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A Theory of Measuring, Electing and Ranking

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Résumé: Le modèle traditionnel du choix social admet une multitude d'impossibilités et n'a abouti à aucune méthode incontestable pour élire un candidat ou ranger des compétiteurs. Un nouveau modèle permet de contourner les impossibilités avec une méthode éminemment pratique, "la valeur-majoritaire".

Abstract: The traditional model of the theory of social choice admits a host of impossibility theorems and has led to no satisfactory methods for electing candidates or ranking competitors. A new model allows the impossibilities to be avoided with an eminently practical method: "the majority-value."

Mots clés : Choix social, théorème d'impossibilité d'Arrow, non-manipulabilité, fonctions de notation-sociale, fonctions de rangement-social, note-majoritaire, rangement majoritaire, valeur-majoritaire.

Key Words : Social choice, Arrow's impossibility theorem, strategy-proofness, social-grading functions, social-ranking functions, majority-grade, majority-ranking, majority-value.

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Introduction

The theory of social choice concerns methods for amalgamating the appreciations or evaluations of many individuals into one collective appreciation or evaluation. It has two principal applications. (1) Voting: electors in a democracy choose one among several candidates, or committee members decide on one among several courses of action. (2) Jury decisions: judges evaluate competitors—figure skaters, gymnasts, pianists, wines, ...—and rank them or classify them by level of excellence.[1]

The fundamental problem is to find a *social decision function (SDF)* whose *inputs* are *messages* of judges or voters and whose *outputs* are the jury or electoral *decisions*, usually rank-orderings of competitors and winners. Much of the theory of social choice has blurred the distinction between a judge's complex aims, ends, purposes and wishes—in short, his or her *preferences* or *utilities*—and the messages he or she is allowed to send.[2]

In the traditional model, consecrated by some seven centuries of use,[3] each individual judge's or voter's rank-ordering of the competitors is at once his or her message *and* his or her preferences. Does it mean that the judge prefers this

[1] Our thesis is capsuled in this article. A complete account, including proofs, many other results and references, is given in the forthcoming book, Michel Balinski and Rida Laraki, *One-Value, One-Vote: Measuring, Electing and Ranking.*

[2] The word "preferences" contains a misleading connotation: voters do not merely express what they prefer, they may well express what they believe is right (*e.g.*, R. Goodin and K. Roberts, "The ethical voter," *American Political Science Review* 69 (1975) 926-928); a judge in a court of justice is supposed to evaluate conformity with the law, not merely express his preferences. The real, deep preferences of a judge or voter cannot come close to being expressed in a rank-ordering of competitors.

[3] Ramon Lull proposed a refinement of Condorcet's method in 1297: it is known today as Copeland's method. Nicolaus Cusanus put forth what is today known as Borda's method in 1433. See, I. McLean, "The Borda and Condorcet principles: three medieval applications," *Social Choice and Welfare* 7 (1990) 99-108; G. Hägele and F. Pukelsheim, "Llull's writings on electoral systems," *Studia Lulliana* 41 (2001) 3-38; Gunter Hägele and Friedrich Pukelsheim, "The electoral systems of Nicolaus Cusanus," to appear in Hg. G. Christianson, T.M. Izbicki, and C.M. Bellitto, eds., *The Church, the Councils and Reform: Lessons from the Fifteenth Century*, Catholic University of America Press, Washington, DC.

rank-ordering above all others; or, that the judge wishes the first competitor on his list to be the winner, the second to be the winner if the first is not, the third to be the winner if the first two are not, and so on down the list; or, is the rank-ordering required and chosen strategically by the judge given his or her "true" rank-ordering.

In the real world, a judge's message is simply a message, nothing more. It depends on the judge's preferences, but it is not and cannot be his or her preferences. And in the real world, a judge's or a voter's preferences or utilities depends on a host of factors that include the decision (or output), the messages of the other judges (a judge or voter may wish to differ from the others, or on the contrary resemble the others), the social decision function that is used (a judge may prefer a decision given by "democratic" function to one rendered by an "oligarchique" function, or the contrary) and the message he or she thinks is the right one (a judge may prefer honest behavior, or not).

Kenneth Arrow, in the first deep theoretical analysis of the theory of social choice, uses the traditional model: each judge's input message is a rank-ordering, routinely interpreted to be a complete expression of his "preferences" (strategic considerations are absent); the output is a rank-ordering and a winner (the first-ranked competitor of the order). His celebrated "impossibility" theorem shows that there exists no *social welfare function SWF* satisfying three reasonable properties for obtaining a decision given any inputs (unless there are only two competitors).[4] Donald Saari's algebraic analysis explains why and how the traditional model cannot but lead to paradoxical outcomes.[5] Amartya Sen models each judge's inputs as a numerical "utility" over the competitors, *i.e.*, the judge assigns a real number to every competitor; the output is a rank-ordering whose utility to a judge is not specified. The model has theoretical interest but no practical significance because a voter's individual utility is a much more complex concept. In any case, Arrow's theorem emerges again unless the utilities are assumed to be comparable (that bugabear of economists!).[6] The model used to prove the well-known Gibbard-Satterthwaite impossibility theorem assumes the output is a winner (indeed, how could preferences be modelled if the output were a rank-ordering?); each judge has "true" preferences expressed as a rank-ordering; but a judge's input is a strategically chosen rank-ordering. The theorem states that there exists no SWF that makes it a dominant strategy for every judge to report his true preferences.[7]

Refined, extended, and reformulated in many variants, the traditional approach has continued to produce a host of related impossibility theorems. We add to this list a negative theorem of a new kind: a fundamental incompatibility between winners and rank-orderings as outputs of the traditional model. It

[4]Kenneth J. Arrow, *Social Choice and Individual Values*, John Wiley & Sons, Inc., 1951.

[5]Donald Saari, "Mathematical structure of voting paradoxes. I. Pairwise votes, and II. Positional voting," *Economic Theory* 15 (2000) 1-53 and 55-102.

[6]Amartya K. Sen, *Collective Choice and Social Welfare*, Holden-Day, 1970.

[7]Allan Gibbard, "Manipulation of voting schemes: a general result," *Econometrica* 41 (1973) 587-601; and Mark A. Satterthwaite, "Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions," *Journal of Economic Theory* 10 (1973) 187-217.

devolves from a simple observation: if the output is to be a rank-ordering and inputs are interpreted as preferences, should not an individual's input message be his preferences over rank-orderings rather than a single rank-ordering?

Given all of these negative results, it is not surprising that the debate over what method of voting *should* be used in practice goes on unabated. By and large, it may be said to pit the supporters of Lull (alias Condorcet) against those of Cusanus (alias Borda), though some argue for a new method, "approval voting,"[8] and diverse hybrids are regularly proposed.

We contend: (1) Arrow's and all the other impossibility and incompatibility results show that the fundamental problem has no acceptable solution in the context of the traditional model. (2) The traditional approach does not adequately model the messages or the purposes of the judges and voters. (3) A new model is necessary.

Practice, curiously enough, suggests a different formulation of the inputs. Olympic competitions in figure skating and gymnastics, wine competitions, competitions among pianists, flautists or orchestras, ..., all use measures or grades. As Lord Kelvin proclaimed, "If you cannot measure, your knowledge is meager and unsatisfactory." Indeed, Arrow[9] himself states "there are essentially two methods by which social choices can be made, voting,... and the market mechanism": the second uses a measure, *price* expressed in terms of money.

A *measure* or *grade* is a message that has strictly nothing to do with a utility. A judge may dislike a wine yet give it a high grade because of its merits; he or she may also like a wine and yet with great satisfaction give it a low grade because of its demerits. A measure provides a *common language*—be it numerical, ordinal or verbal—to grade and to classify. In this perspective, Arrow's theorem means that without a common language there can be no consistent collective decision.

When the messages are grades expressed in a common language then one method of classifying competitors, candidates or alternatives—the *majority-grade*—and one method for ranking them—the *majority-ranking*—emerge as the only ones that satisfy each of various desirable properties. They are compatible. Moreover, they best resist strategic manipulations of judges and voters under varying assumptions concerning the judges' and voters' preferences or utilities.

1 The traditional model

There is a finite set of *competitors* (alternatives, candidates, performances, or competing goods) $\mathcal{C} = \{A, \ldots, I, \ldots, Z\}$, and a finite set of n *judges* (or voters) $\mathcal{J} = \{1, \ldots, j, \ldots, n\}$. Each judge's input *message* is a rank-ordering of the competitors. Together all of the input messages constitute a *preference profile* (in keeping with traditional terminology we use the word "preference(s)" as a synonym for rank-orderings in this section). A *social welfare function (SWF)* renders an output—a rank-ordering—for any inputs or preference profile.

[8] See S. J. Brams and P. C. Fishburn, *Approval Voting*, Birkhäuser, Boston, 1983.
[9] Arrow, *op. cit.*, p. 1.

Take $A \succ B$ to mean that a judge ranks A ahead of B, and $A \succ_S B$ to mean that the SWF (or "Society") ranks A ahead of B; in examples an integer followed by a rank-ordering is the number of judges sending that input message.

Condorcet is the first to have realized the essential difficulty of the problem. Consider one of his examples with the 60-judge preference profile:

$$23 : A \succ B \succ C \qquad 2 : B \succ A \succ C \qquad 17 : B \succ C \succ A$$

$$10 : C \succ A \succ B \qquad 8 : C \succ B \succ A$$

If majority rule decides the order between each pair of competitors separately, the result is as follows (with numbers in parentheses giving the respective votes):

$$A(33) \succ_S B(27) \qquad B(42) \succ_S C(18) \qquad C(35) \succ_S A(25)$$

Thus majority decision says $A \succ_S B \succ_S C \succ_S A$: this is the *Condorcet-paradox*, no competitor is favored to all others.

Arrow showed that this is an inescapable conundrum. He imposed three conditions that any SWF should satisfy. *Unanimity*: If every judge prefers competitor A to competitor B, *i.e.*, $A \succ B$, then the SWF ranks A ahead of B, *i.e.*, $A \succ_S B$. *Non-dictatorship*: The input of no one judge can determine the output of the SWF whatever the inputs of all the other judges. *Independence of irrelevant alternatives (IIA)*: whether the SWF yields $A \succ_S B$ or the contrary $A \prec_S B$, depends only on the judges' preferences between A and B. His theorem shows that when there are at least three competitors, there is no SWF that satisfies the three conditions for all possible inputs of the judges.

Nevertheless, people vote and judges rank: how? A judge accords k *Borda-points* to a competitor if k opponents are ranked below him.[10] A competitor's *Borda-score* is the sum of his Borda-points over all judges; equivalently, it is the sum of the votes he receives in all pair by pair votes.[11] The *Borda-ranking* ranks the competitors by their Borda-scores, from highest to lowest; the highest designates the *Borda-winner*. The Borda-ranking for Condorcet's 60-judge example is (each competitor's Borda-score is in parentheses): $B(69) \succ_S A(58) \succ_S C(53)$.

Condorcet attacked Borda's method. His argument was that when there exists a *Condorcet-winner*—a competitor who has a majority against every other competitor—then he must be the winner, a property that Borda's method violates, as the following 81-judge example of Condorcet shows:

$$30 : A \succ B \succ C \qquad 1 : A \succ C \succ B \qquad 29 : B \succ A \succ C$$

[10] Laplace justified the Borda-points by imagining that each judge wishes to assign a positive real score in some interval $[0, R]$ to each competitor but is asked instead to rank them. Laplace computed the average of the lowest points, of the next to lowest, on up to the highest, and found them to be proportional to the Borda-points. See, Pierre-Simon, Marquis de Laplace, *Théorie analytique des probabilités*, 3rd ed., in *Œuvres Complètes de Laplace*, t. 7: pp. v and clii-cliii.

[11] Le Chevalier Jean-Charles de Borda, "Mémoire sur les lections au scrutin," *Histoire de l'Académie royale des sciences* (1784) 657-665. A footnote states that the ideas were presented before the Academy on June 16, 1770. Cusanus proposed exactly the same method in 1433.

$$10 : B \succ C \succ A \qquad 10 : C \succ A \succ B \qquad 1 : C \succ B \succ A.$$

Here the Borda-ranking is $B(109) \succ_S A(101) \succ_S C(33)$, yet A is the Condorcet-winner. This argument was widely accepted before the work of Donald Saari.

Consider Condorcet's 81-judge example. 30 of the 81 judges have preferences that constitute a *Condorcet-component*, a perfect symmetry among the rank-orderings of equal numbers of judges:

$$10 : A \succ B \succ C \qquad 10 : B \succ C \succ A \qquad 10 : C \succ A \succ B.$$

The 30 judges together say that the competitors are tied. There is another Condorcet-component as well:

$$1 : A \succ C \succ B \qquad 1 : C \succ B \succ A \qquad 1 : B \succ A \succ C.$$

The 33 judges simply cancel each other out, their messages may be dropped, and the decision should depend on the remaining judges:

$$20 : A \succ B \succ C \qquad 28 : B \succ A \succ C.$$

B is the clear winner. Moral: the Condorcet-winner—when he exists—is certainly *not* the candidate who should win in every case! Borda's method (and any "positional" method) avoids this difficulty because the total number of points awarded to every competitor in a Condorcet-component is the same.

Saari's theory generalizes this fundamental cancellation idea. He asserts that "all election difficulties" come from Condorcet-components and, to a lesser extent, from more intricate symmetries in the preference profiles; and concludes, "the [Borda-count] applied to all n-candidates is the unique ranking which avoids all of the indicated problems."[12] But is Borda's method good for ranking, or for designating winners, or both?

Condorcet proposed a method explicitly for ranking.[13] A voter contributes k *Condorcet-points* to an arbitrary rank-ordering $A \succ_S B \succ_S C \succ_S \ldots \succ_S Z$ if his input agrees in k pair-by-pair comparisons. The *Condorcet-count* of the rank-ordering is the sum of the Condorcet-points over all voters. The *Condorcet-ranking* is the ranking that maximizes the Condorcet-count. It ranks the Condorcet-winner first, and the *Condorcet-loser*—a competitor who loses against every other competitor—last, when either exist.[14] Is it good for ranking, or for designating winners and losers, or both?

The two outputs—a rank-ordering and a winner—have usually been treated as two sides of one coin: given a rank-ordering, the winner is the first-placed

[12] Saari, *op. cit.*, p. 57. However, he claims that to counter manipulation, "Instant-Borda-Runoff" should be used to elect a winner: namely, obtain the Borda-ranking, drop the bottom candidate, and repeat until one candidate remains. This method *always* elects the Condorcet-winner when he exists. See, Donald Saari, *Chaotic Elections!: A Mathematician Looks at Voting*, American Mathematical Society, 2001, p. 103.

[13] Condorcet, *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*, Paris, 1785, l'Imprimerie royale.

[14] The Condorcet-ranking of his 60-judge example is $B \succ_S C \succ_S A$; that of his 81-judge example, $A \succ_S B \succ_S C$. Neither agrees with the Borda-ranking.

competitor; given a mechanism for determining a winner, place him first then apply the mechanism to the remaining competitors and place that winner second, and continue.[15] These are questionable practices in the context of the traditional model.

To see why, consider the preference profile

$$333 : A \succ B \succ C \qquad 333 : B \succ C \succ A \qquad 333 : C \succ A \succ B.$$

The 999 judges constitute a Condorcet-component, so cancel each other out. Borda and Condorcet agree on the winners: A, B and C are tied. Condorcet (reasonably) says the three stated rank-orderings are tied for first; Borda (ridiculously) that all six possible rank-orderings are tied for first. Now consider the situation with one additional judge

$$333 : A \succ B \succ C \qquad 333 : B \succ C \succ A \qquad 333 : C \succ A \succ B \qquad 1 : A \succ C \succ B.$$

Borda (reasonably) declares A the winner and B the loser, but (ridiculously) $A \succ_S C \succ_S B$, since only 1 judge agrees with it, 666 partially agree, 333 totally disagree. Condorcet (reasonably) declares $A \succ_S B \succ_S C$ and $C \succ_S A \succ_S B$ are tied: 333 agree, 667 partially agree. But A and C should certainly not be tied as winners. Borda's method is appropriate for designating winners and losers, Condorcet's for designating rank-orderings, a fact already appreciated by Peyton Young.[16] In fact, the situation is much worse: the two outputs cannot be reconciled.

A SWF is *unanimous* if whenever the inputs of all judges are the same rank-ordering, its output is that one. It is *choice-consistent* if the first-placed and last-placed competitors of the output remain the same when a Condorcet-component of the profile is eliminated. It is *rank-consistent*[17] if removing a first-placed or last-placed competitor of the output does not change the order among the remaining competitors. Borda's method is unanimous and choice-consistent but not rank-consistent; Condorcet's is unanimous and rank-consistent but not choice-consistent.

Theorem 1 (Incompatibility) *There exists no unanimous, choice- and rank-consistent method.*

2 Grading: the basic model

A thorough investigation of practice shows that scores, measures or grades have been invented to classify and to rank in an incredibly wide variety of circumstances. Practical people needing practical solutions have increasingly devised

[15] Alternatively, given a mechanism for determining a loser, place him last then apply the mechanism to the remaining competitors and place that loser next to last, and continue.

[16] H. P. Young, "Optimal ranking and choice from pairwise comparisons," in B. Grofman and G. Owen, eds., *Information Pooling and Group Decision Making*, JAI Press, Greenwich CT, 1986. We give new characterizations of Borda's and Condorcet's methods that make this clear, see Balinski and Laraki, *op. cit.*

[17] This property is equivalent to "local stability" in H. P. Young, "Condorcet's theory of voting,"" *American Political Science Review* 82 (1988) 1231-1244.

mechanisms to transform judges' grades (instead of rank-orderings) into a jury's grades to determine final rank-orderings. A set of grades—numbers from 0 to 20, to 25 or to 100; medal nominations from none to bronze, silver or gold; letters from F to A; or words or phrases from bad to excellent—becomes, in effect, a *common language* used to assess performances, just as grades determine the standing of students in schools and universities.[18]

Formally, a *common language* Λ is a set of *grades* $\alpha, \beta, \ldots$ that are strictly ordered. It may be finite or an interval of the real numbers. $\alpha \succeq \beta$ means that either α is a higher grade than β, in symbols, $\alpha \succ \beta$, or $\alpha = \beta$.

There is a finite set of m *competitors* $\mathcal{C} = \{A, \ldots, I, \ldots, Z\}$, and a finite set of n *judges* (or voters) $\mathcal{J} = \{1, \ldots, j, \ldots, n\}$. A problem is completely specified by an *input* or *profile* $\Phi = \Phi(\mathcal{C}, \mathcal{J})$: an m by n matrix of the grades $\Phi(I, j) \in \Lambda$ assigned by each of the judges $j \in \mathcal{J}$ to each of the competitors $I \in \mathcal{C}$.

A *method of grading* is a function F that assigns to any input or profile Φ one *output or final grade* in the same language for every competitor:

$$F : \Lambda^{m \times n} \to \Lambda^m.$$

Designed to assign grades, it must satisfy certain basic properties.

Axiom 1 *F is* neutral*: $F(\rho\Phi) = \rho F(\Phi)$, for any permutation ρ of the competitors (or rows).*

Axiom 2 *F is* anonymous*: $F(\Phi\tau) = F(\Phi)$, for any permutation τ of the judges (or columns).*

Axiom 3 *F is* unanimous*: If a competitor is given an identical grade α by every judge, then F assigns him the grade α.*

Axiom 4 *F is* monotonic*: If $\Phi = \Phi'$ except that one or more judges give higher grades to competitor I in Φ than in Φ', then $F(\Phi)(I) \succeq F(\Phi')(I)$. Moreover, if all the judges give higher grades to competitor I in Φ than in Φ', then $F(\Phi)(I) \succ F(\Phi')(I)$.*

Axiom 5 *F is* independent of irrelevant alternatives (IIA)*: if the grades assigned by the judges to a competitor $I \in \mathcal{C}$ in two profiles Φ and Φ' are the same then $F(\Phi)(I) = F(\Phi')(I)$.*

A function $f : \Lambda^m \to \Lambda$ that transforms a set of judge's grades into a single grade will be called an *aggregation function* if it satisfies the following three properties:

- *anonymity*: $f(\ldots, \alpha, \ldots, \beta, \ldots) = f(\ldots, \beta, \ldots, \alpha, \ldots)$;
- *unanimity*: $f(\alpha, \alpha, \ldots, \alpha) = \alpha$; and

[18] The history of the use of grades to evaluate students and employees; competitive pianists and flutists; Olympic skaters, divers and gymnasts; cities; and wines proves these claims. See, Balinski and Laraki, *op. cit.*