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Computing uniform convex approximations for convex envelopes and convex hulls

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Abstract: We provide a numerical procedure to compute uniform (convex) approximations \( \{f_{r}\} \) of the convex envelope \( f \) of a rational fraction \( f \), on a compact semi-algebraic set \( D \). At each point \( x \) in \( K=\text{co}(D) \), computing \( f_{r}(x) \) reduces to solving a semidefinite program. We next characterize the convex hull \( K=\text{co}(D) \) in terms of the projection of a semi-infinite LMI, and provide outer convex approximations \( \{K_{r}\} \supseteq K \). Testing whether \( x \) is not in \( K \) reduces to solving finitely many semidefinite programs.

Mots clés : Enveloppe convexe, programme semi-défini, dualité, ensemble semi-algébrique

Key Words : Convex envelope, semidefinite program, duality, semi-algebraic set

Classification AMS:
1. Introduction

Computing the convex envelope \( \hat{f} \) of a given function \( f : \mathbb{R}^n \to \mathbb{R} \) is a challenging problem, and to the best of our knowledge, there is still no algorithm that approximates \( \hat{f} \) by convex functions (except for the simpler univariate case). For instance, for a function \( f \) on a bounded domain \( \Omega \), Brighi and Chipot [4] propose triangulation methods and provide piecewise degree-1 polynomial approximations \( f_h \geq \hat{f} \), and derive estimates of \( f_h - \hat{f} \) (where \( h \) measures the size of the mesh). Another possibility is to view the problem as a particular instance of the general moment problem, and use geometrical approaches as described in e.g. Anastassiou [1] or Kemperman [7]; but, as acknowledged in [1, 7], this approach is only practical for say, the univariate or bivariate cases.

Concerning convex sets, an important issue raised in Ben-Tal and Nemirovski [3], Parrilo and Sturmfels [12], is to characterize the convex sets that have a LMI (or semidefinite) representation, and called SDr sets in e.g. [3]. For instance, the epigraph of a univariate convex polynomial is a SDr set; see [3]. So far, and despite some progress in particular cases (see e.g. the recent proof of the Lax conjecture by Lewis et al [11]), little is known. However, Helton and Vinnikov [6] have proved recently that rigid convexity is a necessary condition for a set to be SDr.

In this paper we consider both convex envelopes and convex hulls for certain classes of functions and sets, namely rational fractions and compact semi-algebraic sets. In both cases we show that one is able to provide relatively simple numerical approximations via semidefinite programming.

Contribution. Our contribution is twofold:

Concerning convex envelopes, we consider the class of rational fractions \( f \) on a compact semi-algebraic set \( \mathbb{D} \subset \mathbb{R}^n \) (and +\( \infty \) outside \( \mathbb{D} \)). We view the problem as a particular instance of the general moment problem, and we provide an algorithm for computing convex and uniform approximations of its convex envelope \( \hat{f} \). More precisely, with \( \mathbb{K} := \text{co}(\mathbb{D}) \) being the convex hull of \( \mathbb{D} \):

- (a) We provide a sequence of convex functions \( \{f_r\} \) that converges to \( \hat{f} \), uniformly on any compact subset of \( \mathbb{K} \) in which \( \hat{f} \) is continuous, as \( r \) increases.
- (b) At each point $x \in \mathbb{R}^n$, computing $f_r(x)$ reduces to solving a semidefinite program $Q_{rx}$.
- (c) For every $x \in \text{int } K$, the SDP dual $Q_r^* x$ is solvable and any optimal solution provides an element of the subgradient $\partial f_r(x)$ at the point $x \in \text{int } K$.

Concerning sets, we consider the class of compact semi-algebraic subsets $D \subset \mathbb{R}^n$, and
- (d) We characterize its convex hull $K := \text{co}(D)$ as the projection of a semi-infinite SDr set $S_\infty$, i.e., a set defined by finitely many LMIs involving matrices of infinite dimension, and countably many variables. Importantly, the LMI representation of $S_\infty$ is simple and given directly in terms of the data defining the original set $D$.
- (e) We provide outer convex approximations of $K$, namely a monotone non-increasing sequence of convex sets $\{K_r\}$, with $K_r \downarrow K$. Each set $K_r$ is the projection of a SDr set $S_r$, obtained from $S_\infty$ by ”finite truncation”. Then, checking whether $x \notin K$ reduces to solving finitely many SDPs based on SDr sets $S_r$, until one is unfeasible, which eventually happens for some $r$. Importantly again, the LMI representation of $S_r$ is simple and given directly in terms of the data defining the original set $D$. Other outer approximations are of course possible, like e.g. convex polytopes $\{\Omega_r\}$ containing $K$, but obtaining such polytopes with $\Omega_r \downarrow K$ is far from trivial.

2. Notation definitions and preliminary results

In the sequel, $\mathbb{R}[y](:= \mathbb{R}[y_1, \ldots, y_n])$ denotes the ring of real-valued polynomials in the variable $y = (y_1, \ldots, y_n)$. Let $\pi_i \in \mathbb{R}[y]$ be the natural projection on the $i$-variable that is for every $x \in \mathbb{R}^n$, $\pi_i(x) = x_i$. For a real-valued symmetric matrix $M$, the notation $M \succeq 0$ stands for $M$ is positive semidefinite.

Let $D \subset \mathbb{R}^n$ be compact, and denote by:
- $K$, the convex envelope of $D$. Hence, by a theorem of Caratheodory, $K$ is convex and compact; see Rockafellar [14].
- $C(D)$, the Banach space of real-valued continuous functions on $D$, equipped with the sup-norm $\|f\| := \sup_{x \in D} |f(x)|$, $f \in C(D)$.
- $M(D)$, its topological dual, i.e., the Banach space of finite signed Borel measures on $K$, equipped with the norm of total variation.
- $M_+(D) \subset M(D)$, the positive cone, i.e., the set of finite Borel measures on $D$.
- $\Delta(D) \subset M_+(D)$, the set of borel probability measures on $D$.
- for $f$ in $C(D)$, let $\tilde{f}$ be its natural extension to $\mathbb{R}^n$ that is

$$\tilde{f}(x) := \begin{cases} f(x) & \text{on } D \\ +\infty & \text{on } \mathbb{R}^n \setminus D. \end{cases}$$

Note that $\tilde{f}$ is lowersemicontinuous (l.s.c.), admits a minimum and its domain is non-empty and compact.
- for $f$ in $C(D)$, let $\hat{f}$ the convex envelope of $\tilde{f}$. This is the greatest convex function smaller than $\tilde{f}$.

Note that the vector spaces $M(D)$ and $C(D)$ are in duality with the duality bracket

$$\langle \sigma, f \rangle := \int_K f \, d\sigma, \quad \sigma \in M(K), \, f \in C(K)$$
Hence, let $\tau^*$ denote the associated weak* topology; this is the coarsest topology on $M(D)$ for which $\sigma \to \langle \sigma, f \rangle$ is continuous for every function $f$ in $C(D)$.

With $f \in C(D)$, and $x \in K = co(D)$ fixed, arbitrary, consider the infinite-dimensional linear program (LP):

$$\text{LP}_x : \begin{cases} \inf_{\sigma \in M_+} \langle \sigma, f \rangle \\ \text{s.t. } \langle \sigma, y_i \rangle = x_i, \quad i = 1, \ldots, n \\ \langle \sigma, 1 \rangle = 1. \end{cases}$$

Its optimal value is denoted by $\inf \text{LP}_x$, and $\min \text{LP}_x$ if the infimum is attained.

Notice that $\text{LP}_x$ is a particular instance of the general moment problem, as described in e.g. Kemperman [7, §2.6]. In particular, the set of $x \in \mathbb{R}^n$ such that $\text{LP}_x$ has a feasible solution, is called the moment space.

**Lemma 2.1.** Let $K = co(D)$, $f \in C(D)$ and $\hat{f}$ be as in (1). Then the convex envelope $\hat{f}$ of $\tilde{f}$ is given by:

$$\hat{f}(x) = \begin{cases} \min \text{LP}_x, & x \in K, \\ +\infty, & x \in \mathbb{R}^n \setminus K, \end{cases}$$

so that the domain of $\hat{f}$ is $K$.

**Proof.** For $x \in K$, let $\Delta_x(D)$ be the set of probability measures $\sigma$ on $D$, that are centered at $x$ (that is $\langle \sigma, y_i \rangle = x_i$ for $i = 1, \ldots, n$). Let $\Delta^*_x(D) \subset \Delta_x(D)$ be the subset of those probability measures that have a finite support. It is well known that $\hat{f}(x) = \inf_{\sigma \in \Delta^*_x(D)} \langle \sigma, f \rangle$, $\forall x \in K$; see Choquet [5] or Laraki [9]. Next, since $\Delta^*_x(D)$ is dense in $\Delta_x(D)$ with respect to the weak* topology, and $\Delta(D)$ is metrizable and compact with respect to the same topology (see Choquet [5]), deduce that for every $x \in K$

$$\hat{f}(x) = \min_{\sigma \in \Delta_x(D)} \langle \sigma, f \rangle = \min \text{LP}_x.$$ 

If $x \notin K$, there is no probability measure on $D$, with finite support, and centered in $x$; therefore $\hat{f}(x) = +\infty$. \qed

Next, let $p, q \in \mathbb{R}[y]$, with $q > 0$ on $D$, and let $f \in C(D)$ be defined as

$$y \mapsto f(y) = p(y)/q(y), \quad y \in D.$$ 

For every $x \in K$, consider the LP,

$$\text{P}_x : \begin{cases} \inf_{\sigma \in M_+} \langle \sigma, p \rangle \\ \text{s.t. } \langle \sigma, y_i/q \rangle = x_i, \quad i = 1, \ldots, n \\ \langle \sigma, q \rangle = 1. \end{cases}$$

A dual of $\text{P}_x$, is the LP

$$\text{P}^*_x : \sup_{\gamma \in \mathbb{R}, \lambda \in \mathbb{R}^n} \{ \gamma + \langle \lambda, x \rangle : p(y) - q(y) \langle \lambda, y \rangle \geq \gamma q(y), \quad \forall y \in D \},$$
where $\langle \lambda, y \rangle := \sum_{i=1}^{n} \lambda_i y_i$ stands for the standard inner product in $\mathbb{R}^n$. The optimal value of $P^*_x$ is denoted by $\sup P^*_x$ (and $\max P^*_x$ if the supremum is attained). Equivalently, as $q > 0$ everywhere on $\mathbb{D}$, and $f = p/q$ on $\mathbb{D}$,

\begin{equation}
(7) \quad P^*_x : \sup_{\gamma \in \mathbb{R}, \lambda \in \mathbb{R}^m} (\gamma + \langle \lambda, x \rangle : f(y) - \langle \lambda, y \rangle \geq \gamma, \ \forall y \in \mathbb{D}).
\end{equation}

In view of the definition (4) of $f^*$, notice that

\[ f(y) - \langle \lambda, y \rangle \geq \gamma, \ \forall y \in \mathbb{D} \iff \tilde{f}(y) - \langle \lambda, y \rangle \geq \gamma, \ \forall y \in \mathbb{R}^n. \]

Hence $P^*_x$ in (7) is just the dual $\text{LP}^*_x$ of $\text{LP}_x$, for every $x \in \mathbb{K}$, and so $\sup P^*_x = \sup \text{LP}^*_x$, for every $x \in \mathbb{K}$. In fact we even have the following result:

**Theorem 2.2.** Let $p, q \in \mathbb{R}[x]$ with $q > 0$ on $\mathbb{D}$, and let $f$ be as in (4). Let $x \in \mathbb{K} = \text{co}(\mathbb{D})$ be fixed, arbitrary, and let $\text{P}_x$ and $P^*_x$ be as in (5) and (7), respectively. Then $\text{P}_x$ and $\text{LP}_x$ are solvable and there is no duality gap, i.e.,

\begin{equation}
(8) \quad \sup P^*_x = \sup \text{LP}^*_x = \min \text{LP}_x = \min \text{P}_x = \tilde{f}(x), \ x \in \mathbb{K}.
\end{equation}

**Proof.** Observe that $P^*_x$ is equivalent to $\text{LP}_x$. Indeed, with $\sigma$ an arbitrary feasible solution of $\text{P}_x$, the measure $d\mu := q\sigma$ is feasible in $\text{LP}_x$, with same value. Similarly, with $\mu$ an arbitrary feasible solution of $\text{LP}_x$, the measure $d\sigma := q^{-1}d\mu$, well defined on $\mathbb{D}$ because $q > 0$ on $\mathbb{D}$, is feasible in $P^*_x$, and with same value. Finally, it is well known that $\tilde{f}$ is the Legendre-Fenchel biconjugate of $f$, and so $\tilde{f}(x) = P^*_x$, for all $x \in \mathbb{K}$. Indeed, let $f^* : \mathbb{R}^n \to \mathbb{R}$ be the Legendre-Fenchel conjugate of $f$, i.e.,

\[ \lambda \mapsto f^*(\lambda) := \sup_{y \in \mathbb{R}^n} : \{\langle \lambda, y \rangle - \tilde{f}(y)\}. \]

In view of the definition of $\tilde{f}$,

\[ f^*(\lambda) = \sup_{y \in \mathbb{D}} : \{\langle \lambda, y \rangle - f(y)\}, \]

and therefore,

\[ \sup P^*_x = \sup_{\lambda \in \mathbb{R}^n} : \{\langle \lambda, x \rangle + \inf_{y \in \mathbb{D}} \{f(y) - \langle \lambda, y \rangle\}\}
\]

\[ = \sup_{\lambda} \{\langle \lambda, x \rangle - \sup_{y \in \mathbb{D}} \{\langle \lambda, y \rangle - f(y)\}\}
\]

\[ = \sup_{\lambda} \{\langle \lambda, x \rangle - f^*(\lambda)\} = (f^*)^*(x) = \tilde{f}(x). \]

\[ \square \]

In fact, we have the following.

**Corollary 2.3.** Let $x \in \mathbb{K}$ be fixed, arbitrary, and let $P^*_x$ be as in (7).

(a) $P^*_x$ is solvable iff $\partial \tilde{f}(x) \neq \emptyset$. In that case, any optimal solution $(\lambda^*, \gamma^*)$ satisfies:

\begin{equation}
(9) \quad \lambda^* \in \partial \tilde{f}(x), \ \text{and} \ \gamma^* = -f^*(\lambda^*).
\end{equation}

(b) If $f$ is a rational fraction on $\mathbb{D}$ as in (4), then $\partial \tilde{f}(x) \neq \emptyset$ for every $x \in \mathbb{K}$ so that, in this case $P^*_x$ is solvable and (a) holds for every $x \in \mathbb{K}$.

\[ ^{1}\text{See Section 2 in Benoist and Hiriart-Urruty [2]} \]

\[ ^{2}\text{\partial f(x) \neq \emptyset at least for every x in the relative interior of \mathbb{K}} \text{(see Rokafellar 1970, Theorem 23.4)} \]
Proof. Suppose that for some \( x \in \mathbb{R} \), \( P_x^* \) is solvable (that is, the supremum is achieved, say at \( \lambda^*(x) \) and \( \gamma^*(x) \)). Then, for every \( y \in \mathbb{R} \),
\[
\hat{f}(x) = \langle \lambda^*(x), x \rangle + \gamma^*(x) \\
\hat{f}(y) = \sup_{\lambda, \gamma} \{ \langle \lambda, y \rangle + \gamma : f(z) - \langle \lambda, z \rangle \geq \gamma, \ \forall z \in D \}, \quad y \in \mathbb{R} \\
\geq \langle \lambda^*(x), y \rangle + \gamma^*(x),
\]
and in view of (3), the latter inequality also holds for every \( y \in \mathbb{R}^n \); therefore,
\[
\hat{f}(y) - \hat{f}(x) \geq \langle \lambda^*(x), y - x \rangle, \quad \forall y \in \mathbb{R}^n.
\]
Hence, \( \lambda^*(x) \in \partial \hat{f}(x) \). Finally, from the standard Legendre-Fenchel equality, we deduce that \( \gamma^*(x) = -f^*(\lambda^*(x)) \) where \( f^* \) is the Legendre-Fenchel conjugate of \( \hat{f} \).

Conversely, if \( \lambda^*(x) \in \partial \hat{f}(x) \) then, by the Legendre-Fenchel equality we have,
\[
\hat{f}(x) = \langle \lambda^*(x), x \rangle - f^*(\lambda^*(x)).
\]
Therefore, we have:
\[
\sup P_x^* = \hat{f}(x) = \langle \lambda^*(x), x \rangle - f^*(\lambda^*(x)). \tag{10}
\]
Next, from
\[
f^*(\lambda^*(x)) = \sup_{y \in D} \langle \lambda^*(x), y \rangle - f(y),
\]
we have
\[
-f^*(\lambda^*(x)) \leq f(y) - \langle \lambda^*(x), y \rangle, \quad \forall y \in D,
\]
which shows that the pair \( (\lambda^*(x), -f^*(\lambda^*(x))) \) is a feasible solution of \( P_x^* \), and in view of (10), an optimal solution.

Now, if \( f \) is a rational fraction on \( D \), then it is differentiable and Lipschitz on \( D \) so that, from Theorem 3.6 in Benoist and Hiriart-Urruty [2], \( \partial \hat{f}(x) \) is uniformly bounded as \( x \) varies on the relative interior of \( \mathbb{K} \). Since \( \hat{f} \) is l.s.c. (see below), we deduce that \( \partial \hat{f}(x) \neq \emptyset \) for every \( x \in \mathbb{K} \). Actually, let \( x \in \mathbb{K} \) and let \( x_n \) be a sequence in the relative interior of \( \mathbb{K} \) that converges to \( x \) and let \( \lambda_n \in \partial \hat{f}(x_n) \) such that \( \lambda_n \to \lambda \) (which is possible (passing to a subsequence if needed) since \( \partial \hat{f}(x_n) \) is uniformly bounded). Hence, for every \( y \in \mathbb{R}^n \),
\[
\hat{f}(y) \geq \hat{f}(x_n) + \langle \lambda_n, y - x_n \rangle,
\]
so that,
\[
\hat{f}(y) \geq \liminf_{n \to \infty} \hat{f}(x_n) + \langle \lambda, y - x \rangle \geq \hat{f}(x) + \langle \lambda, y - x \rangle
\]
consequently, \( \lambda \in \partial \hat{f}(x) \), the desired result. \( \square \)

In other words, any optimal solution of \( P_x^* \) provides an element of the subgradient of \( \hat{f} \) at the point \( x \). Corollary 2.3 should be viewed as a refinement for convex envelopes of rational fractions, of Theorem 2.20 in Kemperman [7, p. 28] for the general moment problem, where strong duality results are obtained for the interior of the moment space (here \( \text{int} \mathbb{K} \)) only.
3. Preservation of continuity

Later we will construct a sequence $f_r$ that approximates $\hat{f}$ uniformly at each compact set on which $\hat{f}$ is continuous. Hence it is natural to first define conditions on the data, to ensure that $\hat{f}$ is continuous. The question was solved in Laraki [9] when $\mathbb{D}$ is convex $(\mathbb{K} = \text{co}(\mathbb{D}))$. Here we extend this to our general framework.

**Definition 3.1.** The compact set $\mathbb{D}$ of $\mathbb{R}^n$ is Splitting-Continuous if and only if $x \mapsto \Delta_x(\mathbb{D})$ is continuous when $\Delta(\mathbb{D})$ is equipped with the weak* topology.

**Examples:** this is exactly the Splitting-Continuous condition defined in Laraki [9] when $\mathbb{D}$ is convex. Note that if $\mathbb{D}$ is a polytope or is strictly-convex then it is Splitting-Continuous (see [9][Theorem 1.16]). In fact, if $\mathbb{D}$ is convex then there are equivalence between Splitting-continuous and the more natural condition faces-closed. The latter concept means (when $\mathbb{D}$ is convex) that any Hausdorff converging sequence of faces of $\mathbb{D}$ is also a face of $\mathbb{D}$ (see Laraki 2004, Theorem 1.16).

**Lemma 3.2.** Let $f$ be continuous on the compact $\mathbb{D}$ of $\mathbb{R}^n$. Let $\hat{f}$ be as in (3). Then, $\hat{f}$ is always l.s.c. with respect to $\mathbb{K} = \text{co}(\mathbb{D})$ and is continuous on any compact $\mathbb{K}$ that is strictly included in the relative interior of $\mathbb{K}$. Moreover, $\mathbb{D}$ is Splitting-Continuous if and only if $\hat{f}$ is continuous on $\mathbb{K}$, for every $f$ which is the restriction on $\mathbb{D}$ of some continuous function on $\mathbb{K}$.

*Proof.* From Theorem 10.1 in Rockafellar [14], since $\hat{f}$ is convex, it is always continuous in the relative interior of $\mathbb{K}$. Also, since the correspondence $x \mapsto \Delta_x(\mathbb{D})$ is always uppersemicontinuous (u.s.c.), and since $f$ is continuous, $\hat{f}$ is always l.s.c.; see e.g. Theorem 6 in Laraki and Sudderth [8].

Again, from Theorem 6 in Laraki and Sudderth [8], $x \mapsto \Delta_x(\mathbb{D})$ is continuous if and only if $\hat{f}$ is continuous on $\mathbb{K}$, for every $f$ which is the restriction on $\mathbb{D}$ of some continuous function on $\mathbb{K}$. □

4. Uniform convex approximations of $\hat{f}$ by SDP-relaxations

In this section, we assume that $f$ is defined as in (4) for some polynomials $p, q \in \mathbb{R}[x]$, with $q > 0$ on $\mathbb{K}$, where $\mathbb{D} \subset \mathbb{R}^n$ is the convex and compact semi-algebraic set defined as

$$\mathbb{K} := \{x \in \mathbb{R}^n : |g_j(x)| \geq 0, \ j = 1, \ldots, m\},$$

for some polynomials $\{g_j\} \subset \mathbb{R}[x]$. Depending on its parity, let $2r_j - 1$ or $2r_j$ be the total degree of $g_j$, for all $j = 1, \ldots, m$. Similarly, let $2r_p, 2r_q$ or $2r_p - 1, 2r_q - 1$ be the total degree of $p$ and $q$ respectively.

We next provide a sequence $\{f_r\}_r$ of functions such that for all $r$

- $f_r$ is convex;
- the domain of $f_r$ is $\mathbb{K}_r \supset \mathbb{K} = \text{co}(\mathbb{D})$;
- $f_r \leq \hat{f}$ and for every $x \in \mathbb{K}$, $f_r(x) \uparrow \hat{f}(x)$ as $r \to \infty$. In fact, we even have

$$\lim_{r \to \infty} : \|\hat{f} - f_r\|_{\mathbb{K}} \to 0,$$

that is, $f_r$ converges to $\hat{f}$, uniformly on any compact $\mathbb{K}_r \subset \mathbb{K}$ in which $\hat{f}$ is continuous. Consequently, if $\mathbb{D}$ is Splitting-Continuous and if $q > 0$ on $\mathbb{K}$, then we obtain uniform convergence on $\mathbb{K}$. Also, if $\mathbb{K}$ is strictly included in the relative interior of $\mathbb{K}$ then we also obtain uniform convergence.

To do this we first introduce some additional notation.
4.1. Notation and definitions. Let \( y = \{y_α\}_{α ∈ \mathbb{N}^n} \) be a sequence indexed in the canonical basis \( \{z^α\} \) of \( \mathbb{R}[z] \), and let \( L_y : \mathbb{R}[z] \to \mathbb{R} \) be the linear functional defined by

\[
h(\alpha) = \sum_{α ∈ \mathbb{N}^n} h_α z^α \mapsto L_y(h) := \sum_{α ∈ \mathbb{N}^n} h_α y_α.
\]

Let \( \mathcal{P}_k \subset \mathbb{R}[z] \) be the space of polynomials of total degree less than \( k \), and let \( r_0 := \max\{r_p, r_q + 1, r_1, \ldots, r_m\} \). Then for \( r ≥ r_0 \), consider the optimization problem:

\[
\begin{aligned}
\inf_{y} L_y(p) \\
\text{s.t.} \quad L_y(z_i^i) = x_i, \quad i = 1, \ldots, n, \\
L_y(h^2) ≥ 0, \quad ∀ h ∈ \mathcal{P}_r, \\
L_y(h^2 g_j) ≥ 0, \quad ∀ h ∈ \mathcal{P}_{r-j}, \quad j = 1, \ldots, m, \\
L_y(q) = 1.
\end{aligned}
\]

(12) \( Q_rx \)

Problem \( Q_rx \) is a convex optimization problem, in fact, a so-called semidefinite programming problem, called a SDP-relaxation of \( P_x \). For more details on semidefinite programming and its applications, the reader is referred to Vandenberghe and Boyd [16].

Indeed, given \( y = \{y_α\} \), let \( M_r(y) \) be the moment matrix associated with \( y \), that is, the rows and columns of \( M_r(y) \) are indexed in the canonical basis of \( \mathcal{P}_r \), and the entry \( (α, β) \) is defined by \( M_r(y)(α, β) = L_y(z^{α+β}) = y_{α+β} \), for all \( α, β ∈ \mathbb{N} \), with \( |α|, |β| ≤ r \). Then

\[
L_y(h^2) ≥ 0, \quad ∀ h ∈ \mathcal{P}_r \quad ⇔ \quad M_r(y) ≥ 0.
\]

Similarly, writing

\[
z \mapsto g_j(z) := \sum_{γ ∈ \mathbb{N}^n} (g_j)_γ z^γ, \quad j = 1, \ldots, m,
\]

the localizing matrix \( M_r(g_j y) \) associated with \( y \) and \( g_j ∈ \mathbb{R}[z] \), is the matrix also indexed in the canonical basis of \( \mathcal{P}_r \), and whose entry \( (α, β) \) is defined by

\[
M_r(g_j y)(α, β) = L_y(g_j(z)z^{α+β}) = \sum_{γ ∈ \mathbb{N}^n} y_{α+β+γ}(g_j)_γ,
\]

for all \( α, β ∈ \mathbb{N} \), with \( |α|, |β| ≤ r \). Then, for every \( j = 1, \ldots, m \),

\[
L_y(g_j h^2) ≥ 0, \quad ∀ h ∈ \mathcal{P}_r \quad ⇔ \quad M_r(g_j y) ≥ 0.
\]

Observe that if \( y \) has a representing measure \( µ_y \), i.e. if

\[
y_α = \int x^α dµ_y, \quad ∀ α ∈ \mathbb{N}^n,
\]

then, with \( h ∈ \mathcal{P}_r \), and also denoting by \( h = \{h_α\} ∈ \mathbb{R}^{r(γ)} \) its vector of coefficients in the canonical basis,

\[
(h, M_r(g_j y)h) = \int h^2 g_j dµ_y.
\]

Therefore, if \( µ_y \) has its support contained in the level set \( \{x ∈ \mathbb{R}^n : g_j(x) ≥ 0\} \), we must have \( M_r(g_j y) ≥ 0 \).

One also denotes by \( M_∞(y) \) and \( M_∞(g_j y) \) the (obvious) respective "infinite" versions of \( M_r(y) \) and \( M_r(g_j y) \), i.e., moment and localizing matrices with countably
many rows and columns indexed in the canonical basis \( \{ z_\alpha \} \), and involving all the variables \( y \) (as opposed to finitely many in \( M_r(y) \) and \( M_{r-j}(g_jy) \)).

For more details on moment and localizing matrices, the reader is referred to Lasserre [10].

4.2. SDP-relaxations. Hence, using the above notation, the optimization problem \( Q_{rx} \) defined in (12) is just the SDP

\[
Q_{rx} : \begin{cases} 
\inf_p \, L_y(p) \\
\text{s.t.} \\
L_y(z_iq) = x_i, \quad i = 1, \ldots, n \\
M_r(y) \succeq 0, \\
M_{r-j}(g_jy) \succeq 0, \quad j = 1, \ldots, m, \\
L_y(q) = 1,
\end{cases}
\]

with optimal value denoted \( \inf Q_{rx} \), and \( \min Q_{rx} \) if the infimum is attained.

If we write

\[
M_r(y) = \sum_{\alpha} B_\alpha y_\alpha \quad \text{and} \quad M_{r-j}(g_jy) = \sum_{\alpha} C_{j\alpha} y_\alpha,
\]

for appropriate symmetric matrices \( \{ B_\alpha, C_{j\alpha} \} \), then the dual of \( Q_{rx} \) is the SDP

\[
Q^*_{rx} : \begin{cases} 
\sup_{\lambda, \gamma, X, Z} \gamma + \langle \lambda, x \rangle \\
\text{s.t.} \\
\langle B_\alpha, X \rangle + \sum_{j=1}^m \langle C_{j\alpha}, Z_j \rangle + \gamma q_\alpha + \sum_{i=1}^n \lambda_i(z_iq)_\alpha = p_\alpha, \quad |\alpha| \leq 2r \\
X, Z_j \succeq 0,
\end{cases}
\]

In fact, \( Q^*_{rx} \) is the same as (letting \( g_0 \equiv 1 \))

\[
Q^*_{rx} : \begin{cases} 
\sup_{\gamma \in \mathbb{R}, \lambda \in \mathbb{R}^n, u_j \in \mathbb{R}[z]} \gamma + \langle \lambda, x \rangle \\
\text{s.t.} \\
p - \gamma q - \langle \lambda, z \rangle q = \sum_{j=0}^m u_j g_j \\
u_j \text{ s.o.s., } \deg u_j g_j \leq 2r, \quad j = 0, \ldots, m
\end{cases}
\]

(15)

(where s.o.s. stands for sum of squares).

We next make the following assumption on the polynomials \( \{ g_j \} \subset \mathbb{R}[z] \) in the definition of the set \( D \) in (11).

**Assumption 4.1.** There is a polynomial \( u \in \mathbb{R}[z] \), positif on \( D \), which can written

\[
u = u_0 + \sum_{j=1}^m u_j g_j,
\]

for a family of s.o.s. polynomials \( \{ u_j \}^{m}_{j=0} \subset \mathbb{R}[z] \), and the level set \( \{ z \in \mathbb{R}^n \mid u(z) \geq 0 \} \) is compact.

Assumption 4.1 is not very restrictive. For instance, it is satisfied if:

- all the \( g_j \)'s are linear (and so, \( D \) is a polytope), or if
- the level set \( \{ z \in \mathbb{R}^n \mid g_j(z) \geq 0 \} \) is compact, for some \( j \in \{ 1, \ldots, m \} \).

Moreover, if one knows that the compact set \( D \) is contained in a ball \( \{ z \in \mathbb{R}^n \mid \| z \| \leq M \} \), for some \( M \in \mathbb{R} \), then it suffices to add the redundant quadratic constraint \( M^2 - \| z \|^2 \geq 0 \) in the definition (11) of \( D \), and Assumption 4.1 holds true.
Under Assumption 4.1, every polynomial \( v \in \mathbb{R}[x] \), strictly positive on \( \mathbb{D} \), can be written as

\[
v = v_0 + \sum_{j=1}^{m} v_j g_j,
\]

for some family of s.o.s. polynomials \( \{v_j\}_{j=0}^{m} \subset \mathbb{R}[x] \). This is Putinar’s Positivstellensatz, an particular version of Schmüdgen’s Positivstellensatz (see Putinar [13]).

Then we have the following result:

**Theorem 4.2.** Let \( D \) be as in (11), and let Assumption 4.1 hold. Let \( f \) be as in (4) with \( p, q \in \mathbb{R}[x] \), and with \( q > 0 \) on \( \mathbb{D} \). Let \( \hat{f} \) be as in (3), and with \( x \in \mathbb{K} = \text{co}(\mathbb{D}) \) fixed, consider the SDP-relaxations \( \{Q_{r,x}\} \) defined in (12) (or equivalently in (13)). Then:

(a) For every \( x \in \mathbb{R}^n \),

\[
\inf Q_{r,x} \uparrow \hat{f}(x), \quad \text{as} \quad r \to \infty.
\]

(b) The function \( f_r : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) defined by

\[
x \mapsto f_r(x) := \inf Q_{r,x}, \quad x \in \mathbb{R}^n,
\]

is convex, and as \( r \to \infty \), \( f_r(x) \uparrow \hat{f}(x) \) pointwise, for all \( x \in \mathbb{R}^n \).

(c) If \( \mathbb{K} \) has a nonempty interior \( \text{int} \mathbb{K} \), then

\[
\sup Q_{r,x}^* = \max Q_{r,x}^* = \inf Q_{r,x} = f_r(x), \quad x \in \text{int} \mathbb{K},
\]

and for every optimal solution \( (\lambda^*_r, \gamma^*_r) \) of \( Q_{r,x}^* \),

\[
f_r(y) - f_r(x) \geq \langle \lambda^*_r, y - x \rangle, \quad \forall y \in \mathbb{R}^n,
\]

that is, \( \lambda^*_r \in \partial f_r(x) \).

For a proof see §6.1. As a consequence, we also get:

**Corollary 4.3.** Let \( D \) be as in (11), and let Assumption 4.1 hold. Let \( f \) and \( \hat{f} \) be as in (4) and (3) respectively, and let \( f_r : \mathbb{K} \to \mathbb{R} \), be as in Theorem 4.2. Then \( f_r \) is always l.s.c. Moreover, for every compact \( \overline{\mathbb{K}} \subset \mathbb{K} \) in which \( \hat{f} \) is continuous,

\[
\lim_{r \to \infty} \sup_{x \in \overline{\mathbb{K}}} |\hat{f}(x) - f_r(x)| = 0,
\]

that is, the monotone nondecreasing sequence \( \{f_r\} \) converges to \( \hat{f} \), uniformly on every compact in which \( \hat{f} \) is continuous. Consequently, if in addition \( \mathbb{D} \) is Splitting-continuous and if \( q > 0 \) on \( \mathbb{K} \), then the convergence is uniform on \( \mathbb{K} \).

**Proof.** The lowersemicontinuity of \( f_r \) may be obtained using Laraki & Sudderth [8]. This is due to the facts that:

- the objective function of \( Q_{r,x} \) does not depend on \( x \), and
- the feasible set of \( Q_{r,x} \) as a function of \( x \), is u.s.c., in the sense of Kuratowski.

By Theorem 4.2, we already have that \( f_r \uparrow \hat{f} \) on \( \mathbb{K} \).

(1) The convergence \( f_r \uparrow \hat{f} \) on \( \mathbb{K} \), (2) the fact that \( f_r \) is l.s.c., (3) that the limit \( \hat{f} \) is continuous, and finally (4) that \( \mathbb{K} \) is compact, imply that the convergence is uniform on \( \mathbb{K} \). Actually, by (1) and (2) the function \( \hat{f}(x) - f_r(x) \) is always positive and u.s.c. on \( \mathbb{K} \), and therefore, by (4), it admits a global maximizer \( x_r \in \mathbb{K} \). Let \( \alpha = \limsup_{r \to \infty} \hat{f}(x_r) - f_r(x_r) \). Without loss of generality and using (4), suppose
that the limsup is achieved with the sequence \( \{x_n\} \subset \mathbb{K} \) that converges to some \( x_0 \in \mathbb{K} \). Let \( t \) be some fixed integer. Hence, for any \( n \geq t \), using \( (1) \) one has
\[
\lim_{n \to \infty} f_n(x_n) \geq f_t(x_n), \quad \text{[by monotonicity of \( \{f_n\} \)]}
\]
so that we obtain
\[
\lim_{n \to \infty} f_n(x_n) \geq f_t(x_0), \quad \text{[by lower semicontinuity of \( f_t \).]}
\]
Finally, letting \( t \) go to infinity and using \( (3) \),
\[
\lim_{n \to \infty} f_n(x_n) \geq \hat{f}(x_0),
\]
and so,
\[
\alpha = \lim_{n \to \infty} \hat{f}(x_n) - f_n(x_n) = \hat{f}(x_0) - \lim_{n \to \infty} f_n(x_n) \leq 0.
\]
This implies uniform convergence on \( \mathbb{K} \).

Remark 4.4 If Assumption 4.1 does not hold, then in the SDP-relaxation \( Q_{rx} \) in \( (12) \), one replaces the \( m \) LMI constraints \( L_y(g_J h^2) \geq 0 \) for all \( h \in P_{r-r_J} \), with the \( 2^m \) LMI constraints
\[
L_y(g_J h^2) \geq 0, \quad \forall h \in P_{r-r_J}, \quad \forall J \subseteq \{1, \ldots, m\},
\]
where \( g_J := \prod_{j \in J} g_j \), and \( r_J = \deg g_J \), for all \( J \subseteq \{1, \ldots, m\} \) (and \( g_0 \equiv 1 \)). Indeed, Theorem 4.2 and Corollary 4.3 remain valid with \( f_t(x) := \inf Q_{rx} \), for all \( x \in \mathbb{R}^n \) (with the newly defined \( Q_{rx} \)). In the proof, one now invokes Schmüdgen’s Positivstellensatz [15] (instead of Putinar’s Positivstellensatz [13]) which states that every polynomial \( v \), strictly positive on \( \mathbb{D} \), can be written as
\[
v = \sum_{J \subseteq \{1, \ldots, m\}} v_J g_J, \quad [(\text{compare with (16)})]
\]
for some family of s.o.s. polynomials \( \{v_J\} \subset \mathbb{R}[x] \); see Schmüdgen [15]. \( \diamond \)

4.3. The univariate case. In the univariate case, simplifications occur. Let \( \mathbb{D} \subset \mathbb{R} \) be the interval \([a, b] \), that is, \( \mathbb{D} \) has the representation
\[
(21) \quad \mathbb{D} := \{x \in \mathbb{R}: g(x) \geq 0\}, \quad \text{with } x \mapsto g(x) := (b - x)(x - a), \quad x \in \mathbb{R}.
\]

Theorem 4.5. Let \( \mathbb{D} \) be as in \( (21) \), \( p, q \in \mathbb{R}[x] \), with \( q > 0 \) on \( \mathbb{D} \), and let \( f, \hat{f} \) be as in \( (4) \) and \( (3) \) respectively. Then, with \( 2r \geq \max[\deg p, 1 + \deg q] \), let \( Q_{rx} \) be the SDP-relaxation defined in \( (12) \). Then:
\[
(22) \quad \hat{f}(x) = \inf Q_{rx}, \quad x \in \mathbb{K}.
\]

Proof. Recall that when \( \mathbb{D} \) is convex and compact, then for every \( x \in \mathbb{D} \), \( \hat{f}(x) = \sup P_x^* \), with \( P_x^* \) as defined in \( (6) \). Next, in the univariate case, a polynomial \( h \in \mathbb{R}[x] \) of degree \( 2r \) or \( 2r - 1 \), is nonnegative on \( \mathbb{K} \) if and only if \( h = h_0 + h_1 g \), for some s.o.s. polynomials \( h_0, h_1 \in \mathbb{R}[x] \), with \( \deg h_0, h_1 g \leq 2r \). This is in contrast with the multivariate case, where the degree in Putinar’s representation \( (16) \) is not known in advance. Therefore, let \( 2r \geq \max[\deg p, 1 + \deg q] \). The polynomial \( p - \gamma q - (\lambda, y)q \) (of degree \( \leq 2r \)) is nonnegative on \( \mathbb{D} \) if and only if
\[
p - \gamma q - (\lambda, y)q = u_0 + u_1 g,
\]
for some s.o.s. polynomials \( u_0, u_1 \in \mathbb{R}[x] \), with \( \deg u_0, u_1 g \leq 2r \). Therefore, as \( q > 0 \) on \( \mathbb{K} \), and recalling the definition of \( P_r^* \) in (6),

\[
\begin{align*}
    f(y) - \langle \lambda, y \rangle &\geq \gamma, \quad \forall y \in \mathbb{K} \iff \\
    p(y) - q(y) \langle \lambda, y \rangle &\geq \gamma q(y), \quad \forall y \in \mathbb{K} \iff \\
    p(y) - q(y) \langle \lambda, y \rangle - \gamma q(y) &\geq 0, \quad \forall y \in \mathbb{K} \iff \\
    p(y) - q(y) \langle \lambda, y \rangle - \gamma q(y) &= u_0(y) + u_1(y) g(y),
\end{align*}
\]

for some s.o.s. polynomials \( u_0, u_1 \in \mathbb{R}[x] \), with \( \deg u_0, u_1 g \leq 2r \). Therefore, \( Q_{x_r}^* \) is identical to \( P_r^* \), from which the result follows. \( \square \)

So, in the univariate case, the SDP-relaxation \( Q_{x_1}^* \) is exact, that is, the value at \( x \in \mathbb{K} \) of the convex envelope \( \hat{f} \), is easily obtained by solving a single SDP.

5. The convex hull of a compact semi-algebraic set

An important question stated in Ben-Tal and Nemirovski [3, §4.2 and §4.10.2], Parrilo and Sturmfels [12], and not settled yet, is to characterize the convex subsets of \( \mathbb{R}^n \) that are semidefinite representable (written SDr), or equivalently, have an LMI representation; that is, subsets \( \Omega \subset \mathbb{R}^n \) of the form

\[
\Omega = \{ x \in \mathbb{R}^n : M_0 + \sum_{i=1}^n M_i x_i \succeq 0 \},
\]

for some family \( \{ M_i \}_{i=0} \) of real symmetric matrices. In other words, a SDr set is the feasible set of a system of LMI’s (Linear Matrix Inequalities), and powerful techniques are now available so solve SDPs. For instance, the epigraph of a univariate convex polynomial is SDr; see [3, p. 292]. Recently, Helton and Vinnikov [6] have proved that rigid convexity (as defined in [6]) is a necessary condition for a convex set to be SDr.

In this section, we are concerned with a (large) class of convex sets, namely the convex hull of an arbitrary compact semi-algebraic set, i.e., the convex hull \( \mathbb{K} = \text{co}(\mathbb{D}) \) of a compact set \( \mathbb{D} \) defined by finitely many polynomial inequalities, as in (11). We will show that:

- \( \mathbb{K} \) is the projection of a semi-infinite SDr set \( S_\infty \), that is, \( S_\infty \) is defined by finitely many LMI’s involving matrices with countably many rows and columns, and involving countably many variables. Importantly, the LMI representation of the set \( S_\infty \) is given directly in terms of the data, i.e., in terms of the polynomials \( g_j \)’s that define the set \( \mathbb{D} \).

- \( \mathbb{K} \) can be approximated by a monotone nonincreasing sequence of convex sets \( \{ \mathbb{K}_r \} \) (with \( \mathbb{K}_r \supset \mathbb{K} \) for all \( r \)), that are projections of SDr sets \( S_r \). Each SDr set \( S_r \) is a "finite truncation" of \( S_\infty \), and therefore, also has a specific LMI representation, directly in terms of the data defining the set \( \mathbb{D} \). In other words, \( \{ \mathbb{K}_r \} \) is a converging sequence of outer convex approximations of \( \mathbb{K} \), i.e. \( \mathbb{K}_r \downarrow \mathbb{K} \) as \( r \to \infty \). Detecting whether a point \( x \in \mathbb{R}^n \) belongs to \( \mathbb{K}_r \) reduces to solving a single SDP that involves the SDr set \( \mathbb{K}_r \).

With \( \mathbb{D} \subset \mathbb{R}^n \) as in (11), define the \( 2^m \) polynomials

\[
x \mapsto g_J(x) := \prod_{j \in J} g_j, \quad \forall J \subseteq \{1, \ldots, m\},
\]

of total degree \( 2r_J \) or \( 2r_J - 1 \), and with the convention that \( g_\emptyset \equiv 1 \).
Let $M_r(g_j y) \in \mathbb{R}^{(r) \times (r)}$ be the localizing matrix associated with the polynomial $g_j$, and a sequence $y$, for all $J \subseteq \{1, \ldots, m\}$, and all $r = 0, 1, \ldots$; see also §4.1 for the definition of the infinite matrix $M_\infty(g_j y)$.

Define $S_\infty \subset \mathbb{R}^\infty$ by:

\begin{equation}
S_\infty := \{y \in \mathbb{R}^\infty : y_0 = 1; \ M_\infty(g_j y) \succeq 0, \ \forall J \subseteq \{1, \ldots, m\}\}.
\end{equation}

The set $S_\infty$ is a semi-infinite SDr set as it is defined by $2^n$ LMI's whose matrices have countably many rows and columns, and with countably many variables.

If Assumption 4.1 holds, one may instead use the simpler semi-infinite SDr set

\begin{equation}
S_\infty' := \{y \in \mathbb{R}^\infty : y_0 = 1; \ M_\infty(y) \succeq 0, \ M_\infty(g_j y) \succeq 0, \ \forall j = 1, \ldots, m\}.
\end{equation}

Similarly, define $K_\infty \subset \mathbb{R}^n$ by:

\begin{equation}
K_\infty := \{x \in \mathbb{R}^n : \exists y \in S_\infty \ s.t. \ L_y(z_i) = x_i, \ i = 1, \ldots, n\}.
\end{equation}

**Lemma 5.1.** Let $\mathbb{D} \subset \mathbb{R}^n$ be as in (11) and compact, and let $K_\infty$ be as in (25). Then $K_\infty = K = co(\mathbb{D})$.

**Proof.** If $x \in co(\mathbb{D}) = K$, then $x_i = \int z_i \, d\mu$, $\forall i = 1, \ldots, n$, for some probability measure $\mu$ with support contained in $\mathbb{D}$. Let $y$ be the vector of moments of $\mu$, well defined because $\mu$ has compact support. Then we necessarily have $y_0 = 1$, and $M_r(g_j y) \succeq 0$ for all $r$ and all $J \subseteq \{1, \ldots, m\}$; see §4.1. Equivalently, $M_\infty(g_j y) \succeq 0$, for all $J \subseteq \{1, \ldots, m\}$, and so $y \in S_\infty$. From, $\int z_i \, d\mu = L_y(z_i)$, $\forall i = 1, \ldots, n$, we conclude that $x \in K_\infty$, and so $K \subseteq K_\infty$.

Conversely, let $x \in K_\infty$. Then, there exists $y \in S_\infty$ such that $y_0 = 1$ and $L_y(z_i) = x_i$, for all $i = 1, \ldots, n$. As $M_\infty(g_j y) \succeq 0$, for all $J \subseteq \{1, \ldots, n\}$, then by Schmüdgen Positivstellensatz [15], $y$ is the vector of moments of some probability measure $\mu_y$, with support contained in $\mathbb{D}$. Next, 

\[ L_y(z_i) = x_i, \ \forall i = 1, \ldots, n \quad \iff \quad x_i = \int z_i \, d\mu_y, \ \forall i = 1, \ldots, n, \]

which proves that $x \in co(\mathbb{D}) = K$. Therefore, $K_\infty \subseteq K$, and the result follows. \(\square\)

So, Lemma 5.1 states that the convex hull $K$ of any compact semi-algebraic set $\mathbb{D}$, is the projection on the variables $y_\alpha$ with $|\alpha| = 1$, of the semi-infinite SDr set $S_\infty$ (recall that for every $\alpha \in \mathbb{N}^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$). However, the set $S_\infty$ is not described by finite-dimensional LMI’s, because we have countably many variables $y_\alpha$, and matrices with infinitely many rows and columns.

We next provide outer approximations $\{K_r\}$ of $K$, which are projections of SDr sets $\{S_r\}$, with $S_r \supset S_\infty$, for all $r$, and $K_r \downarrow K$, as $r \to \infty$.

With $r \geq r_0$, let $S_r \subset \mathbb{R}^{(2r)}$ be defined as:

\begin{equation}
S_r := \{y \in \mathbb{R}^{(2r)} : \ y_0 = 1; \ M_{r-r_j}(g_j y) \succeq 0, \ \forall J \subseteq \{1, \ldots, m\}\}.
\end{equation}

Notice that $S_r$ is a SDr set obtained from $S_\infty$ by ”finite” truncation. Indeed, $S_r$ contains finitely many variables $y_\alpha$, namely those with $|\alpha| \leq 2r$. And $M_r(g_j y)$ is a finite truncation of the infinite matrix $M_\infty(g_j y)$; see §4.1.

As for $S_\infty$, under Assumption 4.1, $S_r$ in (26) may be replaced with the (simpler) SDr set

\begin{equation}
S_r' := \{y \in \mathbb{R}^{(2r)} : \ y_0 = 1; \ M_r(y) \succeq 0, \ M_{r-r_j}(g_j y) \succeq 0, \ j = 1, \ldots, m\}.
\end{equation}

Next, let

\begin{equation}
K_r := \{x \in \mathbb{R}^n : \exists y \in S_r \ s.t. \ L_y(z_i) = x_i, \ i = 1, \ldots, n\}.
\end{equation}
Equivalently,
\[ (28) \]
\[ \mathbb{K}_r := \left\{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^{s(2r)} \left\{ \begin{array}{l} L_y(z_i) = x_i, \quad i = 1, \ldots, n \\ M_{r-r_j}(y) \succeq 0, \quad J \subseteq \{1, \ldots, m\} \\ y_0 = 1 \end{array} \right. \right\}. \]

In view of the meaning of \( L_y(z_i) \), \( \mathbb{K}_r \) is the projection on \( \mathbb{R}^n \) of the SDP set \( S_r \subset \mathbb{R}^{s(2r)} \) defined in (26) (on the \( n \) variables \( y_\alpha \), with \( |\alpha| = 1 \)); Obviously, \( \{\mathbb{K}_r\} \) forms a nested sequence of sets, and we have
\[ (29) \]
\[ \mathbb{K}_{r_0} \supset \mathbb{K}_{r_0+1} \supset \cdots \supset \mathbb{K}_r \supset \cdots \supset \mathbb{K}. \]

Let \( f \) be the identity on \( \mathbb{D} \) (\( f = 1 \) on \( \mathbb{D} \)). Then its convex envelope \( \hat{f} \) is given by
\[ \hat{f}(x) = \begin{cases} 1, & x \in \mathbb{K} \\ +\infty, & x \in \mathbb{R}^n \setminus \mathbb{K}. \end{cases} \]

Note that \( \hat{f} \) is clearly a continuous function on \( \mathbb{K} \).

On \( \mathbb{D} \), write \( f = 1 = p/q \) with \( p = q = 1 \), so that with \( r \geq r_0 := \max_j r_j \), the SDP-relaxation \( Q_{rx} \) defined in (12) and in Remark 4.4, now reads
\[ (30) \]
\[ Q_{rx} : \inf_y \{ y_0 : y \in S_r : L_y(z_i) = x_i, \quad i = 1, \ldots, n \}, \quad x \in \mathbb{R}^n, \]

and so, for all \( r \geq r_0 \),
\[ (31) \]
\[ \inf Q_{rx} = \begin{cases} 1, & \text{if } x \in \mathbb{K}_r \\ +\infty, & \text{otherwise}. \end{cases} \]

Next, let \( f_r : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be the function \( x \mapsto f_r(x) := \inf Q_{rx} \), with obvious domain \( \mathbb{K}_r \).

**Corollary 5.2.** Let \( \mathbb{D} \subset \mathbb{R}^n \) be compact and defined as in (11), and let \( \mathbb{K} := \text{co}(\mathbb{D}) \).

(a) If \( x \notin \mathbb{K} \), then \( f_r(x) = +\infty \) whenever \( r \geq r_x \), for some integer \( r_x \).

(b) With \( \mathbb{K}_r \) being as in (28), \( \mathbb{K}_r \uparrow \mathbb{K} \) as \( r \to \infty \).

**Proof.** (a) By Theorem 4.2(b) and Remark 4.4, \( f_r \) is convex and \( f_r(x) \uparrow \hat{f}(x) \), for all \( x \in \mathbb{R}^n \). If \( x \in \mathbb{K} \) then \( f_r(x) = 1 \) for all \( r \). If \( x \notin \mathbb{K} \) then \( f_r(x) = 1 \) if \( x \in \mathbb{K}_r \), and \( +\infty \) outside \( \mathbb{K}_r \). But as \( f_r(x) \uparrow \hat{f}(x) = +\infty \), there is some \( r_x \) such that \( f_r(x) = +\infty \), for all \( r \geq r_x \), as the desired result.

(b) As \( \{\mathbb{K}_r\} \) is a nonincreasing nested sequence and \( \mathbb{K} \subset \mathbb{K}_r \) for all \( r \), one has
\[ \mathbb{K}_r \downarrow \mathbb{K}^* := \bigcap_{r=0}^{\infty} \mathbb{K}_r \supset \mathbb{K}. \]

It suffices to show that \( \mathbb{K}^* \subset \mathbb{K} \), which we prove by contradiction. Let \( x \in \mathbb{K}^* \), and suppose that \( x \notin \mathbb{K} \). By (a), we must have \( f_r(x) = +\infty \) whenever \( r \geq r_x \), for some integer \( r_x \). In other words, \( x \notin \mathbb{K}_r \) whenever \( r \geq r_x \). But then, \( x \notin \mathbb{K}^* \), in contradiction with our hypothesis. \( \square \)

Corollary 5.2 provides us with a means to test whether \( x \notin \mathbb{K} \). Indeed, it suffices to solve the SDP-relaxation \( Q_{rx} \) defined in (30), until \( \inf Q_{rx} = +\infty \) for some \( r \) (which means that \( x \notin \mathbb{K}_r \) for all \( r \geq r_x \), which eventually happens if \( x \notin \mathbb{K} \).
6. Proofs

6.1. Proof of Theorem 4.2. (a) By standard weak duality, sup $Q^*_{rx}$ $\leq$ inf $Q_{rx}$ $\leq$ $\hat{f}(x)$ for all $r \in \mathbb{N}$, and all $x \in \mathbb{R}^n$. Next, let $x \in \mathbb{K}$ be fixed, arbitrary. From Theorem 2.2 and Corollary 2.3, $P_\mathbb{K}^*$ is solvable, and sup $P_\mathbb{K}^*$ $=$ max $P_\mathbb{K}^*$ $=$ $\hat{f}(x)$ for all $x \in \mathbb{K}$. Therefore, from the definition of $P_\mathbb{K}^*$ in (6), there is some $(\gamma^*, \lambda^*)$ $\in \mathbb{R} \times \mathbb{R}^n$ such that

$$p(y) - q(y)(\lambda^*, y) - \gamma^*q(y) \geq 0, \quad y \in \mathbb{D},$$

and $\gamma^* - (\lambda^*, x) = \hat{f}(x)$.

Hence, with $\epsilon > 0$ fixed, arbitrary, $\gamma^* - \epsilon - (\lambda^*, x) = \hat{f}(x) - \epsilon$, and

$$p(y) - q(y)(\lambda^*, y) - (\gamma^* - \epsilon)q(y) \geq \epsilon q(y) > 0, \quad y \in \mathbb{D}.$$ 

Therefore, under Assumption 4.1, the polynomial $p - q(\lambda^*, y) - (\gamma^* - \epsilon)q$, which is strictly positive on $\mathbb{D}$, can be written

$$p(y) - q(y)(\lambda^*, y) - (\gamma^* - \epsilon)q(y) = \sum_{j=0}^m u_j g_j,$$

for some s.o.s. polynomials $\{u_j\}_{j=0}^m \subset \mathbb{R}[x]$. But then, $(\gamma^* - \epsilon, \lambda^*, \{u_j\})$ is a feasible solution of $Q^*_{rx}$ as soon as $r \geq r_\epsilon := \max_{j=0, \ldots, m} \deg (u_j g_j)$, and with value $\gamma^* - \epsilon - (\lambda^*, x) = \hat{f}(x) - \epsilon$. Hence, for every $\epsilon > 0$,

$$\hat{f}(x) - \epsilon \leq \sup Q^*_{rx} \leq \inf Q_{rx} \leq \hat{f}(x), \quad r \geq r_\epsilon.$$ 

This concludes the proof of (17) for $x \in \mathbb{K}$.

Next, let $x \notin \mathbb{K}$ so that $\hat{f}(x) = +\infty$. From the proof of Theorem 2.2, we have seen that $P_\mathbb{K}^* = \hat{f}(x)$ for all $x \in \mathbb{R}^n$. Therefore, with $M > 0$ fixed, arbitrarily large, one may find $\lambda \in \mathbb{R}^n, \gamma \in \mathbb{R}$ such that

$$M \leq (\lambda, x) + \gamma \quad \text{and} \quad f(y) + (\lambda, y) \geq \gamma, \quad \forall y \in \mathbb{D}.$$ 

Hence

$$f(y) + (\lambda, y) - \gamma + \epsilon > 0, \quad \forall y \in \mathbb{D}.$$ 

Therefore, as $q > 0$ on $\mathbb{D}$, the polynomial $g := p + (\lambda, y)q - (\gamma - \epsilon)q$ is positive on $\mathbb{D}$. By Putinar Positivistellensatz [13], it may be written

$$g = u_0 + \sum_{j=1}^m u_j g_j$$

for some family of s.o.s. polynomials $\{u_j\}_{j=0}^m \subset \mathbb{R}[x]$. But then, with $2r_M \geq \max[\deg u_0, \deg u_j g_j]$, the 3-plet $(\lambda, \gamma - \epsilon, \{u_j\})$ is a feasible solution of $Q^*_{rx}$, whenever $r \geq r_M$, and with value $M - \epsilon$. And so, as $M$ was arbitrarily large, sup $P^*_{rx} \rightarrow +\infty = \hat{f}(x)$, as $r \rightarrow \infty$. This concludes the proof of (17) for $x \notin \mathbb{K}$.

(b) That $f_r$ is convex follows from its definition $f_r(x) = \inf Q_{rx}$ for all $x \in \mathbb{R}^n$, and the definition (13) of the SDP $Q_{rx}$. First, observe that for all $r$ sufficiently large, say $r \geq r_0$, inf $Q_{rx} > -\infty$ for all $x \in \mathbb{R}^n$, because sup $Q^*_{rx} \geq -1$, for all $x \in \mathbb{R}^n$. Indeed, with $\gamma = -1, \lambda = 0$, the polynomial $p + q = p - q(\lambda, y)$ is positive on $\mathbb{D}$, and therefore, by Putinar Positivistellensatz [13], $p + q = u_0 + \sum_j u_j g_j$, for some family of s.o.s. polynomials $\{u_j\}_{j=0}^m$. Therefore, $(-1, 0, \{u_j\})$ is feasible for $Q^*_{rx}$ with
value $-1$, whenever $2r_0^j \geq \max\{\deg u_0, \deg u_jg_j\}$. Next, let $x := \alpha u + (1 - \alpha)v$, with $u, v \in \mathbb{R}^n$, and $0 \leq \alpha \leq 1$. As we want to prove
\[ \inf Q_{(\alpha u + (1 - \alpha)v)} \leq \alpha \inf Q_{ru} + (1 - \alpha) \inf Q_{rv}, \quad 0 \leq \alpha \leq 1, \]
we may restrict to $u, v \in \mathbb{R}^n$ such that $\inf Q_{ru}, \inf Q_{rv} < +\infty$. So, let $y_u$ (resp. $y_v$) be feasible for $Q_{ru}$ (resp. $Q_{rv}$), and with respective values $\inf Q_{ru} + \epsilon$, $\inf Q_{rv} + \epsilon$. As the matrices $M_{r}(y), M_{r-}\delta_{r}(y)$ are all linear in $y$, and $y \rightarrow L_{y}(\delta_{r})$ is linear in $y$ as well, $y := \alpha y_u + (1 - \alpha)y_v$ is feasible for $Q_{rx}$, with value $\alpha \inf Q_{ru} + (1 - \alpha) \inf Q_{rv} + \epsilon$. Therefore,
\[ \inf Q_{rx} = \inf Q_{(\alpha u + (1 - \alpha)v)} \leq \alpha \inf Q_{ru} + (1 - \alpha) \inf Q_{rv} + \epsilon, \quad \forall \epsilon > 0, \]
and letting $\epsilon \rightarrow 0$ yields the result. Finally, the pointwise convergence $f_{r}(x) \uparrow \bar{f}(x)$ for all $x \in \mathbb{R}^n$, follows from (17).

(c) Let $K$ be with a nonempty interior int $K$, and let $x \in \text{int} K$. Let $\mu$ be the probability measure uniformly distributed on the ball $B_{x} := \{y \in K : \|y - x\| \leq \delta\} \subseteq K$.

Hence, $\int z d\mu = x_i$ for all $i = 1, \ldots, n$. Next, define the measure $\nu$ to be $d\nu = q^{-1} d\mu$ so that $\int q d\nu = 1$, and $\int z_i d\nu = \int z_i q d\nu = x_i$ for all $i = 1, \ldots, n$. Take for $y = \{y_i\}$, the vector of moments of the measure $\nu$. As $\nu$ has a density, and is supported on $K$, it follows that $M_{r}(y) > 0$ and $M_{r}(y_j) > 0, j = 1, \ldots, m$, for all $r$. Therefore, $y$ is a strictly feasible solution of $Q_{rx}$, i.e., Slater’s condition holds, which in turn implies the absence of a duality gap between $Q_{rx}$ and its dual $Q^{*}_{rx}$ (sup $Q^{*}_{rx} = \inf Q_{rx}$). In addition, as $\inf Q_{rx} > -\infty$, we get $\sup Q^{*}_{rx} = \max Q^{*}_{rx} = \inf Q_{rx}$, which is (19).

So, as $Q^{*}_{rx}$ is solvable, let $(\gamma^{*}_{r}, \lambda^{*}_{r}, \{u^{*}_{j}\})$ be an optimal solution, that is, $f_{r}(x) = \gamma^{*}_{r} - \langle \lambda^{*}_{r}, x \rangle$ and
\[ p(z) - \gamma^{*}_{r} q(z) - q(z) \langle \lambda^{*}_{r}, y \rangle = \sum_{j=0}^{m} u^{*}_{j}(z)g_{j}(z), \quad \forall z \in \mathbb{R}^n. \]

Therefore, one has
\[ f_{r}(x) = \gamma^{*}_{r} + \langle \lambda^{*}_{r}, x \rangle \]
\[ f_{r}(y) = \sup_{\gamma, \lambda} \{ \gamma + \langle \lambda, y \rangle : p(z) - \gamma q(z) - q(z) \langle \lambda, z \rangle = \sum_{j=0}^{m} u_{j}(z)g_{j}(z), \forall z \in \mathbb{R}^n \} \geq \gamma^{*}_{r} + \langle \lambda^{*}_{r}, y \rangle, \quad \forall y \in \mathbb{R}^n, \]
from which we get
\[ f_{r}(y) - f_{r}(x) \geq \langle \lambda^{*}_{r}, y - x \rangle, \quad \forall y \in \mathbb{R}^n, \]
that is, $\lambda^{*}_{r} \in \partial f_{r}(x)$, the desired result. \hfill \Box

References