The node-edge weighted 2-edge connected subgraph problem: linear relaxation, facets and separation

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Résumé: Soit $G=(V,E)$ un graphe non-orienté et *2-arêtes connexe*. Chaque arête et sommet de G est muni d'un poids. Le *problème du sous-graphe 2-arêtes connexe de poids minimum* dans G (2ECSP), est de trouver un sous-graphe 2-arêtes connexe de G tel que la somme des poids sur ses sommets et ses arête soit minimum. Le 2ECSP généralise le problème, bien connu, du sous-graphe Steiner 2-arêtes connexe. Dans cet article, l'enveloppe convexe des vecteurs d'incidences des solutions de 2ECSP est étudiée. Une formulation naturelle du problème par un programme linéaire en nombres entiers est premièrement établie. Il est aussi montré que la relaxation ne suffit pas pour décrire l'enveloppe convexe associée au 2ECSP même dans une classe restreinte comme celle des graphes série-parallèles. Une nouvelle classe d'inégalités valides pour le 2ECSP est introduite. Il est monté qu'une sous-classe de ces inégalités peut être séparée en temps polynomial. Comme conséquence, ces nouvelles inégalités peuvent être séparées en temps polynomial dans la classe des graphes série-parallèles.

Abstract: Let $G=(V,E)$ be an undirected *2-edge connected* graph with weights on its edges and nodes. The *minimum 2-edge connected subgraph problem*, 2ECSP for short, is to find a 2-edge connected subgraph of G, of minimum total weight. The 2ECSP generalizes the well-known Steiner 2-edge connected subgraph problem. In this paper the convex hull of the incidence vectors corresponding to feasible solutions of 2ECSP is studied. First, a natural integer programming formulation is given and it is shown that its linear relaxation is not sufficient to describe the polytope associated with 2ECSP even when G is series-parallel. Then, a class of new valid inequalities is defined together with sufficient conditions for them to be facet-defining. Finally, a polynomial time algorithm is given to separate a subclass of these new inequalities. As a consequence, all these new inequalities may be separated in polynomial time when G is series-parallel.

Mots clés : Optimisation combinatoire, la connexité dans les graphes, polyèdres.

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1 Introduction

Let $G = (V, E)$ be an undirected graph. G is said to be *k-edge* (resp. *k-node*) *connected* if for any pair of nodes $i, j \in V$ there exists at least k edge-disjoint (resp. node-disjoint) paths from i to j. Associate with each edge $e \in E$ a weight c_e and with each node $v \in V$ a weight w_v. The *node-edge weighted 2-edge connected subgraph problem*, denoted by 2ECSP, consists in finding a 2-edge connected subgraph of G (not necessarily spanning all the nodes in V), whose total weight of both nodes and edges is minimized. So the graphs considered in this paper are 2-edge connected. A related problem is to find a 2-node connected subgraph of G whose total weight of both nodes and edges is minimized. This problem is discussed in Section 4, where it is shown how the results obtained for 2ECSP may be applied.

To our knowledge this problem has never been considered in the literature, although some related problems have been studied. For instance, the case where the node weights are large negative numbers for some nodes $v \in T$ (terminals) and 0 for nodes $v \in V \setminus T$, the 2ECSP reduces to the well known Steiner 2-edge connected subgraph problem (STECSP) introduced by Monma, Munson and Pulleyblank in [9]. Given a graph and a set of *terminals* $T \subset V$, the problem is to find a minimum (edge) weight 2-edge connected subgraph of G spanning T. Polyhedral characterizations of the STECSP may be found in [1], [2] and in [8] and [3] when $T = V$. Closely related problems to the STECSP in network design were introduced by Grötschel and Monma in [6]. Stoer [10] surveys related works.

The Steiner 2-edge connected subgraph problem, where the only costs pertain to edges, arise in the design of reliable telecommunication networks: to link (to establish edges between) centers (nodes) that are already determined, at least total cost but that assures that all phone centers (a subset of special nodes) remain connected when one link fails. The 2ECSP is a direct generalization that recognizes that centers are built with costs too, so that a more realistic goal is to minimize the total costs of establishing nodes and links.

Let Z^* be the value of the optimal solution to 2ECSP. In what follows, we fix a node $r \in V$ called the *root*. Consider the problem of finding a 2-edge connected subgraph of G containing r whose total weight, of both nodes and edges, is minimized. We will refer to this problem as the *r-2-edge connected subgraph problem* (r-2ECSP). If Z_r^* denotes the value of the optimal solution of the r-2ECSP, then clearly $Z^* = \min_{r \in V}\{Z_r^*\}$. The idea of fixing a node r was introduced in [4]. It makes it easy to deal with the connectivity of the solutions and leads to a simple formulation of the r-2ECSP as an integer linear program.

We now give some standard definitions used throughout the paper. Consider $F \subseteq E$ and $U \subseteq V$, then $(x^F, y^U) \in \mathbb{R}^{|E|+|V|}$ denotes the *incidence vector* of the subgraph (U, F) of G, i.e., $x_e^F = 1$ if $e \in F$ and 0 otherwise, and $y_v^U = 1$ if $v \in U$ and 0 otherwise. As usual, for any subset of edges (resp. nodes) $F \subseteq E$ (resp. $U \subseteq V$), $x(F) = \sum_{e \in F} x_e$ (resp. $y(U) = \sum_{v \in U} y_v$). The set $E(W)$, for $W \subseteq V$, will denote the set of edges having both end-nodes in W and the set $\delta(W)$, called a *cut*, will denote the edges having one end-node in W and the other in $V \setminus W$. Also, by abuse of notation, $\delta(v) = \delta(\{v\})$ for $v \in V$. $G(W)$ will stand for the subgraph of G induced by W and $V(F)$ the set of nodes incident to the edge set F. If $W \subset S \subseteq V$, the set of edges having one end-node in W and the other in $S \setminus W$ is called an *S-cut* and denoted by $\delta_S(W)$ (i.e., $\delta_S(W)$ is the cut defined by W in the graph $G(S)$). Finally, for any set A, denote its complement by $\bar{A}$.

With the above definitions, the r-2ECSP can be formulated as an integer programming problem:

$$\text{minimize} \sum_{e \in E} w_e x_e + \sum_{v \in V} c_v y_v$$

subject to

$$
\begin{array}{lll}
x(\delta(W)) - 2y_v \geq 0 & \text{for all } W \subset V, r \in W, v \notin W, & (1) \\
x_e \leq y_v, & \text{for all } v \in V, \ e \in \delta(v), & (2) \\
x_e \geq 0 & \text{for all } e \in E, & (3) \\
y_v \leq 1 & \text{for all } v \in V, & (4) \\
x_e, y_v \in \{0, 1\} & \text{for all } e \in E, v \in V. & (5)
\end{array}
$$

Let r-2ECSP(G)=conv$\{(x, y) \in I\!\!R^{|E|+|V|} : (x, y) \text{ satisfies } (1) - (5)\}$ be the polytope associated with the r-2ECSP.

Consider the polytope defined by inequalities (1)-(4), called the linear relaxation of r-2ECSP(G) and denoted by $P(G)$. The projection of $P(G)$ onto the edge variables is given by

$$
\left.
\begin{array}{ll}
0 \leq x_e \leq 1 & \text{for all } e \in E, \\
x(\delta(W)) \geq 2x_e & \text{for all } W \subseteq V, r \in W, e \notin E(W).
\end{array}
\right\} \quad (6)
$$

In [2], it was shown that the above polytope is integral when G is series-parallel. One may be tempted to claim that the same holds for $P(G)$; unfortunately, the following example shows the contrary. Let $H = (V, E)$ be the series-parallel graph defined in Figure 1, where $V = \{r, v_1, v_2, v_3\}$. Let $x_e^* = \frac{1}{2}$, for all $e \in E$, $y_r^* = y_{v_3}^* = 1$ and $y_{v_1}^* = y_{v_2}^* = \frac{1}{2}$: clearly
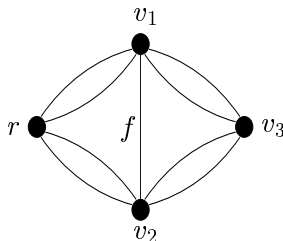
Figure 1: Example: the graph H.

$(x^*, y^*) \in P(H)$. Moreover, (x^*, y^*) is an extreme point of $P(H)$, but it violates the following valid constraint of r-2ECSP(H):

$$y_{v_1} + y_{v_2} - x_f \geq y_{v_3}.$$

In Section 2 we a give a general form for this valid constraint. The above inequality defines, in fact, a facet of r-2ECSP(H), as will be shown in Theorem 4 (in a more general setting).

This paper studies the polytope r-2ECSP(G). First, in Section 2, we introduce a family of valid inequalities and give sufficient conditions for these inequalities to be facets defining

r-2ECSP(G). Section 3 shows that the separation problem associated with a subset of these inequalities is polynomially solvable. Using this result, we obtain a polynomial time algorithm for separating the inequalities in the case of series-parallel graphs. Concluding remarks are given in Section 4.

2 The polytope r-2ECSP(G)

We begin by discussing the dimension of r-2ECSP(G), and introducing classes of valid inequalities.

Lemma 1 *Given a graph $G = (V, E)$ and a fixed node $r \in V$. Let $\sum_{v \in V} \beta_v y_y = 0$ be a valid equality of r-2ECSP(G). If G is 3-edge connected then $\beta_v = 0$ for all $v \in V$.*

Proof. That $\beta_r = 0$ is obvious since r itself constitutes an r-2-edge connected subgraph.

Let $w \in V \setminus \{r\}$. If G-$w \stackrel{\text{def}}{=} G(V \setminus \{w\})$ is 2-edge connected, then $\beta_w = 0$. So suppose the contrary. Let S be a connected component of G-w containing r (S may consist of all the nodes of G-w). Remark that $G(S \cup \{w\})$ is 3-edge connected. S may be partitioned into $S_1, S_2, \ldots, S_p$, where each $G(S_i)$ is a maximal 2-edge connected subgraph of $G(S)$; that is, if $G(W)$ is 2-edge connected for $W \subset S$, then $S_i \not\subset W$.

Let $r \in S_1$ and $T(S)$ be the graph obtained from $G(S)$ by shrinking the components S_i, $i = 1, \ldots, p$, and replacing them by nodes s_i. $T(S)$ is connected and by the maximality of each $G(S_i)$ it contains no cycles. So $T(S)$ is a tree. Moreover, since $T(S)+w$ is 3-edge connected, then there exists three node-disjoint paths P_1, P_2 and P_3 in $T(S)+w$ from s_1 to w. Denote the nodes of each path P_i, by $\{s_1, s_{i_1}, \ldots, s_{i_{k_i}}, w\}$ and let $V_i = \{S_1, S_{i_1}, \ldots, S_{i_{k_i}}, w\}$ for $i = 1, \ldots, 3$. The following subgraphs of G are r-2-edge connected: $G(V_i \cup V_j)$, $i, j = 1, 2, 3$ and $i \neq j$. These graphs have in common only the nodes S_1 and w. This yields the equations:

$$\sum_{v \in (V_i \cup V_j)} \beta_v = 0 \text{ for all } i, j = 1, 2, 3 \text{ and } i \neq j. \tag{7}$$

Also, $G(V_1 \cup V_2 \cup V_3)$ is r-2-edge connected, so

$$\sum_{v \in (V_1 \cup V_2 \cup V_3)} \beta_v = 0. \tag{8}$$

The sum of the equations in (7) minus 2 times the equation (8) gives

$$\sum_{v \in S_1} \beta_v + \beta_w = 0,$$

and since $G(S_1)$ is r-2-edge connected, we obtain β_w=0. $\square$

Theorem 2 *r-2ECSP(G) is of full dimension if and only if G is 3-edge connected.*

Proof. *Necessity.* Suppose that G is not 3-edge connected. If G is not connected or contains a bridge then it is clear that $\dim(r\text{-}2\mathrm{ECSP}(G)) < |E| + |V|$. So suppose that G contains a 2-edge cutset $\delta(W)$; that is, $\delta(W) = \{e_1, e_2\}$. Every r-2-edge connected subgraph of G verifies $x_{e_1} - x_{e_2} = 0$. Thus $\dim(r\text{-}2\mathrm{ECSP}(G)) \leq |E| + |V| - 1$.

Sufficiency. Let G be a 3-edge connected subgraph, and suppose that $\dim(r\text{-}2\mathrm{ECSP}(G)) < |E| + |V|$. Then there must exist at least one valid equality of $r\text{-}2\mathrm{ECSP}(G)$. Denote this equality by $\alpha x + \beta y = \gamma$. Since $(0,0) \in r\text{-}2\mathrm{ECSP}(G)$, it follows that $\gamma = 0$ and as G is 2-edge connected, $\sum_{e \in E} \alpha_e + \sum_{v \in V} \beta_v = 0$. Moreover, $\alpha_e = 0 \; \forall \; e \in E$, since $G \setminus \{e\}$ is r-2-edge connected for all $e \in E$. Thus, we can rewrite our equality as follows:

$$\sum_{v \in V} \beta_v y_v = 0.$$

Now from Lemma 1 it follows that $\beta_v = 0$ for all $v \in V$. Thus the equation is the trivial equation $0 = 0$. $\square$

Given a graph $G = (V, E)$, a root vertex r and $r \in S \subseteq V$. If $G(\bar{S})$ is not connected denote by $\bar{S}_1, \ldots, \bar{S}_k$ the connected components of $G(\bar{S})$; with $\bar{S}_1 = \bar{S}$ when $G(\bar{S})$ is connected. Consider the following inequalities:

$$x(\delta_S(W)) + 2 \sum_{i=1}^{k} \left(y(\bar{S}_i) - x(T_i) \right) \geq 2y_v, \tag{9}$$

$$x(\delta_S(W) \setminus \{e\}) + \sum_{i=1}^{k} \left(y(\bar{S}_i) - x(T_i) \right) \geq y_v, \tag{10}$$

where $T_i \subseteq E(\bar{S}_i)$ is a tree spanning $\bar{S}_i$, $i = 1, \ldots, k$, $W \subset S \subseteq V$ a proper subset of S, $v \in S \setminus W$ and $r \in W$. In inequalities (10), also add the condition that e is any edge in $\delta_S(W)$. Clearly inequalities (9) are a generalization of inequalities (1); they are the same when $S = V$.

Lemma 3 *Given a graph $G = (V, E)$ and a root vertex r then, for all $S \subseteq V$ with $r \in S$, inequalities (9) and (10) are valid for $r\text{-}2ECSP(G)$.*

Proof. One can prove the validity of (9) and (10) by using the fact that $y(\bar{S}_i) - x(T_i) \geq 0$, for all $i = 1, \ldots, k$, and the structure of the r-2-edge connected subgraph of G. Or else they may be obtained by combining inequalities (1)-(4) and rounding up the right hand side; that is, (9) and (10) are obtained as Chvátal-Gomory cuts of rank 1. For the sake of completeness we include this proof for inequalities (10) when $G(\bar{S})$ is connected; that is, we show that

$$x(\delta_S(W) \setminus \{e\}) + y(\bar{S}) - x(T) \geq y_v,$$

is valid for $r \in W \subset S \subseteq V$, $v \in S \setminus W$ and T a spanning tree in $G(\bar{S})$.

If A and B are two node sets, (A, B) denotes the set of edges having one end-node in A and the other in B. Let $e = uw \in \delta_S(W)$ with $w \in S \setminus W$. From (1),

$$x(W, \bar{S}) + x(\delta_S(W)) = x(\delta(W)) \geq 2y_v,$$

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and
$$x(W, \bar{S}) + x(\delta_S(W)) = x(\delta(W)) \geq 2y_w,$$
it follows that $x(W, \bar{S}) + x(\delta_S(W)) \geq y_v + y_w$, and by combining with $y_w \geq x_e$ we obtain $x(W, \bar{S}) + x(\delta_S(W) \setminus \{e\}) \geq y(v)$. Also, inequalities (2) yield $x(W, \bar{S}) \leq \sum_{u \in \bar{S}} \mathrm{dg}_W(u) y_u$, where $\mathrm{dg}_W(u)$ denotes $|(\{u\}, W)|$. Hence

$$x(\delta_S(W) \setminus \{e\}) + \sum_{u \in \bar{S}} \mathrm{dg}_W(u) y_u \geq y_v \tag{11}$$

To complete the proof the following definitions are needed.

- Let v_0 be a special node of $\bar{S}$ and p be the length of the longest path in T.

- Define $L_0 = \{v_0\}$ and $L_i = \{v \in \bar{S} : \exists \; u \in L_{i-1} \text{ with } e = uv \in T\}$, for $i = 1, \ldots, p$.

- Let $v \in L_i$, $i \neq 0$, then the *father* of v, f_v, is the neighbor of v in L_{i-1} with $e = f_v v \in T$.

- Let $v \in L_i$, $i \neq p$, define $s^0(v) = \{v\}$ and $s^l(v) = \{w \in L_{i+l} : \exists \; u \in s_{l-1} \text{ with } e = uw \in T\}$, for $l = 1, \ldots, p - i$. Let $\bar{S}_v = \cup_{l=0}^{p-i} s^l(v)$. $s^1(v)$ may be seen as the *sons* of v and $\bar{S}_v$ as the *progeny* of v.

The inequalities (2) imply

$$(|(W, \bar{S})| - |(W, \bar{S}_u)|) y_u \quad \geq \quad (|(W, \bar{S})| - |(W, \bar{S}_u)|) x_{f_u u} \quad \text{for all } u \in \bar{S} \setminus \{v_0\}, \tag{12}$$
$$|(W, \bar{S}_v)| y_u \quad \geq \quad |(W, \bar{S}_v)| x_{uv} \quad \text{for all } u \in \bar{S} \setminus L_p \text{ and } v \in s^1(u). \tag{13}$$

Summing, carefully, the inequalities (11), (12) and (13) yields

$$|(W, \bar{S})| \big(y(\bar{S}) - x(T) \big) + x(\delta_S(W) \setminus \{e\}) \geq y_v.$$

Hence, $|(W, \bar{S})| \big(y(\bar{S}) - x(T) + x(\delta_S(W) \setminus e) \big) \geq y_v$, and by dividing by $|(W, \bar{S})|$ and rounding up the result is obtained. $\square$

For particular values of S, W, v, e and $F = \bigcup_{i=1}^k T_i$ (F is then a forest spanning $\bar{S}$), we will refer to (9) as (S, W, v, F) and to (10) as (S, W, v, e, F). When we write (S, W, v, T) or (S, W, v, e, T) we mean that $G(\bar{S})$ is connected and T is a spanning tree of $G(\bar{S})$. Note that when (i) $\delta_S(W) = \emptyset$ then inequalities (9) and (10) coincide, (ii) $\delta_S(W) = \{e\}$ then inequalities (9) are implied by (10) and $x_e \geq 0$, (iii) $S = V$ then inequalities (9) and (1) are the same and inequalities (9) are implied by (1) and (2).

Inequalities (9) are a generalization of the well-known *cut inequalities*. In [8], Mahjoub gives necessary and sufficient conditions for the cut inequalities to define facets for the polytope associated with STECSP when $T = V$. One can extend these results to get sufficient conditions for inequalities (9) to define facets of $r\text{-}2\mathrm{ECSP}(G)$. In the following we give sufficient statements under which inequalities (10) are facet defining $r\text{-}2\mathrm{ECSP}(G)$. These conditions may be weakened, but this would require more technical details and longer proofs. Our interest here is to show that inequalities (9) and (10) are necessary in a polyhedral description of $r\text{-}2\mathrm{ECSP}(G)$. We need the following definition.

Let $T \subset E$ be a tree of G spanning $V(T) \subset V$ and $v \notin V(T)$ a fixed node. A path $P = \{v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k\}$ of T has the *2-edge connected property* with respect to v, if there exists graphs G_l, G^l, for all $l = 1, \ldots, k$, such that :

- G_l is an r-2-edge connected subgraph of G containing the subpath $\{v_l, e_l, \ldots, e_{k-1}, v_k\}$ and v, and none of the nodes in $V(T) \setminus \{v_l, \ldots, v_k\}$.

- G^l is an r-2-edge connected subgraph of G containing the subpath $\{v_1, e_1, \ldots, e_{l-1}, v_l\}$ and v, and none of the nodes in $V(T) \setminus \{v_1, \ldots, v_l\}$.

If there exists a collection of paths of T having the 2-edge connected property with respect to v, such that any edge of T is contained in at least one path of that collection, then T has *the 2-edge connected property with respect to v.*

Theorem 4 *Let $G = (V, E)$ be a 3-edge connected graph. Then an inequality (S, W, v', e', T), with $|\delta_S(W)| \leq 1$ defines a facet of r-2ECSP(G) if $G(\bar{S} \cup \{v'\})$ is 2-edge connected and T has the 2-edge connected property with respect to v'.*

Proof. Consider an inequality (S, W, v', e', T) verifying the hypotheses of the theorem. We shall prove that the only valid inequalities satisfied at equality by all incidence vectors, (x, y), of an r-2-edge connected subgraph with $y(\bar{S}) - x(T) = y_{v'}$, are equivalent to (S, W, v', e', T). Assume that $\alpha x + \beta y = \gamma$ for all $(x, y) \in r$-2ECSP(G) with $y(\bar{S}) - x(T) = y_{v'}$. $(0, 0)$ and the incidence vector of G verify $y(\bar{S}) - x(T) = y_{v'}$, which implies, respectively, that $\gamma = 0$, and

$$\sum_{e \in E} \alpha_e + \sum_{v \in V} \beta_v = 0 \tag{14}$$

Also, since $G \setminus \{f\}$, for all $f \notin T$, are r-2-edge connected subgraphs and their incidence vectors verify $y(\bar{S}) - x(T) = y_{v'}$, this implies that $\sum_{e \in E \setminus \{f\}} \alpha_e + \sum_{v \in V} \beta_v = 0$, which combined with (14) yields

$$\alpha_f = 0 \quad \text{for all } f \notin T.$$

Define G^* to be the graph obtained from G by shrinking $\bar{S} \cup \{v'\}$ and let v^* be the resulting node. Note that G^* is 3-edge connected. From above we know that all $(x, y) \in r$-2ECSP(G) with $y(\bar{S}) - x(T) = y_{v'}$ verify

$$\sum_{v \in V} \beta_v y_v + \sum_{e \in T} \alpha_e x_e = 0. \tag{15}$$

We claim that

$$\sum_{v \in V \setminus (\bar{S} \cup \{v'\})} \beta_v y_v^* + \beta_{v^*} y_{v^*}^* = 0 \text{ for all } (x^*, y^*) \in r\text{-2ECSP}(G^*), \tag{16}$$

where $\beta_{v^*} = \sum_{v \in (\bar{S} \cup \{v'\})} \beta_v + \sum_{e \in T} \alpha_e$. In fact, suppose $(x^*, y^*) \in r$-2ECSP(G^*) does not satisfy (16). Define $y_v = y_v^*$ if $v \notin (\bar{S} \cup \{v'\})$, otherwise $y_v = y_{v^*}^*$, and $x_e = y_{v^*}^*$ if $e \in E(\bar{S} \cup \{v'\})$, otherwise $x_e = x_e^*$. Then (x, y) belongs to r-2ECSP(G) (since $G(\bar{S} \cup v')$ is 2-edge connected); moreover $y(\bar{S}) - x(T) = y_{v'}$ and (x, y) does not verify (15), which is a contradiction. Now applying Lemma 1 to G^* and the equality (16) it follows that

$$\beta_v = 0 \quad \text{for all } v \in V \setminus (\bar{S} \cup \{v'\}).$$

Next, we show that $\beta_v = -\alpha_e = -\beta_{v'}$ for all $v \in \bar{S}$, $e \in T$. Let $P = v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k$ be a path of T having the 2-edge connected property with respect to v'. Then the incidence

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vectors of the graphs G_l, (resp. G^l) , for $l = 1, \ldots, k$, verify $y(\bar{S}) - x(T) = y_{v'}$ and thus $\alpha x + \beta y = \gamma$, which implies $\beta_{v_i} + \alpha_{e_i} = 0$ for $i = 1, \ldots, k-1$ and $\beta_{v_k} = -\beta_{v'}$ (resp. $\beta_{v_i} + \alpha_{e_{i-1}} = 0$ for $i = 2, \ldots, k$, $\beta_{v_1} = -\beta_{v'}$). Combining these equalities we obtain, $\beta_{v_i} = -\beta_{v'}$ for $i = 1, \ldots, k$ and $\alpha_{e_i} = \beta_{v'}$ for $i = 1, \ldots, k-1$. Moreover, since any edge of T is contained in a path of T having the 2-edge connected property with respect to v,

$$\beta_v = -\alpha_e = -\beta_{v'} \quad \text{for all } v \in \bar{S} \text{ and } e \in T.$$

We have shown that $\alpha x + \beta y = \gamma$ is $\beta_{v'}$ times $y(\bar{S}) - x(T) = y_{v'}$. This means that (S, W, v', e', T) defines a facet of r-2ECSP(G). $\square$

Corollary 5 *Suppose $G(S \cup \bar{S}_i)$ is 3-edge connected for all i. An inequality $(S, W, v', e', T_{i=1,\ldots,k})$ with $|\delta_S(W)| \leq 1$, defines a facet of r-2ECSP(G) if for $i = 1, \ldots, k$ the following hold: $G(\bar{S}_i \cup \{v'\})$ is 2-edge connected; T_i has the 2-edge connected property with respect to v'.*

Note that Theorem 4 and Corollary 5 may be used to generate a large class of graphs where inequalities (1)-(4) are not sufficient to describe r-2ECSP(G).

Before beginning the next section two subclasses of inequalities (9) and (10) are given and, as shall be seen later, they can be separated in polynomial time.

Given a graph $G = (V, E)$, $S \subseteq V$, W a proper subset of S, $v \in S \setminus W$ and $r \in W$. Consider the following inequalities:

$$x(\delta_S(W)) + 2y(\bar{S}) \geq 2y_v, \tag{17}$$

$$x(\delta_S(W) \setminus \{e\}) + y(\bar{S}) \geq y_v. \tag{18}$$

(17) will be denoted by (S, W, v) and (18) by (S, W, v, e). Inequalities (17) (resp. (18)) are either included in inequalities (9) (resp. (10)) (when $\bar{S}$ is an independent set) or implied by (9) (resp. (10)).

3 Separation

The *separation problem* of a given set of inequalities is to determine whether a given vector satisfies this set of inequalities and, if not, to find an inequality in the set that is violated. It follows from the equivalence between separation and optimization [5] that if the separation problem is solvable in polynomial time then the optimization over this system of inequalities, is also polynomial.

The number of inequalities (2)-(4) is polynomial, thus their separation is straightforward. Also, the separation problem of inequalities (1) can be easily reduced to a min-cut problem and hence can be solved in polynomial time as well. From now on, we are given a point $(\bar{x}, \bar{y})$ satisfying inequalities (1)-(4). First, consider the separation of inequalities (9).

Let $G = (V, E)$ be a graph and $r \in V$ a root vertex. Let $(\bar{x}, \bar{y}) \in \mathbb{R}^{|E|+|V|}$ be a solution verifying inequalities (1)-(4). For $v \in V \setminus \{r\}$ and $S \subseteq V$, let $f^v(S)$ be the function defined as follows:

$$f^v(S) = \begin{cases} +\infty & \text{if } \{r, v\} \not\subset S \\ \min_{r \in W \subset S \setminus \{v\}} \{\bar{x}(\delta_S(W))\} + 2\bar{y}(\bar{S}) - 2 \max_{F \subseteq E(\bar{S})} \{\bar{x}(F) : F \text{ forest}\} & \text{otherwise.} \end{cases}$$

7

Note that given S and v, the value $f^v(S)$ can be computed in polynomial time by a single minimum r-v cut computation in $G(S)$, plus a maximum forest computation in $G(\bar{S})$.

Separating inequalities (9) reduces to the minimization of $f^v(S)$ among all subsets S of V and for every $v \in V$. If one finds $w \in V$ and $\hat{S}$ with $f^w(\hat{S}) < 2\bar{y}_w$ then $(\hat{S}, W, w, F)$ defines a violated inequality of type (9), where $\delta_{\hat{S}}(W)$ is a cut of minimum capacity (equal to $\bar{x}(\delta_{\hat{S}}(W))$) separating r and w, and $F = \cup_{i=1}^{k} T_i$ is a maximum forest (of weight $\bar{x}(F)$) spanning $\bar{\hat{S}}$. Otherwise, there exists no violated inequality of type (9). By the same manner one can define a function whose minimization solves the separation problem associated with inequalities (10). For $S = V$, $f^v(\cdot)$ is submodular but, unfortunately, in general it is not. However, there are some cases where the minimization of f^v can still be done in polynomial time. It turns out that the separation of inequalities (9) and (10) can be done in polynomial time in series-parallel graphs.

3.1 Separation of inequalities (17) and (18)

Construct a network $D_{(\bar{x}, \bar{y})} = (N, A)$ from G and the vectors $\bar{x}$ and $\bar{y}$ as follows. Duplicate every node v of G into two nodes v', v''. Add two arcs (v', v'') with capacity $c(v', v'') = \bar{y}_v$, and (v'', v') with capacity $c(v'', v') = \infty$. Replace every edge $e = vw$ of G by two arcs (v'', w') and (w'', v') both having the capacity $c(v'', w') = c(w'', v') = \bar{x}_e$. An example is shown in Figure 2.
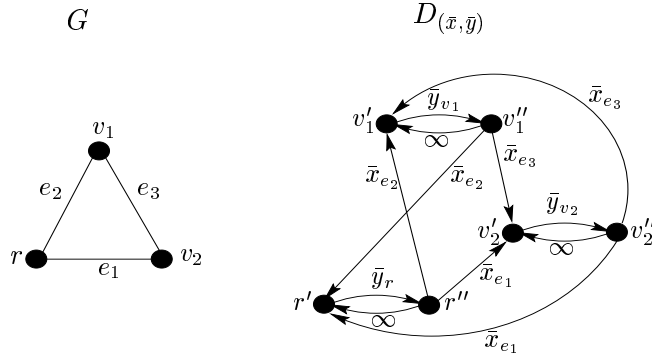

Figure 2: The values associated with the arcs of $D_{(\bar{x}, \bar{y})}$ represent the capacities.

If U is a subset of N, $\delta^+(U) = \{(u, v) \in A : u \in U \text{ and } v \in N \setminus U\}$ is called a *directed-cut*. Let $V' = \{u_1, \ldots, u_k\}$ be a node subset of V. $D_{(\bar{x}, \bar{y})}^{V'}$ is the network obtained from $D_{(\bar{x}, \bar{y})}$ by identifying u_i' with u_i'' and the resulting node is u_i, for $i = 1, \ldots, k$.

Separation of inequalities (17)

Define the function $g^v(\cdot)$:

$$g^v(S) = \begin{cases} +\infty & \text{if } \{r,v\} \not\subset S \\ \displaystyle\min_{r \in W \subset S \setminus \{v\}} \{\bar{x}(\delta_S(W))\} + 2\bar{y}(\bar{S}) & \text{otherwise.} \end{cases}$$

Let $g^v(S^*) = \min_{S \subseteq V} g^v(S)$ for $v \in V$. If $g^v(S^*) \geq 2\bar{y}_v$ for all $v \in V$, then there is no violated inequality (17). Otherwise, we can show a violated inequality. It remains to see how to solve the minimization problem of $g^v(.)$. We show that this reduces to a min-cut problem in the network $D^{\{r,v\}}_{(\bar{x},2\bar{y})}$ defined from G and the vectors $\bar{x}$ and $2\bar{y}$.

Lemma 6 *For all $S \subseteq V$, $W \subset S$, $r \in W$ and $v \in S \setminus W$ there exists a directed-cut $\delta^+(U')$ of $D^{\{r,v\}}_{(\bar{x},2\bar{y})}$ separating r from v such that $c(\delta^+(U')) = \bar{x}(\delta_S(W)) + 2\bar{y}(\bar{S})$.*

Proof. Take $U' = \{r\} \cup (\bigcup_{v \in W}\{v',v''\}) \cup (\bigcup_{v \in \bar{S}}\{v'\})$. $\square$

Lemma 7 *Let $\delta^+(U^*)$ be a minimum capacity directed-cut of $D^{\{r,v\}}_{(\bar{x},2\bar{y})}$ separating r from v. Then there exists $W \subset S' \subseteq V$, $r \in W$ and $v \in S' \setminus W$ with $\bar{x}(\delta_{S'}(W)) + 2\bar{y}(\bar{S}') = c(\delta^+(U^*))$.*

Proof. By Lemma 6, $c(\delta^+(U^*)) \neq \infty$. Hence $v'' \in U^*$ implies $v' \in U^*$. Define $\bar{S}'$ as the set of nodes v such $v' \in U^*$ and $v'' \notin U^*$, and W as the set of nodes v such $v',v'' \in U^*$. Add r to W. Now by the definition of $D^{\{r,v\}}_{(\bar{x},2\bar{y})}$ we have $\bar{x}(\delta_{S'}(W)) + 2\bar{y}(\bar{S}') = c(\delta^+(U^*))$. $\square$

From the two lemmas above follows what had to be shown,

$$g^v(S^*) = c(\delta^+(U')) \geq c(\delta^+(U^*)) \geq g^v(S^*).$$

Separation of inequalities (18)

The separation of inequalities (18) is along the same lines as of inequalities (17).

To separate all inequalities (S,W,v,e) corresponding to a fixed v and e, consider $G' = (V, E \setminus \{e\})$ (i.e. G' is obtained from G by removing e). Then fix v and minimize the function

$$h^v(S) = \begin{cases} +\infty & \text{if } \{r,v\} \not\subset S \\ \displaystyle\min_{r \in W \subset S \setminus \{v\}} \{\bar{x}(\delta_S(W))\} + \bar{y}(\bar{S}) & \text{otherwise,} \end{cases}$$

where $\delta_S(\cdot)$ is taken in G'. As for $g^v(.)$, this problem reduces to a minimum capacity directed-cut problem separating r from v in the network $D^{\{r,v\}}_{(\bar{x},\bar{y})}$ defined from G', the restriction of $\bar{x}$ on G' and the vector $\bar{y}$.

Let $h^v(S^*) = \min_{S \subseteq V} h^v(S)$ for $v \in V$. If $h^v(S^*) \geq \bar{y}_v$, then $\bar{x}(\delta_S(W) \setminus \{e\}) + \bar{y}(\bar{S}) \geq \bar{y}_v$ for all S. Hence there is no violated inequality (S,W,v,e) (for fixed v and e). If on the other hand $h^v(S^*) < \bar{y}_v$, we can exhibit a violated inequality (18). Repeating the procedure for every v and e, the separation problem for inequalities (18) is solved.

Remark. Say that an inequality (S,W,v,F) or (S,W,v,e,F) is of *class 1* if the nodes of $\bar{S}$ are pairwise non-adjacent. It is easy to see that inequalities (S,W,v) and (S,W,v,e) contain inequalities of class 1. It follows that the separation problem for inequalities of class 1 is solvable in polynomial time.

3.2 Another polynomial time separable subclass of (9) and (10)

Consider inequalities (S, W, v, T) or (S, W, v, e, T) with $\delta_S(W) = \emptyset$, $G(\bar{S})$ connected and where T is a path spanning $\bar{S}$. Only the pendant nodes of T are connected with S. This subclass will be called inequalities of *class 2*.

If u and w represent the pendant nodes of T, then $G(V \setminus \{u, w\})$ contains at least two connected components, W_1 containing r and W_2 containing v. The separation problem reduces to finding a path $P = \{u = v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k = w\}$ in $G(V \setminus (W_1 \cup W_2))$ that minimizes

$$\bar{y}_{v_1} + \bar{y}_{v_k} + \sum_{i=2}^{k-1} \bar{y}_{v_i} - \sum_{i=1}^{k-1} \bar{x}_{e_i}. \tag{19}$$

If $\bar{y}_u + \bar{y}_w + \sum_{i=2}^{k-1} \bar{y}_{v_i} - \sum_{i=1}^{k-1} \bar{x}_{e_i} < \bar{y}_v$, then a violated inequality of class 2 is obtained, where $S = V \setminus \{u, v_2, \ldots, v_{k-1}, w\}$, $W = W_1$, $\bar{S} = \{u, v_2, \ldots, v_{k-1}, w\}$ and $T = \{e_1, \ldots, e_{k-1}\}$ is a path spanning $\bar{S}$. Otherwise, there is no violated inequality of class 2, where u and w are the pendant nodes of the path T spanning $\bar{S}$.

How is (19) to be solved? Given a triplet v, u and w such that $G \setminus \{u, w\}$ contains at least two connected components W_1 containing r and W_2 containing v. Construct the network $\bar{D}_{(\bar{x}, \bar{y})}$ from the graph $G' = G(V \setminus (W_1 \cup W_2))$ as follows: replace each edge of G', $e = u_1 u_2$, not incident to u nor to w, by two arcs (u_1, u_2) associated with a cost $c(u_1, u_2) = \bar{y}_{u_1} - \bar{x}_e$ and a reverse arc (u_2, u_1) with cost $c(u_2, u_1) = \bar{y}_{u_2} - \bar{x}_e$. If $e = uu_1$ (resp. $e = u_1 w$) is an edge of G' incident to u (resp. w) then replace e by an arc (u, u_1) (resp. (u_1, w)) having a cost $c(u, u_1) = \bar{y}_u - \bar{x}_e$ (resp. $c(u_1, w) = \bar{y}_{u_1} + \bar{y}_w - \bar{x}_e$).

Problem (19) reduces to a min-cost path problem from u to w in $\bar{D}_{(\bar{x}, \bar{y})}$. $(\bar{x}, \bar{y})$ verifies inequalities (2)-(4). Consequently the cost associated with each arc of $\bar{D}_{(\bar{x}, \bar{y})}$ is nonnegative. One can apply, for example, Dijkstra's algorithm to find such a path.

3.3 Application to series-parallel graphs

A *homeomorph* of K_4 (the complete graph on four nodes) is a graph obtained from K_4 when its edges are subdivided into paths by inserting new nodes of degree two. A graph is called *series-parallel* if it contains no homeomorph of K_4 as a subgraph.

Lemma 8 *Let $G = (V, E)$ be a graph and r a fixed node. Inequalities (S, W, v, F) and (S, W, v, e, F) ($F = \cup_{i=1}^{k} T_i$) define facet of r-2ECSP(G) only if*

(i) $G(W)$ is connected and,

(ii) every pendant node of T_i, for $i = 1, \ldots, k$, is connected to W and to $S \setminus W$; moreover, if $G(S \setminus W)$ is not connected then at least one of its connected components is connected to at least two pendant nodes of T_i, for all $i = 1, \ldots, k$.

Proof. The proof is given for the inequality (S, W, v, F), i.e., an inequality (9). A similar proof holds for an inequality (10).

(i) If $G(W)$ is not connected, let W_1 be the connected component of $G(W)$ containing r, then the inequality (S, W, v, F) is implied by (S, W_1, v, F).

(ii) Let $v_l \in \bar{S}_l$ be a pendant node of T_l and e_l be the edge of T_l incident to v_l, for $1 \leq l \leq k$. Suppose that v_l is not connected to W. Define $S' = S \cup \{v_l\}$; $\bar{S}'_i = \bar{S}_i$ for $i = 1, \ldots, k, i \neq l$;

$\bar{S}'_l = \bar{S}_l \setminus \{v_l\}$; $T'_i = T_i$, for $i = 1, \ldots, k$, $i \neq l$; $T'_l = T_l \setminus \{e_l\}$. Note that T'_l is a tree spanning $\bar{S}'_l$, with $y(\bar{S}'_l) - x(T'_l) \leq y(\bar{S}_l) - x(T_l)$. Hence the inequality $(S, W, v, \cup_{i=1}^k T_i)$ is implied by $(S', W, v, \cup_{i=1}^k T'_i)$. Thus it may be assumed that v_l is connected to W. Now if v_l is not connected to $S \setminus W$ then, by moving v_l to W we show that the inequality $(S, W, v, \cup_{i=1}^k T_i)$ is redundant.

Assume now that $G(S \setminus W)$ is not connected and that each of its connected components is connected to at most one pendant node of T_i, for $i = 1, \ldots, k$. Then, as shown above, every pendant node is connected to $S \setminus W$. Consequently there is a connected component of $G(S \setminus W)$ which does not contain v, denoted by $\bar{W}_1$, that is connected to exactly one of the pendant nodes of T_i, for $i = 1, \ldots, k$, (of course it may be connected to W and some nodes in $\bar{S}$). Let $v_l \in \bar{S}_l$ be such a node. Define $S' = S \cup \{v_l\}$, $W' = W \cup \bar{W}_1 \cup \{v_l\}$ and T'_i, $i = 1, \ldots, k$, as defined above. As v_l is connected to W it follows that $G(W')$ is connected. Consequently, the inequality $(S, W, v, \cup_{i=1}^k T_i)$ is implied by $(S', W', v, \cup_{i=1}^k T'_i)$. $\square$

Theorem 9 *If $G = (V, E)$ is a series-parallel graph, then inequalities (9) and (10) are either of class 1 or of class 2.*

Proof. Let $(S, W, v, \cup_{i=1}^k T_i)$ be an inequality (9). Suppose that there exists T_l, $1 \leq l \leq k$, which is not a path. Then T_l contains at least three pendant nodes. Suppose that $G(S \setminus W)$ is not connected. From Lemma 8 (ii), there exists a connected component $\bar{W}_1$ of $G(S \setminus W)$ connected to at least two pendant nodes of T_l, say v_1 and v_2. Let v_3 be a pendant node of T_l different from v_1 and v_2. By Lemma 8 (ii), the nodes v_1, v_2 and v_3 are connected to W, and by Lemma 8 (i) $G(W)$ is connected, so there is a K_4 that is a subgraph of G, see Figure 3.
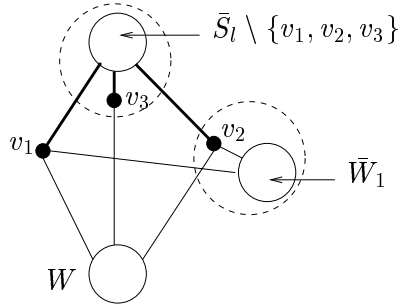

Figure 3: K_4 as subgraph. Bold edges belong to T_l.

K_4 is obtained by shrinking the following connected components: W, $\bar{W}_1 \cup \{v_2\}$ and $\bar{S}_l \setminus \{v_1, v_2\}$. If $G(S \setminus W)$ is connected the same is obtained by replacing $\bar{W}_1$ by $S \setminus W$. Consequently, T_i is a path for all $i = 1, \ldots, k$. Similarly one can show that in this case, $\delta_S(W) = \emptyset$ and $k = 1$ (we have only one path). It follows that $\bar{S}$ is either an independent set of G (class 1), or $k = 1$, T_1 a path and $\delta_S(W) = \emptyset$ (class 2). The same result holds for inequalities (10). $\square$

4 Concluding remarks

Given a graph $G = (V, E)$. The node-edge weighted 2-edge connected subgraph problem has been introduced. This problem reduces to a sequence of $|V|$ r-edge connected subgraph problems (r-2ECSP). Inequalities (1)-(4) define a linear relaxation of the convex hull of the solutions of the r-2ECSP, r-2ECSP(G). These inequalities are based on a direct interpretation of the 2-edge connected property of the solutions. Unfortunately, this linear relaxation does not suffice to solve the problem even in particular classes of graphs (such as series-parallel graphs). Moreover, the graph given in Figure 1 is outer-planar, so more restricted than series-parallel graphs. Valid inequalities (9) and (10) of r-2ECSP(G) have been added in Section 2. We defined two classes among these inequalities, classes 1 and 2, and showed that their separation problem is polynomially solvable. This provide a new linear description, given by (1)-(4) plus inequalities of class 1 and 2, where the optimization can be performed in polynomial time. This linear relaxation provides better lower bounds on the value of the optimal solution of the problem. It has been shown that inequalities (9) and (10) are of class 1 and 2 when the underlying graph is series-parallel. An interesting question arise: *are inequalities (1)-(4) plus (9) and (10) sufficient to describe r-2ECSP(G) when G is series-parallel?* If the answer is positive, then there is a polynomial time algorithm to solve the node-edge weighted 2-edge connected subgraph problem in series-parallel graphs.

Consider a closely related problem to r-2ECSP: find a 2-node connected subgraph of G containing a fixed node r which minimize the overall weight of both edges and nodes. Call this problem r-2NCSP and the associated polytope r-2NCSP(G). Each solution of r-2NCSP(G) is also a solution of r-2ECSP(G). Thus all valid inequalities (1)-(4) and (9)-(10) of r-2ECSP(G) are also valid for r-2NCSP(G). Consider the following valid inequalities for r-2NCSP(G) (which are not valid for r-2ECSP(G))

$$x(\delta_{V \setminus \{v\}}(W)) \geq y_w, \text{ for all } v \in V \setminus \{r\}, r \in W \subset V \setminus \{v\}, w \in (V \setminus \{v\}) \setminus W. \qquad (20)$$

Note that (1)-(5) plus (20) give an integer linear formulation for r-2NCSP. The example of Figure 1 is a fractional extreme point of the linear relaxation of r-2NCSP(G) given by (1)-(4) and (20). But it violates inequalities of class 1 and 2. Thus inequalities (1)-(4), (20) and those of classes 1 and 2 provide a tighter linear relaxation for r-2NCSP(G). Note also that the separation problem of (20) reduces easily to a min-cut problem. The same question may be asked concerning the description of r-2ECSP(G) in series-parallel graphs.

The two linear relaxations associated with r-2ECSP(G) and r-2NCSP(G) may be used to solve the Steiner 2-edge and the Steiner 2-node connected subgraph problems.

We finish by noting that the separation problem of inequalities (9) and (10) is polynomially solvable in series-parallel graphs and that inequalities (17) and (18) can be separated in polynomial time for general graphs. What about the separation of (9) and (10) in the general case?

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