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**Computing uniform convex approximations  
for convex envelopes and convex hulls**

Rida Laraki  
Jean B. Lasserre

*Juillet 2005*

Cahier n° 2005-021

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# Computing uniform convex approximations for convex envelopes and convex hulls

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Juillet 2005

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**Résumé:** Nous établissons une procédure numérique pour approcher uniformément par une suite de fonctions convexes, l'enveloppe convexe d'une fraction rationnelle ayant pour domaine,  $D$ , un ensemble de dimension finie supposé compact et semi-algébrique. Calculer la valeur d'une approximation en un point donné de  $K=\text{co}(D)$  se résume à résoudre un programme semi-défini. En suite, nous caractérisons  $K=\text{co}(D)$  comme projection d'un LMI semi-infini, et en plus nous approximos  $K$  par une suite décroissante d'ensembles convexes. Tester si un point donné n'est pas dans  $K$  se résume à résoudre un nombre fini de programmes semi-définis.

**Abstract:** We provide a numerical procedure to compute uniform (convex) approximations  $\{f_{\{r\}}\}$  of the convex envelope  $f$  of a rational fraction  $f$ , on a compact semi-algebraic set  $D$ . At each point  $x$  in  $K=\text{co}(D)$ , computing  $f_{\{r\}}(x)$  reduces to solving a semidefinite program. We next characterize the convex hull  $K=\text{co}(D)$  in terms of the projection of a semi-infinite LMI, and provide outer convex approximations  $\{K_{\{r\}}\} \supseteq K$ . Testing whether  $x$  is not in  $K$  reduces to solving finitely many semidefinite programs.

**Mots clés :** Enveloppe convexe, programme semi-défini, dualité, ensemble semi-algébrique

**Key Words :** Convex envelope, semidefinite program, duality, semi-algebraic set

**Classification AMS:**

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## 1. INTRODUCTION

Computing the convex envelope  $\widehat{f}$  of a given function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a challenging problem, and to the best of our knowledge, there is still no algorithm that approximates  $\widehat{f}$  by convex functions (except for the simpler univariate case). For instance, for a function  $f$  on a bounded domain  $\Omega$ , Brighi and Chipot [4] propose triangulation methods and provide piecewise degree-1 polynomial approximations  $f_h \geq \widehat{f}$ , and derive estimates of  $f_h - \widehat{f}$  (where  $h$  measures the size of the mesh). Another possibility is to view the problem as a particular instance of the general moment problem, and use geometrical approaches as described in e.g. Anastassiou [1] or Kemperman [7]; but, as acknowledged in [1, 7], this approach is only practical for say, the univariate or bivariate cases.

Concerning convex sets, an important issue raised in Ben-Tal and Nemirovski [3], Parrilo and Sturmfels [12], is to characterize the convex sets that have a *LMI* (or *semidefinite*) *representation*, and called *SDr* sets in e.g. [3]. For instance, the epigraph of a univariate convex polynomial is a SDr set; see [3]. So far, and despite some progress in particular cases (see e.g. the recent proof of the Lax conjecture by Lewis et al [11]), little is known. However, Helton and Vinnikov [6] have proved recently that *rigid convexity* is a necessary condition for a set to be SDr.

In this paper we consider both convex envelopes and convex hulls for certain classes of functions and sets, namely *rational fractions* and compact *semi-algebraic* sets. In both cases we show that one is able to provide relatively simple numerical approximations via semidefinite programming.

**Contribution.** Our contribution is twofold:

Concerning convex envelopes, we consider the class of *rational fractions*  $f$  on a compact semi-algebraic set  $\mathbb{D} \subset \mathbb{R}^n$  (and  $+\infty$  outside  $\mathbb{D}$ ). We view the problem as a particular instance of the general moment problem, and we provide an algorithm for computing *convex* and *uniform* approximations of its convex envelope  $\widehat{f}$ . More precisely, with  $\mathbb{K} := \text{co}(\mathbb{D})$  being the convex hull of  $\mathbb{D}$ :

- (a) We provide a sequence of *convex* functions  $\{f_r\}$  that converges to  $\widehat{f}$ , **uniformly** on any compact subset of  $\mathbb{K}$  in which  $\widehat{f}$  is continuous, as  $r$  increases.

- (b) At each point  $x \in \mathbb{R}^n$ , computing  $f_r(x)$  reduces to solving a semidefinite program  $\mathbf{Q}_{rx}$ .

- (c) For every  $x \in \text{int } \mathbb{K}$ , the SDP dual  $\mathbf{Q}_{rx}^*$  is solvable and any optimal solution provides an element of the subgradient  $\partial f_r(x)$  at the point  $x \in \text{int } \mathbb{K}$ .

Concerning sets, we consider the class of compact semi-algebraic subsets  $\mathbb{D} \subset \mathbb{R}^n$ , and

- (d) We characterize its *convex hull*  $\mathbb{K} := \text{co}(\mathbb{D})$  as the projection of a *semi-infinite* SDr set  $S_\infty$ , i.e., a set defined by finitely many LMIs involving matrices of infinite dimension, and countably many variables. Importantly, the LMI representation of  $S_\infty$  is simple and given *directly* in terms of the data defining the original set  $\mathbb{D}$ .

- (e) We provide outer convex approximations of  $\mathbb{K}$ , namely a monotone nonincreasing sequence of convex sets  $\{\mathbb{K}_r\}$ , with  $\mathbb{K}_r \downarrow \mathbb{K}$ . Each set  $\mathbb{K}_r$  is the projection of a SDr set  $S_r$ , obtained from  $S_\infty$  by "finite truncation". Then, checking whether  $x \notin \mathbb{K}$  reduces to solving finitely many SDPs based on SDr sets  $S_r$ , until one is unfeasible, which eventually happens for some  $r$ . Importantly again, the LMI representation of  $S_r$  is simple and given *directly* in terms of the data defining the original set  $\mathbb{D}$ . Other outer approximations are of course possible, like e.g. convex polytopes  $\{\Omega_r\}$  containing  $\mathbb{K}$ , but obtaining such polytopes with  $\Omega_r \downarrow \mathbb{K}$  is far from trivial.

## 2. NOTATION DEFINITIONS AND PRELIMINARY RESULTS

In the sequel,  $\mathbb{R}[y](:= \mathbb{R}[y_1, \dots, y_n])$  denotes the ring of real-valued polynomials in the variable  $y = (y_1, \dots, y_n)$ . Let  $\bar{y}_i \in \mathbb{R}[y]$  be the natural projection on the  $i$ -variable that is for every  $x \in \mathbb{R}^n$ ,  $\bar{y}_i(x) = x_i$ . For a real-valued symmetric matrix  $M$ , the notation  $M \succeq 0$  stands for  $M$  is positive semidefinite.

Let  $\mathbb{D} \subset \mathbb{R}^n$  be compact, and denote by:

- $\mathbb{K}$ , the convex envelope of  $\mathbb{D}$ . Hence, by a theorem of Caratheodory,  $\mathbb{K}$  is convex and compact; see Rockafellar [14].
- $C(\mathbb{D})$ , the Banach space of real-valued continuous functions on  $\mathbb{D}$ , equipped with the sup-norm  $\|f\| := \sup_{x \in \mathbb{D}} |f(x)|$ ,  $f \in C(\mathbb{D})$ .
- $M(\mathbb{D})$ , its topological dual, i.e., the Banach space of finite signed Borel measures on  $\mathbb{K}$ , equipped with the norm of total variation.
- $M_+(\mathbb{D}) \subset M(\mathbb{D})$ , the positive cone, i.e., the set of finite Borel measures on  $\mathbb{D}$ .
- $\Delta(\mathbb{D}) \subset M_+(\mathbb{D})$ , the set of borel probability measures on  $\mathbb{D}$ .
- for  $f$  in  $C(\mathbb{D})$ , let  $\tilde{f}$  be its natural extension to  $\mathbb{R}^n$  that is

$$(1) \quad \tilde{f}(x) := \begin{cases} f(x) & \text{on } \mathbb{D} \\ +\infty & \text{on } \mathbb{R}^n \setminus \mathbb{D}. \end{cases}$$

Note that  $\tilde{f}$  is lowersemicontinuous (l.s.c.), admits a minimum and its domain is non-empty and compact.

- for  $f$  in  $C(\mathbb{D})$ , let  $\hat{f}$  the convex envelope of  $\tilde{f}$ . This is the greatest convex function smaller than  $\tilde{f}$ .

Note that the vector spaces  $M(\mathbb{D})$  and  $C(\mathbb{D})$  are in duality with the duality bracket

$$\langle \sigma, f \rangle := \int_{\mathbb{K}} f d\sigma, \quad \sigma \in M(\mathbb{K}), f \in C(\mathbb{K})$$

Hence, let  $\tau^*$  denote the associated weak\* topology; this is the coarsest topology on  $M(\mathbb{D})$  for which  $\sigma \rightarrow \langle \sigma, f \rangle$  is continuous for every function  $f$  in  $C(\mathbb{D})$ .

With  $f \in C(\mathbb{D})$ , and  $x \in \mathbb{K} = \text{co}(\mathbb{D})$  fixed, arbitrary, consider the infinite-dimensional linear program (LP):

$$(2) \quad \text{LP}_x : \begin{cases} \inf_{\sigma \in M_+(\mathbb{D})} & \langle \sigma, f \rangle \\ \text{s.t.} & \langle \sigma, \bar{y}_i \rangle = x_i, \quad i = 1, \dots, n \\ & \langle \sigma, 1 \rangle = 1. \end{cases}$$

Its optimal value is denoted by  $\inf \text{LP}_x$ , and  $\min \text{LP}_x$  if the infimum is attained. Notice that  $\text{LP}_x$  is a particular instance of the general *moment problem*, as described in e.g. Kemperman [7, §2.6]. In particular, the set of  $x \in \mathbb{R}^n$  such that  $\text{LP}_x$  has a feasible solution, is called the *moment space*.

**Lemma 2.1.** *Let  $\mathbb{K} = \text{co}(\mathbb{D})$ ,  $f \in C(\mathbb{D})$  and  $\tilde{f}$  be as in (1). Then the convex envelope  $\hat{f}$  of  $\tilde{f}$  is given by:*

$$(3) \quad \hat{f}(x) = \begin{cases} \min \text{LP}_x, & x \in \mathbb{K}, \\ +\infty, & x \in \mathbb{R}^n \setminus \mathbb{K}, \end{cases}$$

so that the domain of  $\hat{f}$  is  $\mathbb{K}$ .

*Proof.* For  $x \in \mathbb{K}$ , let  $\Delta_x(\mathbb{D})$  be the set of probability measures  $\sigma$  on  $\mathbb{D}$ , that are centered at  $x$  (that is  $\langle \sigma, \bar{y}_i \rangle = x_i$  for  $i = 1, \dots, n$ ). Let  $\Delta_x^*(\mathbb{D}) \subset \Delta_x(\mathbb{D})$  be the subset of those probability measures that have a finite support. It is well know that

$$\hat{f}(x) = \inf_{\sigma \in \Delta_x^*(\mathbb{D})} \langle \sigma, f \rangle, \quad \forall x \in \mathbb{K};$$

see Choquet [5] or Laraki [9]. Next, since  $\Delta_x^*(\mathbb{D})$  is dense in  $\Delta_x(\mathbb{D})$  with respect to the weak\* topology, and  $\Delta(\mathbb{D})$  is metrizable and compact with respect to the same topology (see Choquet [5]), deduce that for every  $x \in \mathbb{K}$

$$\begin{aligned} \hat{f}(x) &= \min_{\sigma \in \Delta_x(\mathbb{D})} \langle \sigma, f \rangle \\ &= \min \text{LP}_x. \end{aligned}$$

If  $x \notin \mathbb{K}$ , there is no probability measure on  $\mathbb{D}$ , with finite support, and centered in  $x$ ; therefore  $\hat{f}(x) = +\infty$ .  $\square$

Next, let  $p, q \in \mathbb{R}[y]$ , with  $q > 0$  on  $\mathbb{D}$ , and let  $f \in C(\mathbb{D})$  be defined as

$$(4) \quad y \mapsto f(y) = p(y)/q(y), \quad y \in \mathbb{D}.$$

For every  $x \in \mathbb{K}$ , consider the LP,

$$(5) \quad \mathbf{P}_x : \begin{cases} \inf_{\sigma \in M_+(\mathbb{D})} & \langle \sigma, p \rangle \\ \text{s.t.} & \langle \sigma, \bar{y}_i q \rangle = x_i, \quad i = 1, \dots, n \\ & \langle \sigma, q \rangle = 1. \end{cases}$$

A dual of  $\mathbf{P}_x$ , is the LP

$$(6) \quad \mathbf{P}_x^* : \sup_{\gamma \in \mathbb{R}, \lambda \in \mathbb{R}^n} \{ \gamma + \langle \lambda, x \rangle : p(y) - q(y) \langle \lambda, y \rangle \geq \gamma q(y), \quad \forall y \in \mathbb{D} \},$$

where  $\langle \lambda, y \rangle := \sum_{i=1}^n \lambda_i y_i$  stands for the standard inner product in  $\mathbb{R}^n$ . The optimal value of  $\mathbf{P}_x^*$  is denoted by  $\sup \mathbf{P}_x^*$  (and  $\max \mathbf{P}_x^*$  if the supremum is attained). Equivalently, as  $q > 0$  everywhere on  $\mathbb{D}$ , and  $f = p/q$  on  $\mathbb{D}$ ,

$$(7) \quad \mathbf{P}_x^* : \quad \sup_{\gamma \in \mathbb{R}, \lambda \in \mathbb{R}^m} \{ \gamma + \langle \lambda, x \rangle : f(y) - \langle \lambda, y \rangle \geq \gamma, \quad \forall y \in \mathbb{D} \}.$$

In view of the definition (4) of  $f$ , notice that

$$f(y) - \langle \lambda, y \rangle \geq \gamma, \quad \forall y \in \mathbb{D} \quad \Leftrightarrow \quad \tilde{f}(y) - \langle \lambda, y \rangle \geq \gamma, \quad \forall y \in \mathbf{R}^n.$$

Hence  $\mathbf{P}_x^*$  in (7) is just the dual  $\text{LP}_x^*$  of  $\text{LP}_x$ , for every  $x \in \mathbb{K}$ , and so  $\sup \mathbf{P}_x^* = \sup \text{LP}_x^*$ , for every  $x \in \mathbb{K}$ . In fact we even have the following result:

**Theorem 2.2.** *Let  $p, q \in \mathbb{R}[x]$  with  $q > 0$  on  $\mathbb{D}$ , and let  $f$  be as in (4). Let  $x \in \mathbb{K} = \text{co}(\mathbb{D})$  be fixed, arbitrary, and let  $\mathbf{P}_x$  and  $\mathbf{P}_x^*$  be as in (5) and (7), respectively. Then  $\mathbf{P}_x$  and  $\text{LP}_x$  are solvable and there is no duality gap, i.e.,*

$$(8) \quad \sup \mathbf{P}_x^* = \sup \text{LP}_x^* = \min \text{LP}_x = \min \mathbf{P}_x = \hat{f}(x), \quad x \in \mathbb{K}.$$

*Proof.* Observe that  $\mathbf{P}_x$  is equivalent to  $\text{LP}_x$ . Indeed, with  $\sigma$  an arbitrary feasible solution of  $\mathbf{P}_x$ , the measure  $d\mu := qd\sigma$  is feasible in  $\text{LP}_x$ , with same value. Similarly, with  $\mu$  an arbitrary feasible solution of  $\text{LP}_x$ , the measure  $d\sigma := q^{-1}d\mu$ , well defined on  $\mathbb{D}$  because  $q > 0$  on  $\mathbb{D}$ , is feasible in  $\mathbf{P}_x$ , and with same value. Finally, it is well known that  $\hat{f}$  is the Legendre-Fenchel biconjugate<sup>1</sup> of  $f$ , and so  $\hat{f}(x) = \sup \mathbf{P}_x^*$ , for all  $x \in \mathbb{K}$ . Indeed, let  $f^* : \mathbb{R}^n \rightarrow \mathbb{R}$  be the Legendre-Fenchel conjugate of  $f$ , i.e.,

$$\lambda \mapsto f^*(\lambda) := \sup_{y \in \mathbb{R}^n} \{ \langle \lambda, y \rangle - f(y) \}.$$

In view of the definition of  $\tilde{f}$ ,

$$f^*(\lambda) = \sup_{y \in \mathbb{D}} \{ \langle \lambda, y \rangle - f(y) \},$$

and therefore,

$$\begin{aligned} \sup \mathbf{P}_x^* &= \sup_{\lambda} \{ \langle \lambda, x \rangle + \inf_{y \in \mathbb{D}} \{ f(y) - \langle \lambda, y \rangle \} \} \\ &= \sup_{\lambda} \{ \langle \lambda, x \rangle - \sup_{y \in \mathbb{D}} \{ \langle \lambda, y \rangle - f(y) \} \} \\ &= \sup_{\lambda} \{ \langle \lambda, x \rangle - f^*(\lambda) \} = (f^*)^*(x) = \hat{f}(x). \end{aligned}$$

□

In fact, we have the following.

**Corollary 2.3.** *Let  $x \in \mathbb{K}$  be fixed, arbitrary, and let  $\mathbf{P}_x^*$  be as in (7).*

(a)  $\mathbf{P}_x^*$  is solvable iff  $\partial \hat{f}(x) \neq \emptyset$ <sup>2</sup>. In that case, any optimal solution  $(\lambda^*, \gamma^*)$  satisfies:

$$(9) \quad \lambda^* \in \partial \hat{f}(x), \quad \text{and} \quad \gamma^* = -f^*(\lambda^*).$$

(b) If  $f$  is a rational fraction on  $\mathbb{D}$  as in (4), then  $\partial \hat{f}(x) \neq \emptyset$  for every  $x$  in  $\mathbb{K}$  so that, in this case  $\mathbf{P}_x^*$  is solvable and (a) holds for every  $x$  in  $\mathbb{K}$ .

<sup>1</sup>See Section 2 in Benoist and Hiriart-Urruty [2]

<sup>2</sup> $\partial \hat{f}(x) \neq \emptyset$  at least for every  $x$  in the relative interior of  $\mathbb{K}$  (see Rokafellar 1970, Theorem 23.4)

*Proof.* Suppose that for some  $x \in \mathbb{K}$ ,  $\mathbf{P}_x^*$  is solvable (that is, the supremum is achieved, say at  $\lambda^*(x)$  and  $\gamma^*(x)$ ). Then, for every  $y \in \mathbb{K}$ ,

$$\begin{aligned}\widehat{f}(x) &= \langle \lambda^*(x), x \rangle + \gamma^*(x) \\ \widehat{f}(y) &= \sup_{\lambda, \gamma} \{ \langle \lambda, y \rangle + \gamma : f(z) - \langle \lambda, z \rangle \geq \gamma, \quad \forall z \in \mathbb{D} \}, \quad y \in \mathbb{K} \\ &\geq \langle \lambda^*(x), y \rangle + \gamma^*(x),\end{aligned}$$

and in view of (3), the latter inequality also holds for every  $y$  in  $\mathbb{R}^n$ ; therefore,

$$\widehat{f}(y) - \widehat{f}(x) \geq \langle \lambda^*(x), y - x \rangle, \quad \forall y \in \mathbb{R}^n.$$

Hence,  $\lambda^*(x) \in \partial \widehat{f}(x)$ . Finally, from the standard Legendre-Fenchel equality, we deduce that  $\gamma^*(x) = -f^*(\lambda^*(x))$  where  $f^*$  is the Legendre-Fenchel conjugate of  $\widehat{f}$ . Conversely, if  $\lambda^*(x) \in \partial \widehat{f}(x)$  then, by the Legendre-Fenchel equality we have,

$$\widehat{f}(x) = \langle \lambda^*(x), x \rangle - f^*(\lambda^*(x)).$$

Therefore, we have:

$$(10) \quad \sup \mathbf{P}_x^* = \widehat{f}(x) = \langle \lambda^*(x), x \rangle - f^*(\lambda^*(x)).$$

Next, from

$$f^*(\lambda^*(x)) = \sup_{y \in \mathbb{D}} \langle \lambda^*(x), y \rangle - f(y),$$

we have

$$-f^*(\lambda^*(x)) \leq f(y) - \langle \lambda^*(x), y \rangle, \quad \forall y \in \mathbb{D},$$

which shows that the pair  $(\lambda^*(x), -f^*(\lambda^*(x)))$  is a feasible solution of  $\mathbf{P}_x^*$ , and in view of (10), an optimal solution.

Now, if  $f$  is a rational fraction on  $\mathbb{D}$ , then it is differentiable and Lipschitz on  $\mathbb{D}$  so that, from Theorem 3.6 in Benoist and Hiriart-Urruty [2],  $\partial \widehat{f}(x)$  is uniformly bounded as  $x$  varies on the relative interior of  $\mathbb{K}$ . Since  $\widehat{f}$  is l.s.c. (see below), we deduce that  $\partial \widehat{f}(x) \neq \emptyset$  for every  $x$  in  $\mathbb{K}$ . Actually, let  $x \in \mathbb{K}$  and let  $x_n$  be a sequence in the relative interior of  $\mathbb{K}$  that converges to  $x$  and let  $\lambda_n \in \partial \widehat{f}(x_n)$  such that  $\lambda_n \rightarrow \lambda$  (which is possible (passing to a subsequence if needed) since  $\partial \widehat{f}(x_n)$  is uniformly bounded). Hence, for every  $y$  in  $\mathbb{R}^n$ ,

$$\widehat{f}(y) \geq \widehat{f}(x_n) + \langle \lambda_n, y - x_n \rangle,$$

so that,

$$\begin{aligned}\widehat{f}(y) &\geq \liminf_{n \rightarrow \infty} \widehat{f}(x_n) + \langle \lambda, y - x \rangle \\ &\geq \widehat{f}(x) + \langle \lambda, y - x \rangle\end{aligned}$$

consequently,  $\lambda \in \partial \widehat{f}(x)$ , the desired result.  $\square$

In other words, any optimal solution of  $\mathbf{P}_x^*$  provides an element of the subgradient of  $\widehat{f}$  at the point  $x$ . Corollary 2.3 should be viewed as a refinement for convex envelopes of rational fractions, of Theorem 2.20 in Kemperman [7, p. 28] for the general moment problem, where strong duality results are obtained for the interior of the moment space (here  $\text{int } \mathbb{K}$ ) only.

## 3. PRESERVATION OF CONTINUITY

Later we will construct a sequence  $f_r$  that approximates  $\widehat{f}$  uniformly at each compact set on which  $\widehat{f}$  is continuous. Hence it is natural to first define conditions on the data, to ensure that  $\widehat{f}$  is continuous. The question was solved in Laraki [9] when  $\mathbb{D}$  is convex ( $\mathbb{K} = \text{co}(\mathbb{D})$ ). Here we extend this to our general framework.

**Definition 3.1.** *The compact set  $\mathbb{D}$  of  $\mathbb{R}^n$  is Splitting-Continuous if and only if  $x \mapsto \Delta_x(\mathbb{D})$  is continuous when  $\Delta(\mathbb{D})$  is equipped with the weak\* topology.*

**Examples :** this is exactly the Splitting-Continuous condition defined in Laraki [9] when  $\mathbb{D}$  is convex. Note that if  $\mathbb{D}$  is a polytope or is strictly-convex then it is Splitting-Continuous (see [9][Theorem 1.16]). In fact, if  $\mathbb{D}$  is convex then there are equivalence between Splitting-continuous and the more natural condition faces-closed. The latter concept means (when  $\mathbb{D}$  is convex) that any Hausdorff converging sequence of faces of  $\mathbb{D}$  is also a face of  $\mathbb{D}$  (see Laraki 2004, Theorem 1.16).

**Lemma 3.2.** *Let  $f$  be continuous on the compact  $\mathbb{D}$  of  $\mathbb{R}^n$ . Let  $\widehat{f}$  be as in (3). Then,  $\widehat{f}$  is always l.s.c. with respect to  $\mathbb{K} = \text{co}(\mathbb{D})$  and is continuous on any compact  $\overline{\mathbb{K}}$  that is strictly included in the relative interior of  $\mathbb{K}$ . Moreover,  $\mathbb{D}$  is Splitting-Continuous if and only if  $\widehat{f}$  is continuous on  $\mathbb{K}$ , for every  $f$  which is the restriction on  $\mathbb{D}$  of some continuous function on  $\mathbb{K}$ .*

*Proof.* From Theorem 10.1 in Rockafellar [14], since  $\widehat{f}$  is convex, it is always continuous in the relative interior of  $\mathbb{K}$ . Also, since the correspondence  $x \mapsto \Delta_x(\mathbb{D})$  is always uppersemicontinuous (u.s.c.), and since  $f$  is continuous,  $\widehat{f}$  is always l.s.c. ; see e.g. Theorem 6 in Laraki and Sudderth [8].

Again, from Theorem 6 in Laraki and Sudderth [8],  $x \mapsto \Delta_x(\mathbb{D})$  is continuous if and only if  $\widehat{f}$  is continuous on  $\mathbb{K}$ , for every  $f$  which is the restriction on  $\mathbb{D}$  of some continuous function on  $\mathbb{K}$ .  $\square$

4. UNIFORM CONVEX APPROXIMATIONS OF  $\widehat{f}$  BY SDP-RELAXATIONS

In this section, we assume that  $f$  is defined as in (4) for some polynomials  $p, q \in \mathbb{R}[x]$ , with  $q > 0$  on  $\mathbb{K}$ , where  $\mathbb{D} \subset \mathbb{R}^n$  is the convex and compact semi-algebraic set defined as

$$(11) \quad \mathbb{K} := \{x \in \mathbb{R}^n : | \quad g_j(x) \geq 0, \quad j = 1, \dots, m\},$$

for some polynomials  $\{g_j\} \subset \mathbb{R}[x]$ . Depending on its parity, let  $2r_j - 1$  or  $2r_j$  be the total degree of  $g_j$ , for all  $j = 1, \dots, m$ . Similarly, let  $2r_p, 2r_q$  or  $2r_p - 1, 2r_q - 1$  be the total degree of  $p$  and  $q$  respectively.

We next provide a sequence  $\{f_r\}_r$  of functions such that for all  $r$

- $f_r$  is convex;
- the domain of  $f_r$  is  $\mathbb{K}_r \supset \mathbb{K} = \text{co}(\mathbb{D})$ ;
- $f_r \leq \widehat{f}$  and for every  $x \in \mathbb{K}$ ,  $f_r(x) \uparrow \widehat{f}(x)$  as  $r \rightarrow \infty$ . In fact, we even have

$$\lim_{r \rightarrow \infty} : \|\widehat{f} - f_r\|_{\overline{\mathbb{K}}} \rightarrow 0,$$

that is,  $f_r$  converges to  $\widehat{f}$ , uniformly on any compact  $\overline{\mathbb{K}} \subset \mathbb{K}$  in which  $\widehat{f}$  is continuous. Consequently, if  $\mathbb{D}$  is Splitting-Continuous and if  $q > 0$  on  $\mathbb{K}$ , then we obtain uniform convergence on  $\mathbb{K}$ . Also, if  $\overline{\mathbb{K}}$  is strictly included in the relative interior of  $\mathbb{K}$  then we also obtain uniform convergence.

To do this we first introduce some additional notation.

**4.1. Notation and definitions.** Let  $y = \{y_\alpha\}_{\alpha \in \mathbb{N}^n}$  be a sequence indexed in the canonical basis  $\{z^\alpha\}$  of  $\mathbb{R}[z]$ , and let  $L_y : \mathbb{R}[z] \rightarrow \mathbb{R}$  be the linear functional defined by

$$h \left( := \sum_{\alpha \in \mathbb{N}^n} h_\alpha z^\alpha \right) \mapsto L_y(h) := \sum_{\alpha \in \mathbb{N}^n} h_\alpha y_\alpha.$$

Let  $\mathcal{P}_k \subset \mathbb{R}[z]$  be the space of polynomials of total degree less than  $k$ , and let  $r_0 := \max[r_p, r_q + 1, r_1, \dots, r_m]$ . Then for  $r \geq r_0$ , consider the optimization problem:

$$(12) \quad \mathbf{Q}_{rx} : \begin{cases} \inf_y L_y(p) \\ \text{s.t. } L_y(z_i q) = x_i, & i = 1, \dots, n, \\ L_y(h^2) \geq 0, & \forall h \in \mathcal{P}_r, \\ L_y(h^2 g_j) \geq 0, & \forall h \in \mathcal{P}_{r-r_j}, j = 1, \dots, m, \\ L_y(q) = 1. \end{cases}$$

Problem  $\mathbf{Q}_{rx}$  is a convex optimization problem, in fact, a so-called *semidefinite programming* problem, called a SDP-relaxation of  $\mathbf{P}_x$ . For more details on semidefinite programming and its applications, the reader is referred to Vandenberghe and Boyd [16].

Indeed, given  $y = \{y_\alpha\}$ , let  $M_r(y)$  be the *moment* matrix associated with  $y$ , that is, the rows and columns of  $M_r(y)$  are indexed in the canonical basis of  $\mathcal{P}_r$ , and the entry  $(\alpha, \beta)$  is defined by  $M_r(y)(\alpha, \beta) = L_y(z^{\alpha+\beta}) = y_{\alpha+\beta}$ , for all  $\alpha, \beta \in \mathbb{N}$ , with  $|\alpha|, |\beta| \leq r$ . Then

$$L_y(h^2) \geq 0, \quad \forall h \in \mathcal{P}_r \quad \Leftrightarrow \quad M_r(y) \succeq 0.$$

Similarly, writing

$$z \mapsto g_j(z) := \sum_{\gamma \in \mathbb{N}^n} (g_j)_\gamma z^\gamma, \quad j = 1, \dots, m,$$

the *localizing* matrix  $M_r(g_j y)$  associated with  $y$  and  $g_j \in \mathbb{R}[z]$ , is the matrix also indexed in the canonical basis of  $\mathcal{P}_r$ , and whose entry  $(\alpha, \beta)$  is defined by

$$M_r(g_j y)(\alpha, \beta) = L_y(g_j(z) z^{\alpha+\beta}) = \sum_{\gamma \in \mathbb{N}^n} y_{\alpha+\beta+\gamma} (g_j)_\gamma,$$

for all  $\alpha, \beta \in \mathbb{N}$ , with  $|\alpha|, |\beta| \leq r$ . Then, for every  $j = 1, \dots, m$ ,

$$L_y(g_j h^2) \geq 0, \quad \forall h \in \mathcal{P}_r \quad \Leftrightarrow \quad M_r(g_j y) \succeq 0.$$

Observe that if  $y$  has a *representing measure*  $\mu_y$ , i.e. if

$$y_\alpha = \int x^\alpha d\mu_y, \quad \forall \alpha \in \mathbb{N}^n,$$

then, with  $h \in \mathcal{P}_r$ , and also denoting by  $h = \{h_\alpha\} \in \mathbb{R}^{s(r)}$  its vector of coefficients in the canonical basis,

$$\langle h, M_r(g_j y) h \rangle = \int h^2 g_j d\mu_y.$$

Therefore, if  $\mu_y$  has its support contained in the level set  $\{x \in \mathbb{R}^n : g_j(x) \geq 0\}$ , we must have  $M_r(g_j y) \succeq 0$ .

One also denotes by  $M_\infty(y)$  and  $M_\infty(g_j y)$  the (obvious) respective "infinite" versions of  $M_r(y)$  and  $M_r(g_j y)$ , i.e., moment and localizing matrices with countably

many rows and columns indexed in the canonical basis  $\{z_\alpha\}$ , and involving *all* the variables  $y$  (as opposed to finitely many in  $M_r(y)$  and  $M_r(g_j y)$ ).

For more details on moment and localizing matrices, the reader is referred to Lasserre [10].

**4.2. SDP-relaxations.** Hence, using the above notation, the optimization problem  $\mathbf{Q}_{rx}$  defined in (12) is just the SDP

$$(13) \quad \mathbf{Q}_{rx} : \begin{cases} \inf_y L_y(p) \\ \text{s.t.} & L_y(z_i q) = x_i, \quad i = 1, \dots, n \\ & M_r(y) \succeq 0, \\ & M_{r-r_j}(g_j y) \succeq 0, \quad j = 1, \dots, m, \\ & L_y(q) = 1, \end{cases}$$

with optimal value denoted  $\inf \mathbf{Q}_{rx}$ , and  $\min \mathbf{Q}_{rx}$  if the infimum is attained.

If we write  $M_r(y) = \sum_\alpha B_\alpha y_\alpha$ , and  $M_{r-r_j}(g_j y) = \sum_\alpha C_\alpha^j y_\alpha$ , for appropriate symmetric matrices  $\{B_\alpha, C_\alpha^j\}$ , then the dual of  $\mathbf{Q}_{rx}$  is the SDP

$$(14) \quad \mathbf{Q}_{rx}^* : \begin{cases} \sup_{\lambda, \gamma, X, Z_j} \gamma + \langle \lambda, x \rangle \\ \text{s.t.} & \langle B_\alpha, X \rangle + \sum_{j=1}^m \langle C_\alpha^j, Z_j \rangle + \gamma q_\alpha + \sum_{i=1}^n \lambda_i (z_i q)_\alpha = p_\alpha, \quad |\alpha| \leq 2r \\ & X, Z_j \succeq 0. \end{cases}$$

In fact,  $\mathbf{Q}_{rx}^*$  is the same as (letting  $g_0 \equiv 1$ )

$$(15) \quad \mathbf{Q}_{rx}^* : \begin{cases} \sup_{\gamma \in \mathbb{R}, \lambda \in \mathbb{R}^n, u_j \in \mathbb{R}[z]} \gamma + \langle \lambda, x \rangle \\ \text{s.t.} & p - \gamma q - \langle \lambda, z \rangle q = \sum_{j=0}^m u_j g_j \\ & u_j \text{ s.o.s.}, \quad \deg u_j g_j \leq 2r, \quad j = 0, \dots, m \end{cases}$$

(where s.o.s. stands for *sum of squares*).

We next make the following assumption on the polynomials  $\{g_j\} \subset \mathbb{R}[z]$  in the definition of the set  $\mathbb{D}$  in (11).

**Assumption 4.1.** *There is a polynomial  $u \in \mathbb{R}[z]$ , positif on  $\mathbb{D}$ , which can written*

$$(16) \quad u = u_0 + \sum_{j=1}^m u_j g_j,$$

for a family of s.o.s. polynomials  $\{u_j\}_{j=0}^m \subset \mathbb{R}[z]$ , and the level set  $\{z \in \mathbb{R}^n \mid u(z) \geq 0\}$  is compact.

Assumption 4.1 is not very restrictive. For instance, it is satisfied if:

- all the  $g_j$ 's are linear (and so,  $\mathbb{D}$  is a polytope), or if
- the level set  $\{z \in \mathbb{R}^n \mid g_j(z) \geq 0\}$  is compact, for some  $j \in \{1, \dots, m\}$ .

Moreover, if one knows that the compact set  $\mathbb{D}$  is contained in a ball  $\{z \in \mathbb{R}^n \mid \|z\| \leq M\}$ , for some  $M \in \mathbb{R}$ , then it suffices to add the redundant quadratic constraint  $M^2 - \|z\|^2 \geq 0$  in the definition (11) of  $\mathbb{D}$ , and Assumption 4.1 holds true.

Under Assumption 4.1, every polynomial  $v \in \mathbb{R}[x]$ , strictly positive on  $\mathbb{D}$ , can be written as

$$v = v_0 + \sum_{j=1}^m v_j g_j,$$

for some family of s.o.s. polynomials  $\{v_j\}_{j=0}^m \subset \mathbb{R}[x]$ . This is Putinar's Positivstellensatz, an particular version of Schmüdgen's Positivstellensatz (see Putinar [13]).

Then we have the following result:

**Theorem 4.2.** *Let  $\mathbb{D}$  be as in (11), and let Assumption 4.1 hold. Let  $f$  be as in (4) with  $p, q \in \mathbb{R}[z]$ , and with  $q > 0$  on  $\mathbb{D}$ . Let  $\hat{f}$  be as in (3), and with  $x \in \mathbb{K} = \text{co}(\mathbb{D})$  fixed, consider the SDP-relaxations  $\{\mathbf{Q}_{rx}\}$  defined in (12) (or equivalently in (13)). Then:*

(a) *For every  $x \in \mathbb{R}^n$ ,*

$$(17) \quad \inf \mathbf{Q}_{rx} \uparrow \hat{f}(x), \quad \text{as } r \rightarrow \infty.$$

(b) *The function  $f_r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by*

$$(18) \quad x \mapsto f_r(x) := \inf \mathbf{Q}_{rx}, \quad x \in \mathbb{R}^n,$$

*is convex, and as  $r \rightarrow \infty$ ,  $f_r(x) \uparrow \hat{f}(x)$  pointwise, for all  $x \in \mathbb{R}^n$ .*

(c) *If  $\mathbb{K}$  has a nonempty interior  $\text{int } \mathbb{K}$ , then*

$$(19) \quad \sup \mathbf{Q}_{rx}^* = \max \mathbf{Q}_{rx}^* = \inf \mathbf{Q}_{rx} = f_r(x), \quad x \in \text{int } \mathbb{K},$$

*and for every optimal solution  $(\lambda_r^*, \gamma_r^*)$  of  $\mathbf{Q}_{rx}^*$ ,*

$$f_r(y) - f_r(x) \geq \langle \lambda_r^*, y - x \rangle, \quad \forall y \in \mathbb{R}^n,$$

*that is,  $\lambda_r^* \in \partial f_r(x)$ .*

For a proof see §6.1. As a consequence, we also get:

**Corollary 4.3.** *Let  $\mathbb{D}$  be as in (11), and let Assumption 4.1 hold. Let  $f$  and  $\hat{f}$  be as in (4) and (3) respectively, and let  $f_r : \mathbb{K} \rightarrow \mathbb{R}$ , be as in Theorem 4.2. Then  $f_r$  is always l.s.c. Moreover, for every compact  $\bar{\mathbb{K}} \subset \mathbb{K}$  in which  $\hat{f}$  is continuous,*

$$(20) \quad \lim_{r \rightarrow \infty} \sup_{x \in \bar{\mathbb{K}}} |\hat{f}(x) - f_r(x)| = 0,$$

*that is, the monotone nondecreasing sequence  $\{f_r\}$  converges to  $\hat{f}$ , uniformly on every compact in which  $\hat{f}$  is continuous. Consequently, if in addition  $\mathbb{D}$  is Splitting-continuous and if  $q > 0$  on  $\mathbb{K}$ , then the convergence is uniform on  $\mathbb{K}$ .*

*Proof.* The lowersemicontinuity of  $f_r$  may be obtained using Laraki & Sudderth [8]. This is due to the facts that:

- the objective function of  $\mathbf{Q}_{rx}$  does not depend on  $x$ , and
  - the feasible set of  $\mathbf{Q}_{rx}$  as a function of  $x$ , is u.s.c., in the sense of Kuratowski.
- By Theorem 4.2, we already have that  $f_r \uparrow \hat{f}$  on  $\bar{\mathbb{K}}$ .

(1) The convergence  $f_r \uparrow \hat{f}$  on  $\bar{\mathbb{K}}$ , (2) the fact that  $f_r$  is l.s.c., (3) that the limit  $\hat{f}$  is continuous, and finally (4) that  $\bar{\mathbb{K}}$  is compact, imply that the convergence is uniform on  $\bar{\mathbb{K}}$ . Actually, by (1) and (2) the function  $\hat{f}(x) - f_r(x)$  is always positive and u.s.c. on  $\bar{\mathbb{K}}$ , and therefore, by (4), it admits a global maximizer  $x_r \in \bar{\mathbb{K}}$ . Let  $\alpha = \limsup_{r \rightarrow \infty} \hat{f}(x_r) - f_r(x_r)$ . Without loss of generality and using (4), suppose

that the limsup is achieved with the sequence  $\{x_n\} \subset \overline{\mathbb{K}}$  that converges to some  $x_0 \in \overline{\mathbb{K}}$ . Let  $t$  be some fixed integer. Hence, for any  $n \geq t$ , using (1) one has

$$f_n(x_n) \geq f_t(x_n), \quad [\text{by monotonicity of } \{f_n\}]$$

so that we obtain

$$\lim_{n \rightarrow \infty} f_n(x_n) \geq f_t(x_0), \quad [\text{by lowersemicontinuity of } f_t].$$

Finally, letting  $t$  go to infinity and using (3),

$$\lim_{n \rightarrow \infty} f_n(x_n) \geq \widehat{f}(x_0),$$

and so,

$$\alpha = \lim_{n \rightarrow \infty} \widehat{f}(x_n) - f_n(x_n) = \widehat{f}(x_0) - \lim_{n \rightarrow \infty} f_n(x_n) \leq 0.$$

This implies uniform convergence on  $\overline{\mathbb{K}}$ .  $\square$

*Remark 4.4* If Assumption 4.1 does not hold, then in the SDP-relaxation  $\mathbf{Q}_{rx}$  in (12), one replaces the  $m$  LMI constraints  $L_y(g_j h^2) \geq 0$  for all  $h \in \mathcal{P}_{r-r_j}$ , with the  $2^m$  LMI constraints

$$L_y(g_J h^2) \geq 0, \quad \forall h \in \mathcal{P}_{r-r_J}, \quad \forall J \subseteq \{1, \dots, m\},$$

where  $g_J := \prod_{j \in J} g_j$ , and  $r_J = \deg g_J$ , for all  $J \subseteq \{1, \dots, m\}$  (and  $g_\emptyset \equiv 1$ ). Indeed, Theorem 4.2 and Corollary 4.3 remain valid with  $f_r(x) := \inf \mathbf{Q}_{rx}$ , for all  $x \in \mathbb{R}^n$  (with the newly defined  $\mathbf{Q}_{rx}$ ). In the proof, one now invokes Schmüdgen's Positivstellensatz [15] (instead of Putinar's Positivstellensatz [13]) which states that every polynomial  $v$ , strictly positive on  $\mathbb{D}$ , can be written as

$$v = \sum_{J \subseteq \{1, \dots, m\}} v_J g_J, \quad [(\text{compare with (16)})]$$

for some family of s.o.s. polynomials  $\{v_J\} \subset \mathbb{R}[x]$ ; see Schmüdgen [15].  $\diamond$

**4.3. The univariate case.** In the univariate case, simplifications occur. Let  $\mathbb{D} \subset \mathbb{R}$  be the interval  $[a, b]$ , that is,  $\mathbb{D}$  has the representation

$$(21) \quad \mathbb{D} := \{x \in \mathbf{R} : g(x) \geq 0\}, \quad \text{with } x \mapsto g(x) := (b-x)(x-a), \quad x \in \mathbb{R}.$$

**Theorem 4.5.** *Let  $\mathbb{D}$  be as in (21),  $p, q \in \mathbf{R}[x]$ , with  $q > 0$  on  $\mathbb{D}$ , and let  $f, \widehat{f}$  be as in (4) and (3) respectively. Then, with  $2r \geq \max[\deg p, 1 + \deg q]$ , let  $\mathbf{Q}_{rx}$  be the SDP-relaxation defined in (12). Then:*

$$(22) \quad \widehat{f}(x) = \inf \mathbf{Q}_{rx}, \quad x \in \mathbb{K}.$$

*Proof.* Recall that when  $\mathbb{D}$  is convex and compact, then for every  $x \in \mathbb{D}$ ,  $\widehat{f}(x) = \sup \mathbf{P}_x^*$ , with  $\mathbf{P}_x^*$  as defined in (6). Next, in the univariate case, a polynomial  $h \in \mathbf{R}[x]$  of degree  $2r$  or  $2r-1$ , is nonnegative on  $\mathbb{K}$  if and only if  $h = h_0 + h_1 g$ , for some s.o.s. polynomials  $h_0, h_1 \in \mathbf{R}[x]$ , and with  $\deg h_0, h_1 g \leq 2r$ . This is in contrast with the multivariate case, where the degree in Putinar's representation (16) is not known in advance. Therefore, let  $2r \geq \max[\deg p, 1 + \deg q]$ . The polynomial  $p - \gamma q - \langle \lambda, y \rangle q$  (of degree  $\leq 2r$ ) is nonnegative on  $\mathbb{D}$  if and only if

$$p - \gamma q - \langle \lambda, y \rangle q = u_0 + u_1 g,$$

for some s.o.s. polynomials  $u_0, u_1 \in \mathbf{R}[x]$ , with  $\deg u_0, u_1 g \leq 2r$ . Therefore, as  $q > 0$  on  $\mathbb{K}$ , and recalling the definition of  $\mathbf{P}_x^*$  in (6),

$$\begin{aligned} f(y) - \langle \lambda, y \rangle &\geq \gamma, \quad \forall y \in \mathbb{K} \Leftrightarrow \\ p(y) - q(y)\langle \lambda, y \rangle &\geq \gamma q(y), \quad \forall y \in \mathbb{K} \Leftrightarrow \\ p(y) - q(y)\langle \lambda, y \rangle - \gamma q(y) &\geq 0, \quad \forall y \in \mathbb{K} \Leftrightarrow \\ p(y) - q(y)\langle \lambda, y \rangle - \gamma q(y) &= u_0(y) + u_1(y)g(y), \end{aligned}$$

for some s.o.s. polynomials  $u_0, u_1 \in \mathbf{R}[x]$ , with  $\deg u_0, u_1 g \leq 2r$ . Therefore,  $\mathbf{Q}_{rx}^*$  is identical to  $\mathbf{P}_x^*$ , from which the result follows.  $\square$

So, in the univariate case, the SDP-relaxation  $\mathbf{Q}_{rx}$  is exact, that is, the value at  $x \in \mathbb{K}$  of the convex envelope  $\widehat{f}$ , is easily obtained by solving a *single* SDP.

## 5. THE CONVEX HULL OF A COMPACT SEMI-ALGEBRAIC SET

An important question stated in Ben-Tal and Nemirovski [3, §4.2 and §4.10.2], Parrilo and Sturmfels [12], and not settled yet, is to characterize the convex subsets of  $\mathbb{R}^n$  that are *semidefinite representable* (written SDr), or equivalently, have an *LMI representation*; that is, subsets  $\Omega \subset \mathbb{R}^n$  of the form

$$\Omega = \{x \in \mathbb{R}^n : M_0 + \sum_{i=1}^n M_i x_i \succeq 0\},$$

for some family  $\{M_i\}_{i=0}^n$  of real symmetric matrices. In other words, a SDr set is the *feasible set* of a system of LMI's (Linear Matrix Inequalities), and powerful techniques are now available to solve SDPs. For instance, the epigraph of a univariate convex polynomial is SDr; see [3, p. 292]. Recently, Helton and Vinnikov [6] have proved that *rigid convexity* (as defined in [6]) is a necessary condition for a convex set to be SDr.

In this section, we are concerned with a (large) class of convex sets, namely the convex hull of an arbitrary compact semi-algebraic set, i.e., the convex hull  $\mathbb{K} = \text{co}(\mathbb{D})$  of a compact set  $\mathbb{D}$  defined by finitely many polynomial inequalities, as in (11). We will show that:

- $\mathbb{K}$  is the *projection* of a *semi-infinite* SDr set  $S_\infty$ , that is,  $S_\infty$  is defined by finitely many LMIs involving matrices with countably many rows and columns, and involving countably many variables. Importantly, the LMI representation of the set  $S_\infty$  is given directly in terms of the data, i.e., in terms of the polynomials  $g_j$ 's that define the set  $\mathbb{D}$ .

- $\mathbb{K}$  can be approximated by a monotone nonincreasing sequence of convex sets  $\{\mathbb{K}_r\}$  (with  $\mathbb{K}_r \supset \mathbb{K}$  for all  $r$ ), that are *projections* of SDr sets  $S_r$ . Each SDr set  $S_r$  is a "finite truncation" of  $S_\infty$ , and therefore, also has a specific LMI representation, directly in terms of the data defining the set  $\mathbb{D}$ . In other words,  $\{\mathbb{K}_r\}$  is a converging sequence of *outer convex approximations* of  $\mathbb{K}$ , i.e.  $\mathbb{K}_r \downarrow \mathbb{K}$  as  $r \rightarrow \infty$ . Detecting whether a point  $x \in \mathbb{R}^n$  belongs to  $\mathbb{K}_r$  reduces to solving a single SDP that involves the SDr set  $S_r$ .

With  $\mathbb{D} \subset \mathbb{R}^n$  as in (11), define the  $2^m$  polynomials

$$(23) \quad x \mapsto g_J(x) := \prod_{j \in J} g_j, \quad \forall J \subseteq \{1, \dots, m\},$$

of total degree  $2r_J$  or  $2r_J - 1$ , and with the convention that  $g_\emptyset \equiv 1$ .

Let  $M_r(g_J y) \in \mathbb{R}^{s(r) \times s(r)}$  be the localizing matrix associated with the polynomial  $g_J$ , and a sequence  $y$ , for all  $J \subseteq \{1, \dots, m\}$ , and all  $r = 0, 1, \dots$ ; see also §4.1 for the definition of the infinite matrix  $M_\infty(g_J y)$ .

Define  $S_\infty \subset \mathbb{R}^\infty$  by:

$$(24) \quad S_\infty := \{y \in \mathbb{R}^\infty : y_0 = 1; M_\infty(g_J y) \succeq 0, \quad \forall J \subseteq \{1, \dots, m\}\},$$

The set  $S_\infty$  is a *semi-infinite* SDr set as it is defined by  $2^m$  LMIs whose matrices have countably many rows and columns, and with countably many variables.

If Assumption 4.1 holds, one may instead use the simpler semi-infinite SDr set

$$S'_\infty := \{y \in \mathbb{R}^\infty : y_0 = 1; M_\infty(y) \succeq 0, M_\infty(g_j y) \succeq 0, \quad \forall j = 1, \dots, m\}.$$

Similarly, define  $\mathbb{K}_\infty \subset \mathbb{R}^n$  by:

$$(25) \quad \mathbb{K}_\infty := \{x \in \mathbb{R}^n : \exists y \in S_\infty \text{ s.t. } L_y(z_i) = x_i, i = 1, \dots, n\}.$$

**Lemma 5.1.** *Let  $\mathbb{D} \subset \mathbb{R}^n$  be as in (11) and compact, and let  $\mathbb{K}_\infty$  be as in (25). Then  $\mathbb{K}_\infty = \mathbb{K} = \text{co}(\mathbb{D})$ .*

*Proof.* If  $x \in \text{co}(\mathbb{D}) = \mathbb{K}$ , then  $x_i = \int z_i d\mu, \forall i = 1, \dots, n$ , for some probability measure  $\mu$  with support contained in  $\mathbb{D}$ . Let  $y$  be the vector of moments of  $\mu$ , well defined because  $\mu$  has compact support. Then we necessarily have  $y_0 = 1$ , and  $M_r(g_J y) \succeq 0$  for all  $r$  and all  $J \subseteq \{1, \dots, m\}$ ; see §4.1. Equivalently,  $M_\infty(g_J y) \succeq 0$ , for all  $J \subseteq \{1, \dots, m\}$ , and so  $y \in S_\infty$ . From,  $\int z_i d\mu = L_y(z_i), \forall i = 1, \dots, n$ , we conclude that  $x \in \mathbb{K}_\infty$ , and so  $\mathbb{K} \subseteq \mathbb{K}_\infty$ .

Conversely, let  $x \in \mathbb{K}_\infty$ . Then, there exists  $y \in S_\infty$  such that  $y_0 = 1$  and  $L_y(z_i) = x_i$ , for all  $i = 1, \dots, n$ . As  $M_\infty(g_J y) \succeq 0$ , for all  $J \subseteq \{1, \dots, m\}$ , then by Schmüdgen Positivstellensatz [15],  $y$  is the vector of moments of some probability measure  $\mu_y$ , with support contained in  $\mathbb{D}$ . Next,

$$L_y(z_i) = x_i, \forall i = 1, \dots, n \quad \Leftrightarrow \quad x_i = \int z_i d\mu_y, \forall i = 1, \dots, n,$$

which proves that  $x \in \text{co}(\mathbb{D}) = \mathbb{K}$ . Therefore,  $\mathbb{K}_\infty \subseteq \mathbb{K}$ , and the result follows.  $\square$

So, Lemma 5.1 states that the convex hull  $\mathbb{K}$  of any compact semi-algebraic set  $\mathbb{D}$ , is the *projection* on the variables  $y_\alpha$  with  $|\alpha| = 1$ , of the semi-infinite SDr set  $S_\infty$  (recall that for every  $\alpha \in \mathbb{N}^n$ ,  $|\alpha| = \sum_{i=1}^n \alpha_i$ ). However, the set  $S_\infty$  is *not* described by finite-dimensional LMIs, because we have countably many variables  $y_\alpha$ , and matrices with infinitely many rows and columns.

We next provide outer approximations  $\{\mathbb{K}_r\}$  of  $\mathbb{K}$ , which are projections of SDr sets  $\{S_r\}$ , with  $S_r \supset S_\infty$ , for all  $r$ , and  $\mathbb{K}_r \downarrow \mathbb{K}$ , as  $r \rightarrow \infty$ .

With  $r \geq r_0$ , let  $S_r \subset \mathbb{R}^{s(2r)}$  be defined as:

$$(26) \quad S_r := \{y \in \mathbb{R}^{s(2r)} : y_0 = 1; M_{r-r_j}(g_J y) \succeq 0, \forall J \subseteq \{1, \dots, m\}\}.$$

Notice that  $S_r$  is a SDr set obtained from  $S_\infty$  by "finite" truncation. Indeed,  $S_r$  contains finitely many variables  $y_\alpha$ , namely those with  $|\alpha| \leq 2r$ . And  $M_r(g_J y)$  is a finite truncation of the infinite matrix  $M_\infty(g_J y)$ ; see §4.1.

As for  $S_\infty$ , under Assumption 4.1,  $S_r$  in (26) may be replaced with the (simpler) SDr set

$$S'_r := \{y \in \mathbb{R}^{s(2r)} : y_0 = 1; M_r(y) \succeq 0, M_{r-r_j}(g_j y) \succeq 0, \quad j = 1, \dots, m\}.$$

Next, let

$$(27) \quad \mathbb{K}_r := \{x \in \mathbb{R}^n : \exists y \in S_r \text{ s.t. } L_y(z_i) = x_i, i = 1, \dots, n\}.$$

Equivalently,

(28)

$$\mathbb{K}_r := \left\{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^{s(2r)} \left\{ \begin{array}{l} L_y(z_i) = x_i \quad i = 1, \dots, n \\ M_{r-r_J}(g_J y) \succeq 0, \quad J \subseteq \{1, \dots, m\} \\ y_0 = 1. \end{array} \right. \right\}.$$

In view of the meaning of  $L_y(z_i)$ ,  $\mathbb{K}_r$  is the projection on  $\mathbb{R}^n$  of the SDP set  $S_r \subset \mathbb{R}^{s(2r)}$  defined in (26) (on the  $n$  variables  $y_\alpha$ , with  $|\alpha| = 1$ ); Obviously,  $\{\mathbb{K}_r\}$  forms a *nested* sequence of sets, and we have

$$(29) \quad \mathbb{K}_{r_0} \supset \mathbb{K}_{r_0+1} \dots \supset \mathbb{K}_r \dots \supset \mathbb{K}.$$

Let  $f$  be the identity on  $\mathbb{D}$  ( $f = 1$  on  $\mathbb{D}$ ). Then its convex envelope  $\widehat{f}$  is given by

$$\widehat{f}(x) = \begin{cases} 1, & x \in \mathbb{K}, \\ +\infty & x \in \mathbb{R}^n \setminus \mathbb{K}. \end{cases}$$

Note that  $\widehat{f}$  is clearly a continuous function on  $\mathbb{K}$ .

On  $\mathbb{D}$ , write  $f = 1 = p/q$  with  $p = q \equiv 1$ , so that with  $r \geq r_0 := \max_J r_J$ , the SDP-relaxation  $\mathbf{Q}_{rx}$  defined in (12) and in Remark 4.4, now reads

$$(30) \quad \mathbf{Q}_{rx} : \inf_y \{ y_0 : y \in S_r; \quad L_y(z_i) = x_i, : i = 1, \dots, n \}, \quad x \in \mathbb{R}^n,$$

and so, for all  $r \geq r_0$ ,

$$(31) \quad \inf \mathbf{Q}_{rx} = \begin{cases} 1, & \text{if } x \in \mathbb{K}_r \\ +\infty, & \text{otherwise.} \end{cases}$$

Next, let  $f_r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be the function  $x \mapsto f_r(x) := \inf \mathbf{Q}_{rx}$ , with obvious domain  $\mathbb{K}_r$ .

**Corollary 5.2.** *Let  $\mathbb{D} \subset \mathbb{R}^n$  be compact and defined as in (11), and let  $\mathbb{K} := \text{co}(\mathbb{D})$ .*

- (a) *If  $x \notin \mathbb{K}$ , then  $f_r(x) = +\infty$  whenever  $r \geq r_x$ , for some integer  $r_x$ .*
- (b) *With  $\mathbb{K}_r$  being as in (28),  $\mathbb{K}_r \downarrow \mathbb{K}$  as  $r \rightarrow \infty$ .*

*Proof.* (a) By Theorem 4.2(b) and Remark 4.4,  $f_r$  is convex and  $f_r(x) \uparrow \widehat{f}(x)$ , for all  $x \in \mathbb{R}^n$ . If  $x \in \mathbb{K}$  then  $f_r(x) = 1$  for all  $r$ . If  $x \notin \mathbb{K}$  then  $f_r(x) = 1$  if  $x \in \mathbb{K}_r$ , and  $+\infty$  outside  $\mathbb{K}_r$ . But as  $f_r(x) \uparrow \widehat{f}(x) = +\infty$ , there is some  $r_x$  such that  $f_r(x) = +\infty$ , for all  $r \geq r_x$ , the desired result.

(b) As  $\{\mathbb{K}_r\}$  is a nonincreasing nested sequence and  $\mathbb{K} \subset \mathbb{K}_r$  for all  $r$ , one has

$$\mathbb{K}_r \downarrow \mathbb{K}^* := \bigcap_{r=0}^{\infty} \mathbb{K}_r \supset \mathbb{K}.$$

It suffices to show that  $\mathbb{K}^* \subseteq \mathbb{K}$ , which we prove by contradiction. Let  $x \in \mathbb{K}^*$ , and suppose that  $x \notin \mathbb{K}$ . By (a), we must have  $f_r(x) = +\infty$  whenever  $r \geq r_x$ , for some integer  $r_x$ . In other words,  $x \notin \mathbb{K}_r$  whenever  $r \geq r_x$ . But then,  $x \notin \mathbb{K}^*$ , in contradiction with our hypothesis.  $\square$

Corollary 5.2 provides us with a means to test whether  $x \notin \mathbb{K}$ . Indeed, it suffices to solve the SDP-relaxation  $\mathbf{Q}_{rx}$  defined in (30), until  $\inf \mathbf{Q}_{rx} = +\infty$  for some  $r$  (which means that  $x \notin \mathbb{K}_r$  for all  $r \geq r_x$ ), which eventually happens if  $x \notin \mathbb{K}$ .

## 6. PROOFS

6.1. **Proof of Theorem 4.2.** (a) By standard weak duality,  $\sup \mathbf{Q}_{rx}^* \leq \inf \mathbf{Q}_{rx} \leq \widehat{f}(x)$  for all  $r \in \mathbb{N}$ , and all  $x \in \mathbb{R}^n$ . Next, let  $x \in \mathbb{K}$  be fixed, arbitrary. From Theorem 2.2 and Corollary 2.3,  $\mathbf{P}_x^*$  is solvable, and  $\sup \mathbf{P}_x^* = \max \mathbf{P}_x^* = \widehat{f}(x)$  for all  $x \in \mathbb{K}$ . Therefore, from the definition of  $\mathbf{P}_x^*$  in (6), there is some  $(\gamma^*, \lambda^*) \in \mathbb{R} \times \mathbb{R}^n$  such that

$$p(y) - q(y)\langle \lambda^*, y \rangle - \gamma^* q(y) \geq 0, \quad y \in \mathbb{D},$$

and  $\gamma^* - \langle \lambda^*, x \rangle = \widehat{f}(x)$ .

Hence, with  $\epsilon > 0$  fixed, arbitrary,  $\gamma^* - \epsilon - \langle \lambda^*, x \rangle = \widehat{f}(x) - \epsilon$ , and

$$p(y) - q(y)\langle \lambda^*, y \rangle - (\gamma^* - \epsilon)q(y) \geq \epsilon q(y) > 0, \quad y \in \mathbb{D}.$$

Therefore, under Assumption 4.1, the polynomial  $p - q\langle \lambda^*, y \rangle - (\gamma^* - \epsilon)q$ , which is strictly positive on  $\mathbb{D}$ , can be written

$$p(y) - q(y)\langle \lambda^*, y \rangle - (\gamma^* - \epsilon)q(y) = \sum_{j=0}^m u_j g_j,$$

for some s.o.s. polynomials  $\{u_j\}_{j=0}^m \subset \mathbb{R}[x]$ . But then,  $(\gamma^* - \epsilon, \lambda^*, \{u_j\})$  is a feasible solution of  $\mathbf{Q}_{rx}^*$  as soon as  $r \geq r_\epsilon := \max_{j=0,1,\dots,m} \deg(u_j g_j)$ , and with value  $\gamma^* - \epsilon - \langle \lambda^*, x \rangle = \widehat{f}(x) - \epsilon$ . Hence, for every  $\epsilon > 0$ ,

$$\widehat{f}(x) - \epsilon \leq \sup \mathbf{Q}_{rx}^* \leq \inf \mathbf{Q}_{rx} \leq \widehat{f}(x), \quad r \geq r_\epsilon.$$

This concludes the proof of (17) for  $x \in \mathbb{K}$ .

Next, let  $x \notin \mathbb{K}$  so that  $\widehat{f}(x) = +\infty$ . From the proof of Theorem 2.2, we have seen that  $\mathbf{P}_x^* = \widehat{f}(x)$  for all  $x \in \mathbb{R}^n$ . Therefore, with  $M > 0$  fixed, arbitrarily large, one may find  $\lambda \in \mathbb{R}^n, \gamma \in \mathbb{R}$  such that

$$M \leq \langle \lambda, x \rangle + \gamma \quad \text{and} \quad f(y) + \langle \lambda, y \rangle \geq \gamma, \quad \forall y \in \mathbb{D}.$$

Hence

$$f(y) + \langle \lambda, y \rangle - \gamma + \epsilon > 0, \quad \forall y \in \mathbb{D}.$$

Therefore, as  $q > 0$  on  $\mathbb{D}$ , the polynomial  $g := p + \langle \lambda, y \rangle q - (\gamma - \epsilon)q$  is positive on  $\mathbb{D}$ . By Putinar Positivstellensatz [13], it may be written

$$g = u_0 + \sum_{j=1}^m u_j g_j$$

for some family of s.o.s. polynomials  $\{u_j\}_{j=0}^m \subset \mathbb{R}[x]$ . But then, with  $2r_M \geq \max[\deg u_0, \deg u_j g_j]$ , the 3-uplet  $(\lambda, \gamma - \epsilon, \{u_j\})$  is a feasible solution of  $\mathbf{Q}_{rx}^*$ , whenever  $r \geq r_M$ , and with value  $M - \epsilon$ . And so, as  $M$  was arbitrarily large,  $\sup \mathbf{P}_{rx}^* \rightarrow +\infty = \widehat{f}(x)$ , as  $r \rightarrow \infty$ . This concludes the proof of (17) for  $x \notin \mathbb{K}$ .

(b) That  $f_r$  is convex follows from its definition  $f_r(x) = \inf \mathbf{Q}_{rx}$  for all  $x \in \mathbb{R}^n$ , and the definition (13) of the SDP  $\mathbf{Q}_{rx}$ . First, observe that for all  $r$  sufficiently large, say  $r \geq r'_0$ ,  $\inf \mathbf{Q}_{rx} > -\infty$  for all  $x \in \mathbb{R}^n$ , because  $\sup \mathbf{Q}_{rx}^* \geq -1$ , for all  $x \in \mathbb{R}^n$ . Indeed, with  $\gamma = -1, \lambda = 0$ , the polynomial  $p + q = p - q\gamma - q\langle \lambda, y \rangle$  is positive on  $\mathbb{D}$ , and therefore, by Putinar Positivstellensatz [13],  $p + q = u_0 + \sum_j u_j g_j$ , for some family of s.o.s. polynomials  $\{u_j\}_0^m$ . Therefore,  $(-1, 0, \{u_j\})$  is feasible for  $\mathbf{Q}_{rx}^*$  with

value  $-1$ , whenever  $2r'_0 \geq \max[\deg u_0, \deg u_j g_j]$ . Next, let  $x := \alpha u + (1 - \alpha)v$ , with  $u, v \in \mathbb{R}^n$ , and  $0 \leq \alpha \leq 1$ . As we want to prove

$$\inf \mathbf{Q}_{r(\alpha u + (1 - \alpha)v)} \leq \alpha \inf \mathbf{Q}_{ru} + (1 - \alpha) \inf \mathbf{Q}_{rv}, \quad 0 \leq \alpha \leq 1,$$

we may restrict to  $u, v \in \mathbb{R}^n$  such that  $\inf \mathbf{Q}_{ru}, \inf \mathbf{Q}_{rv} < +\infty$ . So, let  $y_u$  (resp.  $y_v$ ) be feasible for  $\mathbf{Q}_{ru}$  (resp.  $\mathbf{Q}_{rv}$ ), and with respective values  $\inf \mathbf{Q}_{ru} + \epsilon$ ,  $\inf \mathbf{Q}_{rv} + \epsilon$ . As the matrices  $M_r(y)$ ,  $M_{r-r_j}(g_j y)$  are all linear in  $y$ , and  $y \mapsto L_y(\bullet)$  is linear in  $y$  as well,  $y := \alpha y_u + (1 - \alpha)y_v$  is feasible for  $\mathbf{Q}_{rx}$ , with value  $\alpha \inf \mathbf{Q}_{ru} + (1 - \alpha) \inf \mathbf{Q}_{rv} + \epsilon$ . Therefore,

$$\inf \mathbf{Q}_{rx} = \inf \mathbf{Q}_{r(\alpha u + (1 - \alpha)v)} \leq \alpha \inf \mathbf{Q}_{ru} + (1 - \alpha) \inf \mathbf{Q}_{rv} + \epsilon, \quad \forall \epsilon > 0,$$

and letting  $\epsilon \rightarrow 0$  yields the result. Finally, the pointwise convergence  $f_r(x) \uparrow \widehat{f}(x)$  for all  $x \in \mathbb{R}^n$ , follows from (17).

(c) Let  $\mathbb{K}$  be with a nonempty interior  $\text{int } \mathbb{K}$ , and let  $x \in \text{int } \mathbb{K}$ . Let  $\mu$  be the probability measure uniformly distributed on the ball  $B_x := \{y \in \mathbb{K} : \|y - x\| \leq \delta\} \subset \mathbb{K}$ .

Hence,  $\int z_i d\mu = x_i$  for all  $i = 1, \dots, n$ . Next, define the measure  $\nu$  to be  $d\nu = q^{-1} d\mu$  so that  $\int q d\nu = 1$ , and  $\int z_i d\mu = \int z_i q d\nu = x_i$  for all  $i = 1, \dots, n$ . Take for  $y = \{y_\alpha\}$ , the vector of moments of the measure  $\nu$ . As  $\nu$  has a density, and is supported on  $\mathbb{K}$ , it follows that  $M_r(y) \succ 0$  and  $M_r(q_j y) \succ 0$ ,  $j = 1, \dots, m$ , for all  $r$ . Therefore,  $y$  is a *strictly* feasible solution of  $\mathbf{Q}_{rx}$ , i.e., Slater's condition holds, which in turn implies the absence of a duality gap between  $\mathbf{Q}_{rx}$  and its dual  $\mathbf{Q}_{rx}^*$  ( $\sup \mathbf{Q}_{rx}^* = \inf \mathbf{Q}_{rx}$ ). In addition, as  $\inf \mathbf{Q}_{rx} > -\infty$ , we get  $\sup \mathbf{Q}_{rx}^* = \max \mathbf{Q}_{rx}^* = \inf \mathbf{Q}_{rx}$ , which is (19).

So, as  $\mathbf{Q}_{rx}^*$  is solvable, let  $(\gamma_r^*, \lambda_r^*, \{u_j^*\})$  be an optimal solution, that is,  $f_r(x) = \gamma_r^* - \langle \lambda_r^*, x \rangle$  and

$$p(z) - \gamma_r^* q(z) - q(z) \langle \lambda_r^*, y \rangle = \sum_{j=0}^m u_j^*(z) g_j(z), \quad \forall z \in \mathbf{R}^n.$$

Therefore, one has

$$\begin{aligned} f_r(x) &= \gamma_r^* - \langle \lambda_r^*, x \rangle \\ f_r(y) &= \sup_{\gamma, \lambda, u} \{ \gamma + \langle \lambda, y \rangle : p(z) - \gamma q(z) - q(z) \langle \lambda, z \rangle = \sum_{j=0}^m u_j(z) g_j(z), \forall z \in \mathbf{R}^n \} \\ &\geq \gamma_r^* - \langle \lambda_r^*, y \rangle, \quad \forall y \in \mathbb{R}^n, \end{aligned}$$

from which we get

$$f_r(y) - f_r(x) \geq \langle \lambda_r^*, y - x \rangle, \quad \forall y \in \mathbb{R}^n,$$

that is,  $\lambda_r^* \in \partial f_r(x)$ , the desired result.  $\square$

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