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FINITE SIZE SCALING FOR HOMOGENEOUS PINNING MODELS

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ABSTRACT. Pinning models are built from discrete renewal sequences by rewarding (or penalizing) the trajectories according to their number of renewal epochs up to time N , and N is then sent to infinity. They are statistical mechanics models to which a lot of attention has been paid both because they are very relevant for applications and because of their *exactly solvable character*, while displaying a non-trivial phase transition (in fact, a localization transition). The order of the transition depends on the tail of the inter-arrival law of the underlying renewal and the transition is continuous when such a tail is sufficiently heavy: this is the case on which we will focus. The main purpose of this work is to give a mathematical treatment of the *finite size scaling limit* of pinning models, namely studying the limit (in law) of the process close to criticality when the system size is proportional to the correlation length.

Keywords: Renewal Theory, Pinning Models, Criticality, Subordinators, Regenerative Sets

AMS 2000 Subject Classification Numbers: 60K35, 82B41

1. INTRODUCTION

The model of directed polymers interacting with a one dimensionnal defect turns out to be quite satisfactory to describe various biological or physical phenomena, such as $(1+1)$ interface wetting [7], the problem of depinning of flux lines from columnar defects in type II superconductors [18] and the denaturation transition of DNA in the Poland Sheraga approximation [15]. The first results obtained by the physical community have been summarized in [10], a paper which received a lot of attention from the mathematicians. Since then, this model has been under close scrutiny, including the last few years (the recent mathematical monograph [12], but also [4], [14] or [8]). Here, we are going to deal with the homogeneous version of the model; this version is mathematically remarkable in the sense that it is essentially completely solvable. For example, the model can have a transition of any given order [12] depending on the value of α , a parameter in the definition of the system ($1+\alpha$ is sometimes called loop exponent in the literature).

A central role in statistical mechanics is played by the correlation length, a quantity that we will denote by ξ . Properly defining this quantity in an (infinite volume) model requires some work and in most of the cases, such a concept does not correspond to a unique mathematical object. All the same, it is essential that any “reasonably defined” correlation length behaves in the same way close to criticality. And precisely close to criticality, for a large class of models, ξ becomes the only “relevant” scale in the system (see below for a more precise explanation). This idea is one of the basic concepts of the

so called “finite size scaling” theory (see [6]). Here we illustrate in very concrete terms this concept via the scaling limit of pinning systems, obtaining a limiting behavior if the parameter is in a small, size dependent window near criticality as the size of the system goes to ∞ .

In this first part, we will try to see in which way the above, rather unprecise statements, can be made quantitative. The second part should be seen as a warm up for the two last parts; actually, in the one dimensionnal wetting, the existing literature makes the statement quite easy. In the third part, we introduce our general return model, and make the precise statements of our result in terms of scaling limits of the zero level set of our system near criticality.

In this note, α will always be a real number in $(0, 1)$. For positive sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$, we will write $a_n \sim b_n$ if $\lim_{n \rightarrow \infty} a_n/b_n = 1$, and for random variables $(X_n)_{n \geq 0}$ and X with values in a Polish space (E, d) , we will write $X_n \Rightarrow X$ if the sequence $(X_n)_{n \geq 0}$ converges in law towards X .

1.1 A first model. Let τ be a recurrent renewal process with law \mathbf{P} , that is $\tau = \{\tau_n\}_{n \geq 0}$ where $\tau_0 = 0$ and $(\tau_{i+1} - \tau_i)_{i \geq 0}$ are iid with common law \mathbf{P} , where \mathbf{P} is \mathbf{N} valued and verifying, for $k \rightarrow \infty$,

$$\mathbf{P}(\tau_{i+1} - \tau_i = k) \sim \frac{C}{k^{1+\alpha}}. \quad (1.1)$$

We will actually consider a slightly more general model in section 3, allowing in particular τ to be transient, but for simplicity, we will restrict ourselves temporarily to this setup. The set $\tau \cap [0, N]$ can be considered as a random subset of $\{0, 1, \dots, N\}$. We define the law $\mathbf{P}_{N,\beta}$ on the subsets of $\{0, 1, \dots, N\}$ by:

$$\frac{d\mathbf{P}_{N,\beta}}{d\mathbf{P}}(\tau) = \frac{1}{Z_{N,\beta}} \exp(\beta \mathcal{N}_N(\tau)), \quad (1.2)$$

where $\mathcal{N}_N(\tau)$ is the cardinality of the set $\tau \cap [0, N]$. In (1.2), $Z_{N,\beta}$ is the partition function of the model and of course

$$Z_{N,\beta} = \mathbf{E}[\exp(\beta \mathcal{N}_N(\tau))]. \quad (1.3)$$

It is not difficult to show that the limit of the quantity $F_N(\beta) := \frac{1}{N} \log Z_{N,\beta}$ exists and is non negative. We denote it by $F(\beta)$. $F(\cdot)$ is a convex, non decreasing and non negative function, and there exists a critical value of β such that $\beta \leq \beta_c$ implies $F(\beta) = 0$ and $\beta > \beta_c$ implies the positivity of $F(\beta)$. If $F(\beta) = 0$, the system is said to be *delocalized*, and it is *localized* otherwise. As the underlying renewal is recurrent, we actually have $\beta_c = 0$ (see [12], chapter 2 and section 3 below).

1.2 Finite size scaling. We define the correlation length of our system by

$$\xi(\beta) := \frac{1}{F(\beta)} \quad (1.4)$$

if $\beta > 0$, and $\xi(\beta) = \infty$ otherwise. This definition has first been introduced in the physical litterature in [10], and its mathematical relevance with respect to other quantities (in particular with the *correlation function*) is discussed in [13]. Also note that the concept of correlation length has been considered in depth in the inhomogeneous case (that is when the reward-penalty become site-dependent and random), see e.g. [20]. It has been shown

in [12] that

$$\xi(\beta) \stackrel{\beta \searrow \beta_c}{\sim} c_\alpha (\beta - \beta_c)^{-1/\alpha}, \quad (1.5)$$

for some (explicit) constant $c_\alpha > 0$.

We expect finite size effects to appear only if the system is of the scale of the correlation length, that is only if $\xi(\beta)/N$ stays fixed (say equal to q). It is quite clear (considering (1.5)) that this is equivalent to keeping $(\beta - \beta_c)^{1/\alpha} N$ fixed (at least close to criticality). In this case, some of the observables of our system have an interesting behavior near criticality (in particular different from their behavior in each of the two phases in the infinite volume limit). To describe it, we will need two basic objects in the sequel. The first one is the *stable subordinator* with index α . Recall that a subordinator is an increasing Lévy process; a subordinator $(\sigma_s^{(\alpha)})_{s \geq 0}$ is said to be α *stable* (for $\alpha \in (0, 1)$) if, for all $M \in \mathbf{R}^+$, the process $(\sigma_{Ms}^{(\alpha)})_{s \geq 0}$ has the same law than the process $(M^{1/\alpha} \sigma_s^{(\alpha)})_{s \geq 0}$. The second one is its local time $(L_t^{(\alpha)})_{t \geq 0}$, which is simply the generalized inverse of $(\sigma_s^{(\alpha)})_{s \geq 0}$, that is

$$L_t^{(\alpha)} := \inf \left\{ s \geq 0, \sigma_s^{(\alpha)} \geq t \right\} \quad (1.6)$$

It will be shown in the last part (Lemma 1) that there exists $C_K > 0$ such that

$$\left(C_K \frac{\mathcal{N}_N}{N^\alpha} \right)_{N \geq 0} \Rightarrow L_1^{(\alpha)} \quad (1.7)$$

and moreover the sequence $\left(C_K \frac{\mathcal{N}_N}{N^\alpha} \right)_{N \geq 0}$ is uniformly integrable (Lemma 2). Using these two facts and (1.5), one gets that the *finite size correlation length* $\xi_N(\beta) := 1/F_N(\beta)$ verifies

$$\frac{\xi_N(\beta)}{N} \stackrel{N \rightarrow \infty}{\sim} \left(\log \mathbf{E} \left[\exp \left(q^\alpha c_\alpha^\alpha \frac{\mathcal{N}_N}{N^\alpha} \right) \right] \right)^{-1} \sim f(q) \quad (1.8)$$

where $f(q) = \left(\log \mathbf{E} \left[\exp \left(\frac{q^\alpha c_\alpha^\alpha}{C_K} L_1^{(\alpha)} \right) \right] \right)^{-1}$.

Interesting examples broadly used by physicists could also include the (finite volume) *contact fraction* $\rho_N(\beta)$ (that is the expectation of \mathcal{N}_N/N with respect to the finite polymer measure) and the (once again finite volume) *specific heat* $\chi_N(\cdot)$ (that is the second derivative of the finite volume free energy). In our setup, these quantities can be written in an analogous way as above. Specifically, we get

$$\rho_N(\beta) \sim N^{\alpha-1} \frac{1}{C_K} \frac{\mathbf{E} \left[L_1^{(\alpha)} \exp \left(\frac{q^\alpha c_\alpha^\alpha}{C_K} L_1^{(\alpha)} \right) \right]}{\mathbf{E} \left[\exp \left(\frac{q^\alpha c_\alpha^\alpha}{C_K} L_1^{(\alpha)} \right) \right]} := N^{\alpha-1} \hat{\rho}(q). \quad (1.9)$$

In an equivalent way, introducing $\tilde{\rho}(x) := x^{1-\alpha} \hat{\rho}(q)$, we get $\rho_N(\beta) \sim \xi(\beta)^{\alpha-1} \tilde{\rho}(q)$, that is we have a quite precise behavior of $\rho_N(\cdot)$ as $N \rightarrow \infty$ as a function of $\xi(\cdot)$. Analogously, we have

$$\frac{\partial^2}{\partial^2 \beta} F_N(\beta) = \chi_N(\beta) \sim N^{2\alpha-1} \frac{1}{C_K^2} \frac{\mathbf{E} \left[(L_1^{(\alpha)} - \mathbf{E}[L_1^{(\alpha)}])^2 \exp \left(\frac{q^\alpha c_\alpha^\alpha}{C_K} L_1^{(\alpha)} \right) \right]}{\mathbf{E} \left[\exp \left(\frac{q^\alpha c_\alpha^\alpha}{C_K} L_1^{(\alpha)} \right) \right]}. \quad (1.10)$$

so that, once again, we get the following equivalence:

$$\chi_N(\beta) \sim \xi(\beta)^{2\alpha-1} \hat{\chi}(q). \quad (1.11)$$

It then becomes quite natural to ask in which way other quantities of the system get affected by the constraint $\xi(\beta)/N = q$, that is of considering the system at a correlation

length scale. The aim of this paper is to give in a certain sense an exhaustive description of what happens in an infinite volume limit under this constraint. For this, we actually compute the scaling limit of the system in terms of a Radon Nykodym derivative with respect to the α regenerative set.

2. A LOOK AT THE WETTING MODEL

As a warm-up, we will deal with the case in which the underlying law of our polymer model is the simple symmetric random walk conditioned to stay non negative, that is a sequence $(S_n)_{n \geq 0}$ where $S_0 = 0$, the variables $(S_i - S_{i-1})_{i \geq 1}$ are iid and such that $\mathbf{P}[S_i - S_{i-1} = 1] = \mathbf{P}[S_i - S_{i-1} = -1] = \frac{1}{2}$. We introduce the following probability law on \mathbf{Z}^N :

$$\frac{d\mathbf{P}_{\beta,N}}{d\mathbf{P}}(x) = \frac{1}{Z_{\beta,N}} \exp(H_{N,\beta}(x)) \mathbb{1}_{x_1 \geq 0, \dots, x_N \geq 0} \quad (2.1)$$

where the Hamiltonian of our system is given by

$$H_{N,\beta}(x) = \beta \sum_{i=1}^N \mathbb{1}_{x_i=0} = \beta \mathcal{N}_N. \quad (2.2)$$

Here also, it is not hard to prove the existence of $\lim_{N \rightarrow \infty} \frac{1}{N} \log(Z_{\beta,N}) = F(\beta)$. We have the inequality

$$Z_{\beta,N} = \mathbf{E}[\exp(H_{N,\beta}(S)) \mathbb{1}_{S_1 \geq 0, \dots, S_N \geq 0}] \quad (2.3)$$

$$\geq \mathbf{E}[\exp(H_{N,\beta}(S)) \mathbb{1}_{S_1 > 0, \dots, S_N > 0}] = \mathbf{P}[S_1 > 0, \dots, S_N > 0] \sim \frac{1}{\sqrt{2\pi N}} \quad (2.4)$$

where the last equivalence is well known (cf [9]), which entails that for every $\beta \in \mathbf{R}$, $F(\beta) \geq 0$. It is quite easy to show that $F(\cdot)$ verifies the same properties as in the last part.

In this model, it is possible to show that β_c is actually equal to $\log(2)$ (see [12] chapter 1 or [14]). We are interested in the scaling limit of the system near criticality. To be more explicit, we define the application: $X^N : \mathbf{R}^N \rightarrow C([0, 1])$ by:

$$(X_t^N(x))_{t \in [0,1]} = \frac{x_{[Nt]}}{N^{1/2}} + (Nt - [Nt]) \frac{x_{[Nt]+1} - x_{[Nt]}}{N^{1/2}}. \quad (2.5)$$

and we introduce the sequence of measures $Q_{\beta,N} := \mathbf{P}_{\beta,N} \circ (X^N)^{-1}$, $N \geq 0$. Of course, in (2.5), $[x]$ denotes the integer part of the real number x .

Let $(B_t)_{t \geq 0}$ be a standard Brownian motion defined on our probability space $(\Omega, \mathcal{F}, \mathcal{P})$ in the sequel, and $(L_t)_{t \geq 0}$ denote its local time at zero. It has been proven in [8] that the sequence $(Q_{\beta_c, N})_{N \geq 0}$ converges weakly to the law of the process $(|B_t|)_{t \geq 0}$. The authors actually considered the case where the underlying process is much more general than the simple random walk.

The next result is quite easy to prove, but gives some intuition about the more general fact we are going to prove in the next section in terms of zero level sets only.

Theorem 1. *Let $\varepsilon \in \mathbf{R}$. The sequence of measures $(Q_{\frac{\varepsilon}{\sqrt{N}}, N})_{N \geq 0}$ converges weakly to the law of a process whose law is absolutely continuous with respect to the law of the process $(|B_t|)_{t \geq 0}$ with Radon-Nykodym density given by $e^{\varepsilon L_1} / \mathbf{E}[e^{\varepsilon L_1}]$.*

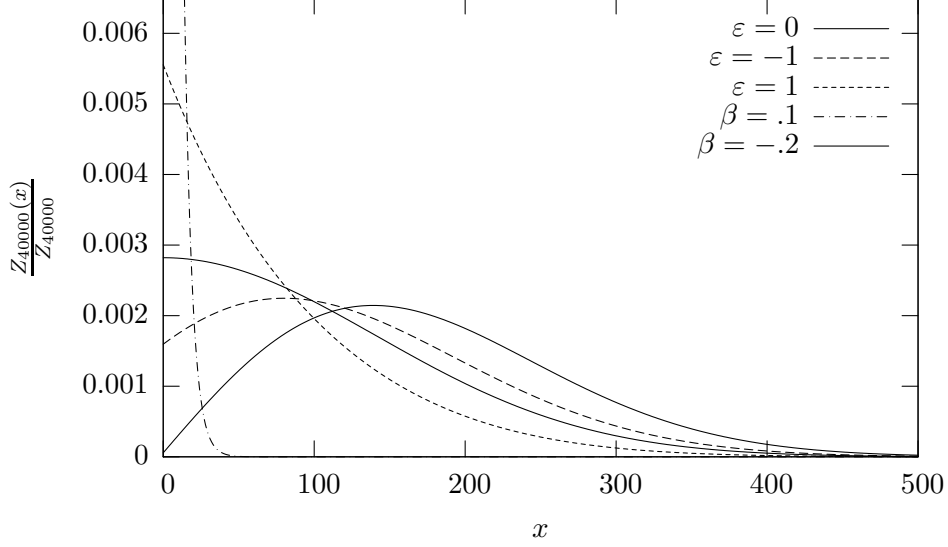


FIGURE 1. The distribution of the last point of the system for $N = 40000$, where $Z_N(x) := \mathbf{E}[\exp(\beta \mathcal{N}_N(\tau)) \mathbf{1}_{S_N=x}]$ for $\beta \in \{.1, -.2, \varepsilon/\sqrt{40000}\}$ and $\varepsilon \in \{-1; 0; 1\}$.

Proof. A proof of the tightness of the sequence $(Q_{\frac{\varepsilon}{\sqrt{N}}, N})_{N \geq 0}$ can be found in [5].

The following convergence in law is classical (see for example ([19],Th. 10.1)):

$$\frac{\mathcal{N}_N}{\sqrt{N}} \Rightarrow L_1. \quad (2.6)$$

Furthermore, in the Remark 3 of the same theorem, the author noted that it is possible to construct the random walk $(S_n)_{n \geq 0}$ such that besides $B_n - S_n = \mathcal{O}(\log(n))$ almost surely, which immediately implies the convergence of the joint law $(S_{[nt]}/\sqrt{n}, \mathcal{N}_n/\sqrt{n}) \Rightarrow (B_t, L_1)$. Since the tightness is already proven, we just have to show the finite dimensional convergence in law, that is we have to show that, for every $n \in \mathbf{N}$ and for every continuous bounded function $F(\cdot)$ from $[0, 1]^n$ to \mathbf{R} , the following convergence holds:

$$Q_{\frac{\varepsilon}{\sqrt{N}}, N} [F(\omega_{t_1}, \dots, \omega_{t_n})] \xrightarrow{N \rightarrow \infty} \frac{\mathbf{E}[e^{\varepsilon L_1} F(|B_{t_1}|, \dots, |B_{t_n}|)]}{\mathbf{E}[e^{\varepsilon L_1}]}. \quad (2.7)$$

We know by [8] that (2.7) holds for $\varepsilon = 0$. The term in the left hand side of (2.7) can be written:

$$\frac{1}{Z_{\frac{\varepsilon}{\sqrt{N}}, N}} \mathbf{E} \left[F \left(\frac{S_{[Nt_1]}}{\sqrt{N}}, \dots, \frac{S_{[Nt_n]}}{\sqrt{N}} \right) e^{\frac{\varepsilon \mathcal{N}_N}{\sqrt{N}}} \mathbb{1}_{S_1 > 0, \dots, S_N > 0} \right]. \quad (2.8)$$

Then we have the convergence $e^{\frac{\varepsilon \mathcal{N}_N}{\sqrt{N}}} \Rightarrow e^{\varepsilon L_1}$; to prove that, we just have to show that the family $(e^{\frac{\varepsilon \mathcal{N}_N}{\sqrt{N}}})_{N \geq 0}$ is uniformly integrable. This is obvious if $\varepsilon \leq 0$, and will be proved in the last section in a more general setup (see Lemma 2) for $\varepsilon > 0$, although it could be also proven directly using the exact distribution of \mathcal{N}_N , which is actually well known in the SRW setup (see for example [19]).

□

3. THE RENEWAL SETUP

2.1 The model. As we did in the introduction, we denote by τ the points of a renewal process, that is the image set of a sequence of iid positive random variables $(l_i)_{i \geq 0}$ whose law is \mathbf{P} . More precisely, $\tau := \{\tau_0, \tau_1, \dots\}$ where $\tau_k = \sum_{i=1}^k l_i$. In what follows, \mathbf{P} is an integer valued law verifying

$$K(n) := \mathbf{P}(l_1 = n) = \frac{L(n)}{n^{1+\alpha}} \quad (3.1)$$

where $L(\cdot)$ is a slowly varying function. Furthermore, we allow transience for our renewal process, that is $\sum_{n \geq 1} K(n) \leq 1$ (or equivalently $\mathbf{P}(l_1 = \infty) \geq 0$). For $N \in \mathbf{N}$, we introduce $\bar{K}(N) := \sum_{n > N} K(n)$. We recall that a positive measurable function $L(\cdot)$ is called slowly varying if for all $c > 0$, we have $\lim_{x \rightarrow \infty} L(cx)/L(x) = 1$. Such functions are well known, see [3]. One basic example of slowly varying function (apart from the trivial case of constants) at infinity is $x \mapsto \log(1+x)$, $x \in (0, \infty)$; but logarithmic functions are not the only examples, $x \mapsto \exp(a(\log(1+x))^c)$ is also slowly varying for every $a \in \mathbf{R}$ and $c < 1$.

We then define the homogeneous pinning polymer law exactly as in (1.2). In all what follows, we will denote by $\tau_{(N)}$ the set $(\tau \cap [0, N])/N \subset [0, 1]$. When we use this notation, it will be implicit in the sequel that the set τ has the law $\mathbf{P}_{N, \beta}$, where β will be obvious from the context.

This model has been essentially completely solved, see [12]; basic facts about it are the existence of a localization delocalization transition for a critical β , whose value is given by $\beta_c = -\log(\Sigma_K)$, where $\Sigma_K := \sum_{n \geq 1} K(n)$. In particular, we have the equivalence $\beta_c = 0$ iff the underlying renewal process is recurrent. This transition is actually defined in terms of the free energy, which is defined exactly as in the previous sections. It is actually possible to give a very intuitive description of both phases in terms of the scaling limit of the zero level set. Intuitively, if the system is localized, we expect the set $\tau_{(N)}$ to converge in a suitable way to the full interval $[0, 1]$. Similarly, if $\beta < \beta_c$, we expect it to converge to $\{0\}$.

To make such statements quantitative, we introduce \mathcal{F} the set of closed subsets of \mathbf{R}^+ . We endow it with the topology of Matheron, which can be described as follows. For $F \in \mathcal{F}$ and $t \in \mathbf{R}^+$, we set $d_t(F) := \inf(F \cap (t, \infty))$. Notice that $t \mapsto d_t(F)$ is a right-continuous function and that F can actually be identified with $d_{(\cdot)}(F)$ because $F = \{t \in \mathbf{R}^+ : d_{t-}(F) = t\}$. Then in terms of $d_{(\cdot)}(F)$, the Matheron topology is the standart Skorohod topology on càdlàg functions taking values in $\mathbf{R}^+ \cup \{+\infty\}$. We point out that with this topology, the space \mathcal{F} is metrizable, separable and compact, hence in particular Polish. We note $\rho(\cdot)$ the distance metrizing this topology, the Hausdorff metric, and we have, for $F_1, F_2 \in \mathcal{F}$, $\rho(F_1, F_2) := \max_{i \in \{(1,2), (2,1)\}} \sup_{t \in F_{i_1}} \inf_{s \in F_{i_2}} |t - s|$. Using this topology, both of the above statements have actually been proven in [12]. The scaling limit at criticality (that is when $\beta = \beta_c$) is actually much richer than that. To describe it, we will need the notion of *regenerative sets*.

2.2 Regenerative sets and subordinators. Let $(\Omega, \mathbf{P}, (\mathcal{M}_t)_{t \geq 0})$ be a probability space endowed with a complete filtration. Let \mathcal{S} be a progressively measurable subset of $[0, \infty)$ (that is a random subset of $[0, \infty)$ such that $\mathcal{S} \cap [0, t]$ is measurable with respect to $\mathcal{M}_t \otimes \mathcal{B}(\mathbf{R})$ for $t \geq 0$) containing 0 and almost surely without isolated points. The set \mathcal{S} is a regenerative set if, $\forall s \geq 0$, and conditionally on the event $\{\mathcal{D}_s := \inf\{t > s; t \in \mathcal{S}\} < \infty\}$, the set $\mathcal{S} \circ \Theta_{\mathcal{D}_s}$ is independent of $\mathcal{M}_{\mathcal{D}_s}$ and is equal in

law to \mathcal{S} .

Recall that a subordinator is a non decreasing Lévy process. A well-known fact about a subordinator $(\sigma_t)_{t \geq 0}$ is the existence of the so-called Lévy Khintchine exponent $\Phi(\cdot)$, that is a measurable function verifying $\forall \lambda > 0, \forall t \geq 0, \mathbf{E} [e^{-\lambda \sigma_t}] = e^{-t\Phi(\lambda)}$, and there exist non negative real numbers k, d and a measure Π (its Lévy measure) on $(0, \infty)$ verifying $\int_{(0, \infty)} \min(1, x) \Pi(dx) < \infty$ such that $\Phi(t) = k + dt + \int_0^\infty (1 - e^{-xt}) \Pi(dx)$. Furthermore, k, d and Π are unique. An important class of subordinators are the *stable* α -subordinators where $\alpha \in (0, 1)$, that is the subordinator whose Lévy Khintchine exponent is given by $\Phi(t) = t^\alpha$; its Lévy measure is then given by $\alpha/\Gamma(1 - \alpha)x^{-1-\alpha}dx \mathbb{1}_{x>0}$. We will denote such a subordinator by $(\sigma_s^{(\alpha)})_{s \geq 0}$ in what follows. We already introduced its *local time* in the introduction (see (1.6)).

It was shown in [17] that there is a one-to-one correspondence between regenerative sets and the set of subordinators whose Lévy exponent verifies $\Phi(1) = 1$, that is given such a subordinator $(\sigma_s)_{s \geq 0}$, the set $\overline{\{\sigma_s, s \geq 0\}}$ is regenerative (where of course for a subset $A \subset \mathbf{R}, \overline{A}$ is the closure of A), and every regenerative set can be obtained in such a way. We then define the set $\mathcal{A}_\alpha (\in \mathcal{F})$ the *regenerative set* of index α , which corresponds to $(\sigma_s^{(\alpha)})_{s \geq 0}$.

2.3 Scaling limits at and near criticality. In [12], it was shown that, if $\beta = \beta_c$, we have the convergence:

$$\tau_{(N)} \Rightarrow \mathcal{A}_\alpha \quad (3.2)$$

in the recurrent case (that is $K(\infty) = 0$), and in the transient case

$$\tau_{(N)} \Rightarrow \tilde{\mathcal{A}}_\alpha \quad (3.3)$$

where $\tilde{\mathcal{A}}_\alpha$ is a random subset of $[0, 1]$ whose law is absolutely continuous with respect to the law of \mathcal{A}_α with Radon-Nykodym density equal to $(\alpha\pi / \sin(\alpha\pi)) (1 - \max(\mathcal{A}_\alpha \cap [0, 1]))^\alpha$.

A first step towards this convergence has been made in [14] in a discrete random walk set-up; they actually proved the convergence of the entire process at criticality towards the brownian motion. This work has been strongly generalized in [8], actually being the most important part of the proof of the convergence of more general pinning models towards the reflected brownian motion at criticality. This was in turn generalized in [4] using powerful renewal techniques, on which the present work is based.

Let us focus for the moment on the recurrent case, that is we assume for the moment $K(\infty) = 0$. First of all, we recall the following result (see [9], XIII.6 Theorem 2): it is possible to choose a sequence $(a_n)_{n \geq 0}$ such that

$$\frac{n\Gamma(1 - \alpha)L(a_n)}{\alpha a_n^\alpha} \rightarrow 1 \quad (3.4)$$

and furthermore, as soon as a sequence $(a_n)_{n \geq 0}$ verifies (3.4), we have the convergence :

$$\frac{\sum_{i=1}^n l_i}{a_n} \Rightarrow \sigma_1^{(\alpha)} \quad (3.5)$$

where the $(l_i)_{i \geq 0}$ are iid with common law $K(\cdot)$ satisfying (3.1). We then define the sequence $(b_n)_{n \geq 0}$ to be the inverse sequence of the $(a_n)_{n \geq 0}$, that is the unique sequence, up to asymptotic equivalence, which verifies

$$a_{[b_n]} \sim b_{[a_n]} \sim n \quad (3.6)$$

as $n \rightarrow \infty$. Its existence is ensured by Theorem 1.5.12, page 28 of [3]. It is rather easy to see that $b_n \sim (\Gamma(1 - \alpha)\overline{K}(n))^{-1}$ as $n \rightarrow \infty$. Let us quickly prove this fact. Using the definition of $(a_n)_{n \geq 0}$, we get

$$\frac{b_n \Gamma(1 - \alpha) L(a_{[b_n]})}{\alpha a_{[b_n]}^\alpha} \sim \frac{b_n \Gamma(1 - \alpha) L(n)}{\alpha n^\alpha} \sim 1 \quad (3.7)$$

and then we note that $\overline{K}(N) \sim L(N)(\alpha N^\alpha)^{-1}$ (see [12], P.201). We also used the fact that $x_n \sim y_n$ yields $L(x_n) \sim L(y_n)$ as $n \rightarrow \infty$ for $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ positive sequences. For n large enough, $x_n/y_n \in [1 - \eta; 1 + \eta]$ for a given $\eta > 0$, and the fact that the convergence $L(cx)/L(x) \rightarrow 1$ is uniform in c on every compact subset of \mathbf{R}^+ (a basic result on slowly varying functions, see [3]) yields the assumption.

To deal with the transient case, we introduce the recurrent law $\tilde{\mathbf{P}}$ given by

$$\tilde{\mathbf{P}}(l_1 = n) := \tilde{K}(n) = \frac{L(n)e^{\beta c}}{n^{1+\alpha}} = \frac{L(n)}{n^{1+\alpha}\Sigma_K}. \quad (3.8)$$

We have then exactly the same statements as in Eq.(3.4) and (3.5) making the obvious changes, and we introduce the sequence $(\tilde{b}_n)_{n \geq 0}$, the analogous of $(b_n)_{n \geq 0}$ for the new sequence $(\tilde{a}_n)_{n \geq 0}$. We are now ready to state our main result:

Theorem 2. *Let $\varepsilon \in \mathbf{R}$. The following statements hold:*

- (1) *if $K(\infty) = 0$, if τ is distributed according to $\mathbf{P}_{\beta c + \varepsilon/b_N, N}$, then*

$$\tau_{(N)} \Rightarrow \mathcal{B}_{\alpha, \varepsilon} \quad (3.9)$$

where $\mathcal{B}_{\alpha, \varepsilon}(\subset [0, 1])$ is the random set whose law is absolutely continuous with respect to the law of \mathcal{A}_α with Radon-Nykodym density equal to $\exp(\varepsilon L_1^{(\alpha)})/\mathbf{E}[\exp(\varepsilon L_1^{(\alpha)})]$.

- (2) *if $K(\infty) > 0$, if τ is distributed according to $\mathbf{P}_{\beta c + \varepsilon/\tilde{b}_N, N}$, then*

$$\tau_{(N)} \Rightarrow \tilde{\mathcal{B}}_{\alpha, \varepsilon} \quad (3.10)$$

where $\tilde{\mathcal{B}}_{\alpha, \varepsilon}(\subset [0, 1])$ has a Radon-Nykodym density given by

$$\left(\exp(\varepsilon L_1^{(\alpha)}) (1 - \max(\mathcal{A}_\alpha \cap [0, 1]))^\alpha \right) / \left(\mathbf{E}[\exp(\varepsilon L_1^{(\alpha)}) (1 - \max(\mathcal{A}_\alpha \cap [0, 1]))^\alpha] \right).$$

4. PROOFS

Lemma 1. *If $K(\infty) = 0$, the following convergence holds:*

$$\frac{|\tau_{(N)}|}{b_N} \Rightarrow L_1^{(\alpha)}. \quad (4.1)$$

Proof. For $M > 0$, we have the equality:

$$\mathbf{P}\left(\frac{|\tau_{(N)}|}{b_N} \leq M\right) = \mathbf{P}\left(\frac{1}{a_{[Mb_N]}} \sum_{i=1}^{[Mb_N]} l_i \geq \frac{N}{a_{[Mb_N]}}\right). \quad (4.2)$$

The right hand side of (4.2) is easily seen to converge towards $\mathbf{P}\left(M^{\frac{1}{\alpha}} \sigma_1^{(\alpha)} \geq 1\right)$. Actually, we have $N/a_{[Mb_N]} \rightarrow M^{-\frac{1}{\alpha}}$ as $N \rightarrow \infty$ (since $Mb_N \sim b_{[M^{\frac{1}{\alpha}} N]}$), and it is easy to see that, if $X_n \Rightarrow X$, if a deterministic sequence $u_n \rightarrow u$ and the distribution of X is continuous,

then $\mathbf{P}(X_n \geq u_n) \rightarrow \mathbf{P}(X \geq u)$. So we just have to recall that the distribution of $\sigma_1^{(\alpha)}$ is continuous, which is actually a well known fact about stable subordinators (see [1], P.271).

Because of the scaling property of the stable subordinator, we get

$$\mathbf{P}\left(M^{\frac{1}{\alpha}}\sigma_1^{(\alpha)} \geq 1\right) = \mathbf{P}(\sigma_M^{(\alpha)} \geq 1) \quad (4.3)$$

$$= \mathbf{P}(L_1^{(\alpha)} \leq M) \quad (4.4)$$

which proves our claim. \square

Lemma 2. *For $K(\infty) = 0$ and for all $\varepsilon > 0$, the family $(\exp(\frac{\varepsilon|\tau(N)|}{b_N}))_{N \geq 0}$ is uniformly integrable.*

Proof. It is enough to show that, for all $a > 0$, we have

$$\sup_{N \in \mathbf{N}} \mathbf{E} \left[e^{a|\tau(N)|/b_N} \right] < \infty. \quad (4.5)$$

We note that $\mathbf{E} \left[e^{a|\tau(N)|/b_N} \right] = \int_1^\infty \mathbf{P} \left[|\tau(N)|/b_N > 1/a \log(x) \right] dx$. We have, for $y \geq 0$:

$$\mathbf{P} \left(\frac{|\tau(N)|}{b_N} > y \right) = \mathbf{P} \left(\sum_{i=1}^{\lfloor yb_N \rfloor} l_i < N \right) \quad (4.6)$$

Then, for every $\lambda > 0$, using the Markov inequality, we see that the last term is smaller than:

$$\mathbf{E} \left[e^{-\lambda l_i} \right]^{yb_N} e^{\lambda N}. \quad (4.7)$$

For $\lambda \searrow 0$, we have $-\log(\mathbf{E}[e^{-\lambda l_1}]) \sim \Gamma(1-\alpha)/\alpha \lambda^\alpha L(1/\lambda)$. This was shown in ([12]) to prove (A.48). This entails that there exists $c(\alpha) > 0$ such that for all $\lambda \in (0, 1)$, we have $-\log(\mathbf{E}[e^{-\lambda l_1}]) \leq c(\alpha)\lambda^\alpha L(1/\lambda)$. Then, we get, for $N \rightarrow \infty$,

$$\mathbf{P} \left(\frac{|\tau(N)|}{b_N} > y \right) \leq \exp \left(-c(\alpha)\lambda^\alpha L(1/\lambda)y \frac{N^\alpha}{L(N)} + \lambda N \right) \quad (4.8)$$

possibly modifying $c(\alpha)$ in doing so, and of course using the fact that $(b_N L(N) N^{-\alpha})_{N \geq 0}$ is a positive bounded sequence. Using the slowly varying character of $L(\cdot)$, we get that for any $C_1 > 0$, there exists $C_2 > 0$ such that the term in the right hand side of (4.8) is smaller than $C_2 \exp(-C_1 y)$ as soon as N is large enough, which entails (4.5). \square

Lemma 3. *For $K(\infty) = 0$, we have the convergence:*

$$(|\tau(N)|/b_N, \tau(N)) \Rightarrow (L_1^{(\alpha)}, \mathcal{A}_\alpha). \quad (4.9)$$

Proof. We introduce the process $(N_{\gamma_N}(t))_{t \geq 0}$, a Poisson process of rate γ_N where we define $\gamma_N := \left(\sum_{n \geq 1} (1 - e^{-n/N}) K(n) \right)^{-1}$. We take $(N_{\gamma_N}(t))_{t \geq 0}$ independent from τ . It is easy to see that the process $(\tau_{N_{\gamma_N}(t)}/N)_{t \geq 0}$ is a subordinator. In [12], it was shown that the Lévy exponent of $(\tau_{N_{\gamma_N}(t)}/N)_{t \geq 0}$ converges to the one of $(\sigma_s^{(\alpha)})_{s \geq 0}$, which implies $\overline{\{\tau_{N_{\gamma_N}(s)}/N, s \geq 0\}} \Rightarrow \mathcal{A}_\alpha$ as well as the convergence in law of the entire process $(\tau_{N_{\gamma_N}(t)}/N)_{t \geq 0}$ towards $(\sigma_s^{(\alpha)})_{s \geq 0}$ (see [11]) in the Polish space \mathcal{D} . Here, \mathcal{D} denotes the space of càdlàg functions endowed with the standart Skorokhod topology (that is

$d(f, g) := \inf_{\lambda \in \Lambda} \{\max(\|\lambda - id\|, \|f \circ \lambda - g\|\}$ where Λ is the set of homeomorphisms from $[0, 1]$ to $[0, 1]$.

We define the function $(F, G) : \mathcal{D} \rightarrow \mathbf{R} \times \mathcal{F}$ by

$$(F, G)(f) : f \mapsto (\sup \{s \geq 0 \mid f(s) \leq 1\}, \text{Im}(f)) \quad (4.10)$$

where $\text{Im}(f)$ is the image set of f . For the space $\mathbf{R} \times \mathcal{F}$, we use the distance $\max(\|\cdot\|, \rho)$ where ρ is the Hausdorff metric. We will show that, for all continuous bounded $\mathcal{G} : \mathbf{R} \times \mathcal{F} \rightarrow \mathbf{R}$, we have:

$$\mathbf{E} \left[\mathcal{G} \left(\frac{|\tau_{(N)}|}{b_N}, \tau_{(N)} \right) \right] = \mathbf{E} \left[\mathcal{G} \left((F, G) \left(\left(\frac{\tau_{N\gamma_N(s)}}{N} \right)_{s \geq 0} \right) \right) \right] + (o_N(1)), \quad (4.11)$$

that is that we can consider the joint law in the left hand side of (4.9) as a function of $(\tau_{N\gamma_N(s)}/N)_{s \geq 0}$ up to negligible corrections. Note that the second arguments of \mathcal{G} in (4.11) are the same almost surely with respect to the law of $(N_{\gamma_N(t)})_{t \geq 0}$.

Assume for the moment that (4.11) is true. Taking into account the convergence $(\tau_{N\gamma_N(t)}/N)_{t \geq 0} \Rightarrow (\sigma_s^{(\alpha)})_{s \geq 0}$ and the fact that $(F, G) \left((\sigma_s^{(\alpha)})_{s \geq 0} \right)$ is exactly the limiting law we want, it is enough to show that $(F, G)(\cdot)$ is almost surely continuous at $(\sigma_s^{(\alpha)})_{s \geq 0}$ to apply the continuous mapping theorem (see [2]) to prove our lemma. So let us show this almost sure continuity first, then we will turn to the proof of (4.11).

For $\eta \in (0, 1)$, we denote by \mathcal{B}_η the subset of \mathcal{D} defined by

$$\mathcal{B}_\eta := \{f \in \mathcal{D} \mid [1 - \eta; 1 + \eta] \cap \text{Im}(f) = \emptyset\}. \quad (4.12)$$

It is then well known that $\mathbf{P}(1 \in \mathcal{A}_\alpha) = 0$ (see [16]), so that with probability one, $(\sigma_s^{(\alpha)})_{s \geq 0} \in \cup_{\eta > 0} \mathcal{B}_\eta$. We show that (F, G) restricted to $\cup_{\eta > 0} \mathcal{B}_\eta$ is continuous. As the topology put on the image set is the product topology $(\mathcal{B}(\mathbf{R}) \otimes \mathcal{F})$, we just have to show the continuity for each component of our function.

Let $f \in \mathcal{B}_\eta$ with $\eta > 0$, let us show that F is continuous at f . We introduce $s_f := \sup \{s \geq 0 \mid f(s) \leq 1\}$. Let $g \in \mathcal{D}$ such that $d(f, g) < \mu/2$ where $\mu \in (0, \eta)$, and in the same way we define $s_g := \sup \{s \geq 0 \mid g(s) \leq 1\}$. We consider $\lambda \in \Lambda$ such that $\|f - g \circ \lambda\| < \mu$ and $\|\lambda - id\| < \mu$. Furthermore, for all $\xi > 0$, and as $f \in \mathcal{B}_\eta$, we have $f(s_f - \xi) < 1 - \eta$ and $f(s_f) > 1 + \eta$. Then we get

$$g(\lambda(s_f - \xi)) \leq f(s_f - \xi) + \mu \leq 1 - \eta + \mu < 1, \quad (4.13)$$

and similarly

$$g(\lambda(s_f)) \geq f(s_f) - \mu > 1 + \eta - \mu > 1, \quad (4.14)$$

which gives

$$\lambda(s_f - \xi) \leq s_g \leq \lambda(s_f) \quad (4.15)$$

and this entails the desired continuity since ξ is arbitrarily small, λ is continuous and $\|\lambda - id\| < \mu$.

Note that, by the virtue of the continuous mapping's theorem, this implies the convergence

$$\sup \left\{ s \geq 0, \frac{\tau_{N\gamma_N(s)}}{N} \leq 1 \right\} \Rightarrow L_1^{(\alpha)}. \quad (4.16)$$

For the second component, it is very easy also; for g as above, we have:

$$\rho(\text{Im}(f), \text{Im}(g)) = \max \left(\sup_{s \in \mathbf{R}} \inf_{t \in \mathbf{R}} |f(t) - g(s)|, \sup_{s \in \mathbf{R}} \inf_{t \in \mathbf{R}} |f(s) - g(t)| \right). \quad (4.17)$$

And for λ as above and $s \in \mathbf{R}$, we have $\inf_{t \in \mathbf{R}} |f(s) - g(t)| \leq |f(s) - g(\lambda(s))| \leq \mu$, which achieves the proof of the continuity of the couple (F, G) on $\cup_{\eta > 0} \mathcal{B}_\eta$.

We now go to the proof of (4.11). Let $\delta > 0$, there exists a compact set $K_\delta \subset \mathbf{R}^+$ such that for all $N \in \mathbf{N}$, we have:

$$\max \left(\mathbf{P} \left[\frac{|\tau(N)|}{b_N} \in K_\delta^c \right], \mathbf{P} \left[\sup \left\{ s \geq 0 \mid \frac{\tau_{N\gamma_N}(s)}{N} \leq 1 \right\} \in K_\delta^c \right] \right) < \delta. \quad (4.18)$$

This is due to the fact that the sequences $\left(\sup \left\{ s \geq 0 \mid \tau_{N\gamma_N}(s)/N \leq 1 \right\} \right)_{N \geq 0}$ and $\left(\sup \left\{ \frac{n}{b_N} \mid \frac{|\tau_n|}{N} \leq 1 \right\} \right)_{N \geq 0}$ are tight because they converge in law (see (4.16) and Lemma 1). Similarly, the convergence of the sequence $(\tau(N))_{N \geq 0}$ implies the existence of a compact set $L_\delta \subset \mathcal{F}$ such that for every $N \in \mathbf{N}$,

$$\mathbf{P} [\tau(N) \in L_\delta^c] < \delta. \quad (4.19)$$

For $N \in \mathbf{N}$, we introduce the event

$$\mathcal{H}_{N,\delta} := \left\{ \frac{|\tau(N)|}{b_N} \in K_\delta, \sup \left\{ s \geq 0 \mid \frac{\tau_{N\gamma_N}(s)}{N} \leq 1 \right\} \in K_\delta, \tau(N) \in L_\delta \right\}. \quad (4.20)$$

Eqs. (4.18) and (4.19) say that $\mathbf{P} [\mathcal{H}_{N,\delta}^c] < \delta$.

Let $M > 0$ be an upper bound for $|\mathcal{G}|$. For every $N \in \mathbf{N}$ and taking into account the fact that $\gamma_N \sim b_N$ (see [12], (A.48)), we have:

$$\begin{aligned} & \left| \mathbf{E} \left[\mathcal{G} \left(\sup \left\{ s \geq 0 \mid \frac{\tau_{N\gamma_N}(s)}{N} \leq 1 \right\}, \tau(N) \right) - \mathcal{G} \left(\frac{|\tau(N)|}{\gamma_N}, \tau(N) \right) \right] \right| \\ & \leq \mathbf{E} \left[\left| \mathcal{G} \left(\sup \left\{ s \geq 0 \mid \frac{\tau_{N\gamma_N}(s)}{N} \leq 1 \right\}, \tau(N) \right) - \mathcal{G} \left(\frac{|\tau(N)|}{\gamma_N}, \tau(N) \right) \right| \mathbb{1}_{\mathcal{H}_{N,\delta}} \right] + 2M\delta \\ & \leq \mathbf{E} \left[\left| \mathcal{G} \left(\sup \left\{ s \geq 0 \mid \frac{\tau_{N\gamma_N}(s)}{N} \leq 1 \right\}, \tau(N) \right) \right. \right. \\ & \quad \left. \left. - \mathcal{G} \left(\sup \left\{ \frac{N\gamma_N(s)}{\gamma_N} - s + s \mid \frac{\tau_{N\gamma_N}(s)}{N} \leq 1 \right\}, \tau(N) \right) \right| \mathbb{1}_{\mathcal{H}_{N,\delta}} \right] + 2M\delta. \end{aligned} \quad (4.21)$$

For a given $\kappa > 0$, we introduce the event $\mathcal{J}_{\kappa,\delta}$ defined by:

$$\mathcal{J}_{\kappa,\delta} := \left\{ \sup_{s \in K_\delta} \left| \frac{N\gamma_N(s)}{\gamma_N} - s \right| \geq \kappa \right\}. \quad (4.22)$$

Note that the process $\left(\frac{N\gamma_N(s)}{\gamma_N} - s \right)_{s \geq 0}$ is a martingale, so that, applying Doob's inequality, we get:

$$\begin{aligned} \mathbf{P} [\mathcal{J}_{\kappa,\delta}] &= \mathbf{P} \left[\sup_{s \in K_\delta} \left| \frac{N\gamma_N(s)}{\gamma_N} - s \right|^2 \geq \kappa^2 \right] \\ &\leq \frac{1}{\kappa^2} \sup_{s \in K_\delta} \mathbf{E} \left[\left| \frac{N\gamma_N(s)}{\gamma_N} - s \right|^2 \right] = \frac{\max K_\delta}{\gamma_N \kappa^2}. \end{aligned} \quad (4.23)$$

Cutting once again the expectation appearing in Eq.(4.21) with respect to the fact that $\mathcal{J}_{\kappa,\delta}$ occurs or not, we get:

$$\begin{aligned}
& \left| \mathbf{E} \left[\mathcal{G} \left(\sup \left\{ s \geq 0 \left| \frac{\tau_{N\gamma_N}(s)}{N} \leq 1 \right. \right\}, \tau_{(N)} \right) - \mathcal{G} \left(\frac{|\tau_{(N)}|}{\gamma_N}, \tau_{(N)} \right) \right] \right| \\
& \leq \mathbf{E} \left[\sup_{|u| \leq \kappa} \left| \mathcal{G} \left(\sup \left\{ s \geq 0 \left| \frac{\tau_{N\gamma_N}(s)}{N} \leq 1 \right. \right\}, \tau_{(N)} \right) \right. \right. \\
& \quad \left. \left. - \mathcal{G} \left(u + \sup \left\{ s \geq 0 \left| \frac{\tau_{N\gamma_N}(s)}{N} \leq 1 \right. \right\}, \tau_{(N)} \right) \right| \right] \mathbb{1}_{\mathcal{H}_{N,\delta}} \mathbb{1}_{\mathcal{J}_{\kappa,\delta}^c} \\
& \quad + 2M \left(\delta + \frac{\max K_\delta}{\gamma_N \kappa^2} \right).
\end{aligned} \tag{4.24}$$

In the last inequality, we consider first κ very small, which deals with the expectation term due to the uniform continuity of \mathcal{G} on the compact set $K_\delta \times L_\delta$. Then, we consider δ small enough, and this achieves the proof of (4.11), since $\gamma_N \xrightarrow{N \rightarrow \infty} \infty$. Thus the lemma is proved. \square

We are now ready to show the main part of our theorem.

Proof of Theorem 2. We have just shown the following convergence: under the law $K(\cdot)$, assuming $K(\infty) = 0$, for all F continuous bounded function on \mathcal{F} and for all $\varepsilon \in \mathbf{R}$, we have:

$$\mathbf{E} \left[F(\tau_{(N)}) e^{\varepsilon |\tau_{(N)}|/b_N} \right] \xrightarrow{N \rightarrow \infty} \mathbf{E} \left[F(\mathcal{A}_\alpha) e^{\varepsilon L_1^{(\alpha)}} \right]. \tag{4.25}$$

Note that the case where $\varepsilon > 0$ actually uses the fact (see Lemma 2) that the sequence $(e^{\varepsilon |\tau_{(N)}|/b_N})_{N \geq 0}$ is uniformly integrable.

We then have

$$\mathbf{E} [F(\tau_{(N)})] = \frac{Z_{N,\beta_c}}{Z_{N,\beta_c + \varepsilon/b_N}} \times \frac{1}{Z_{N,\beta_c}} \mathbf{E} \left[F(\tau_{(N)}) e^{\varepsilon |\tau_{(N)}|/b_N} \right]. \tag{4.26}$$

As in the case $K(\infty) = 0$, \mathbf{P}_{N,β_c} is actually \mathbf{P} (because $\beta_c = -\log(\Sigma_K) = 0$) and thanks to the easy remark that $Z_{N,\beta_c + \varepsilon/b_N} = \mathbf{E} \left[e^{\varepsilon |\tau_{(N)}|/b_N} \right] \xrightarrow{N \rightarrow \infty} \mathbf{E} \left[e^{\varepsilon L_1^{(\alpha)}} \right]$, we have actually shown the first part of the claim. We also used the fact that $Z_{N,\beta_c} = 1$ for all $N \in \mathbf{N}$ see ([12], (2.17)).

We now deal finally with the case $K(\infty) > 0$; we can write:

$$\mathbf{E}_{N,\beta_c + \varepsilon/\tilde{b}_N} [F(\tau_{(N)})] = \tilde{\mathbf{E}} \left[F(\tau_{(N)}) e^{\varepsilon |\tau_{(N)}|/\tilde{b}_N} \frac{\overline{K}(N(1 - \max(\tau_{(N)}))) + K(\infty)}{Z_{N,\beta_c + \varepsilon/\tilde{b}_N} \sum_{j > N(1 - \max(\tau_{(N)}))} \tilde{K}(j)} \right]. \tag{4.27}$$

The justification for (4.27) can be found in [12], equality (2.48). Then, on the event $\max \tau_{(N)} < 1 - \eta$ for a given $\eta \in (0, 1)$, it is known that ((2.50) of [12])

$$\frac{\overline{K}(N(1 - \max(\tau_{(N)}))) + K(\infty)}{Z_{N,\beta_c} \sum_{j > N(1 - \max(\tau_{(N)}))} \tilde{K}(j)} \underset{N \rightarrow \infty}{\sim} \frac{\alpha\pi}{\sin(\alpha\pi)} (1 - \max(\tau_{(N)}))^\alpha. \tag{4.28}$$

uniformly in the trajectories of τ . Using once again the convergence of the joint law $(\tau_{(N)}, |\tau_{(N)}|/\tilde{b}_N)$, where this time τ is distributed according to $\tilde{\mathbf{P}}$, we get :

$$\mathbf{E}_{N, \beta_c + \varepsilon \tilde{b}_N} \left[F(\tau_{(N)}) \mathbb{I}_{\max \tau_{(N)} < 1 - \eta} \right] \xrightarrow{N \rightarrow \infty} \mathbf{E} \left[F(\mathcal{A}_\alpha) e^{\varepsilon L_1^{(\alpha)}} \mathbb{I}_{\max \mathcal{A}_\alpha < 1 - \eta} \frac{\alpha \pi}{\sin(\alpha \pi)} (1 - \max(\mathcal{A}_\alpha))^\alpha \right]. \quad (4.29)$$

The fact that $\max(\mathcal{A}_\alpha) < 1$ almost surely allows us to take the above limit for $\eta \searrow 0$. Taking $F \equiv 1$ in (4.29), we get

$$\frac{Z_{N, \beta_c + \varepsilon \tilde{b}_N}}{Z_{N, \beta_c}} \xrightarrow{N \rightarrow \infty} \mathbf{E} \left[e^{\varepsilon L_1^{(\alpha)}} \frac{\alpha \pi}{\sin(\alpha \pi)} (1 - \max(\mathcal{A}_\alpha))^\alpha \right]. \quad (4.30)$$

This achieves the proof because we can write

$$\begin{aligned} & \mathbf{E}_{N, \beta_c + \varepsilon \tilde{b}_N} [F(\tau_{(N)})] \\ &= \frac{Z_{N, \beta_c}}{Z_{N, \beta_c + \varepsilon \tilde{b}_N}} \tilde{\mathbf{E}} \left[F(\tau_{(N)}) e^{\varepsilon |\tau_{(N)}|/\tilde{b}_N} \frac{\overline{K}(N(1 - \max(\tau_{(N)}))) + K(\infty)}{Z_{N, \beta_c} \sum_{j > N(1 - \max(\tau_{(N)}))} \tilde{K}(j)} \right] \end{aligned} \quad (4.31)$$

and the term in the right hand side converges towards the desired quantity. \square

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