence a.e. for $t \in [0,T]$, (3.31) can be rewritten as

$$\phi_h^\varepsilon(t) = \xi + \sqrt{\varepsilon}\int_0^t \sigma(\phi_h^\varepsilon(s))dW_s + \int_0^t \big[F(\phi_h^\varepsilon(s)) + \tilde{\sigma}(\phi_h^\varepsilon(s))h(s)\big]ds. \quad (3.39)$$

Furthermore, (i), (iv) and (3.19) for $p = 2$ imply that

$$E\Big(\int_0^T \|\phi_h^\varepsilon(t)\|^2\, dt\Big) \leq \sup_n \mathbb{E}\int_0^T \|\phi_{n,h}^\varepsilon(t)\|^2 dt \leq C\big(1 + E|\xi|^4\big), \quad (3.40)$$
$$E\Big(\sup_{0\leq t\leq T} |\phi_h^\varepsilon(t)|^4\Big) \leq \sup_n \mathbb{E}\Big(\sup_{0\leq t\leq T} |\phi_{n,h}^\varepsilon(t)|^4\Big) \leq C\,\big(1 + E|\xi|^4\big). \quad (3.41)$$

Since $|x|^2 \leq 1 \vee |x|^4$ for any $x \in \mathbb{R}$, this completes the proof of (3.5).

Step 4: To complete the proof of Theorem 3.1, we show that ϕ_h^ε has a $C([0,T],H)$-valued modification and that the solution to (3.39) is unique in $X := C([0,T],H) \cap L^2([0,T],V)$. Note that (3.5) implies that if $\tilde{\tau}_N = \inf\{t \geq 0 : |\phi_h^\varepsilon(t)| \geq N\} \wedge T$ for $N > 0$, $\mathbb{P}(\tilde{\tau}_N < T) \leq CN^{-2}$. The Borel-Cantelli lemma yields $\tilde{\tau}_N \to T$ a.s. when $N \to \infty$. We at first prove uniqueness. Let $\psi = (v,\eta) \in X$ be another solution to (3.39). Then if $\bar{\tau}_N = \inf\{t \geq 0 : |\psi(t)| \geq N\} \wedge T$ for $N > 0$, since $|\psi(.)|$ is a.s. bounded on $[0,T]$, as $N \to \infty$, $\bar{\tau}_N \to T$ a.s. and $\tau_N = \tilde{\tau}_N \wedge \bar{\tau}_N \to T$, a.s.

Let $\phi_h^\varepsilon = (u_h^\varepsilon, \theta_h^\varepsilon)$, $\Phi = (U,\Theta) = \phi_h^\varepsilon - \psi$, $\delta = \frac{\nu\wedge\kappa}{2}$ and $a = \frac{1}{\delta} = \frac{2}{\nu\wedge\kappa}$. Itô's formula, Lemmas 3.5 and 3.7, and (A.3) imply that if $0 \leq \varepsilon \leq \varepsilon_0 \leq \frac{\delta}{L}$,

$$\mathbb{E}e^{-a\int_0^{t\wedge\tau_N}\|\psi(r)\|^2dr}|\Phi(t\wedge\tau_N)|^2 = -a\,\mathbb{E}\int_0^{t\wedge\tau_N} \|\psi(s)\|^2|\Phi(s)|^2 e^{-a\int_0^s\|\psi(r)\|^2dr}\, ds$$
$$+ \mathbb{E}\int_0^{t\wedge\tau_N} e^{-a\int_0^s\|\psi(r)\|^2dr}\Big\{-2\nu\|U(s)\|^2 - 2\kappa\|\Theta(s)\|^2$$
$$- 2\langle B_1(u_h^\varepsilon(s), u_h^\varepsilon(s)) - B_1(v(s),v(s)), U(s)\rangle - 2\langle B_2(\Phi(s)) - B_2(\psi(s)), \Theta(s)\rangle$$

$$
\begin{aligned}
&+\varepsilon|\sigma(\phi_h^\varepsilon(s))-\sigma(\psi(s))|_{L_Q}^2+2\big([\tilde\sigma(\phi_h^\varepsilon(s))-\tilde\sigma(\psi(s))]h(s),\Phi(s)\big)+4|U(s)|\,|\Theta(s)|\Big\}\,ds\\
&\le\mathbb{E}\int_0^{t\wedge\tau_N}e^{-a\int_0^s\|\psi(r)\|^2dr}\Big\{-a\|\psi(s)\|^2|\Phi(s)|^2-2\big[\nu\|U(s)\|^2+\kappa\|\Theta(s)\|^2\big]\\
&\quad+\delta\left[\|U(s)\|^2+\|\Phi(s)\|^2\right]+\frac{1}{\delta}\big[|U(s)|^2\|v(s)\|^2+|\Phi(s)|^2\,\|\eta(s)\|^2\big]\\
&\quad+\big[\varepsilon L+\delta\big]\|\Phi(s)\|^2+L\,\delta^{-1}\,|\Phi(s)|^2|h(s)|_0^2+2\,|\Phi(s)|^2\Big\}ds\\
&\le\mathbb{E}\int_0^t e^{-a\int_0^{s\wedge\tau_N}\|\psi(r)\|^2dr}\,\left(L\,\delta^{-1}\,|h(s\wedge\tau_N)|_0^2+2\right)\,|\Phi(s\wedge\tau_N)|^2\,ds.
\end{aligned}
$$

Since $\sup_{0\le s\le T}e^{-a\int_0^{s\wedge\tau_N}\|\psi(r)\|^2dr}|\Phi(s\wedge\tau_N)|^2\le 2N$ and $s\mapsto L\,\delta^{-1}\,|h(s)|_0^2$ belongs to $L^1(\Omega_T)$, by the Gronwall lemma, we conclude that

$$
\sup_{0\le s\le T}\ \mathbb{E}\,e^{-a\int_0^{s\wedge\tau_N}\|\psi(r)\|^2dr}\,|\Phi(s\wedge\tau_N)|^2=0. \tag{3.42}
$$

Since $\lim_{N\to\infty}\tau_N=T$ a.s., we thus have $|\Phi(s,\omega)|=0$ a.s. on Ω_T. Thus if ϕ_h^ε is in $C([0,T],H)$, we conclude that $\phi_h^\varepsilon(t)=\psi(t)$, a.s., for every $t\in[0,T]$.

Finally, set

$$
\rho(t)=\int_0^t\Big[\frac{3^3}{2^3(\nu\wedge\kappa)^3}\,\left(|v(s)|_{L^4}^4+|\eta(s)|_{L^4}^4\right)+2+\Big(\frac{L}{\nu\wedge\kappa}+1\Big)\,|h(s)|_0^2\Big]\,ds. \tag{3.43}
$$

Set $\Phi_{n,h}^\varepsilon=\phi_{n,h}^\varepsilon-\phi_h^\varepsilon$ and let $0\le\varepsilon\le\varepsilon_0\le\frac{\nu\wedge\kappa}{2L}$. By the Itô formula, (A3) , (3.13) and Schwarz's inequality, for every $t\in[0,T]$,

$$
\begin{aligned}
&\mathbb{E}\big(e^{-\rho(t)}|\Phi_{n,h}^\varepsilon(t)|^2\big)\le\mathbb{E}\int_0^t e^{-\rho(s)}\Big[-\rho(s)|\Phi_{n,h}^\varepsilon(s)|^2-2(\nu\wedge\kappa)\,\|\Phi_{n,h}^\varepsilon(s)\|^2\\
&+\Big(\frac{3^3(|v(s)|_{L^4}^4+|\eta(s)|_{L^4}^4)}{2^3(\nu\wedge\kappa)^3}+2\Big)\,|\Phi_{n,h}^\varepsilon(s)|^2+2\varepsilon L\|\Phi_{n,h}^\varepsilon(s)\|^2+\varepsilon|P_n\sigma(\phi_h^\varepsilon(s))P_n-\sigma(\phi_h^\varepsilon)|_{L_Q}^2\\
&+2\sqrt{L}\|\Phi_{n,h}^\varepsilon(s)\|\,|h(s)|_0|\Phi_{n,h}^\varepsilon(s)|+2|\Phi_{n,h}^\varepsilon(s)|\,|P_n\tilde\sigma(\phi_h^\varepsilon(s))-\tilde\sigma(\phi_h^\varepsilon(s))|_{L_Q}|h(s)|_0\Big]\,ds\\
&\le\mathbb{E}\int_0^t e^{-\rho(s)}\big[\varepsilon|P_n\sigma(\phi_h^\varepsilon(s))P_n-\sigma(\phi_h^\varepsilon(s))|_{L_Q}^2+\,|P_n\tilde\sigma(\phi_h^\varepsilon(s))-\tilde\sigma(\phi_h^\varepsilon(s))|_{L_Q}^2\big]\,ds.
\end{aligned}
$$

Furthermore, for almost every (s,ω), one has $|P_n\sigma(\phi_h^\varepsilon(s))P_n-\sigma(\phi_h^\varepsilon(s))|_{L_Q}\to 0$ and $|P_n\tilde\sigma(\phi_h^\varepsilon(s))-\tilde\sigma(\phi_h^\varepsilon(s))|_{L_Q}\to 0$ as $n\to\infty$. Thus the dominated convergence theorem shows that $\lim_{n\to\infty}I(n)=0$, where

$$
I(n)=\sup_{0\le t\le T}\mathbb{E}\big(e^{-\rho(t)}|\Phi_{n,h}^\varepsilon(t))|^2\big)+\mathbb{E}\int_0^T e^{-\rho(s)}\|\Phi_{n,h}^\varepsilon(s)\|^2\,ds=0.
$$

Using again the Itô formula and the Burkholder-Davies-Gundy inequality, an argument similar to that used in the proof of (3.34) yields for $0\le\varepsilon\le\varepsilon_0\le\frac{\nu\wedge\kappa}{2L}$:

$$
\mathbb{E}\Big(\sup_{0\le t\le T}e^{-\rho(t)}|\Phi_{n,h}^\varepsilon(t)|^2\Big)\le\frac{1}{2}\mathbb{E}\Big(\sup_{0\le t\le T}e^{-\rho(t)}|\Phi_{n,h}^\varepsilon(t)|^2\Big)
$$

$$
\begin{aligned}
&+ 18\varepsilon\mathbb{E}\int_0^T |\sigma_n(\phi^\varepsilon_{n,h}(s))P_n - \sigma(\phi^\varepsilon_h(s))|^2_{L_Q} ds \\
&+ \mathbb{E}\sup_{0\le t\le T}\int_0^t e^{-\rho(s)}\Big[-\rho(s)|\Phi^\varepsilon_{n,h}(s)|^2 - 2\,(\nu\wedge\kappa)\,\|\Phi^\varepsilon_{n,h}(s)\|^2 \\
&+ \Big(\frac{3^3(|v(s)|^4_{L^4}+|\eta(s)|^4_{L^4})}{2^3(\nu\wedge\kappa)^3}+2\Big)|\Phi^\varepsilon_{n,h}(s)|^2 + 2\varepsilon\,L\,\|\Phi^\varepsilon_{n,h}(s)\|^2 + \varepsilon|P_n\sigma(\phi^\varepsilon_h(s))P_n - \sigma(\phi^\varepsilon_h)|^2_{L_Q} \\
&+ 2\sqrt{L}\|\Phi^\varepsilon_{n,h}(s)\|\,|h(s)|_0|\Phi^\varepsilon_{n,h}(s)| + 2|\Phi^\varepsilon_{n,h}(s)|\,|P_n\tilde\sigma(\phi^\varepsilon_h(s)) - \tilde\sigma(\phi^\varepsilon_h(s))|_{L_Q}|h(s)|_0\Big] ds \\
&\le CI(n).
\end{aligned}
$$

Therefore, $\phi^\varepsilon_{n,h}$ has a subsequence converging a.s. uniformly to ϕ^ε_h in H. Because $\phi^\varepsilon_{n,h}\in C([0,T],H)$, we conclude that ϕ^ε_h has a modification in $C([0,T],H)$.

4. Large deviations

We consider large deviations via a weak convergence approach [1, 2], based on variational representations of infinite dimensional Wiener processes. The solution to the stochastic Bénard model (2.7) is denoted as $\phi^\varepsilon = \mathcal{G}^\varepsilon(\sqrt{\varepsilon}W)$ for a Borel measurable function $\mathcal{G}^\varepsilon : C([0,T],H)\to X$. The space $X = C([0,T];H)\cap L^2((0,T);V)$ endowed with the metric associated with the norm defined in (3.4) is Polish. Let $\mathcal{B}(X)$ denote its Borel $\sigma-$field. We recall some classical definitions.

Definition 4.1. *The random family $\{\phi^\varepsilon\}$ is said to satisfy a large deviation principle on X with the good rate function I if the following conditions hold:*

*I **is a good rate function.** The function function $I : X\to[0,\infty]$ is such that for each $M\in[0,\infty[$ the level set $\{\phi\in X : I(\phi)\le M\}$ is a compact subset of X. For $A\in\mathcal{B}(X)$, set $I(A) = \inf_{\phi\in A} I(\phi)$.*

***Large deviation upper bound.** For each closed subset F of X:*

$$\limsup_{\varepsilon\to 0}\ \varepsilon\log\mathbb{P}(\phi^\varepsilon\in F)\le -I(F).$$

***Large deviation lower bound.** For each open subset G of X:*

$$\liminf_{\varepsilon\to 0}\ \varepsilon\log\mathbb{P}(\phi^\varepsilon\in G)\ge -I(G).$$

To establish the large deviation principle, we need to strengthen the hypothesis on the growth condition and Lipschitz property of σ as follows:

Assumption A Bis There exist positive constants K and L such that

(A4) $|\sigma(t,\phi)|^2_{L_Q}\le K\,(1+|\phi|^2)$, $\forall t\in[0,T]$, $\forall\phi\in V$.

(A5) $|\sigma(t,\phi)-\sigma(t,\psi)|^2_{L_Q}\le L\,|\phi-\psi|^2$, $\forall t\in[0,T]$, $\forall\phi,\psi\in V$.

Note that due to the continuous embedding $V\hookrightarrow H$, the assumptions (A4-A5) imply (A2-A3). Thus the conclusions of Theorem 3.1 hold under the assumptions (A4-A5).

The proof of the large deviation principle will use the following technical lemma which studies time increments of the solution to the stochastic control equation. For

any integer $k = 0, \cdots, 2^n - 1$, and $s \in [kT2^{-n}, (k+1)T2^{-n}[$, set $\underline{s}_n = kT2^{-n}$ and $\bar{s}_n = (k+1)T2^n$. Given $N > 0$, $h \in \mathcal{A}_M$, $\varepsilon \geq 0$ small enough, let ϕ_h^ε denote the solution to (3.2) given by Theorem 3.1, and for $t \in [0,T]$, let

$$G_N(t) = \Big\{\omega : \Big(\sup_{0\leq s\leq t} |\phi_h^\varepsilon(s)(\omega)|^2\Big) \vee \Big(\int_0^t \|\phi_h^\varepsilon(s)(\omega)\|^2 ds\Big) \leq N\Big\}.$$

Lemma 4.2. *Let $M, N > 0$, σ and $\tilde{\sigma}$ satisfy the Assumptions (A1),(A4) and (A5), $\xi \in L^4(H)$. Then there exists a positive constant $C := C(\nu, \kappa, K, L, T, M, N, \varepsilon_0)$ such that for any $h \in \mathcal{A}_M$, $\varepsilon \in [0, \varepsilon_0]$,*

$$I_n(h,\varepsilon) := \mathbb{E}\Big[1_{G_N(T)} \int_0^T |\phi_h^\varepsilon(s) - \phi_h^\varepsilon(\bar{s}_n)|^2 \, ds\Big] \leq C\, 2^{-\frac{n}{2}}. \tag{4.1}$$

Proof. Let $h \in \mathcal{A}_M$, $\varepsilon \geq 0$; Itô's formula yields $I_n(h,\varepsilon) = \sum_{1\leq i\leq 6} I_{n,i}$, where

$$\begin{aligned}
I_{n,1} =& 2\sqrt{\varepsilon}\, \mathbb{E}\Big(1_{G_N(T)} \int_0^T ds \int_s^{\bar{s}_n} \big(\sigma(\phi_h^\varepsilon(r))dW_r \,,\, \phi_h^\varepsilon(r) - \phi_h^\varepsilon(s)\big)\Big),\\
I_{n,2} =& \varepsilon\, \mathbb{E}\Big(1_{G_N(T)} \int_0^T ds \int_s^{\bar{s}_n} |\sigma(\phi_h^\varepsilon(r))|_{L_Q}^2 \, dr\Big),\\
I_{n,3} =& 2\mathbb{E}\Big(1_{G_N(T)} \int_0^T ds \int_s^{\bar{s}_n} \big(\tilde{\sigma}(\phi_h^\varepsilon(r))\, h(r) \,,\, \phi_h^\varepsilon(r) - \phi_h^\varepsilon(s)\big)\, dr\Big),\\
I_{n,4} =& -2\mathbb{E}\Big[1_{G_N(T)} \int_0^T ds \int_s^{\bar{s}_n} \big[\nu\big(A_1 u_h^\varepsilon(r), u_h^\varepsilon(r) - u_h^\varepsilon(s)\big) + \kappa\big(A_2\theta_h^\varepsilon(r), \theta_h^\varepsilon(r) - \theta_h^\varepsilon(s)\big)\big] dr\Big),\\
I_{n,5} =& -2\mathbb{E}\Big(1_{G_N(T)} \int_0^T ds \int_s^{\bar{s}_n} \big[\big(B_1(u_h^\varepsilon(r)), u_h^\varepsilon(r) - u_h^\varepsilon(s)\big) + \big(B_2(\phi_h^\varepsilon(r)), \theta_h^\varepsilon(r) - \theta_h^\varepsilon(s)\big)\big] dr\Big),\\
I_{n,6} =& 2\mathbb{E}\Big(1_{G_N(T)} \int_0^T ds \int_s^{\bar{s}_n} \big[\big(u_{h,2}^\varepsilon(r), \theta_h^\varepsilon(r) - \theta_h^\varepsilon(s)\big) + \big(\theta_h^\varepsilon(r), u_{h,2}^\varepsilon(r) - u_{h,2}^\varepsilon(s)\big)\big] dr\Big).
\end{aligned}$$

Clearly $G_N(T) \subset G_N(r)$ for $r \in [0,T]$. The Burkholder-Davis-Gundy inequality, (A4) and the definition of $G_N(r)$ yield for $0 \leq \varepsilon \leq \varepsilon_0$

$$\begin{aligned}
|I_{n,1}| &\leq 2\sqrt{\varepsilon}\, \mathbb{E} \int_0^T ds \Big| \int_s^{\bar{s}_n} \big(\sigma(\phi_h^\varepsilon(r))dW_r \,,\, \phi_h^\varepsilon(r) - \phi_h^\varepsilon(s)\big) 1_{G_N(r)}\Big|\\
&\leq 6\sqrt{\varepsilon} \int_0^T ds\, \mathbb{E}\Big(\int_s^{\bar{s}_n} |\sigma(\phi_h^\varepsilon(r))|_{L_Q}^2 |\phi_h^\varepsilon(r) - \phi_h^\varepsilon(s)|^2 1_{G_N(r)} dr\Big)^{\frac{1}{2}}\\
&\leq 12\sqrt{\varepsilon}\, \sqrt{KN(1+N)} \int_0^T ds\, (T2^{-n})^{\frac{1}{2}} \leq C(\varepsilon_0, K, N, T) 2^{-\frac{n}{2}}.
\end{aligned} \tag{4.2}$$

The property (A4) implies that for $\varepsilon \leq \varepsilon_0$,

$$|I_{n,2}| \leq \varepsilon K \mathbb{E}\Big(1_{G_N(T)} \int_0^T ds \int_s^{\bar{s}_n} \big(1 + |\phi_h^\varepsilon(r)|^2\big) dr\Big) \leq \varepsilon_0 K(1+N)T^2\, 2^{-n}. \tag{4.3}$$

Schwarz's inequality, Fubini's theorem, (A4) and the definition of $\mathcal{A}_M$ yield

$$\begin{aligned} |I_{n,3}| &\leq 2\sqrt{K}\,\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\int_s^{\bar{s}_n}\big(1+|\phi_h^\varepsilon(r)|^2\big)^{\frac{1}{2}}\,|h(r)|_0|\,\phi_h^\varepsilon(r)-\phi_h^\varepsilon(s)|\,dr\Big) \\ &\leq 2\sqrt{2KN(1+N)}\,\mathbb{E}\int_0^T |h(r)|_0\,dr\int_{\underline{r}_n}^r ds \leq C(K,N,M,T)2^{-n}. \end{aligned} \tag{4.4}$$

Schwarz's inequality implies that for some constant $\tilde{C} := C(\varepsilon_0,\nu,\kappa,K,T)$ from equation (3.5),

$$\begin{aligned} I_{n,4} &\leq 2\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\int_s^{\bar{s}_n} dr\big[-\nu\|u_h^\varepsilon(r)\|^2-\kappa\|\theta_h^\varepsilon(r)\|^2+\nu\|u_h^\varepsilon(r)\|\|u_h^\varepsilon(s)\| \\ &\qquad +\kappa\|\theta_h^\varepsilon(r)\|\|\theta_h^\varepsilon(s)\|\big]\Big) \\ &\leq \frac{\nu+\kappa}{2}\,\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\|\phi_h^\varepsilon(s)\|^2\int_s^{\bar{s}_n} dr\Big) \leq T\,\tilde{C}\,\frac{\nu+\kappa}{2}\,2^{-n}. \end{aligned} \tag{4.5}$$

The inequalities (3.8) and (3.9), and Schwarz's inequality imply

$$\begin{aligned} |I_{n,5}| &\leq 2\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\int_s^{\bar{s}_n} dr\Big[|u_h^\varepsilon(r)|\|u_h^\varepsilon(r)\|\big(\|u_h^\varepsilon(r)\|+\|u_h^\varepsilon(s)\|\big]\big) \\ &\qquad +|\phi_h^\varepsilon(r)|\|\phi_h^\varepsilon(r)\|\big(\|\theta_h^\varepsilon(r)\|+\|\theta_h^\varepsilon(s)\|\big)\Big] \\ &\leq 3\sqrt{N}\mathbb{E}\int_0^T dr\big(\|u_h^\varepsilon(r)\|^2+\|\phi_h^\varepsilon(r)\|^2\big)\int_{\underline{r}_n}^r ds \\ &\qquad +\sqrt{N}\mathbb{E}\int_0^T ds\big(\|u_h^\varepsilon(s)\|^2+\|\phi_h^\varepsilon(s)\|^2\big)\int_s^{\bar{s}_n} dr \leq 8T\sqrt{N}\tilde{C}2^{-n}. \end{aligned} \tag{4.6}$$

Finally, Schwarz's inequality implies that

$$|I_{n,6}| \leq 4\mathbb{E}\Big(1_{G_N(T)}\int_0^T ds\int_s^{\bar{s}_n}\big(|u_h^\varepsilon(r)|+|u_h^\varepsilon(s)|\big)\big(|\theta_h^\varepsilon(r)|+|\theta_h^\varepsilon(s)|\big)dr\Big) \leq \frac{16T^2N}{2^n}. \tag{4.7}$$

Collecting the upper estimates from (4.2)-(4.7), we conclude the proof of (4.1). □

Let ε_0 be defined as in Theorem 3.1 and $(h_\varepsilon, 0<\varepsilon\leq\varepsilon_0)$ be a family of random elements taking values in $\mathcal{A}_M$. Let ϕ_{h_ε}, or strictly speaking, $\phi^\varepsilon_{h_\varepsilon}$, be the solution of the corresponding stochastic control equation with initial condition $\phi_{h_\varepsilon}(0)=\xi\in H$:

$$d\phi_{h_\varepsilon}+[A\phi_{h_\varepsilon}+B(\phi_{h_\varepsilon})+R\phi_{h_\varepsilon}]dt=\sigma(\phi_{h_\varepsilon})h_\varepsilon dt+\sqrt{\varepsilon}\,\sigma(\phi_{h_\varepsilon})dW(t). \tag{4.8}$$

Note that $\phi_{h_\varepsilon}=\mathcal{G}^\varepsilon\Big(\sqrt{\varepsilon}\big(W_\cdot+\frac{1}{\sqrt{\varepsilon}}\int_0^\cdot h_\varepsilon(s)ds\big)\Big)$ due to the uniqueness of the solution.

For all $h\in L^2([0,T],H_0)$, let ϕ_h be the solution of the corresponding control equation (4.20) with initial condition $\phi_h(0)=\xi$:

$$d\phi_h+[A\phi_h+B(\phi_h)+R\phi_h]dt=\sigma(\phi_h)hdt. \tag{4.9}$$

Let $\mathcal{C}_0 = \{\int_0^\cdot h(s)ds \,:\, h \in L^2([0,T],H_0)\} \subset C([0,T],H_0)$. Define $\mathcal{G}^0 : C([0,T],H_0) \to X$ by $\mathcal{G}^0(g) = \phi_h$ for $g = \int_0^\cdot h(s)ds \in \mathcal{C}_0$ and $\mathcal{G}^0(g) = 0$ otherwise.

Proposition 4.3. *(Weak convergence)*
Suppose that σ satisfies the Assumptions (A1), (A4) and (A5). Let ξ be $\mathcal{F}_0$-measurable such that $E|\xi|_H^4 < +\infty$, and let h_ε converge to h in distribution as random elements taking values in $\mathcal{A}_M$. (Note that here $\mathcal{A}_M$ is endowed with the weak topology induced by the norm (3.4)). Then as $\varepsilon \to 0$, ϕ_{h_ε} converges in distribution to ϕ_h in $X = C([0,T];H) \cap L^2((0,T);V)$ endowed with the norm (3.4). That is, $\mathcal{G}^\varepsilon\Big(\sqrt{\varepsilon}\big(W_\cdot + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot h_\varepsilon(s)ds\big)\Big)$ converges in distribution to $\mathcal{G}^0\big(\int_0^\cdot h(s)ds\big)$ in X, as $\varepsilon \to 0$.

Proof. Since $\mathcal{A}_M$ is a Polish space (complete separable metric space), by the Skorokhod representation theorem, we can construct processes $(\tilde{h}_\varepsilon, \tilde{h}, \tilde{W})$ such that the joint distribution of $(\tilde{h}_\varepsilon, \tilde{W})$ is the same as that of (h_ε, W), the distribution of $\tilde{h}$ coincides with that of h, and $\tilde{h}_\varepsilon \to \tilde{h}$, a.s., in the (weak) topology of S_M. Hence a.s. for every $t \in [0,T]$, $\int_0^t \tilde{h}_\varepsilon(s)ds - \int_0^t \tilde{h}(s)ds \to 0$ weakly in H_0. Let $\Phi_\varepsilon = \phi_{h_\varepsilon} - \phi_h$, or in component form $\Phi_\varepsilon = (U_\varepsilon, \Theta_\varepsilon) = (u_{h_\varepsilon} - u_h, \theta_{h_\varepsilon} - \theta_h)$; then

$$\begin{aligned} &d\Phi_\varepsilon + \big[A\Phi_\varepsilon + B(\phi_{h_\varepsilon}) - B(\phi_h) + R\Phi_\varepsilon\big]dt \\ &\quad = \big[\sigma(\phi_{h_\varepsilon})h_\varepsilon - \sigma(\phi_h)h\big]dt + \sqrt{\varepsilon}\,\sigma(\phi_{h_\varepsilon})dW(t), \;\; \Phi_\varepsilon(0) = 0. \end{aligned} \tag{4.10}$$

Let ε_0 be defined as in Theorem 3.1. On any finite time interval $[0,t]$ with $t \le T$, Itô's formula, Lemmas 3.7 and 3.5, (A4) and (A5) yield for $0 \le \varepsilon \le \varepsilon_0 \wedge \frac{\nu\wedge\kappa}{4L}$:

$$\begin{aligned} &|\Phi_\varepsilon(t)|^2 + 2\int_0^t \big[\nu\|U_\varepsilon(s)\|^2 + \kappa\,\|\Theta_\varepsilon(s)\|^2\big]\,ds \\ &= 4\int_0^t \big(\Theta_\varepsilon(s), U_{\varepsilon,2}(s)\big)ds - 2\int_0^t \big\langle B_1(u_{h_\varepsilon}(s), u_{h_\varepsilon}(s)) - B_1(u_h(s), u_h(s)), U_\varepsilon(s)\big\rangle ds \\ &\quad - 2\int_0^t \big\langle B_2(\phi_{h_\varepsilon}(s)) - B_2(\phi_h(s)), \Theta_\varepsilon(s)\big\rangle ds \\ &\quad + 2\int_0^t \big(\sigma(\phi_{h_\varepsilon}(s))h_\varepsilon(s) - \sigma(\phi_h(s))\,h(s), \Phi_\varepsilon(s)\big)ds \\ &\quad + 2\sqrt{\varepsilon}\int_0^t \big(\Phi_\varepsilon(s), \sigma(\phi_{h_\varepsilon}(s))dW(s)\big) + \varepsilon\int_0^t |\sigma(\phi_{h_\varepsilon}(s))Q^{\frac{1}{2}}|_{H.S.}^2\,ds \\ &\le 2\int_0^t |\Phi_\varepsilon(s)|^2\,ds + 2\int_0^t \big[|U_\varepsilon(s)|\,\|U_\varepsilon(s)\|\,\|u_h(s)\| + |\Phi_\varepsilon(s)|\,\|\Phi_\varepsilon(s)\|\,\|\theta_h(s)\|\big]\,ds \\ &\quad + 2\int_0^t \sqrt{L}\,|\Phi_\varepsilon(s)|^2\,|h_\varepsilon(s)|_0\,ds + 2\int_0^t \big(\sigma(\phi_h(s))\,\big(h_\varepsilon(s) - h(s)\big)\,,\,\Phi_\varepsilon(s)\big)\,ds \\ &\quad + 2\sqrt{\varepsilon}\int_0^t \big(\Phi_\varepsilon(s), \sigma(\phi_{h_\varepsilon}(s))\,dW(s)\big) + \varepsilon K\int_0^t (1 + |\phi_{h_\varepsilon}(s)|^2)ds \end{aligned}$$

$$\leq \sum_{i=1}^{3} T_i(t,\varepsilon) + 2\int_0^t |\Phi_\varepsilon(s)|^2\, ds + \frac{\nu}{2}\int_0^t \|U_\varepsilon(s)\|^2\, ds + \frac{2}{\nu}\int_0^t \|u_h(s)\|^2\, |U_\varepsilon(s)|^2\, ds$$
$$+ \frac{\nu\wedge\kappa}{2}\int_0^t \|\Phi_\varepsilon(s)\|^2 ds + \int_0^t |\Phi_\varepsilon(s)|^2\Big(\frac{2}{\nu\wedge\kappa}\|\theta_h(s)\|^2 + 2\sqrt{L}\,|h_\varepsilon(s)|_0\Big)\, ds\,, \quad (4.11)$$

where

$$\begin{aligned} T_1(t,\varepsilon) =& 2\sqrt{\varepsilon}\int_0^t \big(\Phi_\varepsilon(s), \sigma(\phi_{h_\varepsilon}(s))\, dW(s)\big)\\ T_2(t,\varepsilon) =& \varepsilon K\int_0^t (1+|\phi_{h_\varepsilon}(s)|^2) ds,\\ T_3(t,\varepsilon) =& 2\int_0^t \big(\sigma(\phi_h(s))\,\big(h_\varepsilon(s)-h(s)\big)\,,\,\Phi_\varepsilon(s)\big)\, ds \end{aligned}$$

Thus, (4.11) yields the following inequality

$$|\Phi_\varepsilon(t)|^2 + (\nu\wedge\kappa)\int_0^t \|\Phi_\varepsilon(s)\|^2 ds \leq \sum_{i=1}^{3} T_i(t,\varepsilon)$$
$$+ 2\int_0^t \Big[1 + \Big(\frac{1}{\nu} + \frac{1}{\nu\wedge\kappa}\Big)\|\phi_h(s)\|^2 + 2\sqrt{L}\,|h_\varepsilon(s)|_0\Big]\, |\Phi_\varepsilon(s)|^2\, ds. \quad (4.12)$$

Our goal here is to show that as $\varepsilon\to 0$, $\sup_{0\leq t\leq T}|\Phi_\varepsilon(t)|^2 + \int_0^T \|\Phi_\varepsilon(s)\|^2 ds \to 0$ in probability, which implies that $\phi_{h_\varepsilon}\to\phi_h$ in distribution in $X := C([0,T];H)\cap L^2((0,T);V)$. Fix $N>0$ and for $t\in[0,T]$ let

$$\begin{aligned} G_N(t) &= \Big\{\sup_{0\leq s\leq t}|\phi_h(s)|^2 \leq N\Big\}\cap\Big\{\int_0^t \|\phi_h(s)\|^2 ds \leq N\Big\},\\ G_{N,\varepsilon}(t) &= G_N(t)\cap\Big\{\sup_{0\leq s\leq t}|\phi_{h_\varepsilon}(s)|^2 \leq N\Big\}\cap\Big\{\int_0^t \|\phi_{h_\varepsilon}(s)\|^2 ds \leq N\Big\}. \end{aligned}$$

Claim 1. For any $\varepsilon_0>0$, $\sup\limits_{0<\varepsilon\leq\varepsilon_0}\ \sup\limits_{h,h_\varepsilon\in\mathcal{A}_M}\mathbb{P}(G_{N,\varepsilon}(T)^c)\to 0$ as $N\to\infty$.

Indeed, for $\varepsilon>0$, $h,h_\varepsilon\in\mathcal{A}_M$, the Markov inequality and the estimate (3.5) imply

$$\begin{aligned} \mathbb{P}(G_{N,\varepsilon}(T)^c) &\leq \mathbb{P}\Big(\sup_{0\leq s\leq T}|\phi_h(s)|^2 > N\Big) + \mathbb{P}\Big(\sup_{0\leq s\leq T}|\phi_{h_\varepsilon}(s)|^2 > N\Big)\\ &\quad + \mathbb{P}\Big(\int_0^T \big(\|\phi_h(s)\|^2 ds > N\Big) + \mathbb{P}\Big(\int_0^T \|\phi_{h_\varepsilon}(s)\|^2\big) ds > N\Big)\\ &\leq \frac{1}{N}\sup_{h,h_\varepsilon\in\mathcal{A}_M}\mathbb{E}\Big(\sup_{0\leq s\leq T}|\phi_h(s)|^2 + \sup_{0\leq s\leq T}|\phi_{h_\varepsilon}(s)|^2 + \int_0^T (\|\phi_h(s)\|^2 + \|\phi_{h_\varepsilon}(s)\|^2) ds\Big)\\ &\leq C_1(\nu,\kappa,K,L,T,M)\,\big(1+E|\xi|^4\big)N^{-1}. \end{aligned}$$

Claim 2. For fixed $N > 0$, $h, h_\varepsilon \in \mathcal{A}_M$ such that as $\varepsilon \to 0$, $h_\varepsilon \to h$ a.s. in the weak topology of $L^2([0,T], H_0)$, one has as $\varepsilon \to 0$

$$\mathbb{E}\Big[1_{G_{N,\varepsilon}(T)}\Big(\sup_{0\le t\le T}|\Phi_\varepsilon(t)|^2 + \int_0^T \|\Phi_\varepsilon(t)\|^2\,dt\Big)\Big] \to 0. \tag{4.13}$$

Indeed, (4.12) and Gronwall's lemma imply that on $G_{N,\varepsilon}(T)$,

$$\sup_{0\le t\le T}|\Phi_\varepsilon(t)|^2 \le \Big[\sup_{0\le t\le T}\big(T_1(t,\varepsilon)+T_3(t,\varepsilon)\big)+\varepsilon KT(1+N)\Big]\,\exp\Big(2T+\Big[\frac{2}{\nu}+\frac{2}{\nu\wedge\kappa}\Big]N+2\sqrt{M}\Big).$$

Thus, using again (4.12) we deduce that for some constant $\tilde{C} = C(\nu,\kappa,K,T,M,N)$, one has for every $\varepsilon > 0$:

$$\mathbb{E}\big(1_{G_{N,\varepsilon}(T)}\,|\Phi_\varepsilon|_X^2\big) \le \tilde{C}\Big(\varepsilon KTN + \mathbb{E}\Big[1_{G_{N,\varepsilon}(T)}\sup_{0\le t\le T}\big(T_1(t,\varepsilon)+T_3(t,\varepsilon)\big)\Big]\Big). \tag{4.14}$$

Since the sets $G_{N,\varepsilon}(.)$ decrease, $\mathbb{E}\big(1_{G_{N,\varepsilon}(T)}\sup_{0\le t\le T}|T_1(t,\varepsilon)|\big) \le \mathbb{E}(\lambda_\varepsilon)$, where

$$\lambda_\varepsilon := 2\sqrt{\varepsilon}\sup_{0\le t\le T}\Big|\int_0^t 1_{G_{N,\varepsilon}(s)}\big(\Phi_\varepsilon(s),\sigma(\phi_{h_\varepsilon}(s))dW(s)\big)\Big|.$$

The scalar-valued random variables λ_ε converge to 0 in L^1 as $\varepsilon \to 0$. Indeed, by the Burkholder-Davis-Gundy inequality, (A4) and the definition of $G_{N,\varepsilon}(s)$, we have

$$\begin{aligned}
\mathbb{E}(\lambda_\varepsilon) &\le 6\sqrt{\varepsilon}\,\mathbb{E}\Big\{\int_0^T 1_{G_{N,\varepsilon}(s)}\,|\Phi_\varepsilon(s)|^2\,|\sigma(\phi_{h_\varepsilon}(s))|_{L_Q}^2 ds\Big\}^{\frac12}\\
&\le 6\sqrt{\varepsilon}\,\mathbb{E}\Big[\Big\{4N\int_0^T 1_{G_{N,\varepsilon}(s)}\,K\,(1+|\phi_{h_\varepsilon}(s)|^2)ds\Big\}^{\frac12}\Big]\\
&\le 12\,\sqrt{\varepsilon}\,\sqrt{KT}\,(1+N).
\end{aligned} \tag{4.15}$$

For $k = 0,\cdots,2^n$ set $t_k = kT2^{-n}$; for $s \in]t_k, t_{k+1}]$, set $\bar{s}_n = t_{k+1}$ and $\underline{s}_n = t_k$. Then for any $n \ge 1$,

$$\mathbb{E}\Big(1_{G_{N,\varepsilon}(T)}\sup_{0\le t\le T}|T_3(t,\varepsilon)|\Big) \le 2\,\sum_{i=1}^3 \tilde{T}_i(N,n,\varepsilon) + 2\,\mathbb{E}\big(\bar{T}_4(N,n,\varepsilon,\omega)\big),$$

where

$$\tilde{T}_1(N,n,\varepsilon) = \mathbb{E}\Big[1_{G_{N,\varepsilon}(T)}\sup_{0\le t\le T}\Big|\int_0^t\Big(\sigma(\phi_h(s))\big(h_\varepsilon(s)-h(s)\big)\,,\,\big[\Phi_\varepsilon(s)-\Phi_\varepsilon(\bar{s}_n)\big]\Big)ds\Big|\Big],$$

$$\tilde{T}_2(N,n,\varepsilon) = \mathbb{E}\Big[1_{G_{N,\varepsilon}(T)}\sup_{0\le t\le T}\Big|\int_0^t\Big(\big[\sigma(\phi_h(s))-\sigma(\phi_h(\bar{s}_n))\big]\big(h_\varepsilon(s)-h(s)\big)\,,\,\Phi_\varepsilon(\bar{s}_n)\Big)ds\Big|\Big],$$

$$\tilde{T}_3(N,n,\varepsilon) = \mathbb{E}\Big[1_{G_{N,\varepsilon}(T)}\sup_{1\le k\le 2^n}\sup_{t_{k-1}\le t\le t_k}\Big|\big(\sigma(\phi_h(t_k))\int_{t_{k-1}}^t (h_\varepsilon(s)-h(s))\,ds\,,\,\Phi_\varepsilon(t_k)\big)\Big|\Big]$$

$$\bar{T}_4(N,n,\varepsilon) = 1_{G_{N,\varepsilon}(T)}\sum_{k=1}^{2^n}\Big|\Big(\sigma(\phi_h(t_k))\int_{t_{k-1}}^{t_k}\big(h_\varepsilon(s)-h(s)\big)\,ds\,,\,\Phi_\varepsilon(t_k)\Big)\Big|.$$

Using Schwarz's inequality, (A4) and Lemma 4.2, we deduce that for some constant $\bar{C}_1 := C(\nu, \kappa, K, T, M, N)$ and any $\varepsilon \in]0, \varepsilon_0]$,

$$\begin{aligned}\tilde{T}_1(N,n,\varepsilon) &\le \sqrt{K}\mathbb{E}\Big[1_{G_{N,\varepsilon}(T)} \int_0^T \big(1+|\phi_h(s)|^2\big)^{\frac{1}{2}} |h_\varepsilon(s)-h(s)|_0 \big|\Phi_\varepsilon(s)-\Phi_\varepsilon(\bar{s}_n)\big|\, ds\Big] \\ &\le \sqrt{2K(1+N)}\,\Big(\mathbb{E}\int_0^T |h_\varepsilon(s)-h(s)|_0^2\, ds\Big)^{\frac{1}{2}} \\ &\quad \times \Big(\mathbb{E}\Big[1_{G_{N,\varepsilon}(T)}\int_0^T \big\{|\phi_{h_\varepsilon}(s)-\phi_{h_\varepsilon}(\bar{s}_n)|^2 + |\phi_h(s)-\phi_h(\bar{s}_n)|^2\big\}\, ds\Big]\Big)^{\frac{1}{2}} \\ &\le \bar{C}_1\, 2^{-\frac{n}{4}}. \end{aligned} \tag{4.16}$$

A similar computation based on (A5) and Lemma 4.2 yields for some constant $\bar{C}_2 := C(\nu, \kappa, K, T, M, N)$ and any $\varepsilon \in]0, \varepsilon_0]$

$$\begin{aligned}\tilde{T}_2(N,n,\varepsilon) &\le \sqrt{L}\Big(\mathbb{E}\Big[1_{G_{N,\varepsilon}(T)}\int_0^T |\phi_h(s)-\phi_h(\bar{s}_n)|^2\, ds\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\int_0^T |h_\varepsilon(s)-h(s)|_0^2\, 2N\, ds\Big)^{\frac{1}{2}} \\ &\le \bar{C}_2\, 2^{-\frac{n}{4}}. \end{aligned} \tag{4.17}$$

Using Schwarz's inequality and (A4) we deduce for $\bar{C}_3 = C(K, N, M)$ and any $\varepsilon \in]0, \varepsilon_0]$

$$\begin{aligned}\tilde{T}_3(N,n,\varepsilon) &\le \sqrt{K}\mathbb{E}\Big[1_{G_{N,\varepsilon}(T)} \sup_{1\le k\le 2^n} \big(1+|\phi_h(t_k)|^2\big)^{\frac{1}{2}} \int_{t_{k-1}}^{t_k} |h_\varepsilon(s)-h(s)|_0\, ds\, |\Phi_\varepsilon(t_k)|\Big] \\ &\le 2\sqrt{KN(1+N)}\,\mathbb{E}\Big(\sup_{1\le k\le 2^n}\int_{t_{k-1}}^{t_k} |h_\varepsilon(s)-h(s)|_0\, ds\Big) \\ &\le 4\sqrt{KN(1+N)}\,\sqrt{M}\, 2^{-\frac{n}{2}} = \bar{C}_3\, 2^{-\frac{n}{2}}. \end{aligned} \tag{4.18}$$

Finally, note that the weak convergence of h_ε to h implies that for any $a, b \in [0, T]$, $a < b$, the integral $\int_a^b h_\varepsilon(s)ds \to \int_a^b h(s)ds$ in the weak topology of H_0. Therefore, since for $\phi \in H$ the operator $\sigma(\phi)$ is compact from H_0 to H, we deduce that $\big|\sigma(\phi)\big(\int_a^b h_\varepsilon(s)ds - \int_a^b h(s)ds\big)\big|_H \to 0$ as $\varepsilon \to 0$. Hence a.s. for fixed n as $\varepsilon \to 0$, $\bar{T}_4(N, n, \varepsilon, \omega) \to 0$. Furthermore, $\bar{T}_4(N, n, \varepsilon, \omega) \le \sqrt{K}\sqrt{1+N}\sqrt{2N}\sqrt{M}$ and hence the dominated convergence theorem proves that for any fixed n, $\mathbb{E}(\bar{T}_4(N, n, \varepsilon)) \to 0$ as $\varepsilon \to 0$.

Thus, given $\alpha > 0$, we may choose n_0 large enough to have $(\bar{C}_1 + \bar{C}_2)2^{-\frac{n}{4}} + \bar{C}_3 2^{-\frac{n}{2}} \le \alpha$ for $n \ge n_0$. Then for fixed $n \ge n_0$, let $\varepsilon_1 \in]0, \varepsilon_0]$ be such that for $0 < \varepsilon \le \varepsilon_1$, $\mathbb{E}\big[\bar{T}_4(N, n, \varepsilon)\big] \le \alpha$. Using (4.16)-(4.18), we deduce that for $\varepsilon \in]0, \varepsilon_1]$,

$$\mathbb{E}\Big[1_{G_{N,\varepsilon}(T)} \sup_{0\le t\le T} |T_3(t, \varepsilon)|\Big] \le 2\alpha. \tag{4.19}$$

Claim 2 is a straightforward consequence of the inequalities (4.14), (4.15) and (4.19).

To conclude the proof of the Proposition 4.3, let $\delta > 0$ and $\alpha > 0$ and set

$$\Lambda_\varepsilon := |\Phi_\varepsilon|_X^2 = \sup_{0\le t\le T} |\Phi_\varepsilon(t)|^2 + \int_0^T \|\Phi_\varepsilon(s)\|^2 ds.$$

Then the Markov inequality implies that

$$\mathbb{P}(\Lambda_\varepsilon > \delta) = \mathbb{P}(G_{N,\varepsilon}(T)^c) + \frac{1}{\delta}\mathbb{E}\Big(1_{G_{N,\varepsilon}(T)}|\Phi_\varepsilon|_X^2\Big)$$

Using *Claim 1*, one can choose N large enough to make sure that $\mathbb{P}(G_{N,\varepsilon}(T)^c) < \alpha$ for every ε. Fix N ; *Claim 2* shows that for ε small enough, $\mathbb{E}\Big(1_{G_{N,\varepsilon}(T)}|\Phi_\varepsilon|_X^2\Big) < \delta\alpha$. This concludes the proof of the proposition. □

The following compactness result will show that the rate function of the LDP satisfied by the solution to (4.8) is a good rate function. The proof is similar to that of Proposition 4.3 and easier.

Proposition 4.4. *(Compactness)*
Let M be any fixed finite positive number. Define

$$K_M = \{\phi_h \in C([0,T];H) \cap L^2((0,T);V) : h \in S_M\},$$

where ϕ_h is the unique solution of the deterministic control equation:

$$d\phi_h(t) + \big[A\phi_h(t) + B(\phi_h(t)) + R\phi_h(t)\big]dt = \sigma(\phi_h(t))h(t)dt, \;\; \phi_h(0) = \xi, \quad (4.20)$$

and σ satisfies (A1), (A4) and (A5). Then K_M is a compact subset of X.

Proof. Let (ϕ_n) be a sequence in K_M, corresponding to solutions of (4.20) with controls (h_n) in S_M:

$$d\phi_n(t) + \big[A\phi_n(t) + B(\phi_n(t)) + R\phi_n(t)\big]dt = \sigma(\phi_n(t))h_n(t)dt, \;\; \phi_n(0) = \xi. \quad (4.21)$$

Since S_M is a bounded closed subset in the Hilbert space $L^2((0,T);H_0)$, it is weakly compact. So there exists a subsequence of (h_n), still denoted as (h_n), which converges weakly to a limit h in $L^2((0,T);H_0)$. Note that in fact $h \in S_M$ as S_M is closed. We now show that the corresponding subsequence of solutions, still denoted as (ϕ_n), converges in X to ϕ which is the solution of the following "limit" equation

$$d\phi(t) + [A\phi(t) + B(\phi(t)) + R\phi(t)]dt = \sigma(\phi(t))h(t)dt, \;\; \phi(0) = \xi. \quad (4.22)$$

This will complete the proof of the compactness of K_M. To ease notation we will often drop the time parameters s, t, ... in the equations and integrals.

Let $\Phi_n = \phi_n - \phi$, or in component form $\Phi_n = (U_n, \Theta_n) = (u_n - u, \theta_n - \theta)$; then

$$d\Phi_n + [A\Phi_n + B(\phi_n) - B(\phi) + R\Phi_n]dt = [\sigma(\phi_n)h_n - \sigma(\phi)h]dt, \;\; \Phi_n(0) = 0. \quad (4.23)$$

Using Lemmas 3.5 and 3.7, (A5) and Young's inequality, we deduce that for $t \in [0,T]$,

$$|\Phi_n(t)|^2 + 2\int_0^t \big(\nu\|U_n(s)\|^2 + \kappa\|\Theta_n(s)\|^2\big)ds = 4\int_0^t \big(\Theta_n(s), U_{n,2}(s)\big)ds$$
$$- 2\int_0^t \Big\{\big\langle B_1(u_n(s),u_n(s)) - B_1(u(s),u(s)), U_n(s)\big\rangle + \big\langle B_2(\phi_n(s)) - B_2(\phi(s)), \Theta_n(s)\big\rangle\Big\}ds$$

$$
\begin{aligned}
&+2\int_0^t \Big\{\Big(\big[\sigma(\phi_n(s))-\sigma(\phi_h(s))\big]h_n(s),\Phi_n(s)\Big)ds+\big(\sigma(\phi_h(s))\big(h_n(s)-h(s)\big)\,,\,\Phi_n(s)\big)\Big\}ds\\
&\le 2\int_0^t |\Phi_n(s)|^2\,ds+\frac{\nu}{2}\int_0^t \|U_n(s)\|^2 ds+\frac{\nu\wedge\kappa}{2}\int_0^t \|\Phi_n(s)\|^2\,ds\\
&+\frac{2}{\nu}\int_0^t \|u(s)\|^2\,|U_n(s)|^2\,ds+\frac{2}{\nu\wedge\kappa}\int_0^t \|\theta(s)\|^2\,|\Phi_n(s)|^2\,ds+2L\int_0^t |\Phi_n(s)|^2\,|h_n(s)|_0\,ds\\
&+2\int_0^t \big(\sigma(\phi(s))\,[h_n(s)-h(s)]\,,\,\Phi_n(s)\big)\,ds. \qquad (4.24)
\end{aligned}
$$

Thus (4.24) yields the following integral inequality

$$
\begin{aligned}
|\Phi_n(t)|^2+(\nu\wedge\kappa)\int_0^t \|\Phi_n(s)\|^2 ds &\le 2\int_0^t \big(\sigma(\phi(s))\,[h_n(s)-h(s)]\,,\,\Phi_n(s)\big)\,ds\\
&+\int_0^t \Big\{2+2L|h_n(s)|_0+\Big(\frac{2}{\nu}+\frac{2}{\nu\wedge\kappa}\Big)\|\phi(s)\|^2\Big\}\,|\Phi_n(s)|^2 ds. \qquad (4.25)
\end{aligned}
$$

For $N\ge 1$ and $k=0,\cdots,2^N$, set $t_k=k2^{-N}$. For $s\in]t_{k-1},t_k]$, $1\le k\le 2^N$, let $\bar{s}_N=t_k$. The inequality (3.7) implies that there exists a constant $\bar{C}>0$ such that

$$
\sup_n\Big[\sup_{0\le t\le T}\big(|\phi(t)|^2+|\phi_n(t)|^2\big)+\int_0^T\big(\|\phi(s)\|^2+\|\phi_n(s)\|^2\big)ds\Big]=\bar{C}<+\infty.
$$

Thus Gronwall's inequality implies

$$
\sup_{t\le T}|\Phi_n(t)|^2\le\exp\Big(\int_0^T\Big\{2+2L|h_n(s)|_0+\Big(\frac{2}{\nu}+\frac{2}{\nu+\kappa}\Big)\|\phi(s)\|^2\Big\}ds\Big)\sum_{i=1}^4 I^i_{n,N}, \quad (4.26)
$$

where

$$
\begin{aligned}
I^1_{n,N} &= \int_0^T \big|\big(\sigma(\phi(s))\,[h_n(s)-h(s)]\,,\,\Phi_n(s)-\Phi_n(\bar{s}_N)\big)\big|\,ds,\\
I^2_{n,N} &= \int_0^T \Big|\Big(\big[\sigma(\phi(s))-\sigma(\phi(\bar{s}_N))\big][h_n(s)-h(s)]\,,\,\Phi_n(\bar{s}_N)\Big)\Big|\,ds,\\
I^3_{n,N} &= \sup_{1\le k\le 2^N}\sup_{t_{k-1}\le t\le t_k}\Big|\Big(\sigma(\phi(t_k))\int_{t_{k-1}}^t (h_\varepsilon(s)-h(s))ds\,,\,\Phi_\varepsilon(t_k)\Big)\Big|,\\
I^4_{n,N} &= \sum_{k=1}^{2^N}\Big(\sigma(\phi(t_k))\int_{t_{k-1}}^{t_k}[h_n(s)-h(s)]\,ds\,,\,\Phi_n(t_k)\Big).
\end{aligned}
$$

Schwarz's inequality, (A4), (A5) and Lemma 4.2 imply that for some constant C which does not depend on n,

$$
\begin{aligned}
I^1_{n,N} &\le \Big(\int_0^T K(1+\bar{C})|h_n(s)-h(s)|_0^2 ds\Big)^{\frac{1}{2}}\Big(2\int_0^T\big(|\phi_n(s)-\phi_n(\bar{s}_N)|^2+|\phi(s)-\phi(\bar{s}_N)|^2\big)ds\Big)^{\frac{1}{2}}\\
&\le C2^{-\frac{N}{4}}, \qquad (4.27)
\end{aligned}
$$

$$I^2_{n,N} \le \Big(L\int_0^T |\phi(s)-\phi(\bar{s}_N)|^2 ds\Big)^{\frac{1}{2}}\Big(\bar{C}\int_0^T |h_n(s)-h(s)|_0^2\, ds\Big)^{\frac{1}{2}} \le C2^{-\frac{N}{4}}\,, \tag{4.28}$$

$$I^3_{n,N} \le K\big(1+\sup_t|\phi(t)|\big)\sup_t\big(|\phi(t)|+\phi_n(t)|\big)2^{-\frac{N}{2}}2M \le C\tilde{C}2^{-\frac{N}{2}}\,. \tag{4.29}$$

Thus, given $\alpha > 0$, one may choose N large enough to have $\sup_n \sum_{i=1}^3 I^i_{n,N} \le \alpha$. Then, for fixed N and $k = 1, \cdots, 2^N$, as $n \to \infty$, the weak convergence of h_n to h implies that of $\int_{t_{k-1}}^{t_k}(h_n(s)-h(s))ds$ to 0 weakly in H_0. Since $\sigma(\phi(t_k))$ is a compact operator, we deduce that for fixed k the sequence $\sigma(\phi(t_k))\int_{t_{k-1}}^{t_k}(h_n(s)-h(s))ds$ converges to 0 strongly in H as $n \to \infty$. Since $\sup_n \sup_k |\Phi_n(t_k)| \le 2\tilde{C}$, we have $\lim_n I^3_{n,N} = 0$. Thus as $n \to \infty$, $\sup_{0\le t\le T}|\Phi_n(t)|^2 \to 0$. Using this convergence and (4.25), we deduce that $\|\Phi_n\|_X \to 0$ as $n \to \infty$. This shows that every sequence in K_M has a convergent subsequence. Hence K_M is a compact subset of X. □

With the above results, we have the following Large Deviation theorem.

Theorem 4.5. *Let σ satisfy (A1), (A4) and (A5), ϕ^ε be the solution of the stochastic Bénard problem (2.7). Then $\{\phi^\varepsilon\}$ satisfies the large deviation principle in $C([0,T];H)\cap L^2((0,T);V)$, with the good rate function*

$$I_\xi(\psi) = \inf_{\{h\in L^2(0,T;H_0):\ \psi=\mathcal{G}^0(\int_0^\cdot h(s)ds)\}}\Big\{\frac{1}{2}\int_0^T |h(s)|_0^2\, ds\Big\}. \tag{4.30}$$

Here the infimum of an empty set is taken as infinity.

Proof. Propositions 4.4 and 4.3 imply that $\{\phi^\varepsilon\}$ satisfies the Laplace principle which is equivalent to the large deviation principle in $X = C([0,T],H)\cap L^2((0,T),V)$ with the above-mentioned rate function; see Theorem 4.4 in [1] or Theorem 4.4 in [2]. □

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