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▶ **To cite this version:**

Abdourrahmane Atto, Dominique Pastor. Central Limit Theorems for Wavelet Packet Decompositions of Stationary Random Processes. IEEE Transactions on Information Theory, 2008, pp.1-30. hal-00241849v1

HAL Id: hal-00241849

https://hal.science/hal-00241849v1

Submitted on 6 Feb 2008 (v1), last revised 17 Apr 2009 (v3)

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Central Limit Theorems for Wavelet Packet Decompositions of Stationary Random Processes

Abdourrahmane M. Atto, Dominique Pastor

hal-00241849, version 1 - 6 Feb 2008

Abstract

This paper provides central limit theorems for the wavelet packet decomposition of stationary band-limited random processes. The asymptotic analysis is performed for the sequences of the wavelet packet coefficients returned at the nodes of any given path of the M-band wavelet packet decomposition tree. It is shown that if the input process is centred and strictly stationary, these sequences converge in distribution to white Gaussian processes when the resolution level increases, provided that the decomposition filters satisfy a suitable property of regularity. For any given path, the variance of the limit white Gaussian process directly relates to the value of the input process power spectral density at a specific frequency. Experimental results are presented to appreciate the convergence rate that can be expected in practical cases when Daubechies filters are used.

Index Terms

Wavelet transforms, Band-limited stochastic processes, Spectral analysis.

I. Introduction

Wavelet Packet Transforms allow flexible representations of functions by providing various Hilbertian bases. Among derivates of the theory, the Discrete Wavelet Packet Transform (DWPT) is simple - in the sense that it can be implemented via a fast algorithm - and has some remarkable properties such as the sparse representation it provides for smooth signals [1], [2] or the 'whitening effect' it asymptotically yields for a large class of random processes [3], [4]. Thus, such wavelet representations are of great interest in signal processing, time-series analysis and communication applications.

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Given a natural number M larger than or equal to 2, the M-Band Discrete Wavelet Packet Transform, hereafter abbreviated as M-DWPT, achieves an orthogonal decomposition of a functional space $\mathbf{U}$ via a double-indexed sequence $\{\mathbf{W}_{j,n}\}_{j\in\mathbb{N},n=0,1,\ldots,M^j-1}$ of nested functional subspaces (see section II), where $\mathbb{N}=\{1,2,\ldots\}$ henceforth stands for the set of natural numbers. Each $\mathbf{W}_{j,n}$ is the closure of a space spanned by wavelet packet functions; j is called the resolution level and the shift parameter n is valued in $\{0,1,\ldots,M^j-1\}$. The standard DWPT corresponds to the particular case where $M=2$.

Throughout, X stands for some Hilbertian and real-valued centred random process, that is some process such that $X(t)\in\mathbb{R}$ where $\mathbb{R}$ is the set of real numbers, $\mathbb{E}[X(t)]=0$ and $\mathbb{E}[X(t)^2]<\infty$ for every $t\in\mathbb{R}$. The set of integers being denoted by $\mathbb{Z}$, let $c_{j,n}=(c_{j,n}[k])_{k\in\mathbb{Z}}$ stand for the coefficients returned by the M-DWPT of X at node (j,n). At each resolution level, the discrete random process $c_{j,n}$ results from the projection of X on $\mathbf{W}_{j,n}$. If the shift parameter n is a constant function of the resolution level j and j tends to infinity, [3] states the following two results. First, if X is Wide-Sense Stationary (WSS) and has power spectral density γ, hereafter called spectrum, it follows from [3, Corollary 5 of Proposition 11] that, when j tends to infinity, the random variables $c_{j,n}[k]$, $k\in\mathbb{Z}$, tend to decorrelate and that the variance of every random variable $c_{j,n}[k]$ tends to $\gamma(0)$; the discrete random process $c_{j,n}$ is said to be asymptotically decorrelated with variance $\gamma(0)$ when j tends to infinity. The second result concerns the case where the random process X is strictly stationary. According to [3, Proposition 12], the sequence $(c_{j,n})_{j\in\mathbb{N}}$, associated with such a process, converges, in the following 'distributional' sense, to a discrete white Gaussian process with standard deviation $\gamma(0)$ when j tends to ∞: given any natural number N and any N-uple $k_1,k_2,\ldots,k_N$ of integers, the distribution of the random vector $(c_{j,n}[k_1],c_{j,n}[k_2],\ldots,c_{j,n}[k_N])$ converges, when j tends to infinity, to the centred N-variate normal distribution $\mathcal{N}(0,\gamma(0)\mathrm{I}_N)$ with covariance matrix $\gamma(0)\mathrm{I}_N$, where I_N is the $N\times N$ identity matrix.

The case where the shift parameter n is constant with j is restrictive because most of the paths of an DWPT tree involve nodes (j,n) whose shift parameters $n=n(j)$ vary with j. Examples will be given below (see section III-A). For such paths, analysing the statistical behaviour of the sequences $c_{j,n(j)}$ returned by the DWPT when j tends to infinity becomes more intricate and the theoretical results of [3] do not apply directly. By using appropriate filters, the asymptotic decorrelation of the random process $c_{j,n}$ returned by the standard DWPT is however established in [4, section 3] for any shift parameter n which may possibly depend on j. More specifically, in [4], the DWPT filters are denoted by $h_0^{[r]}$ and $h_1^{[r]}$. Their Fourier transforms $H_0^{[r]}$ and $H_1^{[r]}$ are assumed to tend almost everywhere (a.e.) to the Fourier transforms of the Shannon DWPT filters when the parameter r, called the *order* of the DWPT filters, increases. Daubechies filters are such filters, r being the number of vanishing moments of the wavelet function associated with $h_0^{[r]}$; Battle-Lemarié filters are also such filters, r being the spline order of the scaling function associated with $h_0^{[r]}$. It is then by increasing the order of such filters with respect to the resolution level that the asymptotic decorrelation is obtained. The asymptotic value of the standard deviation of the sequence $c_{j,n}$ when j

increases is then given by the value of γ at a specific point of $[0, \pi]$. This point depends on the path followed in the DWPT tree.

The present paper extends the results given in [3], [4] in the case where the M-DWPT is achieved by means of M-DWPT filters $h_m^{[r]}$, $m = 0, 1, 2, \ldots, M-1$ that satisfy the same type of properties as those considered in [4]. Namely, the Fourier transforms $H_m^{[r]}$, $m = 0, 1, \ldots, M-1$, of these filters are assumed to tend a.e. to the Fourier transforms of the Shannon M-DWPT filters when the order r increases. These extensions are asymptotic properties of the M-DWPT when the resolution level j and the order r are large enough. They apply to any arbitrary path of the M-DWPT of a Hilbertian random process X with spectrum γ. They hold true when the shift parameter n is any, possibly unbounded, function of j. First, if X is WSS, the asymptotic decorrelation of the process $c_{j,n}$ is obtained, as in [4], by increasing the order r of the filters when the resolution level j is large enough; as in [4], the asymptotic variance of the sequence tends to the value of γ at a specific point of $[0, \pi]$, this point depending on the path followed in the M-DWPT decomposition tree. Second, if X is strictly stationary, the process $c_{j,n}$ is proved to tend in distribution to a discrete white Gaussian process whose variance is given according to the result established in the WSS case. Asymptotic Gaussianity is established by proving that cumulants of order larger than 2 vanish when the resolution level j is large enough and the order r is also large enough with respect to j.

The organization of the paper is the following. Section II recalls basics about the M-DWPT, introduces the Shannon M-DWPT, which will be instrumental for establishing the main results of the paper. After presenting some preliminary results concerning the M-DPWT of random processes, section III states asymptotic properties for the autocorrelation function $R_{j,n}$ of the process $c_{j,n}$. The input process is assumed to be WSS and the asymptotic properties of section III are established when $H_m^{[r]}$, $m = 0, 1, \ldots, M-1$, tend a.e. to the Fourier transforms of the Shannon M-DWPT filters when the order r of these filters tends to infinity. By considering this type of filter again, the convergence in distribution of the sequence $(c_{j,n})_{j \in \mathbb{N}}$ is studied in section IV, for any arbitrary path of the M-DWPT, when the input process is strictly stationary. Section V provides experimental tests, and finally, section VI concludes this work.

II. The M-DWPT

Section II-A presents some basics concerning the M-DWPT. Further details about this kind of transform can be found in [5], among others. Section II-B explains how the paths of the M-DWPT decomposition tree can be represented by M-ary sequences. This representation will be useful for describing wavelet packet subbands. Section II-C introduces the Shannon M-DWPT whose properties are crucial for establishing the results stated in sections III and IV.

A. General formulas

In what follows, j and M are natural numbers and $M \geqslant 2$. An M-DWPT is performed by using M-DWPT filters with impulse responses h_m, $m = 0, 1, 2, \ldots, M-1$, that satisfy the following properties. First, each filter h_m is an element of $\ell^2(\mathbb{Z})$ and its Fourier transform is hereafter defined by

$$H_m(\omega) = \frac{1}{\sqrt{M}} \sum_{\ell \in \mathbb{Z}} h_m[\ell] \exp(-i\ell\omega). \tag{1}$$

Second, the matrix

$$\begin{pmatrix} H_0(\omega) & H_1(\omega) & \cdots & H_{M-1}(\omega) \\ H_0(\omega + \frac{\pi}{M}) & H_1(\omega + \frac{\pi}{M}) & \cdots & H_{M-1}(\omega + \frac{\pi}{M}) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ H_0(\omega + \frac{(M-1)\pi}{M}) & H_1(\omega + \frac{(M-1)\pi}{M}) & \cdots & H_{M-1}(\omega + \frac{(M-1)\pi}{M}) \end{pmatrix},$$

is unitary for every real number ω. The unitary nature of this matrix implies that $|H_m(\omega)| \leqslant 1$ for every $m = 0, 1, \ldots, M-1$ and every $\omega \in \mathbb{R}$. For further details about the computation of M-DWPT filters, the reader is asked to refer to [5].

Let Φ be a function such that $\{\tau_k \Phi \ : k \in \mathbb{Z}\}$ is an orthonormal system of $L^2(\mathbb{R})$, where $\tau_k \Phi \ : t \longmapsto \Phi(t-k)$. Let $\mathbf{U}$ be the closure of the space spanned by this orthonormal system. Define the following sequence of elements of $L^2(\mathbb{R})$ by setting

$$W_m(t) = \sqrt{M} \sum_{\ell \in \mathbb{Z}} h_m[\ell] \Phi(Mt - \ell) \tag{2}$$

and

$$W_{Mn+m}(t) = \sqrt{M} \sum_{\ell \in \mathbb{Z}} h_m[\ell] W_n(Mt - \ell) \tag{3}$$

for $n \in \mathbb{N}$ and $m = 0, 1, 2, \ldots, M-1$. Eq. (2) can be re-written as

$$\mathcal{F}W_m(\omega) = H_m(\omega/M) \mathcal{F}\Phi(\omega/M), \tag{4}$$

where, given $f \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$, $\mathcal{F}f$ henceforth stands for the Fourier transform of f and is given by $\mathcal{F}f(\omega) = \int f(t) \exp(-i\omega t) dt$ if $f \in L^1(\mathbb{R})$. In the same way, it follows from Eq. (3) that

$$\mathcal{F}W_{Mn+m}(\omega) = H_m(\omega/M) \mathcal{F}W_n(\omega/M) \tag{5}$$

for $n \in \mathbb{N}$ and $m = 0, 1, \ldots, M-1$. Note that the function Φ in Eq. (2) is not necessarily the standard scaling function associated with the low-pass filter h_0. If Φ is this scaling function, we have $W_0 = \Phi$. In this case, Eqs. (3) and (5) hold true even if $n = 0$.

For any pair (j, n) of natural numbers, we set

$$W_{j,n}(t) = M^{-j/2} W_n(M^{-j} t) \tag{6}$$

and, according to Eqs. (1), (5) and (6), we have

$$\mathcal{F}W_{j+1,Mn+m}(\omega) = \sqrt{M} H_m(M^j \omega) \mathcal{F}W_{j,n}(\omega). \tag{7}$$

Formulas (4), (5) and (7) hold true pointwise almost everywhere [6]. Set $W_{j,n,k} = \tau_{M^j k} W_{j,n}$, that is

$$W_{j,n,k}(t) = W_{j,n}(t - M^j k) = M^{-j/2} W_n(M^{-j}t - k). \tag{8}$$

Then, the set $\{W_{j,n,k} : k \in \mathbb{Z}\}$ is an orthonormal system of $L^2(\mathbb{R})$. The closure of the functional space spanned by $\{W_{j,n,k} : k \in \mathbb{Z}\}$ will hereafter be called the *wavelet packet space* $\mathbf{W}_{j,n}$. Any $W_{j,n,k}$, and thus, any $W_{j,n}$ since $W_{j,n} = W_{j,n,0}$ and any W_n since $W_n = W_{0,n,0}$, is hereafter called a *wavelet packet function*.

The M-DWPT decomposition of the function space $\mathbf{U}$ consists in the splitting of $\mathbf{U}$ into M orthogonal subspaces:

$$\mathbf{U} = \bigoplus_{m=0}^{M-1} \mathbf{W}_{1,m}, \tag{9}$$

and by recursively applying the following splitting

$$\mathbf{W}_{j,n} = \bigoplus_{m=0}^{M-1} \mathbf{W}_{j+1,Mn+m}, \tag{10}$$

for every natural number j and every $n = 0, 1, 2, \ldots, M^j - 1$. In this decomposition, $\{W_{j+1,Mn+m,k} : k \in \mathbb{Z}\}$ is an orthonormal basis of the vector space $\mathbf{W}_{j+1,Mn+m}$. The 2 and 3-DWPT decomposition trees of figures 1 and 2 are examples of M-DWPTs.

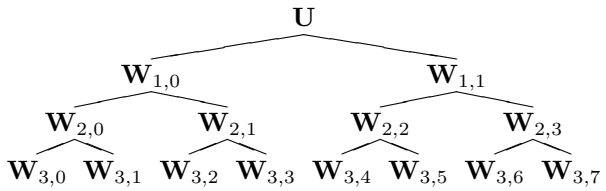

Fig. 1. 2-DWPT decomposition tree of $\mathbf{U}$ down to resolution level $j = 3$.

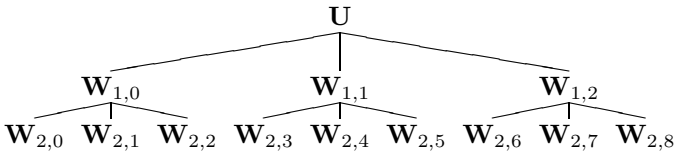

Fig. 2. 3-DWPT decomposition tree of $\mathbf{U}$ down to resolution level $j = 2$.

According to Eqs. (9) and (10), $\mathbf{U}$ can be split into orthogonal sums of wavelet packet spaces. Thus, given an element f of $\mathbf{U}$, the coefficients of the projection of f on a space $\mathbf{W}_{j,n}$ define a sequence $(c_{j,n}[k])_{k\in\mathbb{Z}}$ where

$$c_{j,n}[k] = \int_{\mathbb{R}} f(t) W_{j,n,k}(t) dt. \tag{11}$$

The wavelet function spaces considered in the sequel are always those involved in the computation of the M-DWPT through Eqs. (9) and (10). Therefore, any $\mathbf{W}_{j,n}$ encountered below is such that $n \in \{0, \ldots, M^j - 1\}$.

B. M-ary representations of the paths of the M-DWPT decomposition tree

A given wavelet packet path $\mathcal{P}$ is described by a sequence of nested functional subspaces: $\mathcal{P} = \big(\mathbf{U}, \{\mathbf{W}_{j,n(j)}\}_{j\in\mathbb{N}}\big)$, where $\mathbf{W}_{j,n(j)} \subset \mathbf{W}_{j-1,n(j-1)}$. By construction, each $\mathbf{W}_{j,n(j)}$ is obtained by recursively decomposing $\mathbf{U}$ by means of a particular sequence of filters $(h_{m_\ell})_{\ell=1,2,\cdots,j}$ where each m_ℓ belongs to $\{0,1,\ldots,M-1\}$. Therefore, the shift parameter is

$$n(j) = \sum_{\ell=1}^{j} m_\ell M^{j-\ell} \in \{0,1,\ldots,M^j-1\} \tag{12}$$

at every resolution level j. Thus, path $\mathcal{P}$ can be assigned to the M-ary sequence $(m_\ell)_{\ell\geqslant 1}$ of elements of $\{0,1,\ldots,M-1\}$. This sequence characterizes the packets $\{\mathbf{W}_{j,n(j)}\}_{j\in\mathbb{N}}$, where the shift parameter $n = n(j)$ is given at resolution level j by Eq. (12). Note also the easy relation

$$n(j) = Mn(j-1) + m_j, \tag{13}$$

for $j \in \mathbb{N}$, with the convention $n(0) = 0$.

Conversely, given any arbitrary M-ary sequence $\lambda = (m_\ell)_{\ell\in\mathbb{N}}$ where each m_ℓ is an element of $\{0,1,\ldots,M-1\}$, the finite subsequence $(m_\ell)_{\ell=1,2,\cdots,j}$ formed by the first j terms of λ, defines a unique non negative integer $n(j)$ of the form given by Eq. (12) and is thus associated with a unique wavelet packet space located at node $(j,n(j))$ of the M-DWPT decomposition tree.

According to the discussion above, every M-DWPT decomposition path is associated with a unique sequence $\lambda = (m_\ell)_{\ell\in\mathbb{N}}$ of elements of $\{0,1,\ldots,M-1\}$ and *vice versa*. From now on, any given M-DWPT decomposition path will be represented by an M-ary sequence λ. Since the shift parameter n depends on j and λ via Eq. (12), the notation $n = n_\lambda(j)$ will hereafter be used to indicate this dependence. Therefore, an M-ary sequence λ associated with an M-DWPT decomposition path specifies a unique sequence $(\mathbf{W}_{j,n_\lambda(j)})_{j\in\mathbb{N}}$ of wavelet packets.

Example 1: consider the sequence

$$\lambda_0 = (0,0,0,0,0,0,\cdots).$$

We have $n_{\lambda_0}(j) = 0$ for every natural number j. Thus, λ_0 is associated with the path $\mathcal{P}_{\lambda_0} = (\mathbf{U}, \{\mathbf{W}_{j,0}\}_{j\in\mathbb{N}})$ located at the far left hand side of the wavelet packet tree. This path corresponds to the standard wavelet approximation path since the filter h_0 is used at every resolution level. Now, let us consider the sequence

$$\lambda_1 = (M-1,0,0,0,0,0,\cdots).$$

For this sequence, we have $n_{\lambda_1}(j) = (M-1)M^{j-1}$. This sequence is thus associated with the path $\mathcal{P}_{\lambda_1} = (\mathbf{U}, \{\mathbf{W}_{j,(M-1)M^{j-1}}\}_{j\in\mathbb{N}})$ obtained by using filter h_{M-1} at the first resolution level and filter h_0 at every resolution level $j \geqslant 2$.

We conclude this section with an easy lemma which will prove useful in the sequel.

Lemma 1: For $n = n(j)$ given by Eq. (12), we have

$$\mathcal{F}W_n(\omega) = \left[\prod_{\ell=1}^{j} H_{m_\ell}(\frac{\omega}{M^{j+1-\ell}})\right] \mathcal{F}\Phi(\frac{\omega}{M^j}). \tag{14}$$

Proof: See appendix A. ■

C. Shannon M-DWPT and the Paley-Wiener space of π band-limited functions

The Shannon M-DWPT filters are hereafter denoted h_m^{S} for $m = 0, 1, \ldots, M-1$. These filters are ideal low-pass, band-pass and high-pass filters. The Fourier transform of any h_m^{S} is given by

$$H_m^{\mathsf{S}}(\omega) = \sum_{\ell \in \mathbb{Z}} \mathbb{1}_{\Delta_m}(\omega - 2\pi\ell), \tag{15}$$

where $\Delta_m = \left[-\frac{(m+1)\pi}{M}, -\frac{m\pi}{M}\right] \cup \left[\frac{m\pi}{M}, \frac{(m+1)\pi}{M}\right]$. The scaling function Φ^{S} associated with these filters is defined for every $t \in \mathbb{R}$ by $\Phi^{\mathsf{S}}(t) = \mathsf{sinc}(t) = \sin(\pi t)/\pi t$ with $\Phi^{\mathsf{S}}(0) = 1$. The Fourier transform of this scaling function is

$$\mathcal{F}\Phi^{\mathsf{S}} = \mathbb{1}_{[-\pi,\pi]}, \tag{16}$$

where $\mathbb{1}_K$ denotes the indicator function of a given set K: $\mathbb{1}_K(x) = 1$ if $x \in K$ and $\mathbb{1}_K(x) = 0$, otherwise.

The closure $\mathbf{U}^{\mathsf{S}}$ of the space spanned by the orthonormal system $\{\tau_k \Phi^{\mathsf{S}} : k \in \mathbb{Z}\}$ is then the Paley-Wiener (PW) space of those elements of $L^2(\mathbb{R})$ that are π band-limited in the sense that their Fourier transform is supported within $[-\pi, \pi]$. Any element of this space satisfies Shannon's sampling theorem. Therefore, when the M-DWPT concerns the PW space $\mathbf{U}^{\mathsf{S}}$, the input data for the decomposition of any element f of this functional space are the samples $\{f[k]\}_{k\in\mathbb{Z}}$ of f.

The PW space $\mathbf{U}^{\mathsf{S}}$ will play a crucial role in the sequel for the following reason. Let X be any band-limited WSS random process whose spectrum is supported within $[-\pi, \pi]$. Therefore, we have (see [4, Appendix D])

$$X[k] = \int_{\mathbb{R}} X(t)\Phi^{\mathsf{S}}(t-k)dt, \tag{17}$$

so that the PW space $\mathbf{U}^{\mathsf{S}}$ is the natural representation space of such a process. Any M-DWPT of X can thus be initialized with the samples $X[k]$, $k \in \mathbb{Z}$.

Now, let us consider the Shannon M-DWPT of the PW space $\mathbf{U}^{\mathsf{S}}$. The wavelet packet functions $W_{j,n}^{\mathsf{S}}$ of this M-DWPT can be computed by means of Eqs. (2), (3) and (6) by setting $\Phi = \Phi^{\mathsf{S}}$ and $h_m = h_m^{\mathsf{S}}$, $m = 0, 1, \ldots, M-1$. The Fourier transforms of these wavelet packet functions are given by the following lemma. This lemma extends [7, Proposition 8.2, p. 328] since the latter is obtained by applying the former with $M = 2$.

Proposition 1: For every non-negative integer j and every $n \in \{0, \ldots, M^j - 1\}$,

$$\mathcal{F}W_{j,n}^{\mathsf{S}} = M^{j/2}\mathbb{1}_{\Delta_{j,G(n)}}, \tag{18}$$

where, for any non-negative integer k,

$$\Delta_{j,k} = \left[-\frac{(k+1)\pi}{M^j}, -\frac{k\pi}{M^j}\right] \cup \left[\frac{k\pi}{M^j}, \frac{(k+1)\pi}{M^j}\right] \tag{19}$$

and G is the map defined by recursively setting, for $m = 0, 1, \ldots, M-1$ and $\ell = 0, 1, 2, \ldots$

$$G(M\ell + m) = \begin{cases} MG(\ell) + m & \text{if } G(\ell) \text{ is even,} \\ MG(\ell) - m + M - 1 & \text{if } G(\ell) \text{ is odd,} \end{cases} \tag{20}$$

with $G(0) = 0$.

Proof: See appendix B. In this appendix and throughout the rest of the paper, we set

$$\Delta^+_{j,k} = \left[\frac{k\pi}{M^j}, \frac{(k+1)\pi}{M^j}\right] \tag{21}$$

for any pair (j, k) of non-negative integers. ■

The support of $\mathcal{F}W^{\mathsf{S}}_{j,n}$ is $\Delta_{j,p}$, where $p = G(n)$. The restriction of G to the set $\{0, 1, \ldots, M^j - 1\}$ is a permutation of this set. This permutation induces a frequency re-ordering of the Shannon wavelet packets $\mathcal{F}W^{\mathsf{S}}_{j,n}$, $n = 0, 1, \ldots, M^j - 1$. The frequency ordering yielded by this permutation is given for the Shannon 2-DWPT (the standard DWPT) and 3-DWPT in figures 3 and 4 respectively. In these figures, the positive part $\Delta^+_{j,p}$ of the frequency support of the function $\mathcal{F}W^{\mathsf{S}}_{j,n}$ is given for each resolution level j under consideration and each shift parameter $n = 0, 1, \ldots, M^j - 1$.

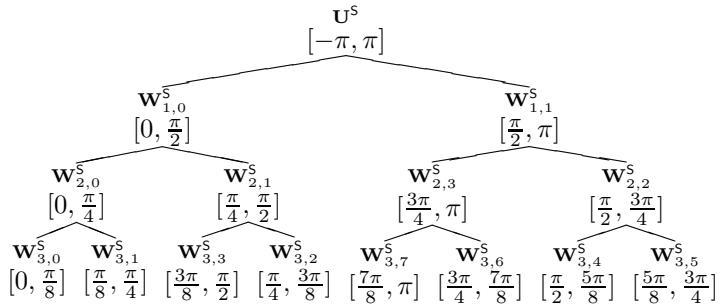

Fig. 3. Re-ordering of the Shannon 2-DWPT decomposition tree. The positive part of the support of $\mathcal{F}W^{\mathsf{S}}_{j,n}$ is indicated below $\mathbf{W}^{\mathsf{S}}_{j,n}$.

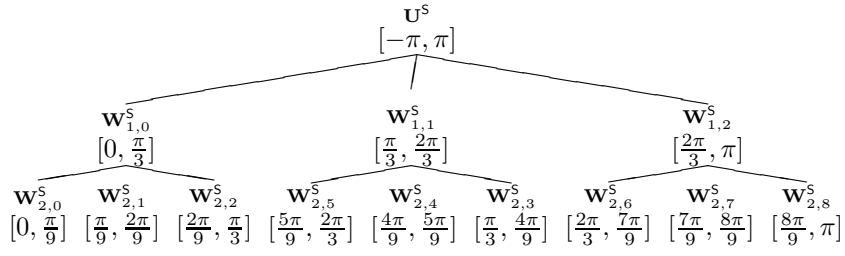

Fig. 4. Re-ordering of the Shannon 3-DWPT decomposition tree. The positive part of the support of $\mathcal{F}W^{\mathsf{S}}_{j,n}$ is indicated below $\mathbf{W}^{\mathsf{S}}_{j,n}$.

III. Asymptotic analysis for the autocorrelation functions of the M-DWPT of second-order WSS random processes

Let X denote a centred second-order real random process assumed to be continuous in quadratic mean. The autocorrelation function of X, denoted by R, is defined by

$$R(t,s) = \mathbb{E}[X(t)X(s)].$$

The projection of X on $\mathbf{W}_{j,n}$ yields a sequence of random variables, the wavelet packet *coefficients* of X:

$$c_{j,n}[k] = \int_{\mathbb{R}} X(t) W_{j,n,k}(t) dt, \quad k \in \mathbb{Z}, \tag{22}$$

provided that the integral

$$\iint_{\mathbb{R}^2} R(t,s) W_{j,n,k}(t) W_{j,n,k}(s) dt ds \tag{23}$$

exists, which will be assumed in the rest of the paper since commonly used wavelet functions are compactly supported or have sufficiently fast decay [8]. The sequence given by Eq. (22) defines the discrete random process $c_{j,n} = (c_{j,n}[k])_{k\in\mathbb{Z}}$ of the wavelet packet *coefficients* of X at any resolution level j and for any shift parameter $n \in \{0, 1, \ldots, M^j - 1\}$.

A. Preliminary results

Let $R_{j,n}$ stand for the autocorrelation function of the random process $c_{j,n}$. We have

$$\begin{aligned} R_{j,n}[k,\ell] &= \mathbb{E}\big[c_{j,n}[k] c_{j,n}[\ell]\big] \\ &= \iint_{\mathbb{R}^2} R(t,s) W_{j,n,k}(t) W_{j,n,\ell}(s) dt ds. \end{aligned} \tag{24}$$

If X is WSS, we write $R(t,s) = R(t-s)$ with some usual and slight abuse of language. From Eq. (24), it then follows that

$$R_{j,n}[k,\ell] = \iint_{\mathbb{R}^2} R(t) W_{j,n,k}(t+s) W_{j,n,\ell}(s) dt ds. \tag{25}$$

In the sequel, the spectrum γ of X, that is, the Fourier transform of R, is assumed to exist. By using Fubini's theorem and Parseval's equality, we can proceed as in [4, Appendix C] to derive from Eqs. (8) and (25) that the autocorrelation function of $c_{j,n}$ can be written:

$$R_{j,n}[k,\ell] = \frac{1}{2\pi} \int_{\mathbb{R}} \gamma(\frac{\omega}{M^j}) |\mathcal{F}W_n(\omega)|^2 \exp\left(i(k-\ell)\omega\right) d\omega. \tag{26}$$

Thus, $c_{j,n}$ is WSS. For any $k, \ell \in \mathbb{Z}$, and with the same abuse of language as above, the value $R_{j,n}[k,\ell]$ of the autocorrelation function of the discrete random process $c_{j,n}$ is $R_{j,n}[k-\ell]$ with

$$R_{j,n}[k] = \frac{1}{2\pi} \int_{\mathbb{R}} \gamma(\frac{\omega}{M^j}) |\mathcal{F}W_n(\omega)|^2 \exp\left(ik\omega\right) d\omega. \tag{27}$$

Let us assume that $\gamma \in L^\infty(\mathbb{R})$ and is continuous at 0. These two assumptions have two easy consequences. First, the integrand on the right hand side (rhs) of Eq. (27) is integrable since its absolute value is upper-bounded by $\|\gamma\|_\infty |\mathcal{F}W_n(\cdot)|^2$, whose integral equals $\|\gamma\|_\infty$; second, the limit of $\gamma(\frac{\omega}{M^j})$ is $\gamma(0)$ when j tends to ∞. Therefore, for every given natural number n, it follows from Lebesgue's dominated convergence theorem applied to Eq. (27) that

$$\begin{aligned}\lim_{j\to+\infty} R_{j,n}[k] &= \frac{1}{2\pi}\int_{\mathbb{R}} \gamma(0)|\mathcal{F}W_n(\omega)|^2 \exp(ik\omega)\, d\omega \\ &= \gamma(0)\delta[k], \end{aligned} \tag{28}$$

where $\delta[\cdot]$ is the standard Kronecker symbol defined for every integer $k \in \mathbb{Z}$ by

$$\delta[k] = \begin{cases} 1 & \text{if} \quad m = 0, \\ 0 & \text{if} \quad m \neq 0. \end{cases}$$

The result thus obtained is given in [3, Corollary 5].

Now, let $\mathcal{P} = \left(\mathbf{U}, \{\mathbf{W}_{j,n_\lambda(j)}\}_{j\in\mathbb{N}}\right)$ be some path of the M-DWPT decomposition tree where $\lambda = (m_\ell)_{\ell\in\mathbb{N}}$ is a sequence of elements of $\{0, 1, \ldots, M-1\}$. At each resolution level j, and as highlighted by the notation used to designate path $\mathcal{P}$, the shift parameter n is the function $n = n_\lambda(j)$ of j. We then have two cases. First, if n_λ is a constant function of j, it derives from Eq. (13) that $\lambda = \lambda_0$ is the null sequence, already considered in example 1. In this case, the shift parameter $n_\lambda(j)$ is 0 at each resolution level j and the M-DWPT of X through path $\mathcal{P} = \mathcal{P}_0$ consists of an infinite sequence of low-pass filters. The decorrelation is then guaranteed by Eq. (28) (see also [3, Corollary 5]). The second case is that of a function n_λ which is not constant with j. For instance, consider the sequence λ_1 given in example 1, or the sequence $\lambda = (1, 1, \ldots)$ for which $n_\lambda(j) = M^j - 1$ so that the nodes $(j, M^j - 1)$ are those of the path located at the extreme rhs of the M-DWPT decomposition tree (see figures 1 and 2). In such cases where n_λ is not a constant function of j, the asymptotic decorrelation of the M-DWPT coefficients at node $(j, n_\lambda(j))$ when j tends to ∞ is no longer a mere consequence of Eqs. (27) and (28).

In order to derive the behaviour of the autocorrelation functions at nodes $(j, n_\lambda(j))$ of a given path when n_λ is not a constant function of j, it is convenient to write the expression given by Eq. (27) for the autocorrelation function in the form

$$R_{j,n}[k] = \frac{1}{2\pi}\int_{\mathbb{R}} \gamma(\omega)|\mathcal{F}W_{j,n}(\omega)|^2 \exp\left(iM^j k\omega\right) d\omega. \tag{29}$$

This equality derives from Eq. (27) after a straightforward change of variable and by taking into account that

$$\mathcal{F}W_{j,n}(\omega) = M^{j/2}\mathcal{F}W_n(M^j\omega), \tag{30}$$

which follows from Eq. (6). The purpose of the next section is then to analyse the behaviour of $R_{j,n}$ when the M-DWPT filters are the Shannon filters or approximate, in a certain sense, these Shannon filters. From now on, the decomposition space is the PW space $\mathbf{U}^{\mathsf{S}}$ for its suitability regarding practical applications (see section II-C).

B. Asymptotic decorrelation achieved by M-DWPT

We start by considering the case where the M-DWPT is performed via the Shannon M-DWPT filters presented in section II-C. The result obtained in this case is lemma 2. This lemma is then useful to establish the main result of this section, namely theorem 1. This theorem applies to M-DWPT filters that approximate, in a sense specified below, the Shannon M-DWPT filters. Lemma 2 as well as theorem 1 concern the case where the process to decompose is band-limited and WSS. Therefore, the decomposition space considered in both statements is the PW space $\mathbf{U}^{\mathsf{S}}$ since it is the natural representation space for band-limited WSS processes (see Eq. (17)).

With the same notations as above, let $\lambda = (m_\ell)_{\ell\in\mathbb{N}}$ be an M-ary sequence of elements of $\{0, 1, \ldots, M-1\}$. Consider the Shannon M-DWPT, that is, the M-DWPT associated with the Shannon M-DWPT filters $(h_m^{\mathsf{S}})_{m=0,1,\ldots,M-1}$. Let $\mathcal{P}_\lambda = (\mathbf{U}^{\mathsf{S}}, \{\mathbf{W}^{\mathsf{S}}_{j,n_\lambda(j)}\}_{j\in\mathbb{N}})$ be the path associated with λ in the Shannon M-DWPT decomposition tree. It follows from proposition 1 that the support of $W^{\mathsf{S}}_{j,n_\lambda(j)}$ is $\Delta_{j,p_\lambda(j)}$, where $p_\lambda(j) = G(n_\lambda(j))$. For $j \in \mathbb{N}$, the sets $\Delta_{j,p^+_\lambda(j)}$ are nested intervals whose diameters tend to 0. Therefore, their intersection contains only one point $a(\lambda)$. It then follows from (19) that

$$a(\lambda) = \lim_{j\to+\infty} \frac{p_\lambda(j)\pi}{M^j}. \tag{31}$$

Let X be some centred second-order WSS random process, continuous in quadratic mean, with spectrum γ. The autocorrelation function $R^{\mathsf{S}}_{j,n}$ resulting from the projection of X on $\mathbf{W}^{\mathsf{S}}_{j,n}$ derives from Eq. (29) and is given by

$$R^{\mathsf{S}}_{j,n}[k] = \frac{1}{2\pi}\int_{\mathbb{R}} \gamma(\omega)|\mathcal{F}W^{\mathsf{S}}_{j,n}(\omega)|^2 \exp\left(iM^j k\omega\right) d\omega. \tag{32}$$

From Eqs. (18) and (32) and taking into account that γ is even, as the Fourier transform of the even function R, it follows that

$$R^{\mathsf{S}}_{j,n}[k] = \frac{M^j}{\pi}\int_{\Delta^+_{j,p}} \gamma(\omega)\cos\left(M^j k\omega\right) d\omega. \tag{33}$$

where $\Delta^+_{j,p}$ is given by Eq. (21) and $p = G(n)$. When X satisfies some additional assumptions, the following lemma 2 states that the Shannon M-DWPT of X yields coefficients that tend to be decorrelated when j tends to infinity. One of these additional assumptions is that X is band-limited in the sense that its spectrum is supported within $[-\pi, \pi]$. When $M = 2$, lemma 2 is equivalent to [4, Proposition 1].

Lemma 2: Let X be a centred second-order WSS random process, continuous in quadratic mean. Assume that the spectrum γ of X is an element of $L^\infty(\mathbb{R})$ and is supported within $[-\pi, \pi]$. Let $\lambda = (m_\ell)_{\ell\in\mathbb{N}}$ be an M-ary sequence of elements of $\{0, 1, \ldots, M-1\}$ and $\mathcal{P}_\lambda = (\mathbf{U}^{\mathsf{S}}, \{\mathbf{W}^{\mathsf{S}}_{j,n_\lambda(j)}\}_{j\in\mathbb{N}})$ be the Shannon M-DWPT decomposition path associated with λ.

If the spectrum γ of X is continuous at point $a(\lambda)$, then

$$\lim_{j\to+\infty} R^{\mathsf{S}}_{j,n_\lambda(j)}[k] = \gamma(a(\lambda))\delta[k] \tag{34}$$

uniformly in $k \in \mathbb{Z}$, where $R^{\mathsf{S}}_{j,n_\lambda(j)}$ is the autocorrelation function of the coefficients resulting from the projection of X on $\mathbf{W}^{\mathsf{S}}_{j,n_\lambda(j)}$.

Proof: See appendix C. ■

The foregoing lemma is mainly of theoretical interest since the Shannon M-DWPT filters have infinite supports and are not really suitable for practical purpose. In order to obtain a result of the same type for filters of practical interest, the M-DWPT is now assumed to be performed by using decomposition filters $h^{[r]}_m$, $m = 0, 1, \ldots, M-1$, that depend on a non-negative integer or real value r such that

$$\lim_{r\to\infty} H^{[r]}_m = H^{\mathsf{S}}_m \quad \text{(a.e.)}, \tag{35}$$

where $H^{[r]}_m$ is the Fourier transform of $h^{[r]}_m$ and H^{S}_m is given by Eq. (15). As mentioned in the introduction, r is called the order of the M-DWPT filters. When r tends to ∞, the M-DWPT filters with impulse responses $\{h^{[r]}_m\}_{m=0,1,\ldots,M-1}$ converge in the sense specified by Eq. (35) to the Shannon M-DWPT filters $\{h^{\mathsf{S}}_m\}_{m=0,1,\ldots,M-1}$. On the other hand, Eq. (35) can be regarded as a property of regularity for the following reasons. According to [9], [10], [11], the Daubechies filters satisfy Eq. (35) for $M = 2$ when r is the number of vanishing moments of the Daubechies wavelet function; according to [12], Battle-Lemarié filters also satisfy Eq. (35) for $M = 2$ when r is the spline order of the Battle-Lemarié scaling function. The existence of such families for $M > 2$ remains an open issue to address in forthcoming work. However, it seems reasonable to expect that general M-DWPT filters of the Daubechies or Battle-Lemarié type converge to Shannon filters in the sense given above.

Theorem 1: Let X be a centred second-order WSS random process, continuous in quadratic mean. Assume that the spectrum γ of X is an element of $L^\infty(\mathbb{R})$ and is supported within $[-\pi, \pi]$. Assume that the M-DWPT of the PW space $\mathbf{U}^{\mathsf{S}}$ is achieved by using decomposition filters $h^{[r]}_m$, $m = 0, 1, \ldots, M-1$, satisfying Eq. (35).

For every natural number j and every $n = 0, 1, \ldots, M^j - 1$, let $R^{[r]}_{j,n}$ stand for the autocorrelation function of the wavelet packet coefficients of X with respect to the packet $\mathbf{W}^{[r]}_{j,n}$.

(i) We have

$$\lim_{r\to+\infty} R^{[r]}_{j,n}[k] = R^{\mathsf{S}}_{j,n}[k], \tag{36}$$

uniformly in $k \in \mathbb{Z}$ and n, where $R^{\mathsf{S}}_{j,n}$ is given by Eq. (33).

(ii) Let $\mathcal{P}_\lambda = (\mathbf{U}^{\mathsf{S}}, \{\mathbf{W}^{[r]}_{j,n_\lambda(j)}\}_{j\in\mathbb{N}})$ be some path of this M-DWPT where λ is a sequence of elements of $\{0, 1, \ldots, M-1\}$. If γ is continuous at $a(\lambda)$, then

$$\lim_{j\to+\infty} \left(\lim_{r\to+\infty} R^{[r]}_{j,n_\lambda(j)}[k] \right) = \gamma(a(\lambda))\delta[k], \tag{37}$$

uniformly in $k \in \mathbb{Z}$, with $a(\lambda)$ given by Eq. (31).

Remark 1: Since the M-DWPT concerns the PW space $\mathbf{U}^{\mathsf{S}}$, we have

$$\mathcal{F}W_n^{\mathsf{S}}(\omega) = \left[\prod_{\ell=1}^{j} H_{m_\ell}^{\mathsf{S}}(\frac{\omega}{M^{j+1-\ell}})\right] \mathcal{F}\Phi^{\mathsf{S}}(\frac{\omega}{M^j}), \tag{38}$$

and

$$\mathcal{F}W_n^{[r]}(\omega) = \left[\prod_{\ell=1}^{j} H_{m_\ell}^{[r]}(\frac{\omega}{M^{j+1-\ell}})\right] \mathcal{F}\Phi^{\mathsf{S}}(\frac{\omega}{M^j}), \tag{39}$$

where n is given by Eq. (12). These equations straightforwardly derive from Eq. (14) of lemma 1. From Eqs. (30), (35), (38) and (39), we obtain, for every given natural number j, that

$$\lim_{r\to+\infty} \mathcal{F}W_{j,n}^{[r]} = \mathcal{F}W_{j,n}^{\mathsf{S}} \quad \text{(a.e.)}, \tag{40}$$

uniformly in n. The three equalities above will prove useful in the sequel.

Proof: (of theorem 1). The autocorrelation function $R_{j,n}^{[r]}$ is given by Eq. (29) and is equal to

$$R_{j,n}^{[r]}[k] = \frac{1}{2\pi}\int_{\mathbb{R}} \gamma(\omega)|\mathcal{F}W_{j,n}^{[r]}(\omega)|^2 \exp\left(iM^j k\omega\right) d\omega. \tag{41}$$

In addition, we have

$$\begin{aligned}&\left|R_{j,n}^{[r]}[k] - R_{j,n}^{\mathsf{S}}[k]\,\right| \\ &\quad\leqslant \frac{1}{2\pi}\int_{\mathbb{R}} |\gamma(\omega)| \left|\,|\mathcal{F}W_{j,n}^{[r]}(\omega)|^2 - |\mathcal{F}W_{j,n}^{\mathsf{S}}(\omega)|^2\right| d\omega,\end{aligned} \tag{42}$$

where $R_{j,n}^{\mathsf{S}}$ is given by Eq. (33). From Eqs. (30), (38) and (39), and by taking into acount that $|H_{m_\ell}^{[r]}(\omega)|$ and $|H_{m_\ell}^{\mathsf{S}}(\omega)|$ are less than or equal to 1, we obtain

$$\left|\,|\mathcal{F}W_{j,n}^{[r]}(\omega)|^2 - |\mathcal{F}W_{j,n}^{\mathsf{S}}(\omega)|^2\right| \leqslant 2M^j \left|\mathcal{F}\Phi^{\mathsf{S}}(\omega)\right|^2. \tag{43}$$

Statement (i) derives from Eqs. (40), (42), (43) and Lebesgue's dominated convergence theorem. Statement (ii) is a consequence of statement (i) and lemma 2. ■

Remark 2: Consider the case where $\mathcal{P}_\lambda = \mathcal{P}_0$ where λ_0 is the null sequence (see example 1). According to Eq. (28), the asymptotic decorrelation is obtained by simply increasing the resolution level j. However, Eq. (37) above suggests that the order of the M-DWPT filters can play some role on the speed of the decorrelation process. This seems to be actually the case. In [4], experimental results carried out for the standard DWPT ($M = 2$) illustrate that the decorrelation process obtained by successive low-pass filtering is accelerated by increasing the order of the filters with respect to the resolution level.

Remark 3: Theorem 1 extends Eq. (28) and [3, Corollary 5] for paths involving low-, band- and high-pass decomposition filters. Such paths are associated with non-null sequences. In this case, theorem 1 shows that, when both j and r increase, the coefficients of the M-DWPT tend to decorrelate and that the variance of the discrete random process returned at node $(j, n_\lambda(j))$ of a given path $\mathcal{P}_\lambda$ tends to $\gamma(a(\lambda))$ when j tends to infinity. This

asymptotic value for the variance depends on $a(\lambda)$ and, thus, may differ from $\gamma(0)$ whenever $a(\lambda) \neq 0$. For instance, for the autoregressive random process used in the experimental results of section V, table II below presents the value $\gamma(a(\lambda))$ for different sequences λ, and thus, different paths of the standard DWPT decomposition tree.

IV. Central limit theorems

In this section, we consider a centred real random process X that has finite cumulants and polyspectra. Denote by

$$\begin{aligned}&\mathsf{cum}(t, s_1, s_2, \cdots, s_N)\\ &\quad = \mathsf{cum}\{X(t), X(s_1), X(s_2), \cdots, X(s_N)\},\end{aligned} \tag{44}$$

the cumulant of order $N+1$ of X. The above cumulant is hereafter assumed to belong to $L^2(\mathbb{R}^{N+1})$ and to be finite for any natural number N (see [13, Proposition 1] for a discussion about the existence of this cumulant). The cumulant of order $N+1$ of the random process $c_{j,n}$ has the integral form given by (see [13, Proposition 1]):

$$\begin{aligned}&\mathsf{cum}_{j,n}[k, \ell_1, \ell_2, \cdots, \ell_N]\\ &\quad = \mathsf{cum}\big\{c_{j,n}[k] c_{j,n}[\ell_1] c_{j,n}[\ell_2] \cdots c_{j,n}[\ell_N]\big\}\\ &\quad = \int_{\mathbb{R}^{N+1}} dt ds_1 ds_2 \cdots ds_N \mathsf{cum}(t, s_1, s_2, \cdots, s_N) W_{j,n,k}(t)\\ &\qquad\qquad W_{j,n,\ell_1}(s_1) W_{j,n,\ell_2}(s_2) \cdots W_{j,n,\ell_N}(s_N).\end{aligned} \tag{45}$$

By setting $s_i = t + t_i$ for $i = 1, 2, \cdots, N$ in Eq. (45) and by assuming that X is strictly stationary so that $\mathsf{cum}(t, t+t_1, t+t_2, \cdots, t+t_N) = \mathsf{cum}(t_1, t_2, \cdots, t_N)$, then

$$\begin{aligned}&\mathsf{cum}_{j,n}[k, \ell_1, \ell_2, \cdots, \ell_N]\\ &\quad = \int_{\mathbb{R}^{N+1}} dt dt_1 dt_2 \cdots dt_N \mathsf{cum}(t_1, t_2, \cdots, t_N) W_{j,n,k}(t)\\ &\qquad\qquad W_{j,n,\ell_1}(t+t_1) W_{j,n,\ell_2}(t+t_2) \cdots W_{j,n,\ell_N}(t+t_N).\end{aligned} \tag{46}$$

In what follows, X is assumed to have a polyspectrum $\gamma_N(\omega_1, \omega_2, \cdots, \omega_N)$ for every natural number N and every $(\omega_1, \omega_2, \cdots, \omega_N) \in \mathbb{R}^N$. The polyspectrum is the Fourier transform of the cumulant $\mathsf{cum}(t_1, t_2, \cdots, t_N)$. When $N = 1$, γ_1 is the spectrum of X and is simply denoted γ as in section III. From now on, it is assumed that γ_N belongs to $L^\infty(\mathbb{R}^N)$ and that

$$\begin{aligned}&\int_{\mathbb{R}^N} d\omega_1 d\omega_2 \cdots d\omega_N\\ &\quad \mathcal{F}W_n(-\omega_1 - \omega_2 - \cdots - \omega_N) \mathcal{F}W_n(\omega_1) \cdots \mathcal{F}W_n(\omega_N)\end{aligned}$$

exists and is finite for any natural number N. A standard and sufficient condition for the existence of this integral concerns the regularity of the function Φ used to generate the wavelet packet functions W_n. This assumption is the

existence of two positive real numbers ε and C such that

$$|\mathcal{F}\Phi(\omega)| \leqslant \frac{C}{1+|\omega|^{1+\varepsilon}}, \quad \omega \in \mathbb{R}. \tag{47}$$

Now, Eq. (46) can be written (see appendix D)

$$\begin{aligned}
&\mathsf{cum}_{j,n}[k,\ell_1,\ell_2,\cdots,\ell_N] \\
&\quad = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} d\omega_1 d\omega_2\cdots d\omega_N \gamma_N(-\omega_1,-\omega_2,\cdots,-\omega_N) \\
&\qquad\qquad \mathcal{F}W_{j,n,k}(-\omega_1-\omega_2-\cdots-\omega_N) \\
&\qquad\qquad \mathcal{F}W_{j,n,\ell_1}(\omega_1)\mathcal{F}W_{j,n,\ell_2}(\omega_2)\cdots\mathcal{F}W_{j,n,\ell_N}(\omega_N).
\end{aligned} \tag{48}$$

In addition, from the first equality of Eq. (8), we have $\mathcal{F}W_{j,n,k}(\omega) = \exp\left(-iM^j k\omega\right)\mathcal{F}W_{j,n}(\omega)$. Thus, by setting $\ell_i = k + k_i$ for $i = 1, 2, \cdots, N$ in Eq. (48), we obtain that $c_{j,n}$ is a strictly stationary random process with cumulants $\mathsf{cum}_{j,n}[k, k+k_1, k+k_2, \cdots, k+k_N] = \mathsf{cum}_{j,n}[k_1, k_2, \cdots, k_N]$ with

$$\begin{aligned}
&\mathsf{cum}_{j,n}[k_1,k_2,\cdots,k_N] \\
&\quad = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} d\omega_1 d\omega_2\cdots d\omega_N \\
&\qquad\qquad \exp\left(-iM^j(k_1\omega_1+k_2\omega_2+\cdots+k_N\omega_N)\right) \\
&\qquad\qquad \gamma_N(-\omega_1,-\omega_2,\cdots,-\omega_N) \\
&\qquad\qquad \mathcal{F}W_{j,n}(-\omega_1-\omega_2-\cdots-\omega_N) \\
&\qquad\qquad \mathcal{F}W_{j,n}(\omega_1)\mathcal{F}W_{j,n}(\omega_2)\cdots\mathcal{F}W_{j,n}(\omega_N).
\end{aligned} \tag{49}$$

The asymptotic Gaussianity of the discrete random process returned at node (j,n) will be analysed through the behaviour of the cumulant $\mathsf{cum}_{j,n}[k_1, k_2, \cdots, k_N]$. In particular, it will be proved that this cumulant tends to zero under additional assumptions on the decomposition filters.

By taking into account Eq. (30), Eq. (49) can also be written, with an easy change of variables:

$$\begin{aligned}
&\mathsf{cum}_{j,n}[k_1,k_2,\cdots,k_N] \\
&\quad = \frac{M^{-j(N-1)/2}}{(2\pi)^N}\int_{\mathbb{R}^N} d\omega_1 d\omega_2\cdots d\omega_N \\
&\qquad\qquad \exp\left(-i(k_1\omega_1+k_2\omega_2+\cdots+k_N\omega_N)\right) \\
&\qquad\qquad \gamma_N(-\omega_1 M^{-j},-\omega_2 M^{-j},\cdots,-\omega_N M^{-j}) \\
&\qquad\qquad \mathcal{F}W_n(-\omega_1-\omega_2-\cdots-\omega_N) \\
&\qquad\qquad \mathcal{F}W_n(\omega_1)\mathcal{F}W_n(\omega_2)\cdots\mathcal{F}W_n(\omega_N).
\end{aligned} \tag{50}$$

Assume that γ_N belongs to $L^\infty(\mathbb{R}^N)$. If the shift parameter is constant, it follows from Lebesgue's dominated convergence theorem that, for any natural number $N > 1$, $\mathsf{cum}_{j,n}[k_1, k_2, \cdots, k_N]$ tends to 0 uniformly in $k_1, k_2, \ldots, k_N$ when j tends to ∞. This is a consequence of [3, Proposition 11].

If the shift parameter n is a non-constant and, thus, non-null function of the resolution level j, the discussion following Eq. (27) has highlighted that Lebesgue's dominated convergence theorem does not make it possible to conclude about the decorrelation of the sequence returned by the M-DWPT at node (j, n). Here, the situation is similar: if $\mathcal{P}_\lambda = (\mathbf{U}^{\mathsf{S}}, \{\mathbf{W}_{j,n_\lambda(j)}\}_{j\in\mathbb{N}})$ is some path of the M-DWPT, the shift parameter $n_\lambda(j)$ depends on j and Lebesgue's dominated convergence theorem does not apply to Eq. (50) to prove the vanishing behaviour of the cumulant at node $(j, n_\lambda(j))$. The main result of this section, namely theorem 2, will however establish this vanishing behaviour by using the same family of filters as above and the following instrumental lemma 3. This lemma concerns the Shannon M-DWPT.

Lemma 3: Let X be a centred second-order strictly stationary random process, continuous in quadratic mean. Assume that the polyspectrum γ_N of X is an element of $L^\infty(\mathbb{R}^N)$ for any $N \geqslant 1$ and that the spectrum γ is supported within $[-\pi, \pi]$. For every natural number $N > 1$, the cumulant of order $N+1$, $\mathsf{cum}^{\mathsf{S}}_{j,n}[k_1, k_2, \cdots, k_N]$, of the discrete random process returned at node (j, n) by the Shannon M-DWPT of X tends to zero uniformly in $n, k_1, k_2, \cdots, k_N$, when j tends to infinity.

Proof: When the wavelet packet functions are the functions $W^{\mathsf{S}}_{j,n}$, it follows from Eqs. (18) and (49) that the cumulant $\mathsf{cum}^{\mathsf{S}}_{j,n}[k_1, k_2, \cdots, k_N]$ of the discrete random process returned at node (j, n) by the Shannon M-DWPT of X satisfies the following inequality

$$\begin{aligned} &|\mathsf{cum}^{\mathsf{S}}_{j,n}[k_1, k_2, \cdots, k_N]| \\ &\quad\leqslant \frac{M^{j(N+1)/2}}{(2\pi)^N}.\|\gamma_N\|_\infty \int_{\Delta^N_{j,p}} d\omega_1 d\omega_2 \cdots d\omega_N \end{aligned} \tag{51}$$

where $\Delta^N_{j,p} = \underbrace{\Delta_{j,p} \times \Delta_{j,p} \times \ldots \times \Delta_{j,p}}_{N \text{ times}}$ and $p = G(n)$.

According to Eq. (19), $\int_{\Delta_{j,p}} d\omega = 2\pi/M^j$. Therefore, we obtain that

$$|\mathsf{cum}^{\mathsf{S}}_{j,n}[k_1, k_2, \cdots, k_N]| \leqslant \|\gamma_N\|_\infty M^{-j(N-1)/2}. \tag{52}$$

Given any natural number $N > 1$, the rhs of the latter inequality does not depend on n, $k_1, \ldots, k_N$ and vanishes when j tends to ∞, which completes the proof. ■

Consider the decomposition filters introduced in section III-B and therefore, satisfying Eq. (35). Let λ be an M-ary sequence of elements of $\{0, 1, \ldots, M-1\}$. The following results describe the asymptotic distribution of the discrete random process $c^{[r]}_{j,n_\lambda(j)}$ returned at node $(j, n_\lambda(j))$ when the resolution level j and the order r of the filters increase.

Theorem 2: Let X be a centred second-order strictly stationary random process, continuous in quadratic mean. Assume that the polyspectrum γ_N of X is an element of $L^\infty(\mathbb{R}^N)$ for every natural number $N \geqslant 1$ and that the spectrum γ is supported within $[-\pi, \pi]$.

For every given natural number j and every $n \in \{0, 1, \ldots, M^j - 1\}$, let $\mathsf{cum}_{j,n}^{[r]}$ stand for the cumulant of order $N+1$ of the wavelet packet coefficients of X with respect to the packet $\mathbf{W}_{j,n}^{[r]}$.

(i) We have

$$\lim_{r \to +\infty} \mathsf{cum}_{j,n}^{[r]}[k_1, k_2, \ldots, k_N] = \mathsf{cum}_{j,n}^{\mathsf{S}}[k_1, k_2, \ldots, k_N], \tag{53}$$

uniformly in $n, k_1, k_2, \ldots, k_N$.

(ii) Let $\lambda = (m_k)_{k \in \mathbb{N}}$ be an M-ary sequence of elements of $\{0, 1, \ldots, M-1\}$ and $\mathcal{P}_\lambda = (\mathbf{U}^{\mathsf{S}}, \{\mathbf{W}_{j,n_\lambda(j)}^{[r]}\}_{j \in \geqslant 1})$ be the sequence of wavelet packets associated with λ. If $N > 1$, we have

$$\lim_{j \to +\infty} \left(\lim_{r \to +\infty} \mathsf{cum}_{j,n_\lambda(j)}^{[r]}[k_1, k_2, \ldots, k_N] \right) = 0, \tag{54}$$

uniformly in $k_1, k_2, \ldots, k_N$.

Proof: We begin with the proof of statement (i). This proof mimics that of the first statement of theorem 1. By applying Eq. (49) to the wavelet packet functions $W_{j,n}^{[r]}$ and, then, to the Shannon wavelet packet functions $W_{j,n}^{\mathsf{S}}$, we obtain the following inequalities

$$\begin{aligned}
&|\mathsf{cum}_{j,n}^{[r]}[k_1, k_2, \cdots, k_N] - \mathsf{cum}_{j,n}^{\mathsf{S}}[k_1, k_2, \cdots, k_N]| \\
&\quad \leqslant \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} d\omega_1 \cdots d\omega_N |\gamma_N(-\omega_1, \cdots, -\omega_N)| \\
&\qquad \Big| \mathcal{F}W_{j,n}^{[r]}(-\omega_1 \cdots - \omega_N) \mathcal{F}W_{j,n}^{[r]}(\omega_1) \cdots \mathcal{F}W_{j,n}^{[r]}(\omega_N) \\
&\qquad - \mathcal{F}W_{j,n}^{\mathsf{S}}(-\omega_1 \cdots - \omega_N) \mathcal{F}W_{j,n}^{\mathsf{S}}(\omega_1) \cdots \mathcal{F}W_{j,n}^{\mathsf{S}}(\omega_N) \Big|, \\
&\quad \leqslant \frac{1}{(2\pi)^N} ||\gamma_N||_\infty \int_{\mathbb{R}^N} d\omega_1 \cdots d\omega_N \\
&\qquad \Big| \mathcal{F}W_{j,n}^{[r]}(-\omega_1 \cdots - \omega_N) \mathcal{F}W_{j,n}^{[r]}(\omega_1) \cdots \mathcal{F}W_{j,n}^{[r]}(\omega_N) \\
&\qquad - \mathcal{F}W_{j,n}^{\mathsf{S}}(-\omega_1 \cdots - \omega_N) \mathcal{F}W_{j,n}^{\mathsf{S}}(\omega_1) \cdots \mathcal{F}W_{j,n}^{\mathsf{S}}(\omega_N) \Big|.
\end{aligned} \tag{55}$$

The integrand on the rhs of the second inequality above can now be upper-bounded by

$$\begin{aligned}
&\left|\mathcal{F}W_{j,n}^{[r]}(-\omega_1\cdots-\omega_N)\right|\left|\mathcal{F}W_{j,n}^{[r]}(\omega_1)\right|\cdots\left|\mathcal{F}W_{j,n}^{[r]}(\omega_N)\right| \\
&+\left|\mathcal{F}W_{j,n}^{\mathsf{S}}(-\omega_1\cdots-\omega_N)\right|\left|\mathcal{F}W_{j,n}^{\mathsf{S}}(\omega_1)\right|\cdots\left|\mathcal{F}W_{j,n}^{\mathsf{S}}(\omega_N)\right| \\
&\quad\leqslant M^{j(N+1)/2}\left|\mathcal{F}W_{n}^{[r]}(-M^j\omega_1\cdots-M^j\omega_N)\right| \\
&\qquad\left|\mathcal{F}W_{n}^{[r]}(M^j\omega_1)\right|\cdots\left|\mathcal{F}W_{n}^{[r]}(M^j\omega_N)\right| \\
&\qquad+\left|\mathcal{F}W_{n}^{\mathsf{S}}(-M^j\omega_1\cdots-M^j\omega_N)\right| \\
&\qquad\left|\mathcal{F}W_{n}^{\mathsf{S}}(M^j\omega_1)\right|\cdots\left|\mathcal{F}W_{n}^{\mathsf{S}}(M^j\omega_N)\right| \\
&\quad\leqslant 2M^{j(N+1)/2}\Phi^{\mathsf{S}}(\omega_1)\Phi^{\mathsf{S}}(\omega_2)\cdots\Phi^{\mathsf{S}}(\omega_N) \qquad (56)
\end{aligned}$$

where we use Eqs. (30), (38), (39), and take into acount that $|H_{m_\ell}^{[r]}(\omega)|$ and $|H_{m_\ell}^{\mathsf{S}}(\omega)|$ are less than or equal to 1. The upper-bound given by Eq. (56) is independent of r and integrable; its integral equals $2M^{j(N+1)/2}(2\pi)^N$. By taking Eq. (40) into account, we derive from Lebegue's dominated convergence theorem that the upper bound in inequality Eq. (55) tends to 0 when r tends to $+\infty$, which completes the proof of statement (i). Statement (ii) follows from lemma 3 and statement (i). ■

Remark 4: Note the following. The first statement of theorem 1 can be proved by applying Eq. (53) with $N = 1$. In fact, when $N = 1$, the cumulant of order 2 is the autocorrelation function; strict stationarity plays no role, wide-sense stationarity is enough and Eq. (36) follows straightforwardly. Two separate proofs have been presented for the sake of making the reading easier.

Remark 5: If $n = n_\lambda(j)$ is a constant function of j, that is, if λ is in fact the null sequence, the vanishing behaviour of the cumulant $\mathsf{cum}_{j,n}^{[r]}[k_1, k_2, \cdots, k_N]$ when j tends to ∞ straightforwardly derives from Eq. (50). However, similarly to remark 2, Eq. (54) suggests that order r may play a role in the convergence to 0 of the cumulant. According to the experimental results of the next section, this convergence seems to accelerate when the order increases.

Corollary 1: With the same assumptions and notations as those of theorems 1 and 2, assume that γ is continuous at $a(\lambda)$. Then, when j and r tend to infinity, the sequence $\left(c_{j,n_\lambda(j)}^{[r]}\right)_{r,j}$ converges in distribution to a white Gaussian process with spectrum $\gamma(a(\lambda))$ in the following sense: For every $x \in \mathbb{R}^N$ and every $\epsilon > 0$, there exists $j_0 = j_0(x, \epsilon) > 0$ and there exists $r_0 = r_0(x, j_0, \epsilon)$ such that, for every $j \geqslant j_0$ and every $r \geqslant r_0$, the absolute value of the difference between the value at x of the probability distribution of the random vector

$$(c_{j,n_\lambda(j)}^{[r]}[k_1], c_{j,n_\lambda(j)}^{[r]}[k_2], \ldots, c_{j,n_\lambda(j)}^{[r]}[k_N])$$

and the value at x of the centred N-variate normal distribution $\mathcal{N}(0, \gamma(a(\lambda))\mathrm{I}_N)$ with covariance matrix $\gamma(a(\lambda))\mathrm{I}_N$ is less than ϵ.

Proof: The result is a straightforward consequence of statement (ii) of theorem 1 and statement (ii) of theorem 2. ■

It is now interesting to study to what extent the results stated above are workable in practical cases. This is what we discuss in the next section on the basis of some experimental results.

V. Experimental results

The experimental results presented in this section concern the standard DWPT, that is the M-DWPT with $M = 2$, when Daubechies filters are used to perform the decomposition. We consider this standard case for the following reasons. To begin with, Daubechies filters are known to converge to the Shannon filters when the number r of vanishing moments of the Daubechies mother wavelet increases, whereas the same type of result has not yet been established for M-band filters with $M > 2$. Since [4] already presents and comments experimental results regarding the asymptotic decorrelation of the DWPT coefficients as the resolution level and the order of the filters increase, the focus is therefore the asymptotic normality described by theorem 2. Inequality (52) then suggests that the larger M is, the faster the convergence to normality. Therefore, by choosing $M = 2$, we not only consider the most standard wavelet packet decomposition but also the decomposition that seems the least favourable with respect to Eq. (52) used to prove lemma 3, the cornerstone of theorem 2.

With respect to the foregoing, and since theorem 2 is of asymptotic nature, the purpose of this section is to experimentally study how well the tendency to Gaussianity is satisfied when the input process is non-Gaussian and the DWPT is performed with finite values for the resolution level and the order.

As above, $X(t)$ stands for the centred Hilbertian random process to decompose. Its spectrum, denoted γ, is assumed to be an element of $L^\infty(\mathbb{R})$ and to have support in $[-\pi, \pi]$. In fact, our experiments concern the case where $X(t)$ is Generalized Gaussian (GG). This means that, for every $t \in \mathbb{R}$, $X(t)$ follows the Generalized Gaussian Distribution (GGD) with *scale* α, *shape* β and zero mean. For each $t \in \mathbb{R}$, the Probability Density Function (PDF) of $X(t)$ is $f_{\alpha,\beta}$ defined for every real value x by:

$$f_{\alpha,\beta}(x) = \frac{\beta}{2\alpha\Gamma(1/\beta)} \exp\left(-(|x|/\alpha)^\beta\right), \tag{57}$$

where Γ is the standard Gamma function. The value of the GGD standard deviation is

$$\sigma = \alpha\sqrt{\Gamma(3/\beta)/\Gamma(1/\beta)}.$$

For our experimentations, we choose

$$\alpha = \sqrt{\Gamma(1/\beta)/\Gamma(3/\beta)}, \tag{58}$$

so that $\sigma = 1$. Note that when the shape parameter β equals 2, the GGD given by Eq. (57) is Gaussian. When β decreases (from 2 to 0), the probability density function of the GGD is sharper, and sharper, at the origin as can be seen in figure 5. When $\beta = 1$, the GGD is the Laplacian distribution.

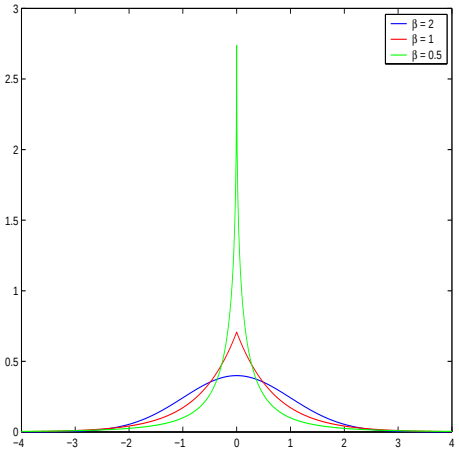

Fig. 5. GGD with unit variance and shape $\beta = 2, 1, 0.5$.

In our experiments, we consider 100 independent random copies of the random vector formed by the N samples $X(1)$, $X(2)$, ..., $X(N)$ with $N = 2^{20}$. Each copy is used as an input of the DWPT. We then consider the four wavelet packet paths associated with the sequences $\lambda_q = (\delta[q-\ell])_{\ell\in\mathbb{N}}$, for $q = 0, 1, 2$ and 3 [4, Example 1]. For these sequences, and taking into account Eq. (12), we have $n_{\lambda_0}(\ell) = 0$ for every natural number ℓ, and for $q = 1, 2, 3$:

$$n_{\lambda_q}(\ell) = \begin{cases} 0 & \text{for} \quad \ell = 1, 2, \cdots, q-1, \\ 2^{\ell-q} & \text{for} \quad \ell = q, q+1, \cdots. \end{cases}$$

It follows that $p_{\lambda_0}(\ell) = G(n_{\lambda_0}(\ell)) = 0$ and that

$$p_{\lambda_q}(\ell) = \begin{cases} 0 & \text{for} \quad \ell = 1, 2, \cdots, q-1, \\ 2^{\ell-q+1} - 1 & \text{for} \quad \ell = q, q+1, \cdots, \end{cases}$$

for $q = 1, 2, 3$. According to Eq. (31), $a(\lambda_0) = 0$ and $a(\lambda_q) = \pi/2^{q-1}$ for $q = 1, 2$ and 3.

For every path λ among those introduced above, the Kolmogorov-Smirnov (KS) test with significant level 5% is used to decide whether the samples $(c^{[r]}_{j,n_\lambda(j)}[k]/\sqrt{\gamma(a(\lambda))})_k$, returned by the DWPT for a given copy, satisfy the null hypothesis (that is, follow the normal distribution $\mathcal{N}(0,1)$), or not (alternative hypothesis).

First, we discuss the case where the samples $X(1)$, $X(2)$, ..., $X(N)$ of the random process $X(t)$ are uncorrelated. In this case, since the GGD is such that $\sigma = 1$, we have $\gamma(a(\lambda)) = 1$ for any of the four paths $(\mathcal{P}_{\lambda_q})_{q=0,1,2,3}$ under consideration. Experimentations show that by increasing the resolution level j up to 6, and by using Daubechies filters of order 7, the convergence to the normal distribution is significant for $0.5 < \beta \leqslant 2$. Table I presents the results obtained for $\beta = 1.5, 1, 0.5$.

Now, we address the case where the samples $X(1)$, $X(2)$, ..., $X(N)$ of the GG process $X(t)$ are correlated. These samples are synthesized by filtering a discrete sequence of independent and identically GG distributed random

variables through an auto-regressive (AR) filter of order 1. The Z-transform of this filter is $1/(1-\mu z^{-1})$ where $0<\mu<1$. If α and β are the parameters of the GG random variables used to synthesize the samples of $X(t)$, we henceforth say that the output discrete process $X(t)$ is an AR(1)-GG process with parameters α, β and μ.

Experimental tests are carried out with $\mu = 0.5,\ 0.75,\ 0.9,\ 0.95$. The spectrum of the GG-AR(1) processes corresponding to these values of μ are given in figure 6. Table II gives the values $\gamma(a(\lambda))$ for the four test sequences.

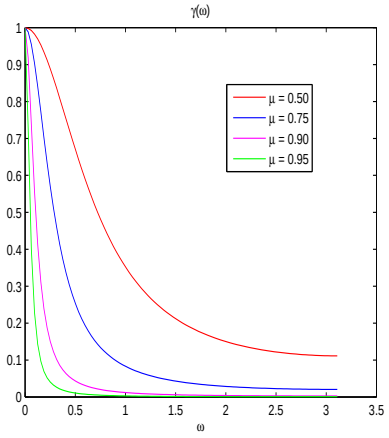

Fig. 6. Coloured GG Spectrum for some values of μ.

The experimental results obtained with the AR(1)-GG process are presented in table III. By increasing the resolution level, from $j=3$ to $j=6$ when the order of the filters is constant and equals $r=1$, the number of times the KS test accepts the Gaussian distribution $\mathcal{N}(0,1)$ for the sequence of random variables $(c^{[r]}_{j,n_\lambda(j)}[k]/\sqrt{\gamma(a(\lambda))})_k$ increases for most of the DWPT paths. When the resolution is fixed to $j=6$, it suffices to increase the order r to also increase the acceptance rate of the Gaussian distribution. If the input coloured AR(1)-GG process $X(t)$ is such that $0.5<\beta\leqslant 2$ and $0<\mu<0.9$, normality can reasonably be considered to be attained when the resolution level j is 6 and the order of the Daubechies filters is $r=7$. The less satisfactory results occur for large values of μ or small values of β. However, when μ is large, the spectrum of the process tends to become 'singular' because it is rather sharp around the null frequency (see figure 6 for $\mu=0.95$); on the other hand, when β becomes small, the GGD becomes 'singular' in the sense that its PDF is still sharper at the origin (see figure 5 for $\beta=0.5$). However, even for large values of μ and small values of β, increasing both the order of the filters and the resolution level leads to better results, as can be seen in table IV.

From tables I, III and IV, we observe a significant acceptance rate by increasing first the resolution level from $j=3$ to $j=6$ when r is fixed to 1. In addition, we observe a significant improvement in the acceptance rate by increasing the order of the filters from 1 to 7, given a fixed resolution level $j=6$. It follows that increasing both the resolution level and the order of the filters yields a high KS test acceptance rate for the normality $\mathcal{N}(0,1)$ of

TABLE I

KS TEST ACCEPTANCE RATES FOR THE NORMAL DISTRIBUTION $\mathcal{N}(0,1)$ OF THE DWPT COEFFICIENTS RETURNED AT RESOLUTION LEVEL $j = 3, 6$ FOR DIFFERENT DWPT PATHS AND WHEN THE DWPT IS PERFORMED WITH DAUBECHIES FILTERS WITH ORDER EQUAL TO EITHER 1 OR 7. THE DWPT INPUT PROCESS IS GG. THE GGD PARAMETERS ARE α GIVEN BY EQ. (58) AND $\beta = 1.5, 1, 0.5$. THE SAMPLES OF THE GG PROCESS ARE UNCORRELATED.

White GG process									
	GGD, with $\beta = 1.5$			GGD, with $\beta = 1$			GGD, with $\beta = 0.5$		
	$j = 3$	$j = 6$	$j = 6$	$j = 3$	$j = 6$	$j = 6$	$j = 3$	$j = 6$	$j = 6$
Path	$r = 1$	$r = 1$	$r = 7$	$r = 1$	$r = 1$	$r = 7$	$r = 1$	$r = 1$	$r = 7$
$\mathcal{P}_{\lambda_0}$	51%	94%	96%	0%	97%	97%	0%	44%	74%
$\mathcal{P}_{\lambda_1}$	53%	97%	98%	0%	96%	93%	0%	44%	71%
$\mathcal{P}_{\lambda_2}$	47%	94%	97%	0%	94%	96%	0%	31%	39%
$\mathcal{P}_{\lambda_3}$	56%	94%	94%	0%	92%	94%	0%	37%	38%

TABLE II

VALUES $\gamma(a(\lambda))$ FOR THE FOUR TEST SEQUENCES.

Path	$\mu = 0.5$	$\mu = 0.75$	$\mu = 0.9$	$\mu = 0.95$
$\mathcal{P}_{\lambda_0}$	1	1	1	1
$\mathcal{P}_{\lambda_1}$	0.1111	0.0204	0.0028	0.0007
$\mathcal{P}_{\lambda_2}$	0.2052	0.0412	0.0057	0.0014
$\mathcal{P}_{\lambda_3}$	0.4798	0.1332	0.0201	0.0048

the samples $(c^{[r]}_{j,n_\lambda(j)}/\sqrt{\gamma(a(\lambda))})_k$.

As an illustration, figure 7 shows histograms of the DWPT coefficients obtained at resolution level 6, by using Daubechies filters of order 7. The decomposition concerns the samples of an AR(1)-GG process with α, given by Eq. (58), $\beta = 1$ and $\mu = 0.75$. These histograms are compared with the PDF of the Gaussian limit distribution.

As already mentioned above (see remark 5), the order of the filters speeds up the convergence to normality when the decomposition path is $\mathcal{P}_{\lambda_0}$ associated with the null sequence $\lambda_0 = (0, 0, \ldots)$, which applies only low-pass filters to the input process. Indeed, for this path, higher order filters yield better approximation of the distribution $\mathcal{N}(0, \gamma(0))$ than lower order ones. This can be noticed by comparing, at resolution level $j = 6$, the acceptance rates obtained for $r = 1$ to those obtained for $r = 7$ in tables I, III and IV for $\mathcal{P}_{\lambda_0}$.

VI. CONCLUSION

In this paper, M-DWPTs of band-limited stationary random processes have been considered. The asymptotic analysis of the distribution of the wavelet packet coefficients is achieved *via* certain families of M-DWPT filters. The specificity of these filters is that their Fourier transforms converge almost everywhere to the Fourier transform of the Shannon filters. Daubechies and Battle-Lemarié filters are examples of such families of filters when $M = 2$

TABLE III

KS TEST ACCEPTANCE RATES FOR THE NORMAL DISTRIBUTION $\mathcal{N}(0,1)$ OF THE DWPT COEFFICIENTS RETURNED AT RESOLUTION LEVEL $j = 3, 6$ FOR DIFFERENT DWPT PATHS AND WHEN THE DWPT IS PERFORMED WITH DAUBECHIES FILTERS WITH ORDER EQUAL TO EITHER 1 OR 7. THE DWPT INPUT PROCESS IS AR(1)-GG WITH α GIVEN BY EQ. (58), $\beta = 1.5, 1, 0.5$ AND $\mu = 0.5, 0.75, 0.9, 0.95$.

Coloured GG process, with $\beta = 1.5$

	AR-1 filtering: $\mu = 0.5$			AR-1 filtering: $\mu = 0.75$			AR-1 filtering: $\mu = 0.9$			AR-1 filtering: $\mu = 0.95$		
	$j=3$	$j=6$	$j=6$	$j=3$	$j=6$	$j=6$	$j=3$	$j=6$	$j=6$	$j=3$	$j=6$	$j=6$
Path	$r=1$	$r=1$	$r=7$	$r=1$	$r=1$	$r=7$	$r=1$	$r=1$	$r=7$	$r=1$	$r=1$	$r=7$
$\mathcal{P}_{\lambda_0}$	0%	95%	98%	0%	42%	99%	0%	0%	19%	0%	0%	0%
$\mathcal{P}_{\lambda_1}$	0%	91%	98%	0%	52%	96%	0%	0%	94%	0%	0%	96%
$\mathcal{P}_{\lambda_2}$	0%	95%	88%	0%	37%	86%	0%	0%	91%	0%	0%	90%
$\mathcal{P}_{\lambda_3}$	0%	0%	86%	0%	14%	65%	0%	0%	53%	0%	0%	52%

Coloured GG process, with $\beta = 1$

	AR-1 filtering: $\mu = 0.5$			AR-1 filtering: $\mu = 0.75$			AR-1 filtering: $\mu = 0.9$			AR-1 filtering: $\mu = 0.95$		
	$j=3$	$j=6$	$j=6$	$j=3$	$j=6$	$j=6$	$j=3$	$j=6$	$j=6$	$j=3$	$j=6$	$j=6$
Path	$r=1$	$r=1$	$r=7$	$r=1$	$r=1$	$r=7$	$r=1$	$r=1$	$r=7$	$r=1$	$r=1$	$r=7$
$\mathcal{P}_{\lambda_0}$	0%	84%	94%	0%	31%	96%	0%	0%	21%	0%	0%	0%
$\mathcal{P}_{\lambda_1}$	0%	94%	96%	0%	67%	93%	0%	0%	92%	0%	0%	94%
$\mathcal{P}_{\lambda_2}$	0%	95%	82%	0%	56%	78%	0%	0%	89%	0%	0%	84%
$\mathcal{P}_{\lambda_3}$	0%	0%	71%	0%	3%	50%	0%	0%	41%	0%	0%	18%

Coloured GG process, with $\beta = 0.5$

	AR-1 filtering: $\mu = 0.5$			AR-1 filtering: $\mu = 0.75$			AR-1 filtering: $\mu = 0.9$			AR-1 filtering: $\mu = 0.95$		
	$j=3$	$j=6$	$j=6$	$j=3$	$j=6$	$j=6$	$j=3$	$j=6$	$j=6$	$j=3$	$j=6$	$j=6$
Path	$r=1$	$r=1$	$r=7$	$r=1$	$r=1$	$r=7$	$r=1$	$r=1$	$r=7$	$r=1$	$r=1$	$r=7$
$\mathcal{P}_{\lambda_0}$	0%	5%	67%	0%	0%	75%	0%	0%	3%	0%	0%	0%
$\mathcal{P}_{\lambda_1}$	0%	66%	70%	0%	93%	93%	0%	1%	89%	0%	0%	78%
$\mathcal{P}_{\lambda_2}$	0%	10%	4%	0%	71%	46%	0%	0%	29%	0%	0%	2%
$\mathcal{P}_{\lambda_3}$	0%	0%	2%	0%	0%	7%	0%	44%	8%	0%	0%	0%

(standard DWPT). Furthermore, since the Daubechies (resp. Battle-Lemarié) scaling function is proved to converge to the Shannon scaling function when the number of vanishing moments of the Daubechies wavelet function (resp. spline order of the Battle-Lemarié scaling function) increases [10] (resp. [12]), we may expect that general M-DWPT decomposition filters of Daubechies or Battle-Lemarié types may converge to the Shannon filters. This remains to be proved. However, if this holds true, it can be expected that the tendency to normality of the wavelet packet coefficients will be accelerated since Eq. (52) suggests that the larger M is, the faster the cumulants vanish. Therefore, designing families of M-DWPT filters that converge to the Shannon filters is of theoretical and practical interest.

Some authors present the wavelet decomposition as an alternative to the Karhunen-Loève (KL) expansion [14].

TABLE IV

KS TEST ACCEPTANCE RATES FOR THE NORMAL DISTRIBUTION $\mathcal{N}(0,1)$ OF THE DWPT COEFFICIENTS RETURNED AT RESOLUTION LEVEL $j = 6, 7$ FOR DIFFERENT DWPT PATHS AND WHEN THE DWPT IS PERFORMED WITH DAUBECHIES FILTERS WITH ORDER EQUAL TO EITHER 7 OR 20. THE DWPT INPUT PROCESS IS AR(1)-GG WITH α GIVEN BY EQ. (58), $\beta = 0.5$ AND $\mu = 0.95$.

Coloured GG process, with $\beta = 0.5$, $\mu = 0.95$			
Path	$j = 6$ $r = 7$	$j = 6$ $r = 20$	$j = 7$ $r = 20$
$\mathcal{P}_{\lambda_0}$	0%	0%	45%
$\mathcal{P}_{\lambda_1}$	78%	86%	91%
$\mathcal{P}_{\lambda_2}$	2%	30%	79%
$\mathcal{P}_{\lambda_3}$	0%	4%	35%

More precisely, [4] highlights the decorrelation that can be achieved in DWPT paths by considering the same families of decomposition filters as those used in the present work. The results given above go beyond the simple decorrelation process of the DWPT by making more precise the asymptotic distribution of the wavelet packet coefficients. This asymptotic distribution is normal with variance equal to the value taken by the input process spectrum at some specific frequency. This frequency can be computed with respect to the nested supports of the Fourier transforms of the wavelet packets associated with the chosen path. The results of this paper may thus be applicable to several signal processing fields, data analysis or communication applications.

APPENDIX A
PROOF OF LEMMA 1

According to Eqs. (5) and (13), we have

$$\begin{aligned} \mathcal{F}W_{n(j)}(\omega) &= \mathcal{F}W_{Mn(j-1)+m_j}(\omega) \\ &= H_{m_j}(\frac{\omega}{M})\,\mathcal{F}W_{n(j-1)}(\frac{\omega}{M}). \end{aligned} \tag{59}$$

An easy recurrence leads to

$$\begin{aligned} &\mathcal{F}W_{n(j)}(\omega) \\ &\quad = H_{m_j}(\frac{\omega}{M})H_{m_{j-1}}(\frac{\omega}{M^2})\ldots H_{m_2}(\frac{\omega}{M^{j-1}})\mathcal{F}W_{m_1}(\frac{\omega}{M^{j-1}}), \end{aligned}$$

since $n(1) = m_1$. It then suffices to apply Eq. (4) to conclude.

APPENDIX B
PROOF OF PROPOSITION 1

The proof is achieved by induction on j. By definition of the Shannon wavelet packet functions $W^{\mathsf{S}}_{j,n}$ (see section II-C), it follows from Eq. (6) that $W^{\mathsf{S}}_{0,0} = W_0 = \Phi^{\mathsf{S}}$, since the Shannon wavelet packet functions are obtained according to Eq. (2) with $\Phi = \Phi^{\mathsf{S}}$. It then follows from Eq. (16) that Eq. (18) holds true for $j = 0$.

According to the definition of the Shannon wavelet packet functions, it follows from Eqs. (7), (15) and the definition of $\Delta_{j,n}$ (see Eq. (19)) that

$$\mathcal{F}W^{\mathsf{S}}_{j+1,Mn+m}(\omega) = \sqrt{M} H^{\mathsf{S}}_m(M^j\omega)\mathcal{F}W^{\mathsf{S}}_{j,n}(\omega), \tag{60}$$

with

$$\begin{aligned} H^{\mathsf{S}}_m(M^j\omega) &= \sum_{\ell\in\mathbb{Z}} \mathbb{1}_{\Delta_{j+1,m}}(\omega - \frac{2\pi\ell}{M^j}), \\ &= \sum_{\ell\in\mathbb{Z}} \mathbb{1}_{\Delta_{j+1,m,\ell}}(\omega), \end{aligned} \tag{61}$$

where, for any non-negative integer j, any integer ℓ and any $m = 0, 1, \ldots, M-1$,

$$\begin{aligned} &\Delta_{j+1,m,\ell} = \\ &\quad \left[\frac{(-m-1+2M\ell)\pi}{M^{j+1}}, \frac{(-m+2M\ell)\pi}{M^{j+1}}\right] \\ &\qquad \bigcup \\ &\quad \left[\frac{(m+2M\ell)\pi}{M^{j+1}}, \frac{(m+1+2M\ell)\pi}{M^{j+1}}\right]. \end{aligned}$$

Suppose that Eq. (18) holds true at resolution level j, for $n \in \{0, 1, \ldots, M^j - 1\}$ and $p = G(n)$. We derive from Eqs. (18), (60) and (61) that

$$\mathcal{F}W^{\mathsf{S}}_{j+1,Mn+m} = M^{(j+1)/2}\mathbb{1}_{\left(\bigcup_{\ell\in\mathbb{Z}}\Delta_{j+1,m,\ell}\right)\bigcap\Delta_{j,p}}. \tag{62}$$

The following lemma gives the intersection between $\left(\bigcup_{\ell\in\mathbb{Z}}\Delta_{j+1,m,\ell}\right)$ and $\Delta_{j,p}$.

Lemma 4:

$$\left(\bigcup_{\ell\in\mathbb{Z}}\Delta_{j+1,m,\ell}\right)\bigcap\Delta_{j,p} = \begin{cases} \Delta_{j+1,Mp+m} & \text{if } p \text{ is even,} \\ \Delta_{j+1,Mp+M-m-1} & \text{if } p \text{ is odd.} \end{cases}$$

Proof: (of lemma 4). Given any subset A of $\mathbb{R}$ and any real value x, let $xA = \{xa : a \in A\}$ be the *dilate* of A by x and $A + x = \{a + x : x \in A\}$ be the *translate* of A by x.

The set $\Delta_{j,p} \cap \left(\bigcup_{\ell\in\mathbb{Z}}\Delta_{j+1,m,\ell}\right)$ is symmetrical with respect to 0 since $\Delta_{j,p}$ is itself symmetrical with respect to the origin and $\bigcup_{\ell\in\mathbb{Z}}\Delta_{j+1,m,\ell} = -\left(\bigcup_{\ell\in\mathbb{Z}}\Delta_{j+1,m,\ell}\right)$. Thus, it suffices to determine $\Delta^+_{j,p} \cap \left(\bigcup_{\ell\in\mathbb{Z}}\Delta_{j+1,m,\ell}\right)$ where $\Delta^+_{j,p} = [p\pi/M^j, (p+1)\pi/M^j]$.

Now, we have

$$\begin{aligned} &\Delta^+_{j,p} \cap \left(\bigcup_{\ell\in\mathbb{Z}}\Delta_{j+1,m,\ell}\right) \\ &\quad = \frac{\pi}{M^{j+1}}\left(J_p \cap \left(\bigcup_{\ell\in\mathbb{Z}}(I_m + 2M\ell)\right)\right), \end{aligned} \tag{63}$$

where $J_p = [pM, (p+1)M]$ and $I_m = [-m-1, -m] \cup [m, m+1]$. To compute $(J_p \cap (\cup_{\ell\in\mathbb{Z}}(I_m + 2M\ell)))$, we determine the sets $J_p \cap (I_m + 2M\ell)$ for any $\ell \in \mathbb{Z}$. Since

$$J_p \cap (I_m + 2M\ell) = ((J_p - 2M\ell) \cap I_m) + 2M\ell, \tag{64}$$

we calculate $(J_p - 2M\ell) \cap I_m$ for any $\ell \in \mathbb{Z}$, which is quite easy. In fact, since $J_p - 2M\ell = [(p-2\ell)M, (p+1-2\ell)M]$, the reader will straightforwardly verify the following facts.

If $\ell < \frac{p-1}{2}$ or $\ell > \frac{p+2}{2}$, then $(J_p - 2M\ell) \cap I_m = \emptyset$,

If p is odd and $\ell = \frac{p-1}{2}$, then

$$(J_p - 2M\ell) \cap I_m = \begin{cases} \emptyset & \text{if} \quad m < M-1, \\ \{M\} & \text{if} \quad m = M-1, \end{cases}$$

If p is even and $\ell = \frac{p}{2}$, then

$$(J_p - 2M\ell) \cap I_m = [m, m+1],$$

If p is odd and $\ell = \frac{p+1}{2}$, then

$$(J_p - 2M\ell) \cap I_m = [-m-1, -m],$$

If p is even and $\ell = \frac{p+2}{2}$, then

$$(J_p - 2M\ell) \cap I_m = \begin{cases} \emptyset & \text{if} \quad m < M-1, \\ \{-M\} & \text{if} \quad m = M-1, \end{cases}$$

We derive from these results and Eq. (64) that

$$(J_p \cap (\cup_{\ell\in\mathbb{Z}}(I_m + 2M\ell))) = \begin{cases} [m + pM, m + 1 + pM] & \text{if } p \text{ is even}, \\ [-m - 1 + pM + M, -m + pM + M] & \text{if } p \text{ is odd}. \end{cases}$$

It then suffices to apply Eq. (63) to complete the proof. ∎

Since $p = G(n)$, we obtain

$$\mathcal{F}W^{\mathsf{S}}_{j+1,Mn+m} = M^{(j+1)/2}\mathbf{1}_{\Delta_{j+1,G(Mn+m)}}$$

where $G(Mn+m)$ is given by Eq. (20). Therefore, Eq. (18) holds true at resolution level $j+1$ and the proof is complete.

APPENDIX C
PROOF OF OF LEMMA 2

Consider an M-ary sequence $\lambda = (m_k)_{k\in\mathbb{N}}$ of elements of $\{0, 1, \ldots, M-1\}$ and let $(\mathbf{W}^{\mathsf{S}}_{j,n_\lambda(j)})_{j\in\mathbb{N}}$ be the sequence of wavelet packets associated with λ. The sequence of autocorrelation functions resulting from the projection of X on $(\mathbf{W}^{\mathsf{S}}_{j,n_\lambda(j)})_{j\in\mathbb{N}}$ is $(R^{\mathsf{S}}_{j,n_\lambda(j)})_{j\in\mathbb{N}}$, where $R^{\mathsf{S}}_{j,n_\lambda(j)}$ is given by Eq. (33).

Now, if γ is continuous at $a(\lambda)$, then for every real number $\eta > 0$, there exists a real number $\alpha > 0$ such that, for every $\omega \in [a(\lambda) - \alpha, a(\lambda) + \alpha]$, we have $|\gamma(\omega) - \gamma(a(\lambda))| < \eta$. In addition, since

$$\lim_{j \to +\infty} \frac{p_\lambda(j)\pi}{M^j} = \lim_{j \to +\infty} \frac{(p_\lambda(j)+1)\pi}{M^j} = a(\lambda),$$

there exists an integer $j_0 = j_0(\alpha)$, such that, for every natural number $j \geqslant j_0$, the values $p_\lambda(j)\pi/M^j$ and $(p_\lambda(j)+1)\pi/M^j$ are within the interval $[a(\lambda) - \alpha, a(\lambda) + \alpha]$. It follows that, for every natural number $j \geqslant j_0$ and every $\omega \in \Delta^+_{j,p_\lambda(j)}$,

$$|\gamma(\omega) - \gamma(a(\lambda))| < \eta.$$

Therefore, for any natural number $j \geqslant j_0$

$$\begin{aligned} &\frac{M^j}{\pi} \int_{\Delta^+_{j,p_\lambda(j)}} |\gamma(\omega) - \gamma(a(\lambda))| \, d\omega \\ &\qquad < \eta \frac{M^j}{\pi} \int_{\Delta^+_{j,p_\lambda(j)}} d\omega = \eta. \end{aligned} \tag{65}$$

On the other hand, it follows from Eq. (33) that for any natural number $j \geqslant j_0$ and every integer k,

$$\begin{aligned} &\left| R^{\mathsf{S}}_{j,n_\lambda(j)}[k] - \frac{M^j}{\pi} \int_{\Delta^+_{j,p_\lambda(j)}} \gamma(a(\lambda)) \cos\left(M^j k\omega\right) d\omega \right| \\ &\quad = \left| \frac{M^j}{\pi} \int_{\Delta^+_{j,p_\lambda(j)}} (\gamma(\omega) - \gamma(a(\lambda))) \cos\left(M^j k\omega\right) d\omega \right|, \\ &\quad \leq \frac{M^j}{\pi} \int_{\Delta^+_{j,p_\lambda(j)}} |\gamma(\omega) - \gamma(a(\lambda))| \, d\omega. \end{aligned} \tag{66}$$

Hence, we derive from Eqs. (65) and (66) that, for every natural number $j \geqslant j_0$,

$$|R^{\mathsf{S}}_{j,n_\lambda(j)}[k] - \frac{M^j}{\pi} \int_{\Delta^+_{j,p_\lambda(j)}} \gamma(a(\lambda)) \cos\left(M^j k\omega\right) d\omega| < \eta$$

uniformly in $k \in \mathbb{Z}$. Since

$$\frac{M^j}{\pi} \int_{\Delta^+_{j,p_\lambda(j)}} \gamma(a(\lambda)) \cos\left(M^j k\omega\right) d\omega = \gamma(a(\lambda))\delta[k],$$

we conclude that, for every natural number $j \geqslant j_0$,

$$\left| R^{\mathsf{S}}_{j,n_\lambda(j)}[k] - \gamma(a(\lambda))\delta[k] \right| < \eta$$

uniformly in $k \in \mathbb{Z}$.

APPENDIX D
PROOF OF EQ. (48)

From (46), and by taking into account that $\mathsf{cum}(t_1, t_2, \cdots, t_N)$ is the inverse Fourier transform of $\gamma(\omega_1, \omega_2, \cdots, \omega_N)$, we have

$$\begin{aligned}
&\mathsf{cum}_{j,n}[k, \ell_1, \ell_2, \cdots, \ell_N] \\
&\quad = \frac{1}{(2\pi)^N} \int_{\mathbb{R}} dt W_{j,n,k}(t) \int_{\mathbb{R}^N} dt_1 dt_2 \cdots dt_N \\
&\quad\quad W_{j,n,\ell_1}(t+t_1) W_{j,n,\ell_2}(t+t_2) \cdots W_{j,n,\ell_N}(t+t_N) \\
&\quad\quad \int_{\mathbb{R}^N} d\omega_1 d\omega_2 \cdots d\omega_N \exp\left(-i(t_1\omega_1 + t_2\omega_2 + \cdots + t_N\omega_N)\right) \\
&\quad\quad \gamma_N(-\omega_1, -\omega_2, \cdots, -\omega_N).
\end{aligned}$$

According to Fubini's theorem,

$$\begin{aligned}
&\mathsf{cum}_{j,n}[k, \ell_1, \ell_2, \cdots, \ell_N] \\
&\quad = \frac{1}{(2\pi)^N} \int_{\mathbb{R}} dt W_{j,n,k}(t) \int_{\mathbb{R}^N} d\omega_1 d\omega_2 \cdots d\omega_N \\
&\quad\quad \gamma_N(-\omega_1, -\omega_2, \cdots, -\omega_N) \int_{\mathbb{R}^N} dt_1 dt_2 \cdots dt_N \\
&\quad\quad \exp\left(-i(t_1\omega_1 + t_2\omega_2 + \cdots + t_N\omega_N)\right) W_{j,n,\ell_1}(t+t_1) \\
&\quad\quad W_{j,n,\ell_2}(t+t_2) \cdots W_{j,n,\ell_N}(t+t_N),
\end{aligned}$$

and then,

$$\begin{aligned}
&\mathsf{cum}_{j,n}[k, \ell_1, \ell_2, \cdots, \ell_N] \\
&\quad = \frac{1}{(2\pi)^N} \int_{\mathbb{R}} dt W_{j,n,k}(t) \int_{\mathbb{R}^N} d\omega_1 d\omega_2 \cdots d\omega_N \\
&\quad\quad \exp\left(it(\omega_1 + \omega_2 + \cdots + \omega_N)\right) \gamma_N(-\omega_1, -\omega_2, \cdots, -\omega_N) \\
&\quad\quad \mathcal{F}W_{j,n,\ell_1}(\omega_1) \mathcal{F}W_{j,n,\ell_2}(\omega_2) \cdots \mathcal{F}W_{j,n,\ell_N}(\omega_N).
\end{aligned}$$

This equation can also be re-written, by using Fubini's theorem again,

$$\begin{aligned}
&\mathsf{cum}_{j,n}[k, \ell_1, \ell_2, \cdots, \ell_N] \\
&\quad = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} d\omega_1 d\omega_2 \cdots d\omega_N \gamma_N(-\omega_1, -\omega_2, \cdots, -\omega_N) \\
&\quad\quad\quad \mathcal{F}W_{j,n,\ell_1}(\omega_1) \mathcal{F}W_{j,n,\ell_2}(\omega_2) \cdots \mathcal{F}W_{j,n,\ell_N}(\omega_N) \\
&\quad\quad\quad \left[\int_{\mathbb{R}} dt W_{j,n,k}(t) \exp\left(it(\omega_1 + \omega_2 + \cdots + \omega_N)\right)\right],
\end{aligned}$$

which leads to Eq. (48) by definition of the Fourier transform.

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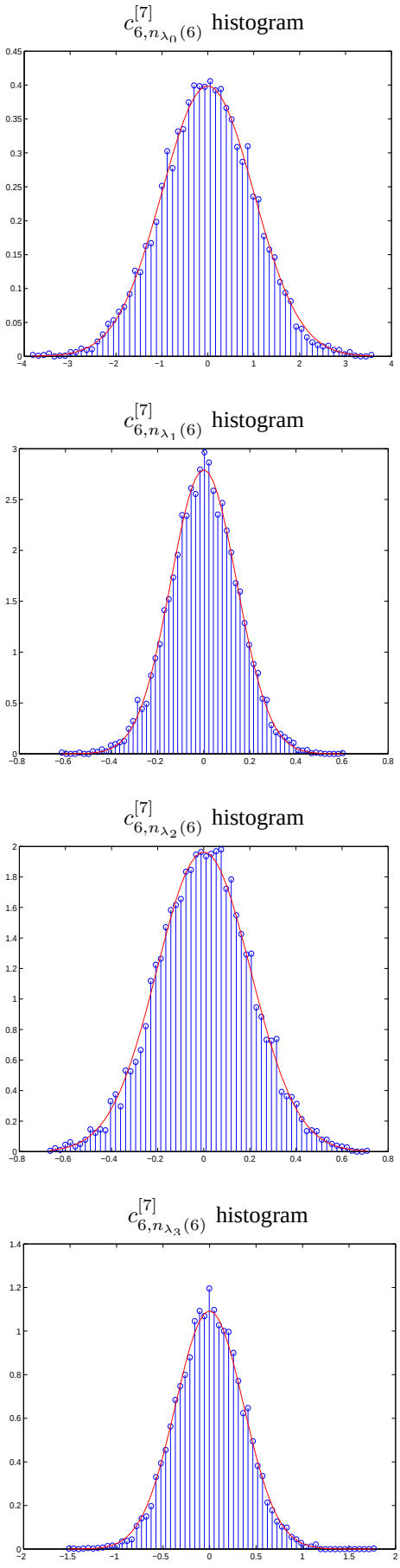

Fig. 7. Histograms of the DWPT coefficients, at resolution level 6, using Daubechies filters of order 7. The decomposition is applied to samples of an AR(1)-GG process with α given by Eq. (58), $\beta = 1$ and $\mu = 0.75$. The limit distribution $\mathcal{N}(0, \gamma(a(\lambda)))$ where $a(\lambda_0) = 0$ and $a(\lambda_q) = \pi/2^{q-1}$ for $q = 1, 2$ and 3 is represented by the (red) continuous line.